The Two-Pillar Policy for the RMB*

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June 4, 2020

Abstract

We document stylized facts about China's recent exchange rate policy for its currency, the Renminbi (RMB). Our empirical findings suggest that a "*two-pillar* policy" is in place, aiming to balance exchange rate flexibility and RMB index stability. Using derivatives data and a reduced-form no-arbitrage model, we assess financial market participants' view about the current exchange rate policy. Based on these empirical results, we develop a flexible-price monetary model for the RMB to evaluate the optimality of the two-pillar policy.

Keywords: Exchange Rate Policy, Two-Pillar Policy, Managed Float, Chinese currency, Renminbi, RMB, Central Parity, RMB Index.

*For helpful comments and suggestions we thank Markus Brunnermeier, Mario Crucini, Wenxin Du (discussant), Jeff Frankel, Pierre-Olivier Gourinchas, Nobuhiro Kiyotaki, Stijn Van Nieuwerburgh, Maurice Obstfeld, Mark Spiegel (discussant), Larry Wall, Shang-Jin Wei, Wei Xiong (editor), Tao Zha, Haoxiang Zhu, an anonymous associate editor, two anonymous referees, and participants at the Atlanta Fed, Federal Reserve Board, Fudan University, George Washington University, Peking University, Tsinghua University, University of Hong Kong, University of Tokyo, University of Western Ontario, 2019 NBER Summer Institute, 2019 American Finance Association meetings, First IMF-Atlanta Fed Workshop on China's Economy, Bank of Canada-University of Toronto China conference. The views expressed in this paper are those of the authors and do not necessarily represent those of the Federal Reserve System.

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1 Introduction

How China manages its currency, the Renminbi (RMB), is one of the most consequential decisions in global financial markets. China has been the world's largest exporter since 2009. The value of the RMB is of paramount importance in determining the prices of China's exports. Many people have argued that China's undervalued currency contributes to the trade surplus that China runs consistently since 1993.[1] Hence it is important to understand how China's monetary authority conducts its exchange rate policy.

Since July 21, 2005 when the RMB was depegged from the U.S. dollar, China has adopted a *managed floating* regime. In the current regime, the People's Bank of China (PBOC) announces the *central parity* (or *fixing*) rate of the RMB against the U.S. dollar before the opening of the market each business day. The central parity rate serves as the midpoint of the daily trading range, and the intraday spot rate is allowed to fluctuate only within a narrow band around it. For a long time, little was revealed about how the central parity was determined. Since August 2015, the PBOC has implemented several reforms to make the formation mechanism of the central parity more transparent and more market-oriented. However, it still remains largely opaque to investors how the policy is implemented.

In this paper, we first document stylized facts about China's recent exchange rate policy. Our empirical findings suggest that a "*two-pillar* policy" is in place, aiming to balance exchange rate flexibility and RMB index stability. According to the PBOC's Monetary Policy Report of the first quarter of 2016, the formation mechanism of the central parity depends on two key factors, or two pillars: the first pillar refers to "*the closing rates of the previous business day to reflect changes in market demand and supply conditions*", while the second pillar is related to changes in the currency basket, "*as a means to maintain the overall stability of the RMB to the currency basket.*" We construct empirical measures of these two pillars and find that they explain as much as 80 percent of the variations in the central parity. We find that both pillars receive roughly equal weights in setting the central parity.

Based on these stylized facts, we develop a reduced-form no-arbitrage model of the RMB under the two-pillar policy. Using derivatives data on the RMB and the U.S. dollar index, we estimate the model to assess financial markets' views about the current exchange rate policy. Our daily estimation results suggest that during our sample period between December 11, 2015 and December 31, 2018 financial market participants attach a high probability to the policy continuing to be in place. The estimated probability of policy continuation fluctuates mostly between 60 percent and 90 percent. We also find that absent the two-pillar policy, the RMB is expected to depreciate about 2 percent on average, which is consistent with recent depreciationary pressure on the RMB. In

[1] For example, since 2003, the United States has been pressuring China to allow the RMB to appreciate and to be more flexible (see Frankel and Wei (2007)). On the other hand, the RMB was assessed in 2015 by the International Monetary Fund (IMF) to be no longer undervalued given its recent appreciation (see the press release at https://www.imf.org/en/News/Articles/2015/09/14/01/49/pr15237).

addition, we find the above estimation results remain largely unchanged even after we incorporate the offshore RMB market. These findings lend further support to the validity of the two-pillar policy we have formulated.

To evaluate China's performance in managing its currency, we develop a flexible-price monetary model of the RMB by extending Svensson (1994). In the model, the government trades off between the variabilities of the exchange rate, the interest rate, and the current account. The central bank optimally chooses the money growth rate and exchange rate policy to achieve the government's objective.

In the theoretical model, the two-pillar policy arises endogenously as an optimal solution to the government's problem. The central parity depends on both pillars because the government's preferences involve two policy targets: minimizing the exchange rate variability and stabilizing the current account. The former policy target incentivizes the government to set the central parity as close as possible to the previous closing rate, which reflects market conditions. The latter policy target requires a stable RMB index which measures the value of the RMB against a basket of currencies of China's trading partners. When the government cares equally about both policy targets, the two pillars carry equal weights in the optimal central parity rule, consistent with our empirical findings.

We calibrate the model and assess quantitatively the trade-off faced by the government. On the one hand, if the central parity were only dependent on a single pillar of the previous day's closing rate, then the exchange rate variability would be minimized, but the current account would be 20 percent more volatile than the data. On the other hand, if the central parity had only depended on the basket pillar, the current account volatility would be very low, but the exchange rate volatility could be as high as 15 percent, almost four times the level in the data.

In the model, the government engages in nonsterilized intervention by optimally setting the money growth rate to trade off between the exchange rate bandwidth and the interest rate volatility. As argued in Svensson (1994), the latter reflects the degree of monetary policy independence, which in turn varies inversely with the exchange rate bandwidth. The effective trading bandwidth, interpreted as three standard deviations of the exchange rate deviation in the data, is roughly 0.75 percent in our sample period. Based on our calibrated parameters, we show that under the 0.75-percent bandwidth, the standard deviation of the interest rate is 1.67 percent, about twice as large as the level in the data. However, if we increase the trading bandwidth to 1.2 percent, the standard deviation of the interest rate drops to 0.78 percent, similar to the level observed in the data, reflecting increased monetary policy independence. The above findings quantify the trade-off faced by the PBOC between the exchange rate bandwidth and the interest rate volatility.

We also extend the model to study direct sterilized government intervention based on Brunnermeier, Sockin, and Xiong (2018). As in Brunnermeier, Sockin, and Xiong (2018), the direct government intervention is an effective tool to "lean against noise traders" when there is noise trading risk in the foreign exchange market. Without government intervention, the foreign exchange market may break down if there is a sufficiently large degree of noise trading. This is because more noise trading leads investors to demand a higher risk premium for providing liquidity to noise traders, which drives up exchange rate volatility. This further raises the risk premium required by investors. When the amount of noise trading is sufficiently large, there does not exist any risk premium that can induce investors to take on any position. As a result, exchange rate volatility explodes and the market breaks down. By "leaning against noise traders", the government could intervene to offset the amount of noise trading and lower the risk premium, forestalling market breakdown.

Our paper is related to the large literature on the Chinese exchange rate.[2] Earlier research papers study the RMB undervaluation or misalignment (e.g., Frankel (2006), Cheung, Chinn, and Fujii (2007), and Yu (2007)), or aim to characterize how China managed its exchange rate (e.g., Frankel and Wei (1994, 2007), Frankel (2009), and Sun (2010)). In particular, Frankel and Wei (1994, 2007) use regression analysis to estimate *unknown* basket weights and reject the notion that an announced basket peg was actually followed by the PBOC in earlier periods. Our empirical analysis of China's recent exchange rate policy follows and goes beyond the tradition established in Frankel and Wei (1994, 2007). Recent research papers that empirically investigated the determinants of the central parity include Cheung, Hui and Tsang (2018), Clark (2018), and McCauley and Shu (2018). To the best of our knowledge, our paper is the first one that empirically characterizes and theoretically evaluates the two-pillar policy.

The paper is also related to the literature on exchange rate target zones, pioneered in Krugman (1991).[3] In a recent study, Jermann (2017) develops a no-arbitrage model in a spirit similar to target zone models to study Switzerland's exchange rate policy. In this paper we first demystify the formation mechanism of the central parity in China. Based on our formulation of the two-pillar policy, we extend Jermann (2017) to assess financial market participants' view about China's exchange rate policy.

Our flexible-price monetary model of the RMB in this paper is most closely related to the model in Svensson (1994) which is used to quantitatively analyze the degree of monetary policy independence for the managed floating system in Sweden. In Svensson (1994) the central bank preferences involve a trade-off between interest rate smoothing and exchange rate variability, and the central parity is assumed to be constant for the case of Sweden. By contrast, in this paper the government optimally chooses the central parity. We find that the optimal central parity rule mimics the two-pillar policy for the case of China as a result of the policy trade-off between minimizing exchange rate variability and stabilizing the current account.

[2] A related literature is on China's monetary policy and capital control, e.g., Prasad et al. (2005), Chang, Liu, and Spiegel (2015).

[3] The target zone literature was initially developed for Europe's path to monetary union. Recently, Bertola and Caballero (1992) and Bertola and Svensson (1993) extend Krugman (1991) to allow for realignment risk that the target cannot be credibly maintained. Dumas et al. (1995), Campa and Chang (1996), Malz (1996), Söderlind (2000), Hui and Lo (2009) utilize options data to estimate the realignment risk.

The extended model of intraday government intervention in this paper is in spirit similar to Jeanne and Rose (2002) and other papers that explicitly model microstructure aspects of foreign exchange markets.[4] Our theoretical findings suggest an important role of direct government intervention via "leaning against the wind", which echoes the findings in Brunnermeier, Sockin, and Xiong (2018). The existing theories of government intervention in the foreign exchange market analyze two channels through which government intervention affects the *level* of the exchange rate: the portfolio balance channel (e.g., Dominguez and Frankel (1993)) and the signalling channel (e.g., Stein (1989), Bhattacharya and Weller (1997), Vitale (1999)).[5] Different from these channels, the channel proposed in this paper is aimed to prevent the *volatility* of the exchange rate from exploding amid market failure.

In the rest of the paper, Section 2 contains the empirical analysis, including derivatives-based estimations. Section 3 presents the theoretical analysis. Section 4 concludes.

2 Empirical Analysis

We start by describing official policies for the RMB in the recent years. We argue that China's exchange rate policy since 2015 can be formulated by a two–pillar approach and provide empirical evidence for our formulation. Furthermore, we assess the financial market participants' view about the implemented two-pillar policy using a reduced-form no-arbitrage model and derivatives data.

2.1 Managed Floating RMB Regime

During the last three decades, China's transition into a market-based economy has been remarkable. However, the Chinese government continues to keep a firm grip on the RMB.

Since July 21, 2005 when the RMB was depegged from the dollar and had a one-time appreciation by 2 percent, China has implemented a managed floating regime for its currency. In the current regime, the PBOC announces the central parity rate of the RMB against the dollar at 9:15AM before the opening of the market each business day. The central parity rate serves as the midpoint of daily trading range in the sense that the intraday spot rate is allowed to fluctuate within a narrow band around it. Figure 1's Panel A displays the RMB central parity and closing rates since 2004. It is evident from the panel that the deviation of the closing rate from the central parity rate is typically very small and falls within the official trading band.

To strengthen the role of demand and supply force, China has gradually widened the trading

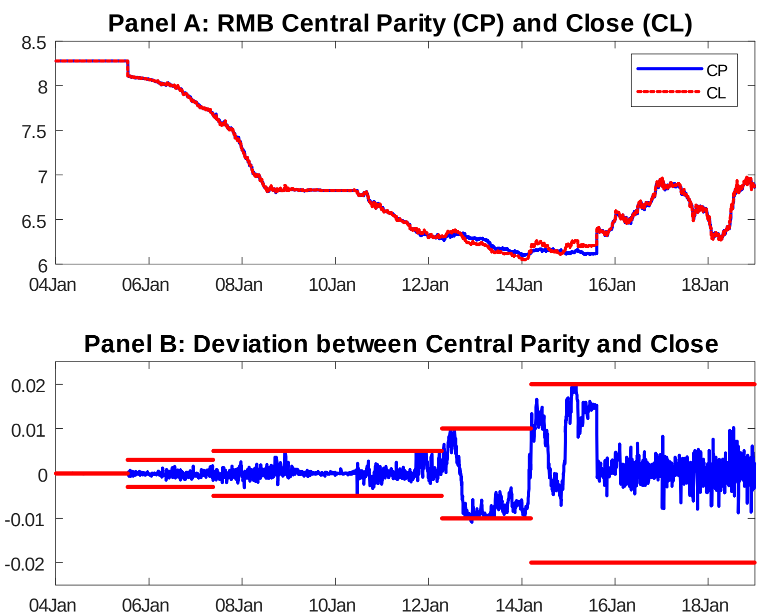
[4] See Lyons (2001) for a comprehensive treatment of both theoretical and empirical work along this line of research, as well as the references therein.

[5] Sarno and Taylor (2001) provide a survey of recent research on government intervention in the foreign exchange market. See, for example, Pasquariello, Roush, and Vega (2020) for research on government intervention in the U.S. treasury market.

band from an initial width of 0.3 percent to the current width of 2 percent.[6] Figure 1's Panel B plots the deviation between the central parity and the close since 2004. It shows that as the trading band widened, the deviation has become more volatile, reflecting more flexibility of the RMB.

Figure 1: RMB Central Parity and Closing Rates between 2004 and 2018

Panel A of this figure plots historical central parity rate (blue solid line) and closing rate (red dashed line) between 2004 and 2018. Panel B of this figure plots in blue solid line the difference between the logarithms of the central parity and closing rates, and in red solid lines the bounds imposed by the PBOC.



However, the PBOC can intervene in the foreign exchange market and control the extent to which the spot rate deviates from the central parity. As a result, the effective width of the trading band can be much smaller than the officially announced width. For example, during the recent financial crisis, the RMB was essentially re-pegged to the dollar. As another example, since August 11, 2015, the band around the central parity has been effectively limited to 0.5 percent, with an exception of a few dates.

On August 11, 2015, China reformed its procedure of setting the daily central parity of the RMB against the dollar. The reformed formation mechanism is meant to be more transparent and more market driven as part of RMB internationalization effort.[7] In particular, "*quotes of the central parity of the RMB to the USD should refer to the closing rates of the previous business day*

[6] Starting from the initial 0.3%, the bandwidth has been widened to 0.5% on May 21, 2007, to 1% on April 16, 2012, and 2% on March 17, 2014.

[7] Over the past decade, China has stepped up its efforts to internationalize the RMB (e.g., its inclusion in the IMF's SDR basket of reserve currencies in October 2016). Some recent studies on RMB internationalization include Chen and Cheung (2011), Cheung, Ma and McCauley (2011), Frankel (2012), Eichengreen and Kawai (2015), and Prasad (2016), among others.

to reflect changes in market demand and supply conditions", according to the PBOC's Monetary Policy Report in the first quarter of 2016. Following the reform the central parity of the RMB against the dollar depreciated 1.9 percent, 1.6 percent, and 1.1 percent, respectively, in the first three trading days under the reformed formation mechanism until the PBOC intervened to halt further depreciation.

On December 11, 2015, the PBOC introduced three trade-weighted RMB indices and reformed the formation mechanism of the central parity. The reform aimed to mitigate depreciation expectations stemming from China's slowing economy and the first possible interest rate liftoff by the Federal Reserve. The three RMB indices are based on China Foreign Exchange Trade System (CFETS), the IMF's Special Drawing Rights (SDR) and Bank for International Settlement (BIS) baskets. We thus refer to them as the CFETS, SDR and BIS indices, respectively, throughout the paper. All three indices have the same base level of 100 in the end of 2014 and are published regularly.

