Andy Abel Celebration Conference

October 14, 2022

Andy teaching in 1983

Spatial Unit Roots

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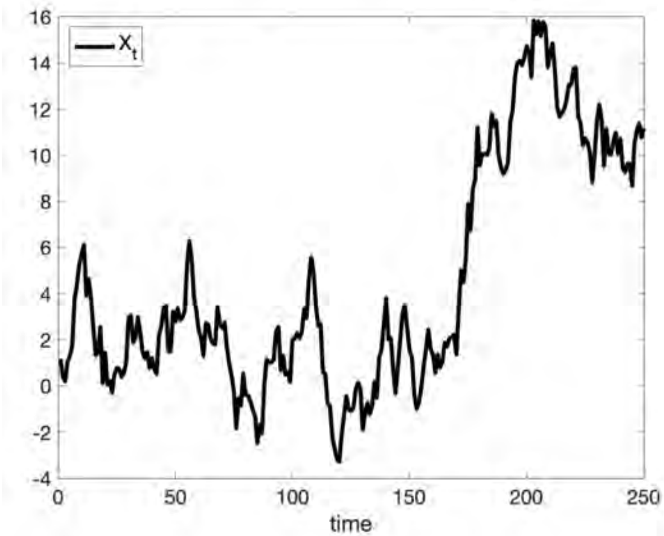
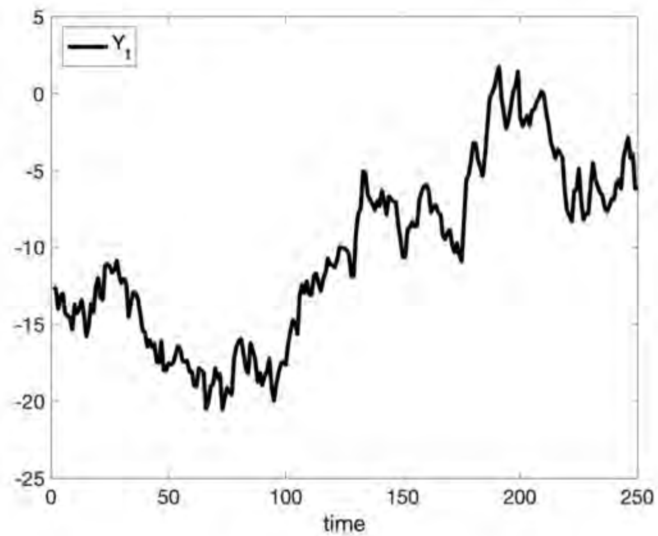
Princeton University

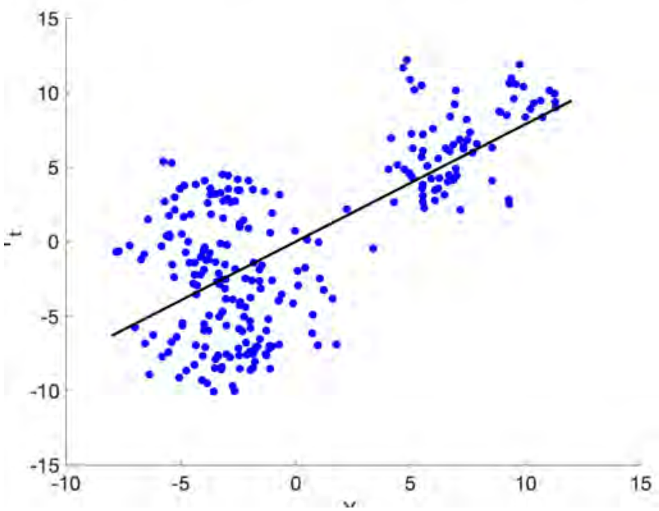
October 14, 2022

Andy Abel Celebration Conference

Time Series: Two independent random walks



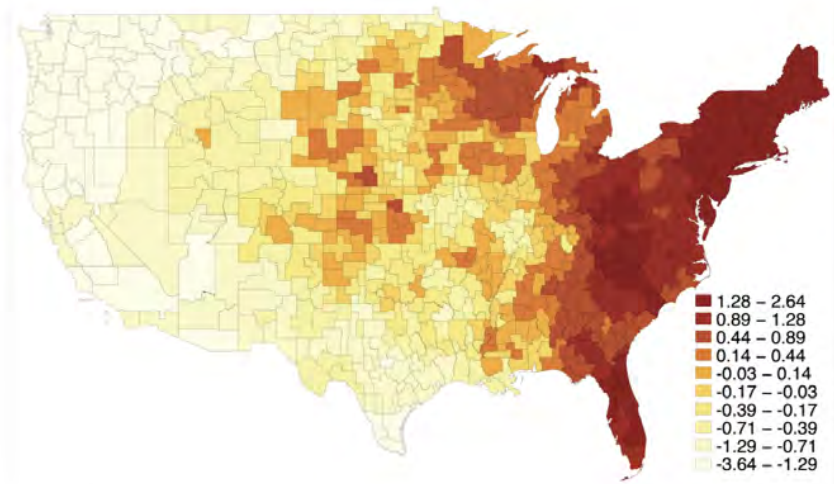


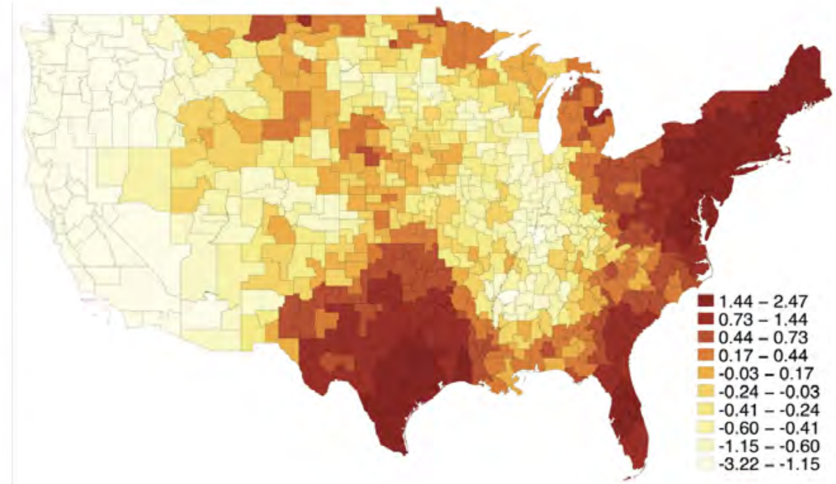


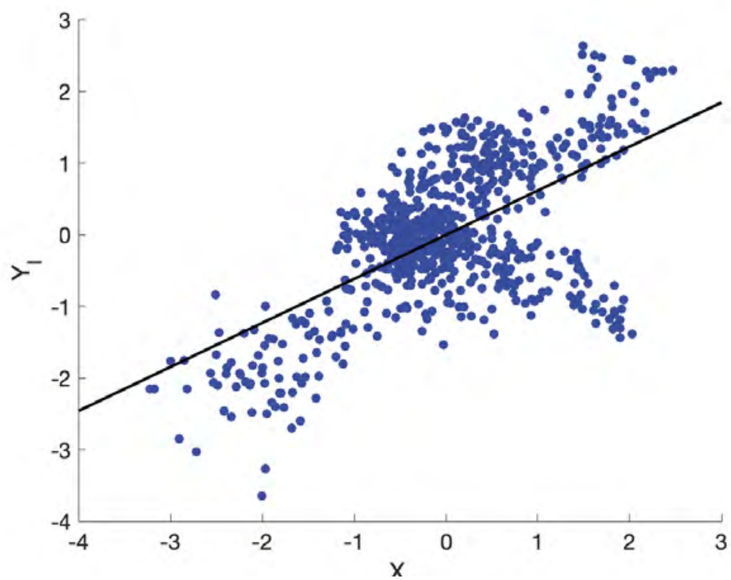
$\hat{\beta} = 0.79$
(0.04) (HR)
[0.09] (Newey-West)

$R^2 = 0.48$

US Commuting Zones: Two independent spatial 'random walks'



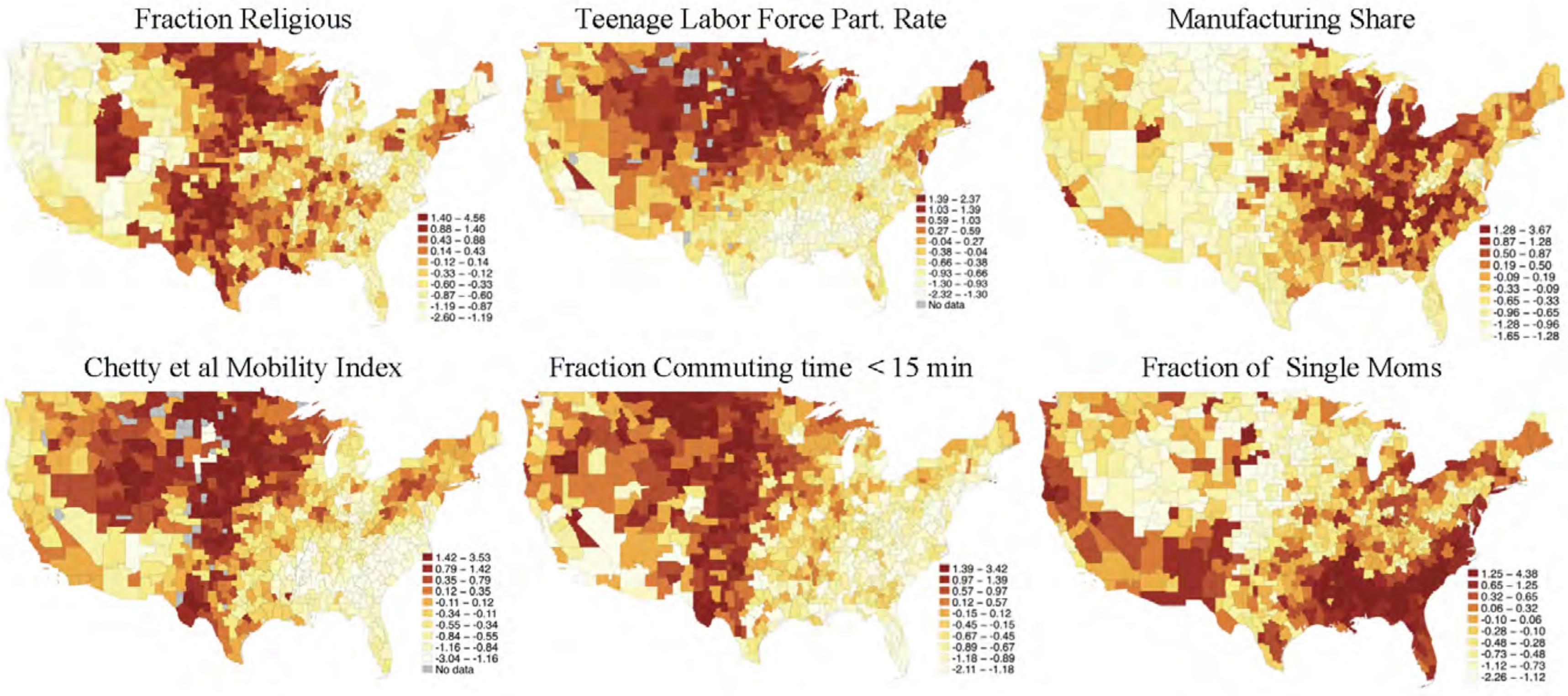




$\hat{\beta} = 0.62\ (0.04)$

$R^2 = 0.38$

US Commuting Zones: Are these variables spatial random walks?