The PBOC's Monetary Policy Report in the first quarter of 2016 provides more details about the new formation mechanism of the central parity. It states that

> "*a formation mechanism for the RMB to the USD central parity rate [consisting] of 'the previous closing rate plus changes in the currency basket' has been preliminarily in place. The 'previous closing rate plus changes in the currency basket' formation mechanism means that market makers must consider both factors when quoting the central parity of the RMB to the USD, namely the 'previous closing rate' and the 'changes in the currency basket'.*"

2.2 The Two-Pillar Policy

Based on the discussion in the previous subsection, we characterize the formation mechanism of the central parity by a two-pillar policy whereby the central parity is a weighted average of the basket target and previous day's close:

$$S_{t+1}^{CP} = \left(S_t^{CL}\right)^{1-w} \left(\bar{S}_{t+1}\right)^{w}, \tag{1}$$

where S_t^{CL} denotes the spot exchange rate of the RMB against the dollar at the close of day t, and $\overline{S}_{t+1}$ denotes the hypothetical rate that achieves basket stability. These two components are the two pillars of the central parity: The former is "*market demand and supply situation*," while the latter reflects the "*the amount of the adjustment in the exchange rate of the RMB to the dollar, as a means to maintain the overall stability of the RMB to the currency basket.*"

Intuitively, the two-pillar policy allows the PBOC to make the RMB flexible and more market-driven through the first pillar, S_t^{CL}, and at the same time to keep it stable relative to the RMB index through the second pillar, $\bar{S}_{t+1}$. At one extreme, when weight w is fixed at zero, the central parity is fully determined by the first pillar and is thus market driven to the extent that the spot exchange rate is permitted to fluctuate within a band around the central parity rate under possible interventions by the PBOC. At the other extreme, when weight w is fixed at 100 percent, the central parity is fully determined by the second pillar; that is, the exchange rate policy is essentially basket pegging and the RMB index does not change over time.

To explicitly represent the pillar associated with the currency basket $\bar{S}_{t+1}$, we first discuss RMB indices. In essence, an RMB index (e.g., CFETS) is a geometric average of a basket of currencies:

$$B_t = C_B \left(S_t^{CP,USD/CNY} \right)^{w_{USD}} \left(S_t^{CP,EUR/CNY} \right)^{w_{EUR}} \left(S_t^{CP,JPY/CNY} \right)^{w_{JPY}} \cdots \tag{2}$$

where C_B is a scaling constant used to normalize the index level to 100 in the end of 2014, $S_t^{CP,i/CNY}$ denotes the central parity rate in terms of the RMB for the currency i in the basket, and w_i the corresponding weight for $i = USD$, EUR, JPY, etc. When the RMB strengthens (or weakens) relative to the currency basket, the RMB index goes up (or down).

The key central parity rate is the one of the RMB against the dollar, denoted as $S_t^{CP} \equiv 1/S_t^{CP,USD/CNY}$. According to the PBOC, once S_t^{CP} is determined, the central parity rates for other non-dollar currencies are determined as the cross rates between S_t^{CP} and the spot exchange rates of the dollar against those currencies. Therefore, we focus on the formation mechanism of the central parity rate S_t^{CP}. For this reason, we refer to it simply as the central parity wherever there is no confusion.

The RMB index can be rewritten in terms of the central parity rate of the RMB against the dollar, S_t^{CP}, and a dollar index of all the non-RMB currencies, X_t:

$$B_t = \chi \frac{X_t^{1-w_{USD}}}{S_t^{CP}}, \tag{3}$$

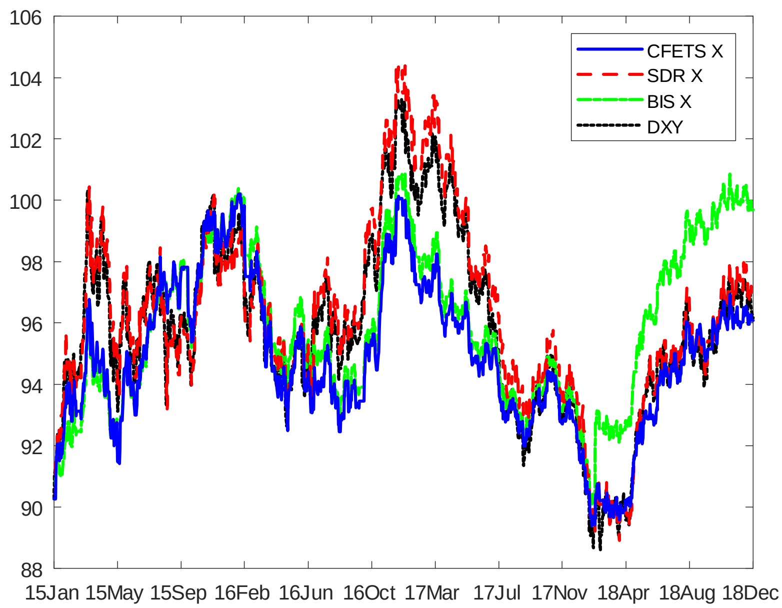
where X_t denotes the index-implied dollar index, defined by

$$X_t \equiv C_X \left(\frac{S_t^{CP,EUR/CNY}}{S_t^{CP,USD/CNY}} \right)^{\frac{w_{EUR}}{1-w_{USD}}} \left(\frac{S_t^{CP,JPY/CNY}}{S_t^{CP,USD/CNY}} \right)^{\frac{w_{JPY}}{1-w_{USD}}} \cdots \tag{4}$$

with a scaling constant C_X, and $\chi \equiv C_B/C_X^{1-w_{USD}}$. The scaling constant C_X is chosen such that X_t coincides in the end of 2014 with the well known U.S. Dollar Index that is actively traded on the Intercontinental Exchange under ticker "DXY". We construct the index-implied basket X_t based on equation (4) (see the online appendix for details). Note that the composition for the CFETS and SDR indices has changed since 2017. Take the CFETS index as an example. On December 29, 2016, the PBOC decided to expand the CFETS basket from 13 currencies to 24 currencies and at the same time reduced the dollar's weight from 26.4 percent to 22.4 percent. We take into account the composition changes of RMB indices when we construct X_t (please see the online appendix for details). The index-implied dollar basket X_t, plotted in Figure 2, is shown to be highly correlated with the U.S. dollar index DXY.

Figure 2: Index-implied Dollar Basket vs. DXY

In this figure we plot the historical dollar index (black dotted line) together with the dollar baskets implied in three indices. The dollar basket implied in the CFETS (respectively, SDR or BIS) index is plotted using blue solid line (respectively, red dashed or green dash-dot lines).



The pillar $\bar{S}_{t+1}$ is determined so as to achieve basket stability. Put differently, it is the value that would keep the RMB index unchanged if the central parity were set at such value. Therefore it is straightforward to show[8]

$$\overline{S}_{t+1} = S_t^{CP} \left(\frac{X_{t+1}}{X_t} \right)^{1-w_{USD}}. \tag{5}$$

The expression of $\overline{S}_{t+1}$ in equation (5) is intuitive. The key idea is that movements in the RMB index are attributable to movements in either the value of the RMB relative to the dollar, or the value of the dollar relative to the basket of non-dollar currencies in the RMB index, or both. The relative contributions of these two types of the movements are determined by w_{USD} and $(1 - w_{USD})$, respectively. As a result, in order for the RMB index to remain unchanged in response to movement in the dollar index, hypothetically, the value of the RMB relative to the dollar should be at a level that exactly offsets such movement.

[8]Specifically, the expression of $\overline{S}_{t+1}$ can be derived as follows. At time t, the RMB index is given by $B_t = \chi \frac{X_t^{1-w_{USD}}}{S_t^{CP}}$. At time $t+1$, if the index-implied dollar baset changes its value to X_{t+1}, the RMB index would become $B_{t+1} = \chi \frac{X_{t+1}^{1-w_{USD}}}{S_{t+1}^{CP}}$ if the central parity were set as S_{t+1}^{CP}. Equalizing B_t and B_{t+1} (i.e., $B_t = \chi \frac{X_t^{1-w_{USD}}}{S_t^{CP}} = \chi \frac{X_{t+1}^{1-w_{USD}}}{S_{t+1}^{CP}}$) determines the hypothetical value of S_{t+1}^{CP}, or $\overline{S}_{t+1}$, which would keep the RMB index unchanged.

Substituting the above equation into equation (1), the two-pillar policy can be described by the following equation:

$$S_{t+1}^{CP} = \left(S_t^{CL}\right)^{(1-w)} \left[S_t^{CP} \left(\frac{X_{t+1}}{X_t}\right)^{1-w_{USD}}\right]^w. \tag{6}$$

In the following subsection we empirically test and present empirical evidence for the above formulation.

2.3 Empirical Evidence

We document here that the RMB central parity has closely tracked our equation (1) summarizing the official policy statements. In addition, we find strong empirical support for a central parity rule that gives equal weights to each of the two pillars (i.e., $w = 1/2$).

To empirically test the two-pillar formulation in equation (1), we run the following regression:

$$\log\left(\frac{S_{t+1}^{CP}}{S_t^{CP}}\right) = \alpha \cdot \log\left(\frac{S_t^{CL}}{S_t^{CP}}\right) + \beta \cdot (1 - w_{USD}) \log\left(\frac{X_{t+1}}{X_t}\right) + \epsilon_{t+1}. \tag{7}$$

That is, the daily change in the log central parity (i.e., $\log\left(S_{t+1}^{CP}/S_t^{CP}\right)$ is regressed on the two pillars scaled by the previous central parity (i.e., $\log\left(S_t^{CL}/S_t^{CP}\right)$ and $(1 - w_{USD})\log\left(X_{t+1}/X_t\right)$). The coefficients α and β correspond to $1-w$ and w, respectively. The R-squared of the regression is a good indicator of the extent to which the actual formation mechanism of the central parity can be explained by our formulation of the two-pillar policy.

The results from the regression (7) for the whole sample period are reported under Column "Whole Sample" in Panel A of Table 1. The regression results support that $w = 1/2$ as both of the coefficients α and β are roughly equal to one half. The PBOC's Monetary Policy Report in the first quarter of 2016 has an example that seems to suggest equal weights for both pillars. Consistent with the report, our empirical analysis provides supportive empirical evidence for $w = 1/2$ for the period following December 11, 2015 when the RMB indices were announced for the first time. Moreover, the regression has a very high R-squared at around 80 percent, which suggests that our formulation of the two-pillar policy has a large explanatory power in describing the formation mechanism of the central parity in practice.

Next, we show that the above results also largely hold after a new "*countercyclical factor*" was introduced in the formation mechanism of the central parity. Specifically, the PBOC confirmed on May 26, 2017 that it had modified the formation mechanism of the central parity by introducing the new countercyclical factor, although no detailed information has been disclosed about how the countercyclical factor is constructed.[9] The modification is believed to "*give authorities more control*

[9] See the statement on the CFETS website: http://www.chinamoney.com.cn/fe/Info/38244066.

over the fixing and restrain the influence of market pricing."[10] The policy change is perceived by many market participants as a tool to address depreciation pressure without draining foreign reserves. However, it undermines earlier efforts to make the RMB more market driven. The countercyclical factor was then subsequently removed as reported by Bloomberg on January 9, 2018. It signals the return to the previous two-pillar policy. The removal of the countercyclical factor in January 2018 reflects the RMB's strength over the past year as well as the dollar's protracted decline.

Accordingly we divide the whole sample period into three subsample periods: regime 1 (12/11/2015 to 5/25/2017), regime 2 (5/26/2017 to 1/8/2018), and regime 3 (1/9/2018 to 12/31/2018).[11] Loosely speaking, regimes 1 and 3 refer to the subperiods without the countercyclical factor, while regime 2 represents the subperiod when the countercyclical factor was added to the determination of the central parity. The regression results for these three subsample periods are reported under Columns "Regime 1", "Regime 2", and "Regime 3" in Table 1, respectively.

Table 1: Empirical Evidence for the Two-Pillar Policy

Panel A: Unconstrained Regressions												
	Whole Sample			Regime 1			Regime 2			Regime 3		
	α	β	R^2	α	β	R^2	α	β	R^2	α	β	R^2
CFETS	0.53	0.47	0.81	0.49	0.51	0.82	0.53	0.40	0.78	0.57	0.40	0.81
SDR	0.55	0.45	0.80	0.49	0.49	0.80	0.53	0.40	0.79	0.60	0.40	0.81
BIS	0.55	0.42	0.78	0.51	0.48	0.78	0.52	0.40	0.79	0.61	0.30	0.79
Panel B: Constrained Regressions												
	Whole Sample			Regime 1			Regime 2			Regime 3		
	α	β	R^2	α	β	R^2	α	β	R^2	α	β	R^2
CFETS	0.53	0.47	–	0.49	0.51	–	0.55	0.45	–	0.58	0.42	–
SDR	0.55	0.45	–	0.51	0.49	–	0.56	0.44	–	0.60	0.40	–
BIS	0.57	0.43	–	0.52	0.48	–	0.55	0.45	–	0.64	0.36	–

Notes: This table reports the results of regression (7) in which the daily change in the log central parity (i.e., $\log\left(S_{t+1}^{CP}/S_t^{CP}\right)$ is regressed on the two pillars scaled by the previous central parity (i.e., $\log\left(S_t^{CL}/S_t^{CP}\right)$ and $(1-w_{USD})\log\left(X_{t+1}/X_t\right)$). The results of unconstrained (constrained) regressions are reported in Panel A (Panel B). The regression is conducted in four different periods: whole sample period (12/11/2015 to 12/31/2018), regime 1 (12/11/2015 to 5/25/2017), regime 2 (5/26/2017 to 1/8/2018), and regime 3 (1/9/2018 to 12/31/2018). All regression coefficients are statistically significant at the 99 percent level.

[10] See the article "China Considers Changing Yuan Fixing Formula to Curb Swings" on Bloomberg News on May 25, 2017.

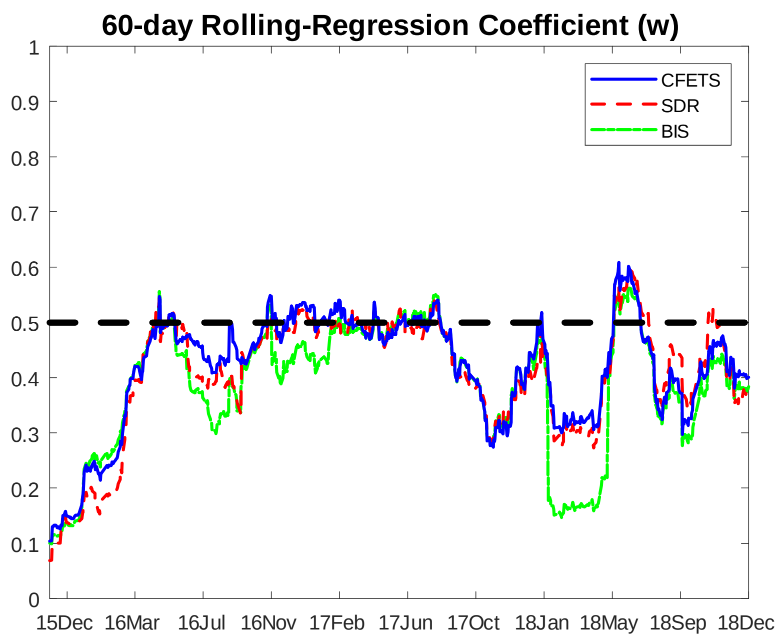
[11] We are grateful to an anonymous referee for suggesting the empirical analysis over these three subsample periods.

Compared to the results in Column "Whole Sample", the results in Column "Regime 1" provide even stronger empirical evidence for our two-pillar policy formulation in that the regression coefficients α or β are even closer to 0.5 with slightly higher R-squared. The stronger results in this subperiod are intuitive, because presumably this is the subperiod when the two-pillar policy holds as closest as possible to our formulation.[12] As a result of the introduction of the countercyclical factor, the R-squared in Column "Regime 2" slightly decreases, but remains large around 80 percent. Once the countercyclical factor was reportedly dropped, the R-squared in Column "Regime 2" increases.

The results reported in Panel A of Table 1 are from unconstrained regressions whereby we do not impose the restriction that the coefficients sum up to one. Interestingly, the regression results suggest it roughly holds in the data (i.e., $\alpha + \beta \approx 1$). We also run constrained regressions by explicitly imposing the above restriction. The results are reported in Panel B of Table 1. The constrained regression results lend further support for $w = 1/2$.

Figure 3: Empirical Evidence for the Two-Pillar Approach

This figure plots the coefficients β from 60-day rolling-window regressions: $\log \frac{S_{t+1}^{CP}}{S_t^{CP}} = \alpha \cdot \log \frac{S_t^{CL}}{S_t^{CP}} + \beta \cdot \left(1 - w_{USD}^{ind}\right) \log \frac{X_{t+1}^{ind}}{X_t^{ind}} + \epsilon_{t+1}$, where superscript "*ind*" indicates one of the three indices CFETS (blue solid line), BIS (red dashed line), and SDR (green dash-dot line).



The above results from running the regression in equation (7) shed light on the *average* weights on the two pillars during a fixed period, but are silent about possible time variation in the weights.

[12]Note that the PBOC also changed the composition of the CFETS basket in the subperiod "regime 1". The new CFETS basket, effective January 1, 2017 includes 13 additional currencies and puts lower weight on the dollar. The adjustment is considered as a signal that the PBOC is likely to maintain the same formation mechanism of the central parity on a trade-weighted basis. In unreported results, we show that focusing on the subperiod between January 1, 2017 and May 25, 2017, the R-squared from the same regression increases to 86% for the CFETS index.

To further investigate how the weights may possibly vary over time, we run the above regression by using 60-day rolling windows, starting from 60 business days after August 11, 2015. Figure 3 plots the estimate of weight w implied by the rolling-window regressions.

Figure 3 shows that the weight w is initially around 0.1 in the period prior to the introduction of the RMB indices. The results suggest that the formation mechanism before December 2015 follows closer to a "one-pillar" policy in the sense that it is almost completely determined by the previous day's close as the dollar basket implied in the RMB index carries very little weight.[13] After the RMB indices were introduced on December 11, 2015, the weight w has since then steadily increased and stabilized around 0.5, suggesting that the two-pillar policy with equal weights is in place. Since May 2017 when a countercyclical factor was introduced, the estimate of weight w exhibits more variability under the modified two-pillar policy with the new countercyclical factor.

2.3.1 Robustness Checks

Lastly we conduct robustness checks by taking into account another modification of the two-pillar policy: On February 20, 2017, the PBOC reduced the reference period for the central parity against the RMB index from 24 hours to 15 hours. According to Monetary Policy Report in the second quarter of 2017, the rational for the adjustment is to avoid "*repeated references to the daily movements of the USD exchange rate in the central parity of the following day*" since the previous close has already incorporated such information to a large extent. This adjustment, however, is widely believed to have limited impact on the RMB exchange rate. Accordingly, we repeat the above empirical analysis for two subsample periods: "*Subsample I*" (12/11/2015 to 2/19/2017) when a 24-hour reference period was used, and "*Subsample II*" (2/20/2017 to 12/31/2018) when a 15-hour reference period was used.

To carry out the analysis, it is important to point out some subtle issues about timing. First, the spot rate S_t^{CL} in equation (7) is the rate at the closing time 5PM New York time or 5AM Beijing time of the next day (or 6AM if not in daylight saving time). To clarify, we denote it $S_{t+1,5AM}^{CL}$ in Beijing time by adding the time stamp in the subscript. Similarly, we denote the central parity at date $t+1$ by $S_{t+1,9:15AM}^{CP}$. Second, even though the central parity $S_{t+1,9:15AM}^{CP}$ is announced at 9:15AM of date $t+1$, it is unclear which 24-hour reference period is used in evaluating the changes in the basket. For now, we denote it by $\log\left(X_{t+1,9:15AM}/X_{t,9:15AM}\right)$. Using high-frequency data we will identify the 24-hour reference period shortly. In the end, we can rewrite the regression (7) below in terms of Beijing time:

$$\log \frac{S_{t+1,9:15AM}^{CP}}{S_{t,9:15AM}^{CP}} = \alpha \cdot \log \frac{S_{t+1,5AM}^{CL}}{S_{t,9:15AM}^{CP}} + \beta \cdot (1 - w_{USD}) \log \frac{X_{t+1,9:15AM}}{X_{t,9:15AM}} + \epsilon_{t+1}. \tag{8}$$

[13] This finding is consistent with the regression results, reported in the online appendix, for the period between August 11, 2015 and December 10, 2015 that the regression-based estimate of w is close to zero and the R-squared is very high (around 0.95) for this period.

To cope with the subtle timing issues, we use Bloomberg BFIX intraday data, which are available every 30 minutes on the hour and half-hour throughout the day. Based on the BFIX data, we can thus construct the index-implied dollar basket X_t and the spot rate S_t^{CL} for all 48 half-hour intervals throughout the day.[14] As shown in the online appendix, we find strong empirical evidence that the formation mechanism of the central parity in Subsample (I) uses the 24-hour reference period starting from 7:30AM to 7:30AM the next day, and the spot rate at 4:30PM close. We also find evidence for the (overnight) 15-hour reference period starting 4:30PM to 7:30AM the next day for Subsample (II).

Table 2: Empirical Evidence for the Two-Pillar Policy: Robustness Checks

Panel A: Unconstrained Regressions						
	Subsample I			Subsample II		
	α	β	R^2	α	β	R^2
CFETS	0.56	0.64	0.82	0.74	0.71	0.86
SDR	0.57	0.58	0.76	0.74	0.61	0.85
BIS	–	–	–	–	–	–
Panel B: Constrained Regressions						
	Subsample I			Subsample II		
	α	β	R^2	α	β	R^2
CFETS	0.41	0.59	–	0.59	0.41	–
SDR	0.46	0.54	–	0.60	0.40	–
BIS	–	–	–	–	–	–

Notes: We use intraday Bloomberg BFIX data to run regressions (9) and (10). The results of unconstrained (constrained) regressions are reported in Panel A (Panel B). Column "Subsample I" reports the results from the regression (9) for Subsample period I (12/11/2015 to 2/19/2017) when the 24-hour reference period starting from 7:30AM to 7:30AM the next day is used. Column "Subsample II" reports the results from the regression (10) for Subsample period II (2/20/2017 to 12/31/2018) when the 15-hour reference period starting from 4:30PM to 7:30AM the next day is used. All regression coefficients are statistically significant at the 99 percent level.

Consequently, using Bloomberg BFIX intraday data, we conduct the empirical tests of the two-pillar policy based on the following two regressions:

$$\log\left(\frac{S^{CP}_{t+1,9:15AM}}{S^{CP}_{t,9:15AM}}\right) = \alpha \cdot \log\left(\frac{S^{CL}_{t,4:30PM}}{S^{CP}_{t,9:15AM}}\right) + \beta \cdot (1 - w_{USD}) \log\left(\frac{X_{t+1,7:30AM}}{X_{t,7:30AM}}\right) + \epsilon_{t+1}, \qquad (9)$$

[14] Because Bloomberg has stopped producing the BFIX data for Venezuela currency, we do not construct intraday spot fixings for the BIS index.

and

$$\log\left(\frac{S^{CP}_{t+1,9:15AM}}{S^{CP}_{t,9:15AM}}\right) = \alpha \cdot \log\left(\frac{S^{CL}_{t,4:30PM}}{S^{CP}_{t,9:15AM}}\right) + \beta \cdot (1 - w_{USD}) \log\left(\frac{X_{t+1,7:30AM}}{X_{t,4:30PM}}\right) + \epsilon_{t+1}. \tag{10}$$

We run the regression (9) using the 24-hour reference period for the relevant subperiod "*Subsample I*". Similarly, we run the regression (10) using the 15-hour reference period for the relevant subperiod "*Subsample II*". The results are reported in Columns "Subsample I" and "Subsample II", respectively, in Table 2.

Consider the unconstrained regression results in Panel A of Table 2. As before, the results suggest that the coefficients α and β are roughly equal, although they do not add up to one in the unconstrained regressions. The R-squared range from 76 percent to 86 percent, suggesting that our two-pillar policy does a good job of characterizing the formation mechanism of the central parity in practice. The constrained regression results in Panel B of Table 2 lend further support to the findings.

2.4 Analysis of the Two-Pillar Policy Using Derivatives Data

We have shown that China's exchange rate policy since December 2015 can be characterized with the two-pillar policy. The regression results in the previous subsection provide strong empirical evidence for such policy and suggest that the two pillars carry about the same weights. In this subsection, using derivatives data from the lens of a reduced-form no-arbitrage model, we analyze how the financial market participants assess the two pillar policy. A priori, it is unclear whether the actual policy, characterized by our two-pillar formulation, is optimal or sustainable from the viewpoint of investors. We will turn to a formal theoretical model to study and evaluate the optimality of the two-pillar policy in Section 3 where the government's objective is explicitly modeled. In this subsection, we empirically assess financial market participants' view about the policy.

In particular, building on Jermann (2017), we assume that there is a probability p that the two-pillar policy regime continues, and a probability $(1-p)$ that the policy ends tomorrow. If it ends, the exchange rate equals the fundamental exchange rate V_t that is arbitrage-free itself, satisfying

$$\frac{1+r^{\$}}{1+r^{C}} E^Q_t\left[V_{t+1}\right] = V_t. \tag{11}$$

The interpretation of V_t can be quite broad; for example, we can interpret it as the exchange rate that would prevail when the RMB had become freely floating.

The equilibrium exchange rate $\widetilde{S}_t$ is given by

$$\begin{aligned}\tilde{S}_t &= \frac{1+r^{\$}}{1+r^C}\left[pE_t^Q H\left(\tilde{S}_{t+1}, S_{t+1}^{CP}, b\right) + (1-p)\, E_t^Q V_{t+1}\right] \\ &= \frac{1+r^{\$}}{1+r^C} pE_t^Q H\left(\tilde{S}_{t+1}, S_{t+1}^{CP}, b\right) + (1-p)\, V_t,\end{aligned} \tag{12}$$

where b denotes the width of the trading band around the central parity,

$$H\left(\tilde{S}_{t+1}, S_{t+1}^{CP}; b\right) = \max\left(\min\left(\tilde{S}_{t+1}, S_{t+1}^{CP}(1+b)\right), S_t^{CP}(1-b)\right), \tag{13}$$

and $E_t^Q[\cdot]$ refers to expectations under the RMB risk-neutral measure, and $r^{\$}$ and r^C are per-period interest rates in the U.S. and China, respectively. Intuitively, the current spot rate is the expected value of the exchange rate in the two regimes, appropriately adjusted for the yields.

If the equilibrium exchange rate $\widetilde{S}_t$ falls within the band around the central parity, the observed spot exchange rate at the close is equal to $\widetilde{S}_t$. Otherwise, the spot exchange rate is equal to $\widetilde{S}_t$ truncated at the (lower or upper) boundary of the band. Therefore, the model-implied spot exchange rate at the close then equals

$$S_t^{CL} = H\left(\tilde{S}_t, S_t^{CP}; b\right). \tag{14}$$

Next, we turn to the formation mechanism of the central parity S_t^{CP}. To make the model tractable enough for estimation, we consider the following two-pillar rule, which represents a good approximation of the two-pillar policy empirically and is also consistent with the formation mechanism of the central parity in the PBOC's report in the broad sense. Specifically, we replace the previous close by $S_t^{CP}(V_{t+1}/V_t)^{\gamma}$, $0 \leq \gamma \leq 1$, while keeping the basket pillar unchanged. That is, the two-pillar rule is modeled as the following:

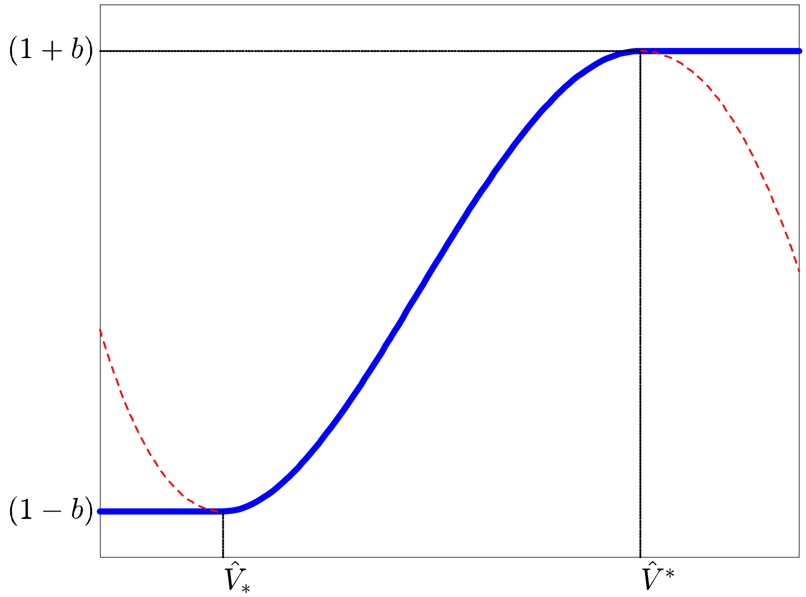
$$\begin{aligned}S_{t+1}^{CP} &= \left[S_t^{CP}\left(\frac{V_{t+1}}{V_t}\right)^{\gamma}\right]^{1-w}\left[S_t^{CP}\left(\frac{X_{t+1}}{X_t}\right)^{(1-w_{USD})}\right]^{w} \\ &\equiv S_t^{CP}\left(\frac{V_{t+1}}{V_t}\right)^{\alpha}\left(\frac{X_{t+1}}{X_t}\right)^{\beta},\end{aligned} \tag{15}$$

where $\alpha \equiv \gamma(1-w)$ and $\beta \equiv (1-w_{USD})\, w$ are some constants bounded between 0 and 1.

The reduced-form model is particularly tractable in continuous time. We show in Appendix B1 that the equilibrium exchange rate $\tilde{S}_t$ has a closed-form solution in continuous time. Specifically, denote by $\widehat{S}_t \equiv \tilde{S}_t/S_t^{CP}$ the equilibrium exchange rate $\tilde{S}_t$ scaled by the central parity S_t^{CP}, and denote by $\widehat{V}_t \equiv V_t/S_t^{CP}$ the scaled fundamental exchange rate. In Proposition 4 in Appendix B1, we prove that the scaled equilibrium exchange rate can be expressed as a univariate function $\widehat{S}(\cdot)$ of the scaled fundamental exchange rate (i.e., $\widehat{S}_t = \widehat{S}(\widehat{V}_t)$), and characterize the solution in closed form. Figure 4 plots the scaled equilibrium exchange rate $\widehat{S}(\widehat{V}_t)$.

Figure 4: Equilibrium Exchange Rate $\widehat{S}(\widehat{V}_t)$

This figure plots in solid line the scaled equilibrium exchange rate $\widehat{S}(\widehat{V})$ as a function of $\widehat{V}$.