Data from Chetty, Hendren, Kline and Saez (2014), 'Land of Opportunity'

Some Questions:

1. What is a spatial random walk?

   (a) What is a spatial $I(1)$ process?

2. Do spatial $I(1)$ processes lead to spurious regressions?

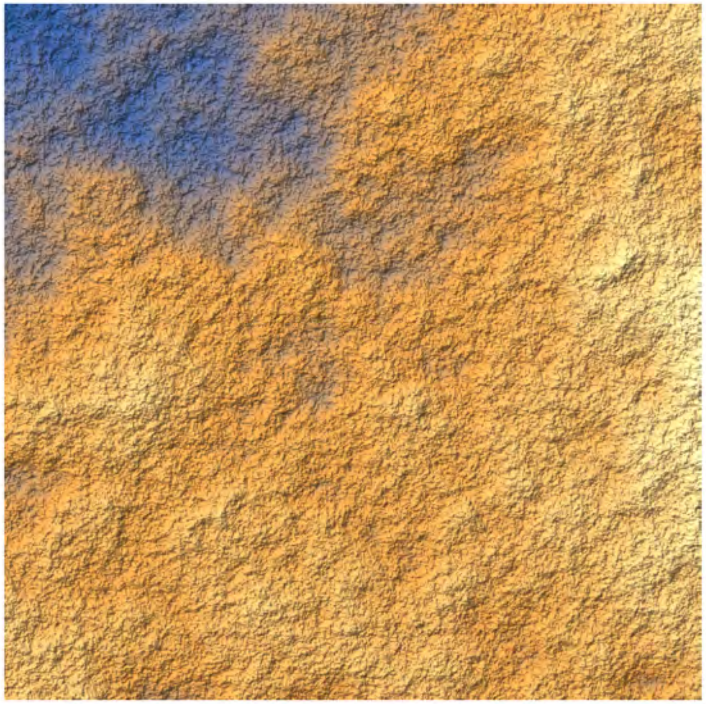
3. Can you test for spatial 'unit roots' (i.e., $I(1)$ processes)? How?

4. Can you eliminate $I(1)$ spatial persistence by 'differencing' the data? How?

5. Is there a large-sample theory that helps answer these questions?

Question 1: What is a spatial random walk (Brownian motion)?

- Location $s \in \mathbb{R}^d$ ($d = 1$ time series, $d = 2$ geography)
- Levy Brownian motion, $L(s)$
  - $\mathbb{E}[L(s)L(r)] = \frac{1}{2}(|s| + |r| - |s - r|)$
    - $\mathrm{var}[L(s) - L(r)] = |s - r|$, etc.

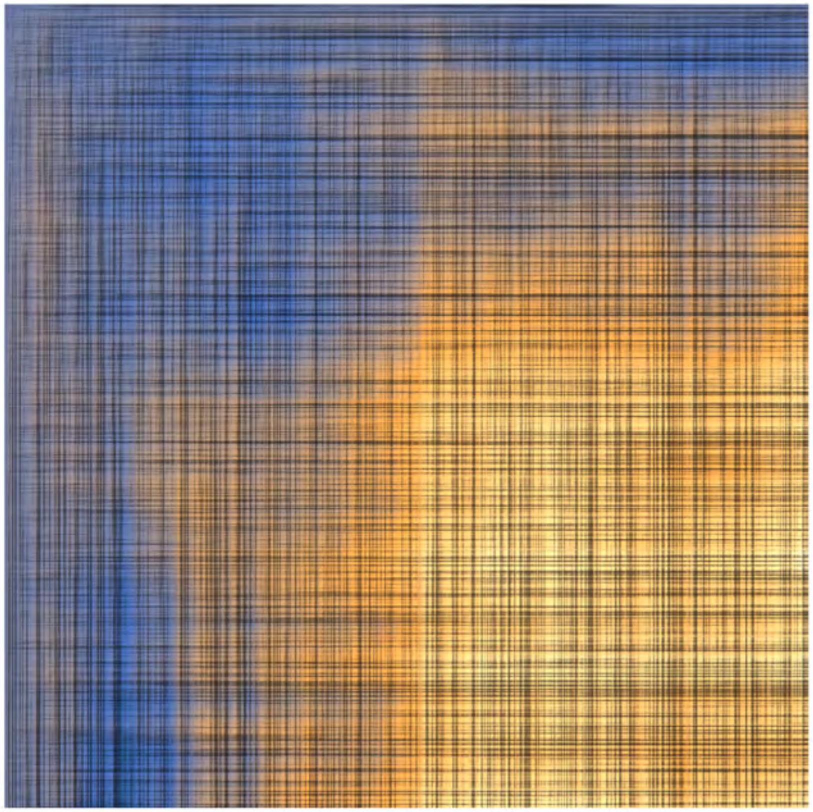


Realization of Levy Brownian Motion, $d = 2$

Notes: Rotation invariant; $d = 1$ Brownian motion along any lines

Question 1 (continued): What is a spatial random walk (Brownian motion)?

- Alternative: Brownian sheet
  - $Y(s)$, $s \geq 0$, $\mathbb{E}[Y(s)Y(r)] = \prod_{i=1}^{d} \min(s_i, r_i)$,



Realization of Brownian Sheet

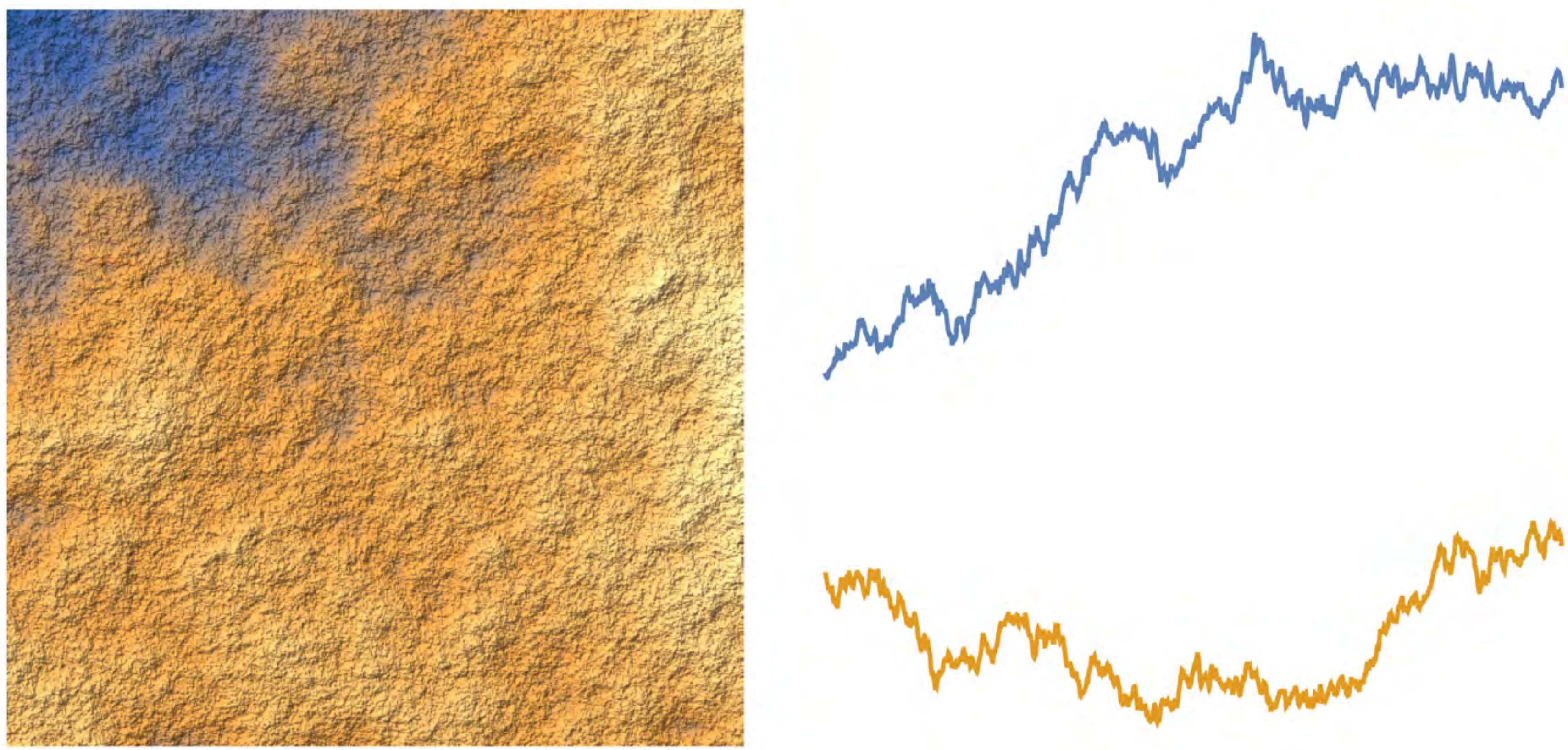
Note: This is a Brownian motion in vertical and horizontal directions. Not otherwise.

Question 1(a) : What is a spatial I(1) process?

- $d > 1$, Spatial $I(1)$ process:
  - Levy Browning motion:
$$L(s) = \int_{\mathbb{R}^d} \theta(s,u) dW(u)$$
    ... (with $\theta(s,u) \propto (|s-u|^{(1-d)/2} - |u|^{1-d)/2})$
  - $I(1)$ process :
$$Y(s) = \int_{\mathbb{R}^d} \theta(s,u) B(u) du$$
    where $B(u)$ is a 'weakly dependent' covariance stationary mean zero process (Condition 1 in paper).

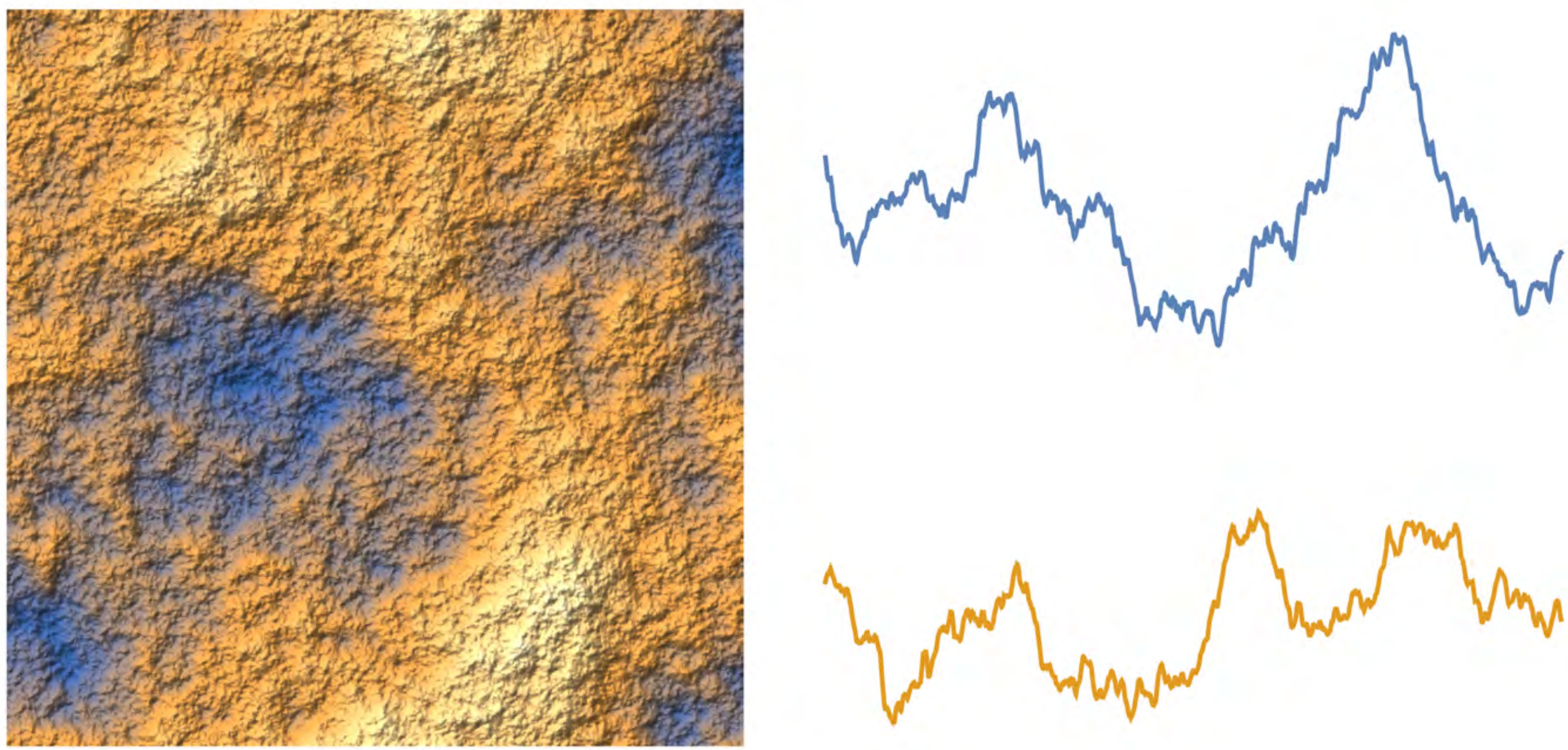
Question 1(a) continued : What is a spatial I(1) process?

- Examples:



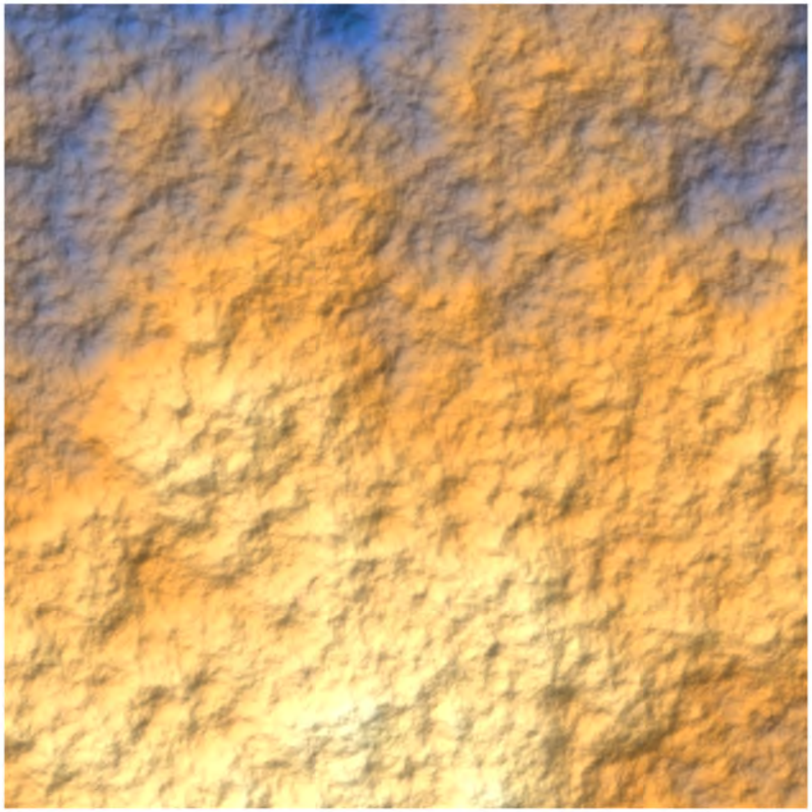
Realization of Levy Brownian Motion

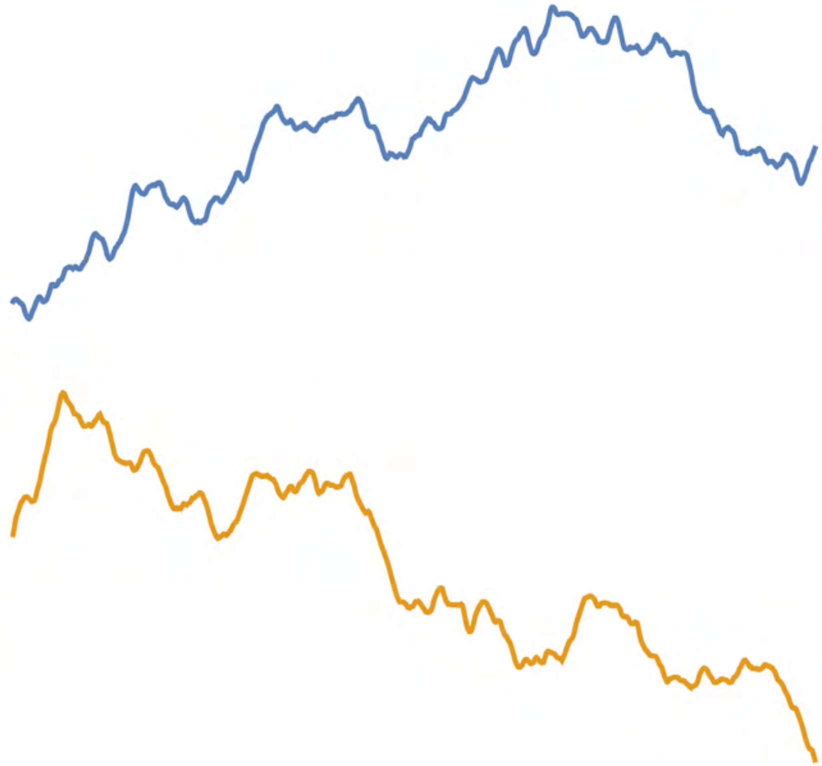
Question 1(a) continued : What is a spatial I(1) process?



I(1) with $B \sim G_{exp}(c_1)$

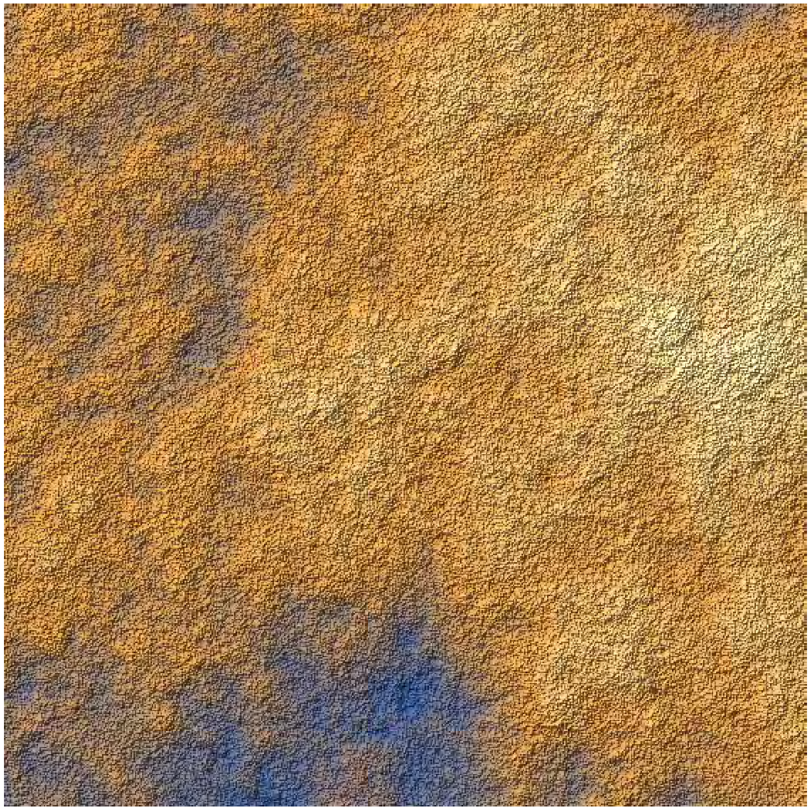
Question 1(a) continued : What is a spatial I(1) process?