As shown in Figure 4, there exist two thresholds $\widehat{V}_*$ and $\widehat{V}^*$ within which the function $\widehat{S}(\widehat{V}_t)$ has a S shape. When $\widehat{V}_t$ hits the threshold $\widehat{V}_*$ or $\widehat{V}^*$, the equilibrium exchange rate $\widetilde{S}_t$ breaches either the lower boundary (i.e., $S_t^{CP}(1-b)$) or the upper boundary (i.e., $S_t^{CP}(1+b)$) of the trading band. When $\widehat{V}_t$ is sufficiently away from both thresholds, the equilibrium exchange rate would change almost one for one in response to changes in the fundamental exchange rate. However, when $\widehat{V}_t$ gets closer to one of the thresholds, government intervention becomes increasingly likely. As a result, the equilibrium exchange rate becomes less sensitive to the fundamental value—a flatter slope of the function $\widehat{S}(\widehat{V}_t)$ near the boundaries of the trading band. This is reminiscent of the exchange rate behavior in a target zone model (see Krugman (1991)) that the expectation of possible intervention affects exchange rate behavior even when the exchange rate lies inside the zone (the so-called "honeymoon effect"). The honeymoon effect implies that the spot exchange rate varies less than the underlying fundamental value. Our estimation explicitly takes into account the non-linear relationship between the observed spot exchange rate and the fundamental value.

We estimate the model to match the closing spot exchange rate and four RMB option prices. We assume that the options market is free of government interventions. The model-implied price of a call option with maturity τ and strike K is given by

$$C(K;\tau) = e^{-r_{CNY}\tau}\left(p^\tau E^Q\left[\max\left(H\left(\widetilde{S}_{t+\tau}, S_{t+\tau}^{CP}, b\right), K\right)\right] + (1-p^\tau)\, E^Q\left[\max\left(V_{t+\tau}, K\right)\right]\right). \quad (16)$$

The price of a put option can be represented in a similar way. The special case with zero trading bandwidth (i.e., $b=0$) is tractable and has closed-form option pricing formula (see the online appendix). However, in the general case with a nonzero trading bandwidth we obtain option prices numerically by simulation.

We estimate (V, p, σ_V) for each day in the sample period between December 11, 2015 and December 31, 2018. The beginning date of the sample period is chosen as December 11, 2015 because the RMB indices were introduced on that date for the first time. During this period, the trading bandwidth is officially 2 percent. However, the effective width is much smaller, around 0.5%. As a result, we choose $b = 0.5\%$ in estimating the model.[15] To simplify the estimation, we fix $\rho = 0$. The parameter γ determines how sensitive the central parity is to the changing market conditions. We choose $\gamma = 1/4$ so that the frequency of the pillar, $S_t^{CP} (V_{t+1}/V_t)^{\gamma}$, staying within the trading band is roughly similar to that of the close S_t^{CL}.

Figure 5: Baseline Parameter Estimates

This figure reports the results of the baseline estimation. In the top panel, we plot the fundamental exchange rate V_t (blue solid line), the central parity (red dashed line), and the close (black dashed line). In the middle panel, we plot the probability of the policy still being in place three months later p_t. In the bottom panel, we plot the fundamental exchange rate volatility σ_V (blue solid line), and the average implied volatilities of 10-delta options (red dashed line) and 25-delta options (black dashed line).

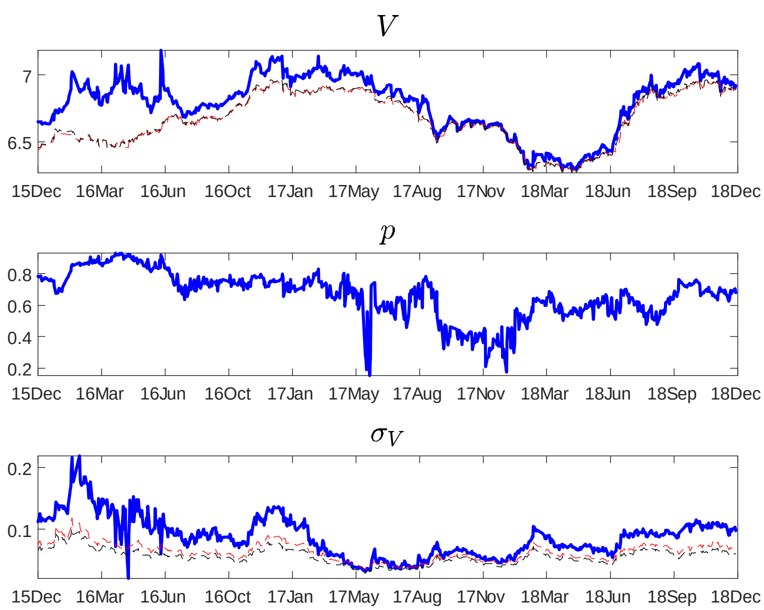


Figure 5 displays the main estimation results. As shown in the top panel, the fundamental exchange rate V_t is estimated to be always greater than both the central parity and spot rates, consistent with depreciation expectations. Implied from the estimate of V_t, the RMB is valued on average about 1.7 percent higher than its fundamental value during the whole sample period. The gap between the spot rate and the estimated fundamental value is particularly elevated in the first

[15]The results for the case of $b = 2\%$, unreported here, are similar and available upon request.

couple of months of the sample period, ranging between 2 percent and 9 percent. The large gap in the early sample period is consistent with the expectation of 10-15 percent depreciation of the RMB.[16] It is possible that the PBOC may intervene in both spot and derivatives markets, which may contribute to the overall small deviations between the estimated fundamental exchange rate and the spot rate.[17] In the next subsection we extend the reduced-form model to the offshore market and provide more discussion on the impact of government intervention.

The middle panel in Figure 5 plots the probability that the current policy would still be in place three months later. It suggests that financial markets attached a high probability to the policy still being in place three months later. The estimated probability of policy continuation fluctuates mostly between 60 percent and 90 percent. In particular, until the countercyclical factor was introduced in May 2017, the average probability of policy continuation is about 76 percent. These findings lend further support to the validity of the two-pillar policy we have formulated.

At the same time our estimation results capture some episodes when financial market participants cast doubt on the sustainability of the two-pillar policy. For example, the model-implied probability of policy continuation drops to the lowest level, around 15 percent, on May 23, 2017 in the week preceding the PBOC's confirmation of adding the new countercyclical factor on May 26, 2017. On January 9, 2018, the PBOC has reportedly removed the countercyclical factor.[18] As a result of the return to the two-pillar policy, the probability of policy continuation has since then increased and stabilized around the average level of 60 percent. Overall, our findings suggest that financial market participants have relatively high confidence in the continuation of the two-pillar policy.

The implied volatility of the fundamental exchange rate shown in the bottom panel of Figure 5 displays a relatively stable pattern in that the implied volatility fluctuates around its average value 8.6 percent during the whole sample period. It is worthwhile to point out that since the U.S. presidential election, the implied volatility has steadily decreased from around 14 percent in mid-December of 2016 to only 4 percent in the end of May, 2017. It suggests "damping currency volatility against the dollar [is] now a bigger priority."[19]

[16] See, for example, the Reuters article "Pressure on China central bank for bigger yuan depreciation" on January 7, 2016.

[17] Consistent with this possibility, we find that the difference between onshore and offshore exchange rates—a gauge for the effect of the government intervention—explains about 5 percent of the variation in the fundamental value scaled by the central parity. This result, unreported but available upon request, also suggests that our fundamental value estimate contains useful information beyond the aforementioned difference.

[18] See the article "China Changes the Way It Manages Yuan After Currency's Jump" on Bloomberg News on January 9, 2018.

[19] See, for example, the article "China Hitches Yuan to the Dollar, Buying Rare Calm" in the Wall Street Journal on May 25, 2017.

2.4.1 Offshore RMB Market

The RMB trades in both onshore (i.e., mainland China) and offshore (e.g., Hong Kong) markets. We extend the reduced-form model to incorporate the joint dynamics between these markets. For greater specificity, we use "CNY" and "CNH" to denote RMB-denominated accounts that trade in onshore and offshore markets, respectively. Similarly, we use "USDCNY" and "USDCNH" to denote the exchange rates of the CNY and the CNH against the USD, respectively. The CNH-CNY basis—the spread between USDCNH and USDCNY—has remained fairly narrow. [20] The small deviations between these two exchange rates also reflect the rising integration of onshore and offshore RMB markets following a series of recent developments.[21]

Figure 6: Onshore vs. Offshore

This figure plots key data series on both onshore CNY (blue solid lines) and offshore CNH (red dashed lines) markets. Panel A plots the spot exchange rates. Panel B plots the interest rates. Panels C and D plot implied volatility quotes for risk reversals for 10%- and 25%-delta options. Panels E and F plot implied volatility quotes for butterfly spreads for 10%- and 25%-delta options.

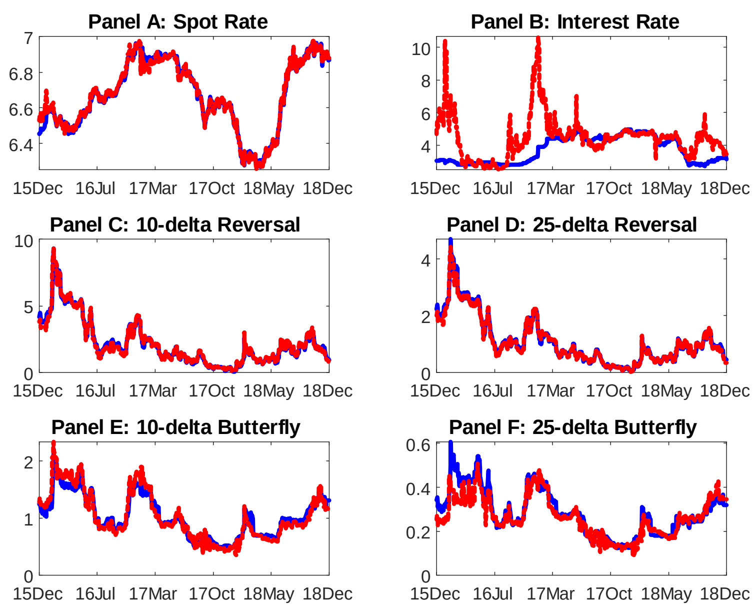


Figure 6 plots key data series on both onshore CNY (blue solid lines) and offshore CNH (red dashed lines) markets for the spot exchange rates (Panel A), the interest rates (Panel B), implied

[20]The IMF viewed the persistent CNH-CNY basis as a technical impediment to the RMB's inclusion in the SDR basket because it implies "the CNH cannot be a perfect hedge for CNY-based exposures." See, for example, the IMF staff report titled "Review of the Method of the Valuation of the SDR—Initial Considerations" on August 3, 2015.

[21]The offshore CNH market started to take off in June 2010 when the authorities in mainland China and Hong Kong introduced a series of reforms that include lifting all restrictions on RMB transfers (regardless of whether the transfer is for trade settlement) and on RMB deposit accounts for corporates, and allowing banks to issue RMB-denominated investment products and loans.

volatility quotes for risk reversals (Panels C and D, respectively) and butterfly spreads (Panels E and F, respectively) for 10%- and 25%-delta options. Panel A of Figure 6 confirms that the onshore and offshore spot rates tend to move in lockstep. The last four panels show that CNY or CNH options have similar prices. As shown in Panel B, the interest rates on these markets occasionally diverge due to the PBOC's intervention.

The PBOC can directly intervene in the offshore spot or forward/swap markets, besides capital controls that limit the RMB flow between the onshore and offshore markets.[22] For the former spot-market intervention, the PBOC can use nonsterilized foreign exchange intervention, for example, to support the CNH by permanently repatriating CNH to mainland China. The cost of such intervention is the immediate run-down of foreign reserves. For instance, offshore PBOC agent banks were believed to have intervened heavily in the spot CNH market, causing the offshore interest rate to sharply rise to more than 10 percent in mid January 2016.[23]

On the other hand, the authorities may also intervene in the offshore forward and swap markets.[24] Forward market intervention through selling-USD-buying-CNH forwards strengthens the CNH because of the covered interest parity; that is, it puts downward pressure on the USDCNH forward rate, mechanically causing the CNH to appreciate against the USD under the covered interest parity. Compared to the spot market intervention, forward market intervention avoids immediately draining foreign reserves.

In the aftermath of the August 2015 devaluation, Chinese commercial banks in Hong Kong acting as agents for the PBOC started intervening in the offshore foreign exchange forward/swap markets. According to the public data on its derivatives holdings disclosed by the PBOC for the first time on March 31, 2016, the central bank held a nominal short position of \$28.9 billion in forwards as of February 29, 2016.[25] From the same data, the PBOC increased its short position to \$45.3 billion in September 2016, indicating another round of intervention in the offshore forward market. The forward market intervention drove up the CNH interest rate in September as shown in Panel B. Later in January and May 2017 the settlement of these forward transactions caused the CNH interest rate to spike again because the counterparty side of the intervention (e.g., investors) had to source CNH for delivery to the agent banks who in turn repatriate the CNH to the PBOC.[26]

[22] Arguably, the PBOC could also intervene in the options market in theory. For example, through the put-call parity relationship, the options intervention may impact the forward rate. However, the PBOC can achieve the same effect by intervening directly in the forward market. The PBOC's derivatives holdings data (discussed below) shows that the PBOC had zero position in options during the period when the data was available.

[23] In January 2016, the CNH was significantly weaker than the CNY amid short selling pressure under the expectations of another imminent devaluation. The elevated implied volatilities for risk reversals in January 2016, shown in Panels C and D of Figure 7, also indicate strong speculative bets on RMB depreciation.

[24] See, for example, a Wall Street Journal article on August 27, 2015, titled "PBOC Uses Unusual Tool to Tame Yuan Fall Expectations."

[25] The same data, however, shows that the PBOC had zero positions in options, signaling no direct intervention in the options market.

[26] Our analysis of the PBOC's derivatives holdings data suggests that as of February 29, 2016 the Chinese central bank held \$28.9 billion short position of forwards at the tenor of around one year, and in September increased the short position to \$45 billion by shorting additionally \$10 billion forward at the tenor of around six months and \$6

In this way, CNH liquidity is reduced in a similar manner as nonsterilized interventions.

We extend the reduced-form model to incorporate the joint dynamics of the onshore and offshore markets. Suppose the cost of arbitraging between these markets is $c \geq 0$. We assume that the onshore spot rate, S_t^{CL}, serves as the "underlying anchor" for the offshore spot rate, denoted T_t^{CL}. Specifically, if we let $\widetilde{T}_t$ denote the equilibrium exchange rate for the offshore CNH market, then the offshore spot rate T_t^{CL} is given by

$$T_t^{CL} = H\left(\widetilde{T}_t, S_t^{CL}; c\right) = \max\left(\min\left(\widetilde{T}_t, S_t^{CL}(1+c)\right), S_t^{CL}(1-c)\right),$$

Similarly as the onshore CNY market, the equilibrium exchange rate for the offshore market, $\widetilde{T}_t$, satisfies the following no-arbitrage condition

$$\begin{aligned} \widetilde{T}_t &= \frac{1+r^{\$}}{1+r^{offshore}} p E_t^Q H\left(\widetilde{T}_{t+1}, S_{t+1}^{CL}; c\right) + (1-p)\frac{1+r^{\$}}{1+r^C} E_t^Q V_{t+1} \\ &= \frac{1+r^{\$}}{1+r^{offshore}} p E_t^Q H\left(\widetilde{T}_{t+1}, S_{t+1}^{CL}; c\right) + (1-p) V_t, \end{aligned} \tag{17}$$

where $r_t^{offshore}$ denotes the offshore interest rate for CNH deposits (i.e., HIBOR rate). We solve and estimate the model in continuous time and set $c = 0.5\%$ to be the same as the trading bandwidth. The equilibrium exchange rate $\widetilde{T}_t$ is a solution to an ordinary differential equation with free boundaries. Because of no closed-form solution, we numerically solve the free-boundary problem using the finite-difference method. The detailed derivation as well as numerical methodology are delegated to Appendix B2.