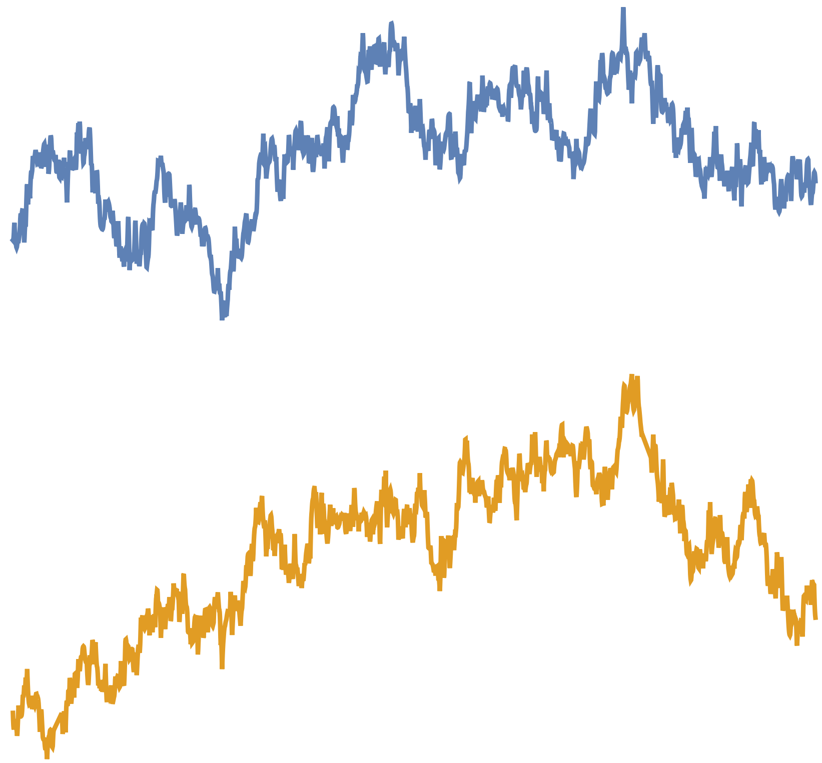




I(1) with $B \sim G_{exp}(c_{(smaller)})$

Question 1(a) continued : What is a spatial I(1) process?





I(1) with $B \sim$ '$ARMA(2,1)$'

Some Questions:

1. ~~What is a spatial random walk?~~
   (a) ~~What is a spatial $I(1)$ process?~~
2. Do spatial $I(1)$ processes lead to spurious regressions?
3. Can you test for spatial 'unit roots' (i.e., $I(1)$ processes)? How?
4. Can you eliminate $I(1)$ spatial persistence by 'differencing' the data? How?
5. **Is there a large-sample theory that helps answer these questions?**

Question 5 : Is there a large-sample theory that helps answer these questions?

- Answer: Yes ... (FCLT) ... Thm 2 in paper

$$\text{If } \lambda_n \to \infty, \text{ then } \lambda_n^{-1/2} Y_n(\cdot) \Rightarrow \omega L(\cdot) \text{ where } \omega^2 = \int_{\mathbb{R}^d} \sigma_B(r) dr$$

Some Questions:

1. ~~What is a spatial random walk?~~
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5. ~~Is there a large-sample theory that helps answer these questions?~~

Question 2 : Do Spatial $I(1)$ processes lead to spurious regressions?

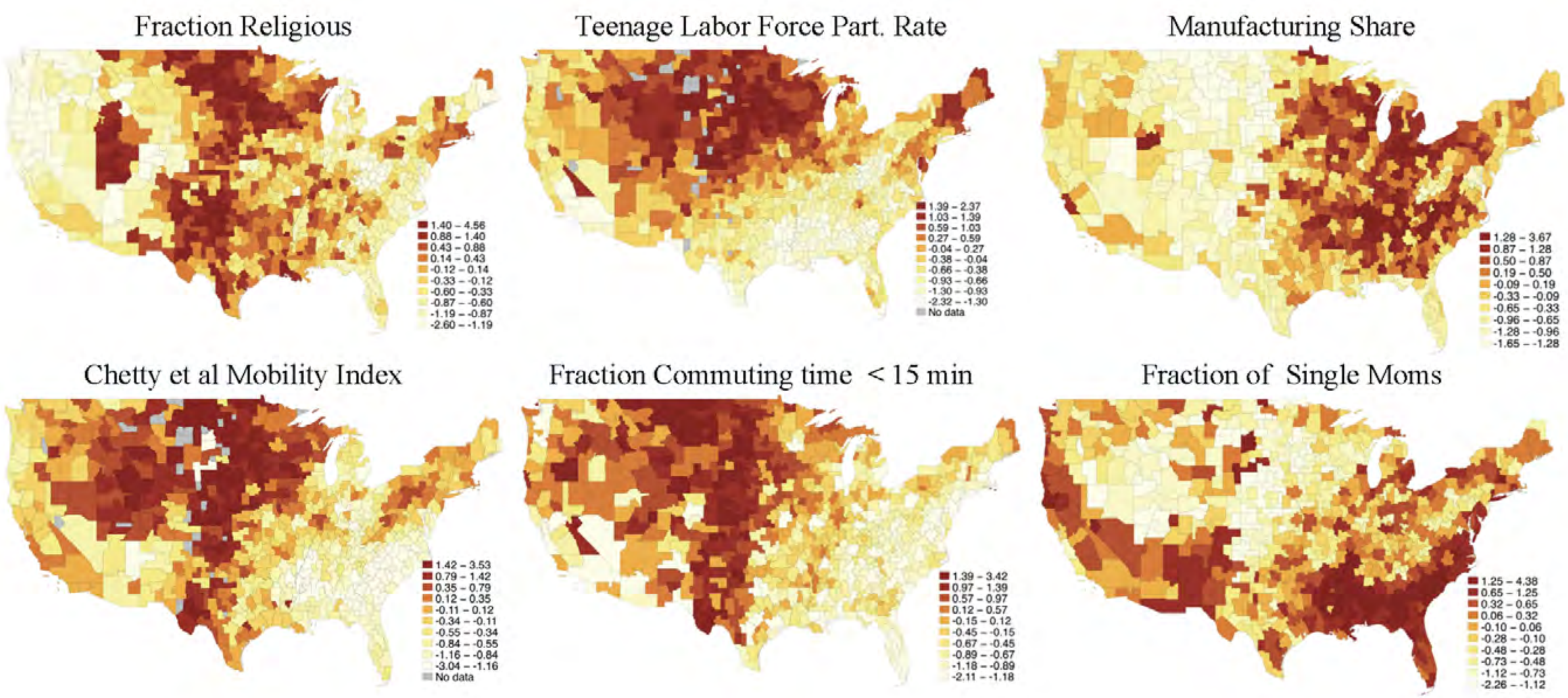
- Answer: Yes ... Thm 3 (like Phillips (1986) time series results) and Thm 4 (with HAC) in paper. Consider

$$y_l = \alpha + \mathbf{x}_l'\beta + u_l$$

  - $\hat{\beta} \Rightarrow RV$
  - $R^2 \Rightarrow RV$
  - $F \rightarrow \infty$
  - $F(HAC) \rightarrow \infty$

Question 2 continued : Spurious Regressions

- Examples (???): Chetty et al 2014.
  - Construct a commuter-zone (CZ) level index of intergenerational mobility (AMI).
  - Regress AMI on various CZ socio-economic variables.



Question 2 continued :

Variable	Spatial Persistence Statistics			Regression of the AMI onto Variable $\hat{\beta}$ [95% CI]	
	p-Value for Test		95% CI for $\bar{\rho}$		
	$I(1)$ Null	$I(0)$ Null		Level [Cluster]	LBM-GLS [CSCPC]
Absolute Mobility Index	0.08	<0.01	[0.14; 1.00]		
Frac. Black Residents	0.02	0.01	[0.02; 0.71]	-0.58 [-0.71; -0.45]	-0.42 [-0.50; -0.34]
Racial Segregation	0.07	0.02	[0.05; 1.00]	-0.36 [-0.45; -0.27]	-0.24 [-0.28; -0.19]
Segregation of Poverty	0.13	0.04	[0.05; 1.00]	-0.41 [-0.54; -0.28]	-0.21 [-0.25; -0.16]
Frac. < 15 Mins to Work	0.69	<0.01	[0.46; 1.00]	0.61 [0.36; 0.85]	0.37 [0.26; 0.48]
Mean Household Income	0.02	0.18	[0.01; 0.61]	0.05 [-0.09; 0.19]	-0.02 [-0.08; 0.04]
Gini	0.56	<0.01	[0.40; 1.00]	-0.58 [-0.76; -0.40]	-0.21 [-0.29; -0.14]
Top 1 Perc. Inc. Share	0.60	0.03	[0.43; 1.00]	-0.19 [-0.33; -0.05]	-0.06 [-0.11; -0.01]
Student-Teacher Ratio	0.03	0.16	[0.04; 0.87]	-0.33 [-0.52; -0.13]	-0.18 [-0.26; -0.09]
Test Scores (Inc. adjusted)	0.40	0.07	[0.27; 1.00]	0.59 [0.42; 0.76]	0.42 [0.34; 0.51]
High School Dropout	0.63	0.02	[0.40; 1.00]	-0.57 [-0.75; -0.40]	-0.31 [-0.42; -0.20]
Social Capital Index	0.73	<0.01	[0.38; 1.00]	0.64 [0.46; 0.82]	0.28 [0.12; 0.44]
Frac. Religious	0.11	0.03	[0.15; 1.00]	0.52 [0.35; 0.69]	0.32 [0.19; 0.45]
Violent Crime Rate	0.52	0.04	[0.38; 1.00]	-0.38 [-0.67; -0.09]	-0.14 [-0.23; -0.06]
Frac. Single Mothers	0.03	<0.01	[0.05; 0.88]	-0.76 [-0.91; -0.62]	-0.60 [-0.69; -0.51]
Divorce Rate	<0.01	0.21	[0.02; 0.53]	-0.49 [-0.68; -0.29]	-0.38 [-0.49; -0.27]
Frac. Married	0.09	0.07	[0.12; 1.00]	0.57 [0.45; 0.69]	0.36 [0.29; 0.43]
Local Tax Rate	0.01	0.25	[0.01; 0.59]	0.32 [0.19; 0.46]	0.07 [0.01; 0.14]
Colleges per Capita	0.57	0.10	[0.00; 1.00]	0.20 [-0.02; 0.42]	0.02 [-0.08; 0.11]
College Tuition	0.21	<0.01	[0.15; 1.00]	-0.02 [-0.15; 0.11]	0.01 [-0.02; 0.04]
Coll. Grad. Rate (Inc. Adjusted)	0.46	0.01	[0.34; 1.00]	0.15 [0.03; 0.28]	0.08 [0.01; 0.15]
Manufacturing Share	0.04	<0.01	[0.10; 1.00]	-0.26 [-0.44; -0.08]	0.06 [-0.03; 0.16]
Chinese Import Growth	0.02	0.07	[0.01; 0.58]	-0.17 [-0.33; -0.02]	0.03 [0.01; 0.04]
Teenage LFP Rate	0.28	<0.01	[0.20; 1.00]	0.63 [0.46; 0.80]	0.25 [0.14; 0.36]
Migration Inflow	0.06	0.11	[0.00; 1.00]	-0.26 [-0.40; -0.11]	-0.13 [-0.18; -0.08]
Migration Outlflow	0.05	0.02	[0.07; 1.00]	-0.16 [-0.30; -0.03]	-0.09 [-0.15; -0.03]
Frac. Foreign Born	0.44	0.02	[0.35; 1.00]	-0.03 [-0.15; 0.10]	-0.12 [-0.24; -0.00]

Some Questions:

1. ~~What is a spatial random walk?~~

    (a) ~~What is a spatial $I(1)$ process?~~

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3. **Can you test for spatial 'unit roots' (i.e., $I(1)$ processes)? How?**

4. Can you eliminate $I(1)$ spatial persistence by 'differencing' the data? How?

5. ~~Is there a large-sample theory that helps answer these questions?~~

Question 3 : Can you test for spatial 'unit roots'?

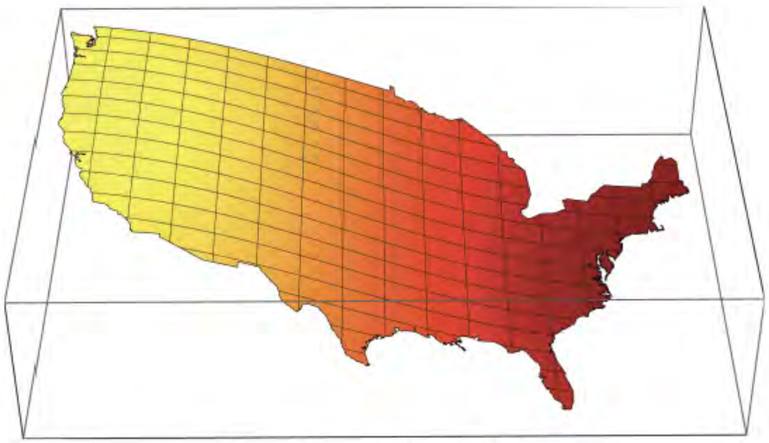
- $d = 1$: $I(1)$ yields unit root in AR representation for process. Dickey-Fuller or related tests.

- $d > 1$: No analogue of AR representation ... whoops ... 'spatial unit root' doesn't make sense (in our context).
  - Alternative approach:
    - Use (population) principal components using eigenvectors from Levy process covariance matrix.
    - Under $I(1)$ model the variance of the PCs decreases sharply. Look for this pattern in data.
      - $d = 1$: variance of j^{th}-PC from detrended random walk has var$(PC_j) \propto 1/j^2$.

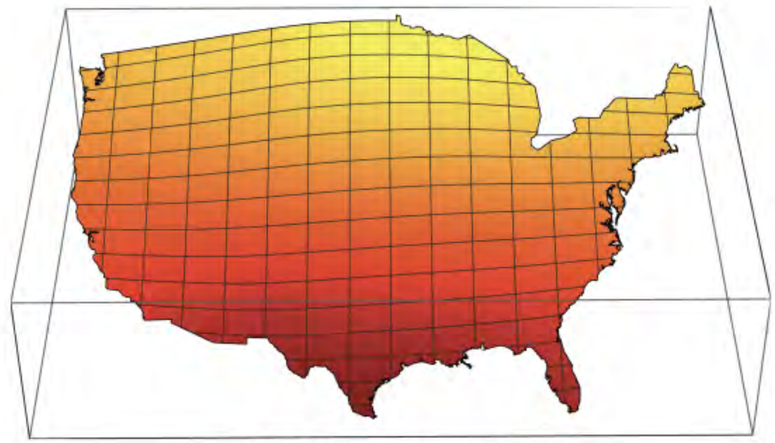
Question 3 continued : Details:

- Let $\widetilde{\Sigma}_L$ denote $n \times n$ covariance matrix for demeaned Levy-BM evaluated at spatial locations $\{s_l\}_{l=1}^n$.
- Let $\mathbf{R}$ denote eigenvectors corresponding to large q eigenvalues.
- $\mathbf{Y}_n$ is $n \times 1$ vector of raw data. $\mathbf{Z}_n = \mathbf{R}_n' \mathbf{Y}_n$ are the q PCs (under Levy-BM).

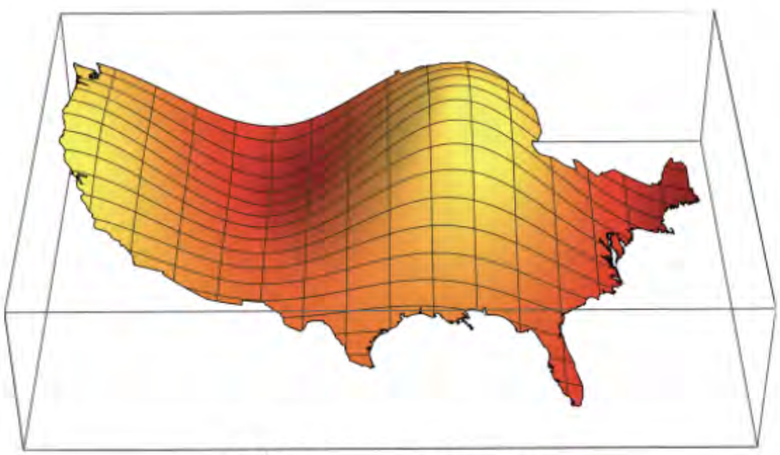
Question 3 continued : Eigenvector weights for PCs, $s \sim Uniform$ over Continental US



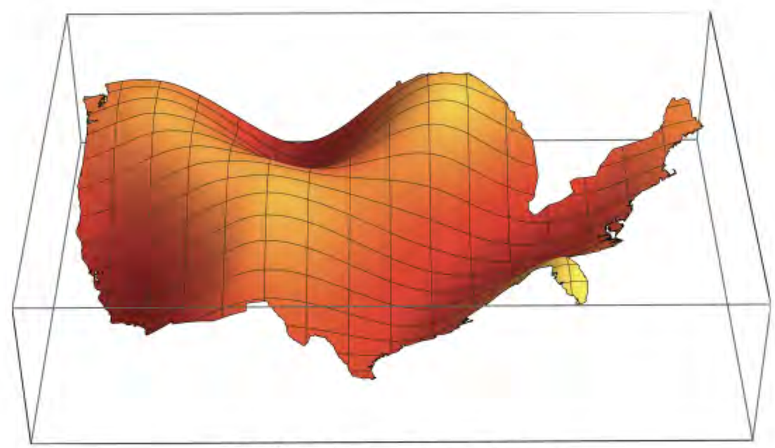
1st Eigenfunction



2nd Eigenfunction



5th Eigenfunction



10th Eigenfunction

Question 3 continued : Details:

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- $\mathbf{Y}_n$ is $n \times 1$ vector or raw data. $\mathbf{Z}_n = \mathbf{R}_n' \mathbf{Y}_n$ are the q PCs (under Levy-BM).
- Null and Alternative:

$$(Y \sim I(1)) \qquad H_0\text{: } \mathbf{Z}_n \sim N(0, \Omega_L) \qquad \text{(Large-sample approximation from FCLT)}$$

$$(Y \sim \mathcal{G}_{exp}(c)) \qquad H_a\text{: } \mathbf{Z}_n \sim N(0, \Omega_{\mathcal{G}}(c)) \qquad \text{(Large-sample approximation from FCLT)}$$

- Details
  - Choice of q, c (see paper)
  - Testing problem is straightforward

Question 3 continued :

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Question 3 continued : Testing for $I(0)$ null and forming confidence interval for c when $Y \sim G_{exp}(c)$

- $I(1)$ Null and Alternative:

$$(Y \sim I(1)) \qquad H_0\colon\ \mathbf{Z}_n \sim N(0, \Omega_L)$$

$$(Y \sim \mathcal{G}_{exp}(c)) \qquad H_a\colon\ \mathbf{Z}_n \sim N(0, \Omega_{\mathcal{G}}(c))$$

- **$I(0)$ Null and Alternative:**

$$(Y \sim I(0)) \qquad H_0\colon\ \mathbf{Z}_n \sim N(0, \Omega_{\mathcal{G}}(c_{large}))$$

$$(Y \sim I(0) + I(1)) \qquad H_a\colon\ \mathbf{Z}_n \sim N(0, \Omega_{\mathcal{G}}(c_{large}) + g_a^2 \Omega_L) \qquad \text{(Large-sample approximation from FCLT)}$$

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High School Dropout	0.63	0.02	[0.40; 1.00]		-0.57 [-0.75; -0.40]	-0.31 [-0.42; -0.20]
Social Capital Index	0.73	<0.01	[0.38; 1.00]		0.64 [0.46; 0.82]	0.28 [0.12; 0.44]
Frac. Religious	0.11	0.03	[0.15; 1.00]		0.52 [0.35; 0.69]	0.32 [0.19; 0.45]
Violent Crime Rate	0.52	0.04	[0.38; 1.00]		-0.38 [-0.67; -0.09]	-0.14 [-0.23; -0.06]
Frac. Single Mothers	0.03	<0.01	[0.05; 0.88]		-0.76 [-0.91; -0.62]	-0.60 [-0.69; -0.51]
Divorce Rate	<0.01	0.21	[0.02; 0.53]		-0.49 [-0.68; -0.29]	-0.38 [-0.49; -0.27]
Frac. Married	0.09	0.07	[0.12; 1.00]		0.57 [0.45; 0.69]	0.36 [0.29; 0.43]
Local Tax Rate	0.01	0.25	[0.01; 0.59]		0.32 [0.19; 0.46]	0.07 [0.01; 0.14]
Colleges per Capita	0.57	0.10	[0.00; 1.00]		0.20 [-0.02; 0.42]	0.02 [-0.08; 0.11]
College Tuition	0.21	<0.01	[0.15; 1.00]		-0.02 [-0.15; 0.11]	0.01 [-0.02; 0.04]
Coll. Grad. Rate (Inc. Adjusted)	0.46	0.01	[0.34; 1.00]		0.15 [0.03; 0.28]	0.08 [0.01; 0.15]
Manufacturing Share	0.04	<0.01	[0.10; 1.00]		-0.26 [-0.44; -0.08]	0.06 [-0.03; 0.16]
Chinese Import Growth	0.02	0.07	[0.01; 0.58]		-0.17 [-0.33; -0.02]	0.03 [0.01; 0.04]
Teenage LFP Rate	0.28	<0.01	[0.20; 1.00]		0.63 [0.46; 0.80]	0.25 [0.14; 0.36]
Migration Inflow	0.06	0.11	[0.00; 1.00]		-0.26 [-0.40; -0.11]	-0.13 [-0.18; -0.08]
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Frac. Foreign Born	0.44	0.02	[0.35; 1.00]		-0.03 [-0.15; 0.10]	-0.12 [-0.24; -0.00]

- $I(1)$ Null and Alternative:

$$(Y \sim I(1)) \qquad H_0\colon\ \mathbf{Z}_n \sim N(0, \Omega_L)$$

$$(Y \sim \mathcal{G}_{exp}(c)) \qquad H_a\colon\ \mathbf{Z}_n \sim N(0, \Omega_{\mathcal{G}}(c))$$

- $I(0)$ Null and Alternative:

$$(Y \sim I(0)) \qquad H_0\colon\ \mathbf{Z}_n \sim N(0, \Omega_{\mathcal{G}}(c_{large}))$$

$$(Y \sim I(0) + I(1)) \qquad H_a\colon\ \mathbf{Z}_n \sim N(0, \Omega_{\mathcal{G}}(c_{large}) + g_a^2 \Omega_L) \qquad \text{(Large-sample approximation from FCLT)}$$

- Note
  - Can use $I(0)$ test to form confidence set for spatial 'cointegrating coefficients': $Y_l - \beta X_l \sim I(0)$.