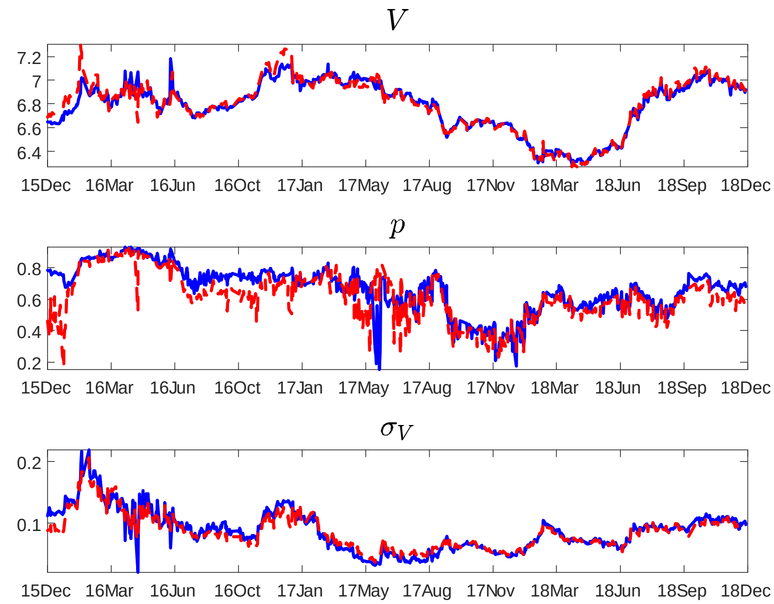
Figure 7 plots the estimation results based on data on both offshore and onshore markets (particularly, the CNH options data). As shown in the figure, the estimation results for the fundamental exchange rate (top panel), the probability of policy continuation (middle panel), and the volatility of the fundamental process (bottom panel) are very close to those in Figure 5 for the onshore market. The similarity in the results is expected given the similar pricing of CNY and CNH options. Furthermore, the PBOC's intervention in the offshore market seems to work through its impact on the offshore funding costs (e.g., spikes in the offshore interest rate as shown in Panel B), which, however, has limited impact on our estimation results. The similarity in the results further validates our reduced-form framework with the two-pillar policy as a plausible approach to study both onshore and offshore RMB markets.

At this point we have empirically assessed financial market participants' view about the two-pillar policy. We next turn to a formal theoretical model to study and evaluate the optimality of the policy.

billion forwards at the tenor of around one year. These forward contracts started to mature in February through May 2017 according to the PBOC's derivatives holdings data.

Figure 7: Offshore CNH Market Estimation Results

This figure reports the estimation results for both offshore (red dashed line) and onshore (blue solid lines) markets. We plot the fundamental exchange rate V_t, the probability of the policy still being in place three months later p_t, and the fundamental exchange rate volatility σ_V in the top, middel, and bottom panels, respectively.



3 Theoretical Model

In this section, we develop a conventional flexible-price monetary model for exchange rate determination based on Svensson (1994).[27] Our model extends the Svensson model along two important dimensions. First, we incorporate the two-pillar policy and provide a theoretical microfoundation for such policy in order to evaluate its optimality. Second, we further extend the model to examine intraday government intervention.

3.1 Setup

There is an infinite number of periods with each period divided into two subperiods: "*AM*" and "*PM*". In each period, the government in the home country (China) chooses the optimal central parity at subperiod *AM*, and the optimal monetary and exchange rate policies at subperiod *PM*.

Specifically, at the *PM* of each period, the government chooses the optimal level of money stock for period t, denoted by m_t, which then determines in equilibrium the domestic interest rate i_t, and the exchange rate e_t under rational expectations. The money market equilibrium condition

[27] We are very grateful to the editor and the associate editor for very constructive suggestions that motivated us to develop the theoretical model for the RMB in this section.

for the home country links the logarithm of the money stock (m_t) deflated by the logarithm of the price level, p_t, to the domestic interest rate, i_t, given by:

$$m_t - p_t = -\alpha i_t. \tag{18}$$

Assuming zero foreign exchange risk premium,[28] the domestic interest rate satisfies the equilibrium condition

$$i_t = i_t^* + E_t\left[e_{t+1} - e_t\right]/\Delta t, \tag{19}$$

where i_t^* denotes the interest rate in the foreign country (the United States), and $E_t\left[\cdot\right]$ denotes the rational expectation. The log of the real exchange rate, q_t, is given by

$$q_t = p_t^* + e_t - p_t, \tag{20}$$

where e_t denotes the spot exchange rate expressed in units of domestic currency (i.e., RMB) per unit of foreign currency (i.e., USD). As normalization we set $p_t^* = 0$.

At the *AM* of each period, the home country (indexed by N) trades with N other countries, indexed by $i = 0, \cdots, N-1$. The U.S. is the numeraire country (indexed by 0), and countries in the rest of the world (RoW) are indexed by 1 through $N-1$. Assume that the price of each country's product is 1 in terms of its currency. Let $R_t^{(i)}$ denote the price of currency i in dollars, then country i's product costs $R_t^{(i)}$ dollar.

Through the balance-of-payments model in Flanders and Helpman (1979), we show that if the government's objective is *solely* minimizing the variability in the trade-balance growth, then the optimal exchange rate policy is a basket peg (see Appendix C). Specifically, let $c_t \equiv \log S_t^{CP,CNY/USD}$ denote the logarithm of the central parity $S_t^{CP,CNY/USD}$, and let TB_t denote the surplus in the home country's balance of trade in terms of dollars defined as exports minus imports. As shown in Appendix C, conditional on the observations of $\Delta \log R_t^{(i)} \equiv \log R_t^{(i)}/R_{t-1}^{(i)}$, for $i = 0, \cdots, N-1$, minimizing the variability in the trade-balance growth, i.e., $\min_{c_t}\left(\Delta \log TB_t\right)^2$, is equivalent to the following:

$$\min_{c_t}\left(\sum_{i=1}^{N-1}\omega_i \Delta \log R_t^{(i)} + c_{t-1} - c_t\right)^2,$$

where ω_i denotes the weight of currency i in a currency basket. Note that by definition $R_t^{(0)}$ is a constant equal to one and thus is dropped from the above objective function. Under additional simplifying assumptions, the optimal weights $\{\omega_i\}_{i=0}^{N-1}$ are shown to equal the share of country i in the home country's exports. The weight for the USD in the basket $\omega_0 = 1 - \sum_{j=1}^{N-1}\omega_j$ is determined

[28] We will relax this assumption later in an extension of the model in which noise trading and intraday government intervention give rise to time-varying foreign exchange risk premium.

as well.[29] Therefore, minimizing the variability in the trade-balance growth is equivalent to

$$\min_{c_t} \left(\left(1-\omega_0\right) \Delta x_t + c_{t-1} - c_t \right)^2,$$

where $x_t \equiv \log X_t$ denotes the logarithm of the basket-implied dollar index X_t. If the government only cares about the stability of the trade-balance growth, the optimal exchange rate policy is thus a basket peg, that is, $c_t = \left(1-\omega_0\right) \Delta x_t + c_{t-1}$, which is equal to the logarithm of the pillar $\overline{S}_t$ in equation (5).

However, the government may also have other policy targets to meet in its objective function, besides the stability of the trade-balance growth. For example, the government may want to minimize the variability of the exchange rate $\left(c_t - e_{t-1}\right)^2$ or c_t^2, where e_{t-1} denotes the logarithm of the spot exchange rate expressed in units of the home currency per unit of numeraire currency (in our case, the RMB against the USD).

Therefore, we extend Svensson (1994) to consider the following government's objective function:[30]

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\xi_d d_t^2 + \xi_i i_t^2 + \xi_{\Delta x} \left(\left(1-\omega_0\right) \Delta x_t - \Delta c_t \right)^2 / \Delta t + \xi_{\Delta e} \left(c_t - e_{t-1}\right)^2 / \Delta t + \xi_c c_t^2 \right] \Delta t, \tag{21}$$

where $d_t \equiv e_t - c_t$ denotes the exchange rate deviation relative to the central parity. The above objective function implies sufficient flexibility for the government to balance among the competing targets of smoothing interest rate and stabilizing the exchange rate and the trade-balance growth.

3.2 The Government's Problem

We now analyze and solve the government's problem using dynamic programming. At the AM of period t, the government observes the realization of Δx_t, c_{t-1}, d_{t-1}, as well as other pre-determined variables. We stack these state variables into the vector $Y_t = \left(\Delta x_t, c_{t-1}, q_{t-1}, i^*_{t-1}, m_{t-1}, d_{t-1}, i_{t-1}\right)'$.

Let $\mathcal{U}\left(Y_t\right)$ denote the government's value function at the AM of period t, that is,

$$\mathcal{U}\left(Y_t\right) = \min_{\{c_s\}} E_t^{AM} \sum_{s=t}^{\infty} \beta^{(s-t)} \left[\begin{array}{c} \left(\xi_d d_s^2 + \xi_i i_s^2\right) \Delta t \\ +\xi_{\Delta x} \left(\left(1-\omega_0\right) \Delta x_s - \Delta c_s \right)^2 + \xi_{\Delta e} \left(c_s - e_{s-1}\right)^2 + \xi_c c_s^2 \end{array} \right], \tag{22}$$

where $E_t^{AM}\left[\cdot\right]$ denotes the expectation conditional on the information set at the AM of period t. Consistent with the data, we assume that the change in the basket-implied dollar index Δx_t

[29] Note that ω_0 corresponds to ω_{USD} in the previous section, and the basket-implied dollar index (see (4) in the previous section) is given by $X_t \equiv C_X \prod_{j=1}^{N-1} \left(R_t^{(j)}\right)^{-\frac{\omega_i}{1-\omega_0}}$.

[30] The model can be easily generalized to include more targets in the government's objective function. In the online appendix, we provide a generalized framework with additional targets such as $\xi_{\Delta d} \left(\Delta d_t\right)^2 / \Delta t$, $\xi_{\Delta i} \left(\Delta i_t\right)^2 / \Delta t$, $\xi_u u_t^2 / \Delta t$ from Svensson (1994).

follows an independent process across time:

$$\Delta x_t = \epsilon_{\Delta x,t},$$

where $\epsilon_{\Delta x,t}$ follows a standard normal distribution with mean zero and variance $\sigma^2_{\Delta x}\Delta t$.

At the PM of period t, besides observing the central parity c_t, the government also observes the realizations of q_t, i^*_t, and other pre-determined variables, which are stacked into the vector:

$$X_t = \left(q_t, i^*_t, m_{t-1}, d_{t-1}, i_{t-1}, c_t\right)'.$$

Using dynamic programming, the government's problem at PM has the following recursive formulation:

$$\mathcal{V}(X_t) = \min_{u_t} \left(\xi_d d_t^2 + \xi_i i_t^2\right)\Delta t + \beta E_t^{PM}\left[\mathcal{U}(Y_{t+1})\right], \tag{23}$$

where $u_t \equiv m_t - m_{t-1}$ denotes the change in the level of money supply, $\mathcal{V}(X_t)$ denotes the government's value function at the PM of period t, and $E_t^{PM}[\cdot]$ denotes the expectation conditional on the information set at the PM of period t. The real exchange rate q_t and the foreign interest rate i^*_t follows exogenous AR(1) processes:

$$\begin{aligned} q_t &= \left(1-\rho_q\Delta t\right) q_{t-1} + \epsilon_{q,t}, \\ i^*_t &= \left(1-\rho_{i^*}\Delta t\right) i^*_{t-1} + \epsilon_{i^*,t}, \end{aligned}$$

where $\epsilon_{q,t} \sim N\left(0, \sigma^2_q\Delta t\right)$ and $\epsilon_{i^*,t} \sim N\left(0, \sigma^2_{i^*}\Delta t\right)$ are independent and normally distributed shocks. In the data, both processes are highly persistent with the AR(1) coefficients close to one.

The government's problem at AM has a similar recursive formulation:

$$\mathcal{U}(Y_t) = \min_{c_t} \xi_{\Delta x}\left(\left(1-\omega_0\right)\Delta x_t - \Delta c_t\right)^2 + \xi_{\Delta e}\left(c_t - e_{t-1}\right)^2 + \xi_c c_t^2 \Delta t + E_t^{AM}\left[\mathcal{V}(X_t)\right]. \tag{24}$$

The Bellman equations (23) and (24) constitute a standard linear-quadratic optimization problem. We show that the value functions $\mathcal{U}(Y_t)$ and $\mathcal{V}(X_t)$ take a quadratic form

$$\begin{aligned} \mathcal{U}(Y_t) &= \left(Y_t' U Y_t + U_0\right)\Delta t, \\ \mathcal{V}(X_t) &= \left(X_t' V X_t + V_0\right)\Delta t, \end{aligned} \tag{25}$$

where the coefficients U, U_0, V, and V_0 are determined endogenously.

3.3 Endogenous Two-Pillar Policy

We now solve the government's problem at the AM period in (24) for the optimal formation mechanism for the central parity. We decompose the state vector X_t into the vector of exogenous shocks $X_t^{(1)} = (q_t, i_t^*)'$, pre-determined variables $X_t^{(2)} = (m_{t-1}, d_{t-1}, i_{t-1})'$, and the endogenous variable $X_t^{(3)} = c_t$. The coefficient matrix V in the value function $\mathcal{V}(X_t)$ is also decomposed accordingly. Similarly, we decompose the state vector Y_t into $Y_t^{(1)} = (\Delta x_t, c_{t-1})'$, $Y_t^{(2)} = \left(q_{t-1}, i_{t-1}^*\right)'$, and $Y_t^{(3)} = (m_{t-1}, d_{t-1}, i_{t-1})'$, and do the similar decomposition to U.

We can then solve the government's problem in (24) for the optimal central parity rule. The proposition below reports the result.

Proposition 1 *Suppose the value function $\mathcal{V}(X_t)$ takes the quadratic form in (25), let $V_{cc} \equiv V^{(3,3)}$, then the optimal central parity has the following two-pillar representation:*

$$c_t = w_1 e_{t-1} + w_2 \left(c_{t-1} + (1 - \omega_0)\, \Delta x_t\right) + h_{t-1}, \tag{26}$$

where $w_1 \equiv \frac{\xi_{\Delta e}/\Delta t}{V_{cc}+\xi_c+(\xi_{\Delta e}+\xi_{\Delta x})/\Delta t}$, $w_2 \equiv \frac{\xi_{\Delta x}/\Delta t}{V_{cc}+\xi_c+(\xi_{\Delta e}+\xi_{\Delta x})/\Delta t}$, and h_{t-1} is related to the hedging demand.

Proof. See Appendix D1. ■

The optimal central parity rule in equation (26) resembles the two-pillar policy we have formulated. In particular, the first two terms correspond to the two pillars. The last term h_{t-1} represents the government's hedging demand as the state of the economy varies over time. As we will characterize the equilibrium more sharply, the hedging term h_{t-1} turns out to depend on the lagged US interest rate only. Note that if the U.S. interest rate is independent over time, the hedging term h_{t-1} is zero. Based on the calibrated parameter values, the hedging term is small in magnitude. So the optimal central parity rule is primarily captured by our two-pillar policy. In addition, we show below that when the government does not care about the interest rate variability (i.e., $\xi_i = 0$), the hedging term is also zero because there is no longer demand for hedging the foreign interest rate risk.