- $I(1)$ Null and Alternative:

$$(Y \sim I(1)) \qquad H_0\colon \mathbf{Z}_n \sim N(0, \Omega_L)$$

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- Note
  - Can use $I(0)$ test to form confidence set for spatial 'cointegrating coefficients': $Y_l - \beta X_l \sim I(0)$.

- **$I(c)$ Null and Alternative for forming confidence interval for c**

$$(Y \sim \mathcal{G}_{exp}(c_0)) \qquad H_0\colon \mathbf{Z}_n \sim N(0, \Omega_{\mathcal{G}}(c_0))$$

$$(Y \sim \text{mixture of } \mathcal{G}_{exp}(c) \text{ processes}) \qquad H_a\colon \mathbf{Z}_n \sim N(0, \Omega_{\mathcal{G}}(c)) \text{ with } c \sim f$$

Question 3 continued :

Variable	Spatial Persistence Statistics			Regression of the AMI onto Variable $\hat{\beta}$ [95% CI]	
	p-Value for Test		95% CI for $\bar{\rho}$		
	$I(1)$ Null	$I(0)$ Null		Level [Cluster]	LBM-GLS [CSCPC]
Absolute Mobility Index	0.08	<0.01	[0.14; 1.00]		
Frac. Black Residents	0.02	0.01	[0.02; 0.71]	-0.58 [-0.71; -0.45]	-0.42 [-0.50; -0.34]
Racial Segregation	0.07	0.02	[0.05; 1.00]	-0.36 [-0.45; -0.27]	-0.24 [-0.28; -0.19]
Segregation of Poverty	0.13	0.04	[0.05; 1.00]	-0.41 [-0.54; -0.28]	-0.21 [-0.25; -0.16]
Frac. < 15 Mins to Work	0.69	<0.01	[0.46; 1.00]	0.61 [0.36; 0.85]	0.37 [0.26; 0.48]
Mean Household Income	0.02	0.18	[0.01; 0.61]	0.05 [-0.09; 0.19]	-0.02 [-0.08; 0.04]
Gini	0.56	<0.01	[0.40; 1.00]	-0.58 [-0.76; -0.40]	-0.21 [-0.29; -0.14]
Top 1 Perc. Inc. Share	0.60	0.03	[0.43; 1.00]	-0.19 [-0.33; -0.05]	-0.06 [-0.11; -0.01]
Student-Teacher Ratio	0.03	0.16	[0.04; 0.87]	-0.33 [-0.52; -0.13]	-0.18 [-0.26; -0.09]
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Divorce Rate	<0.01	0.21	[0.02; 0.53]	-0.49 [-0.68; -0.29]	-0.38 [-0.49; -0.27]
Frac. Married	0.09	0.07	[0.12; 1.00]	0.57 [0.45; 0.69]	0.36 [0.29; 0.43]
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College Tuition	0.21	<0.01	[0.15; 1.00]	-0.02 [-0.15; 0.11]	0.01 [-0.02; 0.04]
Coll. Grad. Rate (Inc. Adjusted)	0.46	0.01	[0.34; 1.00]	0.15 [0.03; 0.28]	0.08 [0.01; 0.15]
Manufacturing Share	0.04	<0.01	[0.10; 1.00]	-0.26 [-0.44; -0.08]	0.06 [-0.03; 0.16]
Chinese Import Growth	0.02	0.07	[0.01; 0.58]	-0.17 [-0.33; -0.02]	0.03 [0.01; 0.04]
Teenage LFP Rate	0.28	<0.01	[0.20; 1.00]	0.63 [0.46; 0.80]	0.25 [0.14; 0.36]
Migration Inflow	0.06	0.11	[0.00; 1.00]	-0.26 [-0.40; -0.11]	-0.13 [-0.18; -0.08]
Migration Outflow	0.05	0.02	[0.07; 1.00]	-0.16 [-0.30; -0.03]	-0.09 [-0.15; -0.03]
Frac. Foreign Born	0.44	0.02	[0.35; 1.00]	-0.03 [-0.15; 0.10]	-0.12 [-0.24; -0.00]

Some Questions:

1. ~~What is a spatial random walk?~~

   (a) ~~What is a spatial $I(1)$ process?~~

2. ~~Do spatial $I(1)$ processes lead to spurious regressions?~~

3. ~~Can you test for spatial 'unit roots' (i.e., $I(1)$ processes)? How?~~

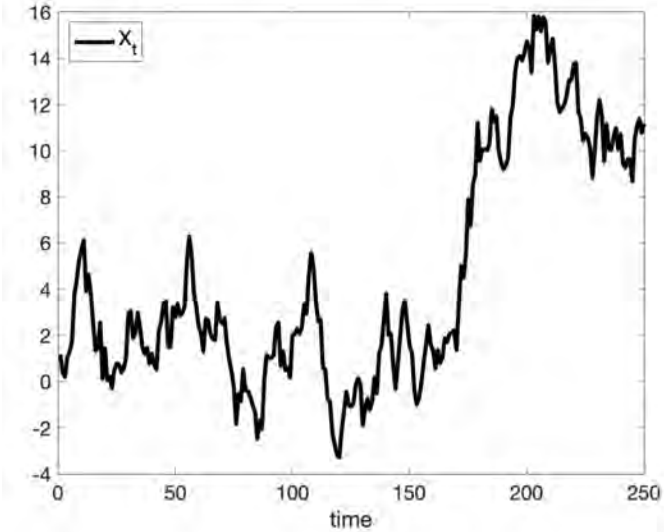
4. **Can you eliminate $I(1)$ spatial persistence by 'differencing' the data? How?**

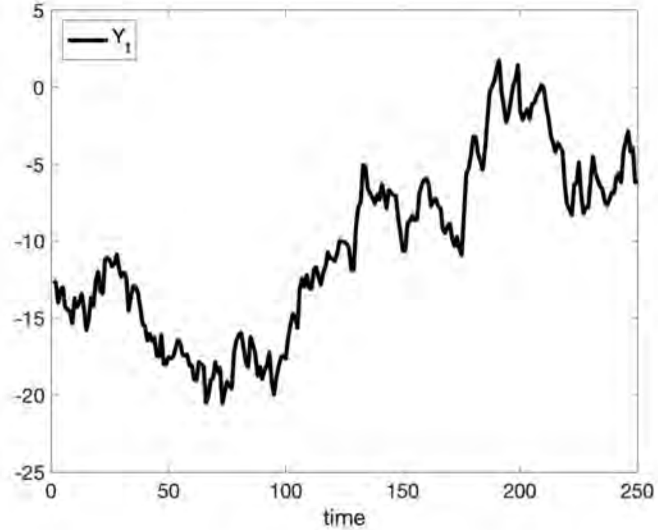
5. ~~Is there a large-sample theory that helps answer these questions?~~

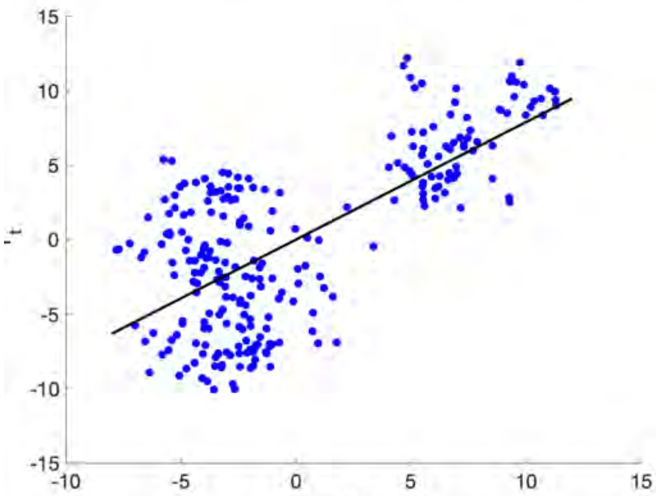
Question 4 : Can you eliminate $I(1)$ spatial persistence in regressions by 'differencing' the data? How?

$$y_l = \alpha + \mathbf{x}_l' \boldsymbol{\beta} + u_l$$

- $d = 1$, discrete time series, (y_t, x_t) in levels and $(\Delta y_t, \Delta x_t)$ as first differences







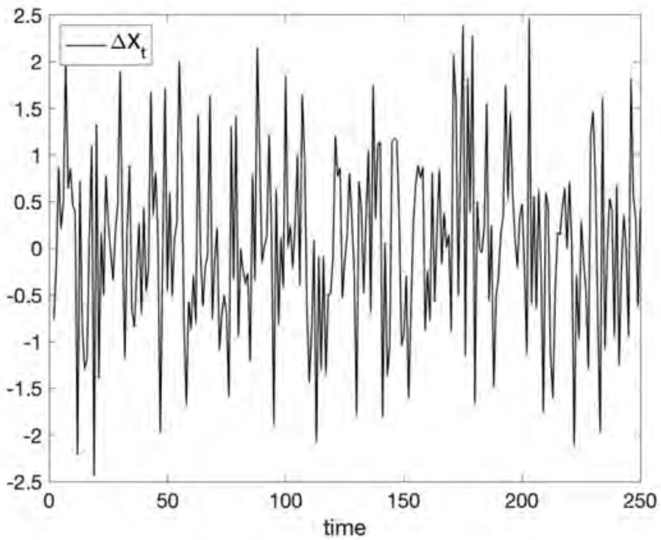
$\hat{\beta} = 0.79$

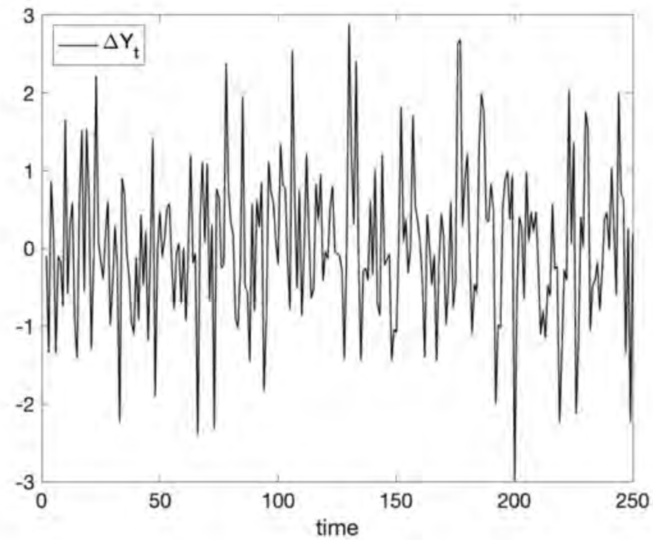
(0.04) (HR)

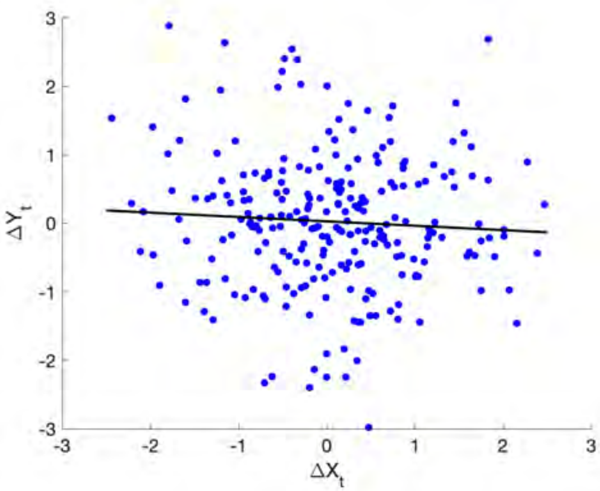
[0.09] (Newey-West)

$R^2 = 0.48$

Question 4 continued :







$\hat{\beta} = -0.06$

(0.06) (HR)

[0.06] (Newey-West)

$R^2 = 0.00$

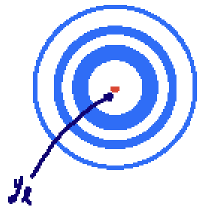
Question 4 continued : $d > 1$ ... Inference methods

- Transformations:

1. Isotropic differences

$$\Delta_{Iso} y_l = y_l - \left[\sum_{\ell \neq l} w_{\ell,l} y_\ell \right]$$

with $w_{\ell,l} = \kappa(|s_\ell - s_l|)/[\sum_{\ell \neq l} \kappa(|s_\ell - s_l|)]$.



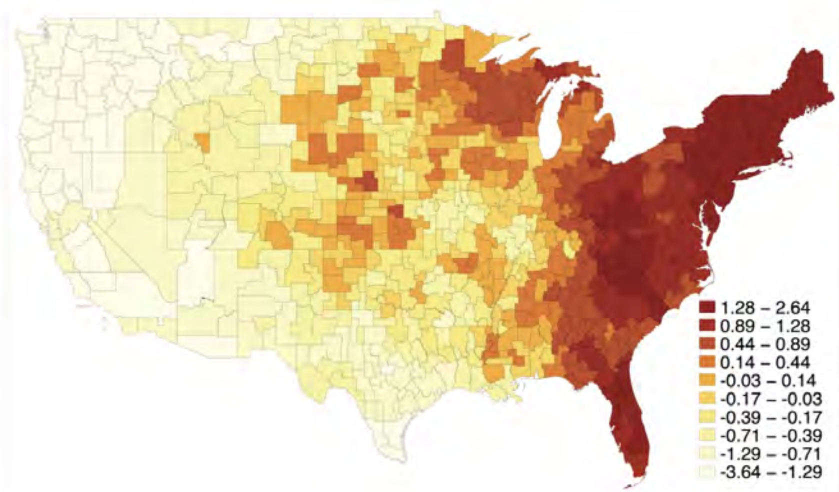
Question 4 continued :

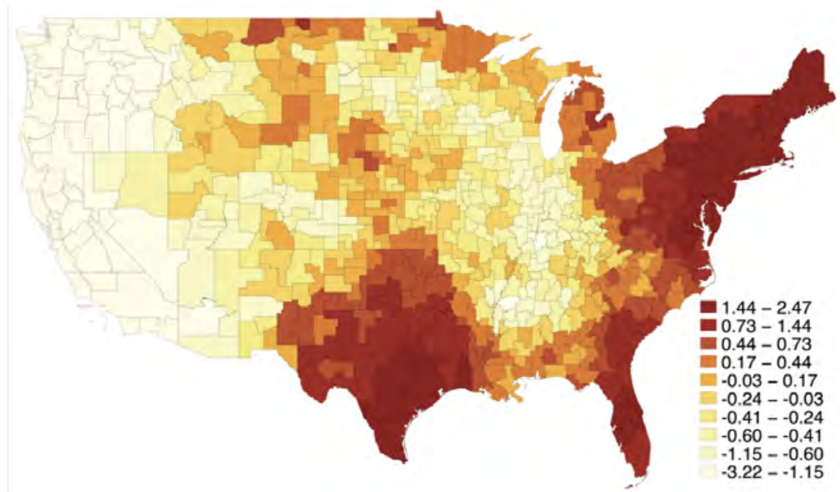
2. Levy-BM GLS:

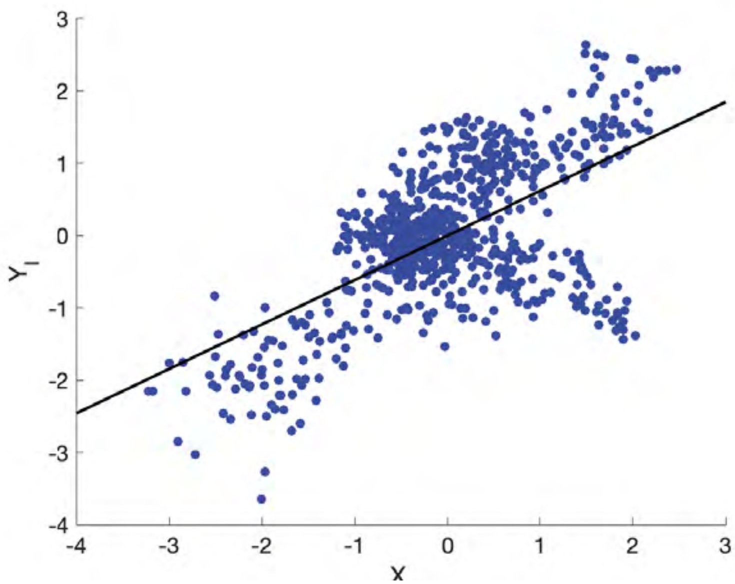
$$\hat{\beta} = (\widetilde{X}'\widetilde{\Sigma}_L^{-1}\widetilde{X})^{-1}(\widetilde{X}'\widetilde{\Sigma}_L^{-1}\widetilde{Y})$$

where ˜ denotes demeaned version.