3.4 Optimal Monetary and Exchange Rate Policies

We now turn to the government's optimization problem at the *PM* of period t. We focus on *discretionary* policies where the central bank reoptimizes each period under discretion. As a consequence, the interventions in each period will only depend on the predetermined variables in that period. In particular, the exchange rate deviation from the central parity, i.e., $d_t = e_t - c_t$, is a forward-looking state variable. In a rational expectations equilibrium private agents' expectations incorporate the restriction that the forward-looking variable d_t is chosen as a function of the predetermined variables X_t in that period. Formally, the restriction is

$$d_t = DX_t,$$

where the *endogenous* matrix D in our case is a 1×6 row vector. We focus on stationary equilibriums.

We stack the state vector X_t and the forward-looking variable d_t into the vector Z_t. That is, the first six elements of Z_t is X_t and the last (seventh) element is d_t. The transition equation can be written as

$$\begin{bmatrix} X_{t+1} \\ E_t\left[d_{t+1}\right] \end{bmatrix} = AZ_t + Bu_t + \epsilon_{Z,t+1}, \tag{27}$$

where $\epsilon_{Z,t+1} \equiv \left(\epsilon_{q,t+1}, \epsilon_{i^*,t+1}, 0, 0, 0, w_2\left(1-\omega_0\right)\Delta x_{t+1}, 0\right)'$ and the expressions for coefficients A and B are given in (56) in Appendix D2.

Given the above linear transition equation, the government's PM problem in (23) can be rewritten as the following linear-quadratic problem:

$$\begin{aligned} \frac{1}{\Delta t}\mathcal{V}\left(X_t\right) &= \min_{u_t} \xi_d d_t^2 + \xi_i i_t^2 + \beta E_t^{PM}\left[\frac{1}{\Delta t}\mathcal{U}\left(Y_{t+1}\right)\right] \\ &= \min_{u_t}\left(X_t'Q^*X_t + X_t'W^*u_t + u_t'W^{*\prime}X_t + u_t'R^*u_t\right) \\ &\quad +\beta\left(X_t'\widetilde{Q}^*X_t + X_t'\widetilde{W}^*u_t + u_t'\widetilde{W}^{*\prime}X_t + u_t'\widetilde{R}^*u_t + U_{11}Var\left(\Delta x\right)\right), \end{aligned} \tag{28}$$

where the expressions of coefficients (e.g., Q^*, $\widetilde{Q}^*$, etc.) are given in Appendix D3. This is a standard linear-quadratic problem to solve. Its solution is reported in the proposition below.

Proposition 2 *In a stationary rational expectations equilibrium, the optimal monetary policy u_t solves the problem in (28), given by*

$$u_t = -\left(R^* + \beta\widetilde{R}^*\right)^{-1}\left(W^* + \beta\widetilde{W}^*\right)' X_t \equiv -F^*X_t, \tag{29}$$

where $F^* \equiv \left(R^* + \beta\widetilde{R}^*\right)^{-1}\left(W^* + \beta\widetilde{W}^*\right)'$. In equilibrium, the exchange rate deviation relative to the central parity must satisfy

$$d_t = HX_t + Gu_t = \left(H - GF^*\right)X_t \equiv DX_t. \tag{30}$$

The value function $\mathcal{V}\left(X_t\right) = \left(X_t'VX_t + V_0\right)\Delta t$ is determined where $V_0 = \beta U_{11}Var\left(\Delta x\right)$ and

$$V = Q^* + \beta\widetilde{Q}^* - \left(W^* + \beta\widetilde{W}^*\right)F^* - F^{*\prime}\left(W^* + \beta\widetilde{W}^*\right)' + F^{*\prime}\left(R^* + \beta\widetilde{R}^*\right)F^*.$$

Proof. See Appendix D3. ∎

The above proposition together with Proposition (26) fully characterize the stationary equilibrium. In the discretionary equilibrium with the rational-expectations restriction $d_t = DX_t$, the transition equation in (27) implies that the exchange rate deviation d_t linearly depends on both the state vector X_t and the control u_t in equilibrium; that is, $d_t = HX_t + Gu_t$ as shown in (30). Under the optimal monetary policy $u_t = F^*X_t$ in (29), equation (30) imposes an explicit constraint on D as a result of rational expectations. We thus need to solve the matrices U and V in the value functions together with D jointly as a fixed point to the system of the Bellman equations and the rational-expectations restriction in equation (30).

We can further sharpen the characterization of the equilibrium. The corollary below summaries the results.

Corollary 3 *In the stationary equilibrium,*

(i) the hedging term h_{t-1} depends on only i^*_{t-1}, i.e., $h_{t-1} = h \cdot i^*_{t-1}$ where h is an endogenous constant coefficient. When $\rho_{i^*} = 1/\Delta t$ or $\xi_i = 0$, it must hold that $h = 0$; i.e., the hedging term is zero.

(ii) Furthermore, the optimal monetary policy $u = -F^* X_t$ satisfies:

$$F^* = [1, F_2, 1, 0, 0, F_6],$$

where the expressions of F_2 and F_6, or the second and sixth elements of the vector F^*, are provided in the proof.

Proof. See Appendix D4. ■

The above corollary shows that in equilibrium, under the optimal monetary policy $u_t = -q_t - m_{t-1} - F_2 i^*_t - F_6 c_t$. In other words, the government chooses the money supply $m_t = m_{t-1} + u_t$ optimally such that the exogenous real exchange rate shock q_t is fully absorbed. This explains why the hedging demand h_{t-1} only depends on i^*_{t-1}, but not q_{t-1}. Intuitively, when the US interest rate is independent over time (i.e., $\rho_{i^*} = 1/\Delta t$) or the government does not care about the interest rate variability (i.e., $\xi_i = 0$), there is no longer demand for hedging the foreign interest rate risk, implying a zero hedging term.

In sum, in the theoretical model, the government optimally chooses its monetary and exchange rate policies. From the lens of the model, we next examine the optimality of the two-pillar policy in practice.

3.5 Results

We calibrate the model and report the calibrated parameter values in Table 3. Because the inflation data needed for constructing the real exchange rate measure are available only at the monthly frequency, we focus on the monthly frequency and convert the daily data to monthly by keeping end-of-month observations. The period length dt is thus one month (i.e., $dt = 1/12$). Following Svensson (1994), the interest-elasticity of the demand for money α is set to 0.5 year and the time discount factor β is set to 0.9913 for a month (or equivalently the annualized discount factor is equal to 0.9). As we focus on 3-month RMB options and forwards, the maturity in periods τ is therefore set equal to three. To calibrate ρ_q, ρ_{i^*}, and $\rho_{\Delta x}$ (respectively, the rates of mean reversion for the real exchange rate q_t, the US interest rate i_t^*, and the index-implied dollar basket Δx_t), we run univariate first-order autoregressions within the sample period between December 2015 and December 2018. We construct the real exchange rate measure using (20) based on the nominal exchange rate data as well as CPI inflation data for China and the U.S. Because of the tightening U.S. monetary policy during this period, we linearly detrend the US interest rate and use detrended interest rate in the autoregression. For the index-implied dollar basket, we use the basket implied from the CFETS index for calibration; the results based on other RMB indices are similar.

The autoregression results suggest large persistence in q_t and i_t^* whose AR(1) coefficients are around 0.9. The calibrated values for ρ_q and ρ_{i^*} are 1.29 and 1.45 per year. By contrast, the growth rate of the index-implied dollar index Δx_t are serially uncorrelated with the AR(1) coefficient nearly zero. We thus set $\rho_{\Delta x} = 1/dt$ so that the process Δx_t is independent over time. From the autoregression results, we infer the values for the standard deviations σ_q, σ_{i^*}, and $\sigma_{\Delta x}$. The calibrated parameter values are reported in Table 3 below.

Table 3: Calibrated Parameters

Parameter	dt	τ	ω_0	α	β	ρ_q	ρ_{i^*}	$\rho_{\Delta x}$	σ_q	σ_{i^*}	$\sigma_{\Delta x}$
Value	1/12	3	0.2240	0.5	0.9913	1.29	1.45	12	0.05	0.0026	0.0535

Notes: This table reports the calibrated values for the parameters in the model. The period of length is one month ($dt = 1/12$) and the horizon is three months ($\tau = 3$). ω_0 denotes the dollar's weight in the RMB index. α denotes the interest-elasticity of the demand for money. β denotes the time discount factor. ρ_q, ρ_{i^*}, and $\rho_{\Delta x}$ denote the rates of mean reversion for the real exchange rate q_t, the US interest rate i_t^*, and the index-implied dollar index Δx_t, respectively, while σ_q, σ_{i^*}, and $\sigma_{\Delta x}$ denote the corresponding standard deviations.

Based on the calibrated parameter values, we provide a quantitative analysis of the model. By varying the weights on various targets (i.e., ξ_d, ξ_i, $\xi_{\Delta x}$, $\xi_{\Delta e}$, ξ_c), we can trace out the multi-dimensional trade-off between the targets. Throughout the analysis, we set $\xi_c = 0.01$ for the sake of numerical stability and normalize the remaining target weights $(\xi_d, \xi_i, \xi_{\Delta e}, \xi_{\Delta x})$ such that they sum up to one. In the analysis below, we set $\xi_{\Delta e} + \xi_{\Delta x} = 0.5$ and $\xi_d + \xi_i = 0.5$.

To quantitatively illustrate the policy trade-off that the government faces, we vary the target weights $\xi_{\Delta e}$ and $\xi_{\Delta x}$ to examine how changes in relative contributions of both pillars influence the central parity. We also shift the target weights on the variability of interest rates and exchange rate deviation, i.e. ξ_d and ξ_i.

Table 4 reports the main results from the quantitative analysis. As shown in Panel A, the interest rate's standard deviation in the data is 0.79 percent. The standard deviation of the exchange rate deviation from the central parity is 0.25 percent. The standard deviation of the difference between the central parity and the previous close (i.e., $c_t - e_{t-1}$) is 4.27 percent. The standard deviation of the difference between the central parity and the basket pillar (i.e., $c_t - (c_{t-1} + (1-\omega_0)\,\Delta x_t))$ is smaller at 3.47 percent. Finally, the central party's standard deviation is 2.94, and the M2 growth rate's standard deviation is 2.32 percent.

Table 4: Model Outcome

	Target		Pillar		Std. Dev. (%)					
	ξ_d	ξ_i	w_1	w_2	d_t	i_t	$\frac{c_t - e_{t-1}}{\sqrt{\Delta t}}$	$\frac{c_t - \bar{s}_t}{\sqrt{\Delta t}}$	c_t	$\frac{u_t}{\sqrt{\Delta t}}$
Panel A: Data										
					0.25	0.79	4.27	3.47	2.94	2.32
Panel B: Dollar Pillar Only $(\xi_{\Delta e} = 0.5, \xi_{\Delta x} = 0)$										
B1	0.5	0	0.99	0	0	0.16	0	4.15	0	5.14
B2	0	0.5	1.00	0	0.05	0.00	0.00	4.16	0.11	5.14
Panel C: Basket Pillar Only $(\xi_{\Delta e} = 0, \xi_{\Delta x} = 0.5)$										
C1	0.5	0	0	0.96	0	1.89	4.04	0.56	4.34	7.11
C2	0	0.5	0	0.96	4.34	0	15.04	0.56	4.34	5.14
Panel D: Both Pillars $(\xi_{\Delta e} = 0.25, \xi_{\Delta x} = 0.25)$										
D1	0.5	0	0.49	0.49	0.08	1.07	2.02	2.16	1.89	5.73
D2	0	0.5	0.39	0.39	0.49	0.09	1.72	2.77	0.51	5.15
D3	0.48	0.02	0.44	0.44	0.26	1.67	1.84	2.43	0.78	6.17
D4	0.44	0.06	0.39	0.39	0.40	0.78	1.66	2.73	0.53	5.45

Notes: Panel A of this table reports, respectively, the standard deviations of the exchange rate deviation d_t, the domestic interest rate i_t, the differences between the central parity and the two pillars scaled by $\sqrt{\Delta t}$ (i.e., $\frac{c_t - e_{t-1}}{\sqrt{\Delta t}}$ and $\frac{c_t - \bar{s}_t}{\sqrt{\Delta t}} = \frac{\Delta c_t - (1-\omega_0)\Delta x_t}{\sqrt{\Delta t}}$), the central parity c_t, and the scaled money growth rate $\frac{u_t}{\sqrt{\Delta t}}$ in the data. The standard deviations are reported in percentage. In Panels B through D, we report the model-implied standard deviations for the same variables with different specifications of target weights $(\xi_d, \xi_i, \xi_{\Delta e}, \xi_{\Delta x})$. Columns "$w_1$" and "$w_2$" reports the coefficients in the optimal central parity rule $c_t = w_1 e_{t-1} + w_2\,(c_{t-1} + (1-\omega_0)\,\Delta x_t) + h i^*_{t-1}$ in the model. The coefficient h is zero in all cases, except Cases $D2$ through $D4$ in which h is equal to 0.13, 0.06, and 0.13, respectively.

First, we consider the case where the government does not care about the current account variability, and thus put zero weight on the basket stability (i.e. $\xi_{\Delta x} = 0$, and $\xi_{\Delta e} = 0.5$). The results are reported in Panel B. In this case, the central parity has a single pillar, namely the pervious close. If the target weight on the interest rate is also zero (see case B1), then it is essentially a dollar peg with a fixed exchange rate. In fact, all the exchange rates (central parity or spot rate) are constant and thus exhibit zero variability. The interest rate's standard deviation is 0.16 percent, or about one fifth of the level observed in the data. However, the lower interest rate volatility comes at the expense of a more volatility current account because the hard dollar peg in this special case passes through all the foreign exchange shocks to the current account. The resulting standard deviation of the difference between the central parity and the basket pillar is as high as 4.16 percent, which is 20 percent above the level in the data.

When the target weights shift completely from the exchange rate deviation to the interest rate (see case B2), the spot exchange rate fluctuates in response to the foreign U.S. interest rate. The central parity is almost equal to the previous close with the weight w_1 close to one. As a result, the standard deviations of both d_t and $(c_t - e_{t-1})$ are relatively small at 0.05 percent and 0.0002 percent, respectively. As the primary target is the interest rate smoothing, the standard deviation of the interest rate is essentially zero. Nevertheless, the standard deviation of the difference between the central parity and the basket pillar remains large as in case B1.

Next, we move on to study the other extreme case where the government does not care about the variability of $(c_t - e_{t-1})$, and thus shifts the weight completely from $\xi_{\Delta e}$ to $\xi_{\Delta x}$ (i.e. $\xi_{\Delta x} = 0.5$, and $\xi_{\Delta e} = 0$). The results are reported in Panel C. In this case, the central parity is solely driven by the basket pillar (i.e., $c_t = w_2 \left(c_{t-1} + (1 - \omega_0) \Delta x_t \right)$). As a result, the domestic government let more volatility go into the central parity, the spot exchange rate as well as the interest rate because of the foreign exchange shocks arising from the currencies in the basket. The results in cases C1 and C2 are qualitatively similar to those in cases B1 and B2, but now have larger volatilities. For example, in case C1 where $\xi_i = 0$, the interest rate's volatility increases to 1.89 percent from only 0.16 percent in case B1. In case C2 where $\xi_d = 0$, the volatility of the exchange rate deviation increases to 4.34 percent from only 0.05 percent in case B2. At the expense of the higher volatilities on these targets, the domestic government, however, achieves a more stable current account whose volatility is now reduced to only 0.56 percent in both cases C1 and C2 from around 4 percent in the previous cases B1 and B2.

Lastly, we examine the case where the domestic government puts equal weights on both pillars. The results are reported in Panel D. This case is supported by the empirical evidence shown in Section 2. The major advantage of having such a two-pillar policy is to balance the targets with respect to both pillars (i.e., $(c_t - e_{t-1})$ and $(c_t - ((c_{t-1} + (1 - \omega_0) \Delta x_t)))$. Focusing on cases D1 and D2, from columns labeled as $\frac{c_t - e_{t-1}}{\sqrt{\Delta t}}$ and $\frac{\Delta c_t - (1-\omega_0)\Delta x_t}{\sqrt{\Delta t}}$ we can see the balance between these two targets. Unlike cases B1 or B2 with a large volatility of the latter target, or cases C1 or C2 with a large volatility of the former target, the volatilities of both targets are now balanced and have similar levels in the range of 1.7 to 3 percent.

Based on the calibrated model, we can quantitatively assess the trade-off faced by the government. Following Svensson (1994), if we interpret the trading bandwidth as three standard deviations of the exchange rate deviation, then the trading bandwidth is roughly 0.75 percent because the standard deviation of the exchange rate deviation is 0.25 percent in the data. Based on our calibrated parameters, we show that under the 0.75-percent bandwidth, the standard deviation of the interest rate is 1.67 percent, about twice as large as the level in the data (see case D3). However, if we increase the trading bandwidth to 1.2 percent, the standard deviation of the interest rate drops to 0.78 percent, similar to the level observed in the data (see case D4), reflecting increased monetary policy independence. Our findings suggest that the effective trading bandwidth is significantly less than the official one.

3.6 Extension: Intraday Government Intervention

In the model so far, the government engages in sterilized intervention to balance between exchange rate volatility and interest rate volatility. The exchange rate volatility reflects shocks in the fundamentals. In reality, exchange rate volatility may include a nonfundamental component, for example, as a result of noise trading out of whims, fads, or sentiment. In this subsection, we augment the main monetary model of the exchange rate determination by the microstructural theory of noise trading (De Long et al. (1990)). In the extended model, the government finds it optimal to intervene intraday in a nonsterlized manner to offset noise trading. As we will show shortly, the nonsterlized intraday government intervention is an effective tool to "lean against noise traders", forestalling market failure. For simplicity of exposition, we assume exogenous variables q_t and i_t^* follow i.i.d. processes.

We now describe the microstructure part of the extended model. Different from the main model, we now assume that there exist two types of investors: informed traders and noise traders. We assume that there is a continuum of informed investors in the foreign exchange market, indexed by $i \in [0,1]$, who trade both the RMB and the dollar. They are assumed to be myopic, and live for only two periods in which they trade in the first one and consume in the second. Each investor born at date t is endowed with wealth $\overline{W}$ and chooses optimal investment-consumption strategy to maximize the expected CARA utility over its next-period wealth W_{t+1}^i:

$$\max_{X_t^i} E\left[-\exp\left(-\gamma W_{t+1}^i\right)\middle|\, \mathcal{F}_t^i\right] \tag{31}$$

$$s.t., W_{t+1}^i = (1 + i^* \Delta t)\, \overline{W} + X_t^i \rho_{t+1},$$

where $\mathcal{F}_t^i$ denotes the information set for informed investors, and ρ_{t+1} denotes the log-linearized excess return on RMB-denominated bonds

$$\rho_{t+1} \equiv (i_t - i_t^*)\,\Delta t - (e_{t+1} - e_t)\,. \tag{32}$$

It is straightforward to show that the optimal demand for the RMB by informed investors is given by

$$X_t^i = \frac{E_t^i\left[\rho_{t+1}\right]}{\gamma Var_t^i\left[\rho_{t+1}\right]},$$

where $E_t^i[\cdot]$ and $Var_t^i[\cdot]$ denote the conditional mean and variance under their information set, respectively. When the RMB is expected to depreciate more relative to the dollar, the expected excess return is negative, prompting informed investors to sell the RMB and buy the dollar.

Assume informed traders hold rational expectations so that

$$\begin{aligned} E_t^i\left[\rho_{t+1}\right] &= E_t\left[\rho_{t+1}\right], \\ Var_t^i\left[\rho_{t+1}\right] &= Var_t\left[\rho_{t+1}\right], \end{aligned}$$

where $E_t[\cdot]$ and $Var_t[\cdot]$ denote the conditional mean and variance under rational expectations, respectively. We will show that the conditional variance is time-invariant and thus simply use $Var\left(\rho_{t+1}\right)$ by dropping the time subscript (see Appendix E).

The second type of investors is noise traders, who trade for exogenous reasons. As is standard in the market microstructure literature, the noise traders submit market orders with quantity N_t which is assumed to be an i.i.d. process:

$$N_t = \epsilon_{N,t} \overset{i.i.d.}{\sim} N\left(0, \sigma_N^2\right),$$

where $\sigma_N > 0$ measures the volatility of noise trading (or the amount of noise trading risk), and $\epsilon_{N,t} \sim N\left(0, \sigma_N^2\right)$ represents independently and identically distributed shocks.

Besides informed and noise traders, the government is another player whose demand for the RMB is denoted by X_t^G. Following Brunnermeier, Sockin, and Xiong (2018), we model the government intervention as follows:

$$\begin{aligned} X_t^G &= \vartheta_{N,t}N_t + \sqrt{Var\left[\vartheta_{N,t}N_t \middle| \mathcal{F}_{t-1}\right]}G_t \\ &= \vartheta_{N,t}N_t + \vartheta_{N,t}\sigma_N G_t. \end{aligned}$$

The first term $\vartheta_{N,t}N_t$ represents the government's intended intervention strategy in trading against the noise traders. The coefficient $\vartheta_{N,t}$ represents the intervention intensity and is endogenously chosen by the government. The second term $\sqrt{Var\left[\vartheta_{N,t}N_t \middle| \mathcal{F}_{t-1}\right]}G_t$ or simply $\vartheta_{N,t}\sigma_N G_t$ captures unintended intervention-induced noise with $G_t = \epsilon_{G,t}$ and $\epsilon_{G,t} \sim N\left(0, \sigma_G^2\right)$ representing independently and identically distributed shocks. Under this specification, the magnitude of the second term scales up with the conditional volatility of the government's intended intervention strategy, which is useful in capturing the notion that the government may face more frictions associated with more intensive intervention.

The market clearing condition $\int_0^1 X_t^i di + X_t^G = N_t$ implies that

$$\frac{E_t\left[\rho_{t+1}\right]}{\gamma Var_t\left[\rho_{t+1}\right]} = \left(1 - \vartheta_{N,t}\right) N_t - \vartheta_{N,t}\sigma_N G_t \equiv v_t,$$

where v_t represents the composite external shock arising from the noise trading as well as intervention-induced noise. Substituting the expression of ρ_{t+1} in equation (32) into the above equation yields

$$\left(i_t - i_t^*\right)\Delta t - E_t\left[e_{t+1} - e_t\right] = \gamma Var_t\left(\rho_{t+1}\right)v_t. \tag{33}$$

The right-hand side of the above equation thus represents the time-varying foreign exchange risk premium. Intuitively, the risk premium required by the risk-averse informed traders increases with the magnitude of the composite external shock.

Next, we describe the government's AM problem where the government sets the central parity rate c_t as before, but now also decides on the intensity of intraday intervention $\vartheta_{N,t}$.

3.6.1 Optimal Intraday Government Intervention Policy

First, we solve the government's AM problem in (24):

$$\mathcal{U}\left(Y_t\right) = \min_{c_t,\vartheta_{N,t}} \xi_{\Delta x}\left(\left(1-\omega_0\right)\Delta x_t - \Delta c_t\right)^2 + \xi_c c_t^2 \Delta t + E_t^{AM}\left[\mathcal{V}\left(X_t\right)\right],$$

where $X_t \equiv \left(q_t, i_t^*, c_t, v_t\right)'$ denotes the state vector. We decompose the state vector X_t into the vectors $X_t^{(1)} = \left(q_t, i_t^*\right)'$, $X_t^{(2)} = c_t$, and $X_t^{(3)} = v_t$, as well as the matrix V accordingly. Note that under the simplifying assumption of i.i.d. processes for q_t and i_t^*, the state vector X_t no longer includes lagged variables and the continuation value can be further simplified as follows

$$\frac{1}{\Delta t}E_t^{AM}\left[\mathcal{V}\left(X_t\right)\right] = V^{(3,3)}E\left[v_t^2\right] + V^{(2,2)}c_t^2 + E\left[X_t^{(1)\prime}V^{(1,1)}X_t^{(1)}\right] + V_0.$$

It is straightforward to see that the optimal central parity remains the same as before: $c_t = w_1 e_{t-1} + w_2\left(c_{t-1} + \left(1-\omega_0\right)\Delta x_t\right)$.[31] Moreover, the optimal intraday intervention policy is determined from the following optimization problem:

$$\begin{aligned} & \min_{\vartheta_{N,t}} V^{(3,3)}E\left[v_t^2\right] \\ = \; & \min_{\vartheta_{N,t}} V^{(3,3)}\sigma_N^2\left(\left(1-\vartheta_{N,t}\right)^2 + \vartheta_{N,t}^2\sigma_G^2\right). \end{aligned}$$

As long as $V^{(3,3)} > 0$, the optimal trading intensity for government intervention is given by:

$$\vartheta_{N,t} = \frac{1}{1+\sigma_G^2}. \tag{34}$$

The optimal intervention strategy is to lean against noise traders to reduce exchange rate volatility. However, the government intervention cannot completely eliminate noise trading because of the noise created by its own intervention. In fact, the optimal intervention intensity is inversely related

[31] Note that under the simplifying assumption of i.i.d. process for the US interest rate, the hedging term h_{t-1} is zero.

to the volatility parameter σ_G. Intuitively, a larger σ_G is associated with more noise or frictions originated from the intervention process, which are internalized by the government through its less intensive intervention. Note also that because the optimal intervention intensity is time invariance, we thus drop the time subscript and simply use ϑ_N from now on.

3.6.2 Model Implications

Through the lens of the extended model we show below that intraday government intervention is an effective tool to "lean against noise traders" so that it helps avoid market breakdown. To simplify presentation, we focus on the freely floating case with $\xi_d = 0$. The detailed derivation is delegated to Appendix E. By way of comparison, we first characterize the equilibrium in the case without government intervention ("Case I"), and then incorporate government intervention into the model ("Case II"). Below we study these two cases one by one.

Case I: No Intraday Government Intervention In this case we fix the intensity of the intervention always at zero (i.e., $\vartheta_N = 0$); that is, there is no government intervention and $v_t = N_t$ consists of noise trading only. This is similar to the model in Jeanne and Rose (2002). We briefly summarize the equilibrium outcome below.

As derived in Appendix E, the equilibrium exchange rate is given by

$$e_t = i_t^* \Delta t + \gamma Var_t\left(\rho_{t+1}\right) N_t. \tag{35}$$

That is, the exchange rate is freely floating, determined by the foreign interest rate and the risk premium. According to equation (33), the government achieves monetary autonomy with a fixed domestic interest rate (i.e., $i_t = 0$). Furthermore, the conditional variance of the excess return satisfies the following quadratic equation:

$$Var_t\left(\rho_{t+1}\right) = Var\left(i_t^*\right)\left(\Delta t\right)^2 + \gamma^2 \sigma_N^2 Var_t\left(\rho_{t+1}\right)^2.$$

The above equation together with equation (35) imply that the conditional variance of the foreign exchange risk premium is not only constant, but also equal to the unconditional variance of the exchange rate. That is, $Var_t\left(\rho_{t+1}\right) = Var\left(e_t\right)$, and $Var\left(e_t\right)$ thus satisfies the same quadratic equation:

$$Var\left(e_t\right) = Var\left(i_t^*\right)\left(\Delta t\right)^2 + \gamma^2 \sigma_N^2 \left(Var\left(e_t\right)\right)^2. \tag{36}$$

It is worthwhile to point out several observations. First, the exchange rate variance depends on both fundamentals and noise. An exogenous increase in the amount of noise trading (i.e., larger σ_N) unambiguously increases the variance of the exchange rate through the risk-premium channel.

Second, the quadratic equation (36) has real roots if and only if

$$\sigma_N \leq \frac{1}{2\gamma\sqrt{Var\left(i_t^*\right)}\Delta t} \equiv \sigma_N^*. \tag{37}$$

If strict inequality holds in the condition, there are two distinct positive roots to the equation. This result is reminiscent of equilibrium multiplicity in Jeanne and Rose (2002).

Lastly, and most importantly, if there is too much noise trading in the market such that the above condition is violated, the equation has no real roots. In this case, there is no equilibrium and the market breaks down. As argued in Brunnermeier, Sockin, and Xiong (2018), the market breakdown occurs because facing more noise trading investors demand a higher risk premium for providing liquidity to noise traders, which drives up the variance of the exchange rate. This further raises the risk premium required by investors. When the amount of noise trading is sufficiently large, there does not exist any risk premium that can induce investors to take on any position, resulting in market breakdown (see also De Long, et al. (1990) for a similar mechanism). In order to avoid market breakdown, the amount of noise trading cannot be too large. That is, the condition (37) must be satisfied.

Case II: Intraday Government Intervention We now introduce government intervention into the model. The equilibrium exchange rate still satisfies (35), but now $v_t = (1-\vartheta_N) N_t - \vartheta_N \sigma_N G_t$ is the composite external risk that is composed of both noise risk as well as the intervention-induced risk. As proved earlier, the optimal intensity of intervention is $\vartheta_N = 1/\left(1+\sigma_G^2\right)$.

Similar as in Case I, the conditional variance of the risk premium $Var_t\left(\rho_{t+1}\right)$ is still constant and equal to $Var\left(e\right)$, which satisfies the following quadratic equation:

$$Var\left(\rho_{t+1}\right) = Var\left(i_t^*\right)\left(\Delta t\right)^2 + \gamma^2\sigma_N^2\left(\left(1-\vartheta_N\right)^2 + \vartheta_N^2\sigma_G^2\right)Var\left(\rho_{t+1}\right)^2.$$

It has at least one positive real solution if the following condition is satisfied:

$$\sigma_N \leq \frac{\sigma_N^*}{\sqrt{\left(1-\vartheta_N\right)^2 + \vartheta_N^2\sigma_G^2}}, \tag{38}$$

where σ_N^* is the volatility threshold defined in (37). By "leaning against noise traders", the government intervenes to offset the amount of noise trading. The reduced noise trading risk thus lowers the risk premium required by investors and increases the threshold from σ_N^* to $\sigma_N^*/\sqrt{\left(1-\vartheta_N\right)^2 + \vartheta_N^2\sigma_G^2}$, forestalling market breakdown.

Jeanne and Rose (2002) study a similar model with endogenous noise trading. They argue that through credible commitment to limit exchange rate volatility, the government can reduce noise trading and indeed pin down the economy on the equilibrium with low exchange rate volatility.

In this paper, we focus on the case with exogenous noise trading and do not assume the existence of such credible commitment. In this case we show that direct government intervention can be another tool to ameliorate noise trading risk.

4 Conclusions

Understanding China's exchange rate policy is a key global monetary issue. China's exchange rate policy not only affects the Chinese economy but also impacts the global financial markets. Our paper is the first academic paper that provides an in-depth analysis of China's recent two-pillar policy for the RMB. We provide empirical evidence for the implementation of the two pillar policy that aims to achieve balance between exchange rate flexibility and stability against a RMB index. Based on that evidence, we develop a reduced-form no-arbitrage model that incorporates the two-pillar exchange rate policy for the RMB. The estimation results based on derivatives data suggest that financial market participants have relatively high confidence in the continuation of the two-pillar policy.