Question 4 continued : 2 spatial random walks (again)... Levels



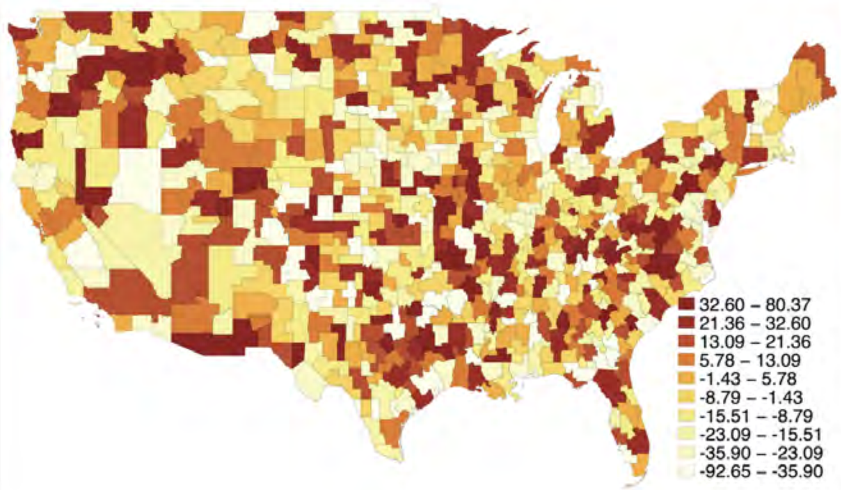


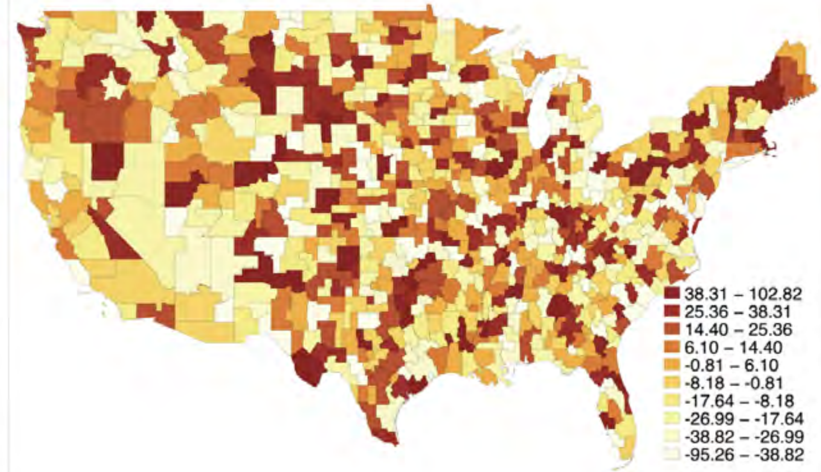


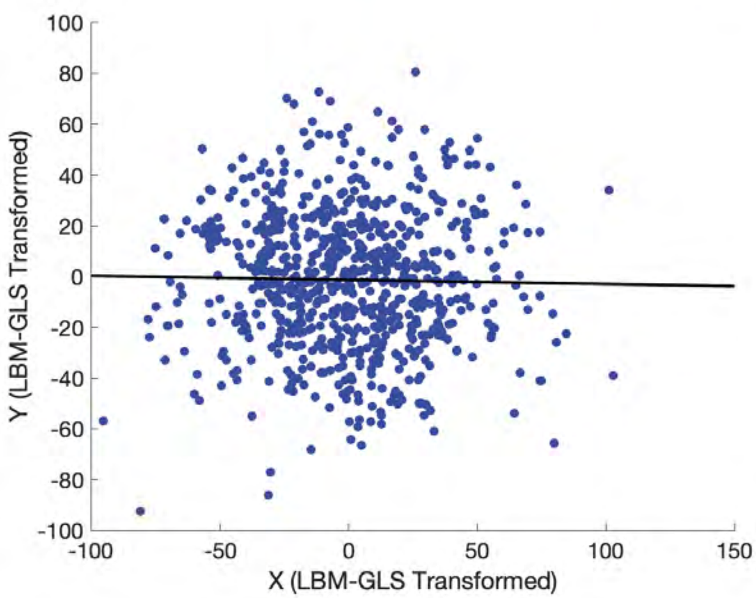
$\hat{\beta} = 0.62\ (0.04)$

$R^2 = 0.38$

Question 4 continued : 2 spatial random walks (again)... Levy-BM GLS transformed



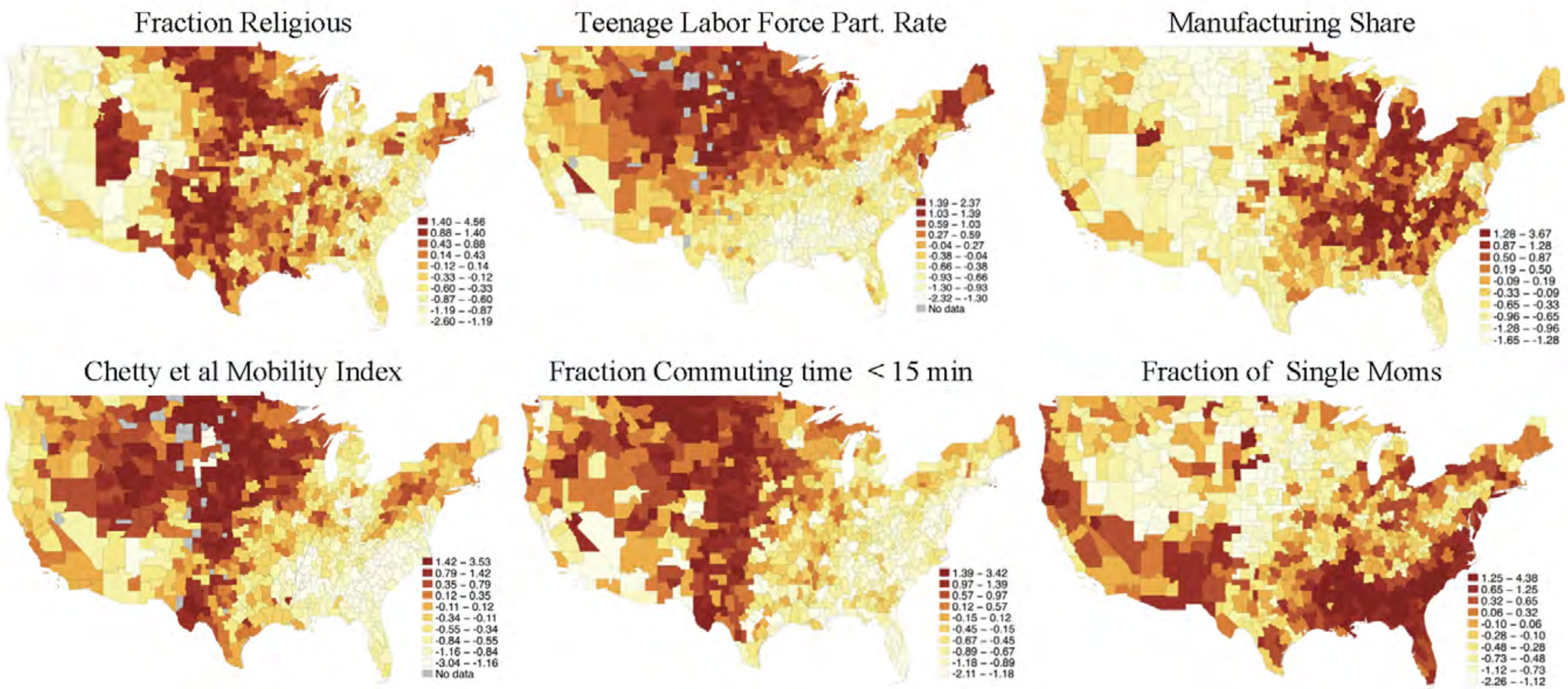




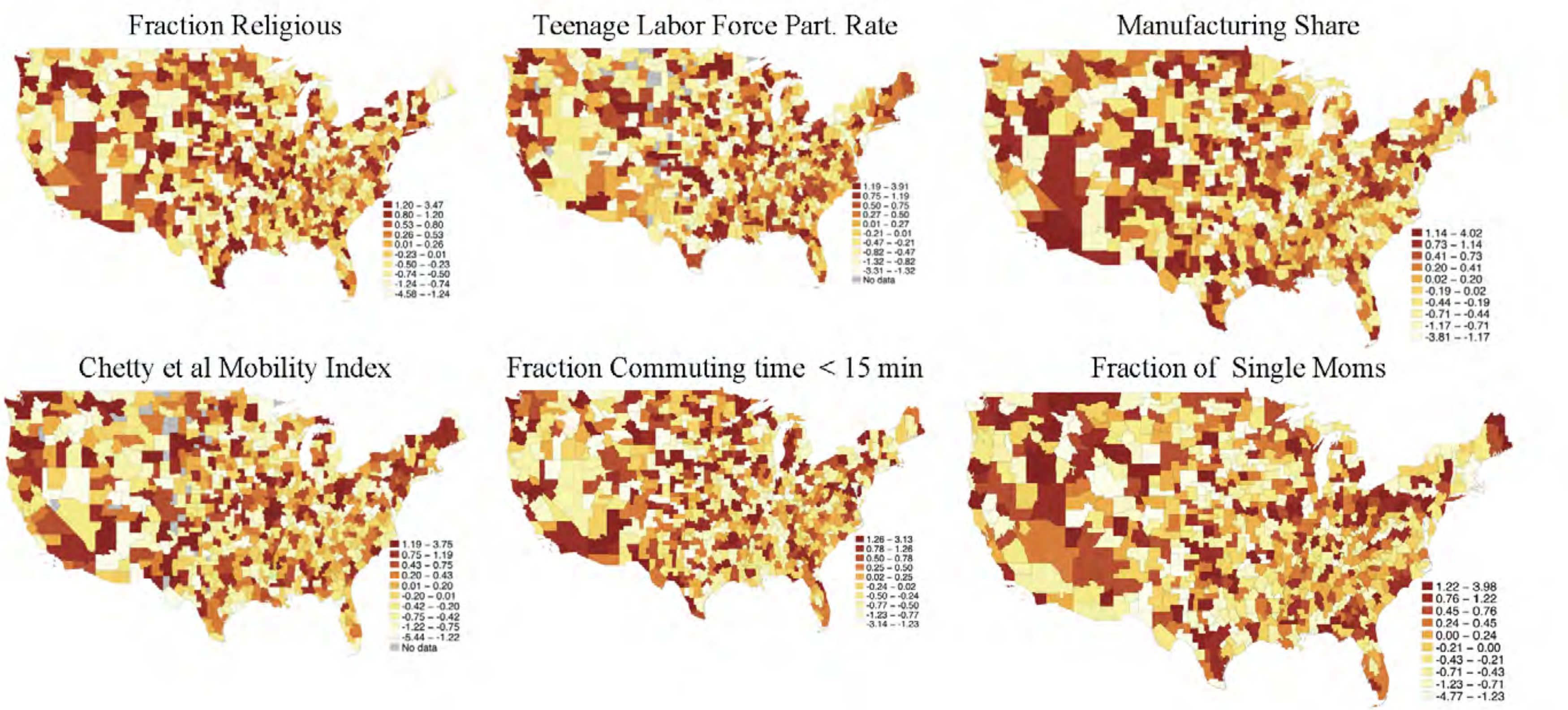
$\hat{\beta}$ = -0.02 (0.03)

$R^2 = 0.00$

Question 4 continued : 6 variables (again)... Levels



Question 4 continued : 6 variables ... Levy-BM GLS transformed



Question 4 continued :

3. Weighted least squares using the largest principal components computed using the Levy-BM eigenvectors.

4. Spatial low-frequency regression.

5. Least squares after deleting the largest principal components computed using the Levy-BM eigenvectors.

6. Spatial high-pass regression.

7. Add 'local-fixed effects' to regression. (This is local demeaning)

8. Run regressions over many non-overlapping regions and average. (IM)

In all cases use HAC/HAR methods to account for $I(0)$ spatial correlation.

Question 4 continued : $d > 1$ ... Experiments

- Spatial Design
  - Choose locations at random in one of 48 US States
- DGPs
  - Variety of $I(1)$ and $I(0)$ DGPs.

Question 4 continued : Selected Results

Rejection Frequency (Median over spatial designs)

	DGP	
Method	$I(1)_{c_{0.03}}$	$J_{c_{0.50}}$
OLS (C-SCPC)	0.35	0.20
Isotropic difference (C-SCPC)	0.07	0.04
Cluster fxed-effects (cluster)	0.35	0.07
Cluster fxed-effects (C-SCPC)	0.12	0.05
LBM-GLS	0.39	0.05
LBM-GLS (C-SCPC)	0.07	0.03
Low-pass Eigenvector	0.05	0.05
High-pass Eigenvector (C-SCPC)	0.13	0.05
Ibragimov-Müller	0.15	0.07

Average Length (median over spatial designs) of (nominal) 95% confidence intervals

	DGP	
Method	$I(1)_{c_{0.03}}$	$J_{c_{0.50}}$
Isotropic difference (C-SCPC)	0.73	0.52
LBM-GLS (C-SCPC)	0.54	0.26
Low-pass Eigenvector	1.51	0.57

Question 4 continued :

Variable	Spatial Persistence Statistics				Regression of the AMI onto Variable $\hat{\beta}$ [95% CI]	
	p-Value for Test		95% CI for $\bar{\rho}$			
	I(1) Null	*I*(0) Null			Level [Cluster]	LBM-GLS [CSCPC]
Absolute Mobility Index	0.08	<0.01	[0.14; 1.00]			
Frac. Black Residents	0.02	0.01	[0.02; 0.71]		-0.58 [-0.71; -0.45]	**-0.42 [-0.50; -0.34]**
Racial Segregation	0.07	0.02	[0.05; 1.00]		-0.36 [-0.45; -0.27]	**-0.24 [-0.28; -0.19]**
Segregation of Poverty	0.13	0.04	[0.05; 1.00]		-0.41 [-0.54; -0.28]	**-0.21 [-0.25; -0.16]**
Frac. < 15 Mins to Work	0.69	<0.01	[0.46; 1.00]		0.61 [0.36; 0.85]	**0.37 [0.26; 0.48]**
Mean Household Income	0.02	0.18	[0.01; 0.61]		0.05 [-0.09; 0.19]	**-0.02 [-0.08; 0.04]**
Gini	0.56	<0.01	[0.40; 1.00]		-0.58 [-0.76; -0.40]	**-0.21 [-0.29; -0.14]**
Top 1 Perc. Inc. Share	0.60	0.03	[0.43; 1.00]		-0.19 [-0.33; -0.05]	**-0.06 [-0.11; -0.01]**
Student-Teacher Ratio	0.03	0.16	[0.04; 0.87]		-0.33 [-0.52; -0.13]	**-0.18 [-0.26; -0.09]**
Test Scores (Inc. adjusted)	0.40	0.07	[0.27; 1.00]		0.59 [0.42; 0.76]	**0.42 [0.34; 0.51]**
High School Dropout	0.63	0.02	[0.40; 1.00]		-0.57 [-0.75; -0.40]	**-0.31 [-0.42; -0.20]**
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Some Questions:

1. **What is a spatial random walk?**
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