In light of the empirical evidence for the two-pillar policy, we quantitatively evaluate China's exchange rate policy using a flexible-price monetary model of the RMB developed in this paper. The theoretical model features policy trade-offs between the variabilities of the exchange rate, the interest rate, and the current account. We show that the two-pillar policy arises endogenously as an optimal solution to the government's problem in which the government tries to minimize the variabilities of exchange rate deviations and the current account. We extend the model further to understand the rationale behind intramarginal government interventions. As in Brunnermeier, Sockin, and Xiong (2018), we show that the direct government intervention is an effective tool to "lean against noise traders" in the presence of noise trading risk.

Our work can be extended along several dimensions. For example, more policy targets can be incorporated. One can also introduce information asymmetry into the theoretical model to study the impact of the government's policy actions on informativeness in the foreign exchange market. We leave these possible extensions to future research.

References

[1] Bertola, Giuseppe, and Ricardo J. Caballero, 1992, Target zones and realignments, *American Economic Review* 82, 520–536.

[2] Bertola, Giuseppe, and Lars E. O. Svensson, 1993, Devaluation risk and the empirical fit of target-zone models, *Review of Economic Studies* 60, 689–712.

[3] Bhattacharya, Utpal, and Paul Weller, 1997, The advantage of hiding one's hand: Speculation and central bank intervention in the foreign exchange market, *Journal of Monetary Economics* 39, 251–277.

[4] Bisesti, Lorenzo, Antonio Castagna, and Fabio Mercurio, 2005, Consistent pricing and hedging of an FX options book, *Kyoto Economic Review* 74, 65–83.

[5] Brooks, Robin, and Gene Ma, 2017, China spotlight: The de facto re-anchoring of the RMB, memo (May 26, 2017), Institute of International Finance.

[6] Brunnermeier, Markus, Michael Sockin, and Wei Xiong, 2018, China's model of managing the financial system, Working Paper.

[7] Campa, José Manuel, and P. H. Kevin Chang, 1996, Arbitrage-based tests of target-zone credibility: Evidence from ERM cross-rate options, *American Economic Review* 86, 726–740.

[8] Chang, Chun, Zheng Liu, and Mark M. Spiegel, 2015, Capital controls and optimal Chinese monetary policy, *Journal of Monetary Economics* 74, 1–15.

[9] Chen, Xiaoli, and Yin-Wong Cheung, 2011, Renminbi going global, *China and World Economy* 19, 1–18.

[10] Cheung, Yin-Wong, Menzie D. Chinn, and Eiji Fujii, 2007, The overvaluation of renminbi undervaluation, *Journal of International Money and Finance* 26, 762–785.

[11] Cheung, Yin-Wong, Guonan Ma, and Robert N. McCauley, 2011, Renminbising China's foreign assets, *Pacific Economic Review* 16, 1–17.

[12] Cheung, Yin-Wong, Cho-Hoi Hui, and Andrew Tsang, 2018, Renminbi central parity: An empirical investigation, *Pacific Economic Review* 23, 164–183.

[13] Clark, John, 2017, China's evolving managed float: An exploration of the roles of the fix and broad dollar movements in explaining daily exchange rate changes, Federal Reserve Bank of New York Staff Reports.

[14] De Long, J. Bradford, Andrei Shleifer, Lawrence H. Summers, and Robert J. Waldmann, 1990, Noise Trader risk in financial markets, *Journal of Political Economy* 98, 703–738.

[15] Dominguez, Kathryn M., and Jeffrey A. Frankel, 1993, Does foreign-exchange intervention matter? The portfolio effect, *American Economic Review* 83, 1356–1369.

[16] Du, Wenxin, Alexander Tepper, and Adrien Verdelhan, 2018, Deviations from covered interest rate parity, *Journal of Finance* 73, 915–957.

[17] Dumas, Bernard, L. Peter Jennergren, Bertil Naslund, 1995, Realignment risk and currency option pricing in target zones, *European Economic Review* 39, 1523–1544,

[18] Eichengreen, Barry, and Masahiro Kawai, 2015, Renminbi internationalization: Achievements, prospects, and challenges, Washington, DC: Brookings Institution Press.

[19] Frankel, Jeffrey A., 2006, On the Yuan: The choice between adjustment under a fixed exchange rate and adjustment under a flexible rate, *CESifo Economic Studies* 52, 246–275.

[20] Frankel, Jeffrey A., 2009, New estimation of China's exchange rate regime, *Pacific Economic Review* 14, 346–360.

[21] Frankel, Jeffrey A., 2012, Internationalization of the RMB and historical precedents, *Journal of Economic Integration* 27, 329–65.

[22] Frankel, Jeffrey A., and Shang-Jin Wei, 1994, Yen bloc or dollar bloc? Exchange rate policies of the east asian economies, in *Macroeconomic Linkage: Savings, Exchange Rates, and Capital Flows*, ed. by Takatoshi Ito and Anne Krueger, pp. 295–333 (Chicago: University of Chicago Press).

[23] Frankel, Jeffrey A., and Shang-Jin Wei, 2007, Assessing China's exchange rate regime, Working paper.

[24] Goldstein, Morris, and Nicholas R. Lardy, 2009, The future of China's exchange rate policy, Washington, DC: Peterson Institute for International Economics.

[25] Hui, C. H., and C. F. Lo, 2009, A note on estimating realignment probabilities – A first-passage-time approach, *Journal of International Money and Finance* 28, 804–812.

[26] Jeanne, Olivier, and Andrew K. Rose, 2002, Noise trading and exchange rate regimes, *Quarterly Journal of Economics* 117, 537–569.

[27] Jermann, Urban J., 2017, Financial markets' views about the euro-swiss franc floor, *Journal of Money, Credit and Banking* 49, 553–565.

[28] Krugman, Paul R., 1991, Target zones and exchange rate dynamics, *Quarterly Journal of Economics* 106, 669–682.

[29] Lyons, Richard K., 2001, *The Microstructure Approach to Exchange Rates*, Cambridge, MA: MIT Press.

[30] Malz, Allan M., 1996, Using option prices to estimate realignment probabilities in the european monetary system: The case of Sterling-Mark, *Journal of International Money and Finance* 15, 717–748.

[31] McCauley, Robert N., and Chang Shu, 2018, Recent RMB policy and currency co-movements, BIS Working Papers No 727.

[32] Meese, Richard A., and Kenneth Rogoff, 1983, Empirical exchange rate models of the seventies: Do they fit out of sample? *Journal of International Economics* 14, 3–24.

[33] Meese, Richard A., and Kenneth J. Singleton, 1982, On unit roots and the empirical modeling of exchange rates, *Journal of Finance* 37, 1029–1035.

[34] Muthuraman, Kumar, and Sunil Kumar, 2006, Multidimensional portfolio optimization with proportional transaction costs, *Mathematical Finance* 16, 301-335.

[35] Pasquariello, Paolo, Jennifer Roush, and Clara Vega, 2020, Government intervention and strategic trading in the U.S. treasury market, *Journal of Financial and Quantitative Analysis* 55, 117–157.

[36] Prasad, Eswar, 2016, Gaining currency: The rise of the Renminbi, Oxford University Press.

[37] Prasad, Eswar, Thomas Rumbaugh, and Qing Wang, 2005, Putting the cart before the horse? Capital account liberalization and exchange rate flexibility in China. IMF Policy Discussion Paper No. 05/1.

[38] Sarno, Lucio, and Mark P. Taylor, 2001, Official intervention in the foreign exchange market: Is it effective and, If so, how does it work? *Journal of Economic Literature* 39, 839–868.

[39] Söderlind, Paul, 2000, Market Expectations in the UK before and after the ERM Crisis, *Economica* 67, 1–18.

[40] Stein, Jeremy, 1989, Cheap talk and the Fed: A theory of imprecise policy announcements, *American Economic Review* 79, 32–42.

[41] Sun, Jie, 2010, Retrospect of the Chinese exchange rate regime after reform: Stylized facts during the period from 2005 to 2010, *China and World Economy* 18, 19–35.

[42] The People's Bank of China, 2016, China Monetary Policy Report Quarter One.

[43] The People's Bank of China, 2017, China Monetary Policy Report Quarter Two.

[44] Vitale, Paolo, 1999, Sterilised central bank intervention in the foreign exchange market, *Journal of International Economics* 49, 245–267.

[45] Yu, Jialin, 2007, Closed-form likelihood approximation and estimation of jump-diffusions with an application to the realignment risk of the Chinese Yuan, *Journal of Econometrics* 141, 1245–1280.

Appendix A: Data

Main sources of our data are the CFETS and Bloomberg.

From the CFETS website (http://www.chinamoney.com.cn), we retrieve the historical data of the central parity rates and the RMB indices.

From Bloomberg, we obtain daily data on spot exchange rates, 3-month SHIBOR and LIBOR interest rates, US dollar index, options and futures data on the RMB and the dollar index (including the data on the implied volatility of options on the dollar index). We also obtain monthly data on China's M2 money supply, CPI, foreign reserves data. In particular, the RMB options data consist of implied volatility quotes for at-the-money options, risk reversals, and butterfly spreads, with a maturity of 3 months. These quotes can then be used to infer implied volatilities by the standard approach (e.g., see Jermann (2017) or Bisesti et al. (2015) for more details).

Lastly, we obtain intraday exchange rate data from the Bloomberg BFIX data, which are available every 30 minutes on the hour and half-hour throughout the day. For each week, the BFIX data begin Sunday 5:30 PM New York time and end Friday 5 PM New York time. We then use the BFIX data to construct intraday values for the U.S. dollar index (DXY), CFETS, and SDR indices.[32] For a given index, we collect the BFIX data for all constituent currencies and then convert the data in China local time, taking into account time-zone difference and daylight saving period. Based on the BFIX data, we can thus construct the index-implied dollar basket and the RMB spot rate for all 48 half-hour intervals throughout the day.

Appendix B: Derivations of the reduced-form model

Appendix B1: The Onshore CNY Market

Denote $\widehat{S}_t \equiv \tilde{S}_t/S_t^{CP}$ and $\widehat{V}_t \equiv V_t/S_t^{CP}$. Due to homogeneity of function $H(x, y; b)$, solving the equilibrium exchange rate $\tilde{S}_t$ boils down to solving the univariate function $\widehat{S}(\widehat{V}_t)$, which satisfies:

$$\widehat{S}(\widehat{V}_t) = \frac{\tilde{S}\left(V_t, S_t^{CP}\right)}{S_t^{CP}} = \frac{1+r^{\$}}{1+r^C} p E_t^Q \left[\frac{S_{t+1}^{CP}}{S_t^{CP}} H\left(\widehat{S}\left(\widehat{V}_{t+1}\right), 1; b\right)\right] + (1-p)\widehat{V}_t. \tag{39}$$

For ease of notation from now on we simply write $H(\widehat{S}(\widehat{V}_{t+1}), 1; b)$ as $H(\widehat{S}(\widehat{V}_{t+1}))$. Substituting the two-pillar rule into the above equation yields

$$\widehat{S}\left(\widehat{V}_t\right) = \frac{1+r^{\$}}{1+r^C} p E^Q \left[\left(\frac{X_{t+1}}{X_t}\right)^{\alpha} \left(\frac{V_{t+1}}{V_t}\right)^{\beta} H\left(\widehat{S}\left(\widehat{V}_{t+1}\right)\right)\right] + (1-p)\widehat{V}_t, \tag{40}$$

[32] Because Bloomberg has stopped producing the BFIX data for Venezuela currency, we do not construct intraday spot fixings for the BIS index.

and

$$\frac{\widehat{V}_{t+1}}{\widehat{V}_t} = \left(\frac{X_{t+1}}{X_t}\right)^{-\alpha} \left(\frac{V_{t+1}}{V_t}\right)^{1-\beta}. \tag{41}$$

We now cast the model in continuous time in which the equilibrium exchange rate $\widehat{S}(\widehat{V}_t)$ is derived in closed form. Let the length of each period Δt tend to zero. The dynamics of the state variables in continuous time under the RMB risk-neutral measure Q is given by

$$\begin{aligned} \frac{dV_t}{V_t} &= (r_{CNY} - r_{USD})\, dt + \sigma_V dW_{V,t} \equiv \mu_V dt + \sigma_V dW_{V,t}, & (42) \\ \frac{dX_t}{X_t} &= \left(r_{DXY} - r_{USD} - \rho\sigma_X\sigma_V + \sigma_X^2\right) dt + \sigma_X \left(\rho dW_{V,t} + \sqrt{1-\rho^2} dW_{X,t}\right) & (43) \\ &\equiv \mu_X dt + \sigma_X \left(\rho dW_{V,t} + \sqrt{1-\rho^2} dW_{X,t}\right), \end{aligned}$$

where $W_{V,t}$ and $W_{X,t}$ are independent Brownian motions under the measure Q, and r_{CNY}, r_{USD}, r_{DXY} are instantaneous interest rates for the RMB, the dollar, and the currency basket in the dollar index DXY, respectively. That is, the per-period interest rates in the preceding discrete-time setup satisfy: $r^{\$} = \exp(r_{USD}\Delta t)$ and $r^C = \exp(r_{CNY}\Delta t)$. Similarly, the per-period probability $p = 1 - \lambda\Delta t$ while we assume that the current managed floating regime will be abandoned upon arrival of a Poisson process with intensity λ. The processes $\{X_t\}$ and $\{V_t\}$ are assumed to have a correlation ρ. Their drifts are specified in the above equations so as to exclude any arbitrage opportunities.

In the proposition below we derive the equilibrium exchange rate $\widehat{S}(\widehat{V}_t)$ in closed form.

Proposition 4 *In the continuous-time model, the scaled equilibrium exchange rate $\widehat{S}(\widehat{V}_t)$ is determined as follows:*

$$\widehat{S}\left(\widehat{V}_t\right) = \begin{cases} 1-b, & \text{if } \widehat{V} \le \widehat{V}_*; \\ C_0\widehat{V} + C_1\widehat{V}^{\eta_1} + C_2\widehat{V}^{\eta_2}, & \text{if } \widehat{V}_* < \widehat{V} < \widehat{V}^*; \\ 1+b, & \text{if } \widehat{V} \ge \widehat{V}^*, \end{cases} \tag{44}$$

where the thresholds $\widehat{V}_*$ and $\widehat{V}^*$ are endogenously determined and the expressions of η_1, η_2, and C_0 through C_2 are given in the proof.

Proof of Proposition 4. Below we use lowercase variables to denote the logarithm of the corresponding uppercase variables. For example, $v_t \equiv \log V_t$, $s_t^{CP} \equiv \log S_t^{CP}$, $\widehat{v}_t \equiv \log \widehat{V}_t$, etc. Under the two-pillar policy in equation (15), by Ito's lemma the dynamics of S_t^{CP} is given by:

$$\frac{dS_t^{CP}}{S_t^{CP}} \equiv \mu_{CP} dt + (\alpha\rho\sigma_X + \beta\sigma_V)\, dW_{V,t} + \alpha\sqrt{1-\rho^2}\sigma_X dW_{X,t}, \tag{45}$$

where $\mu_{CP} \equiv \alpha\left(\mu_X - 1/2\sigma_X^2\right) + \beta\left(\mu_V - 1/2\sigma_V^2\right) + \frac{1}{2}\left(\alpha\rho\sigma_X + \beta\sigma_V\right)^2 + \frac{1}{2}\alpha^2\left(1-\rho^2\right)\sigma_X^2$ denotes the expected growth rate of the central parity; that is, $E_t^Q\left[S_{t+\tau}^{CP}\right] = S_t^{CP}\exp\left(\mu_{CP}\tau\right)$.

Similarly, we derive the dynamics of $\widehat{V}_t$ as follows:

$$\frac{d\widehat{V}_t}{\widehat{V}_t} = \mu_{\widehat{V}}dt + \left(\left(1-\beta\right)\sigma_V - \alpha\sigma_X\rho\right)dW_{V,t} - \alpha\sigma_X\sqrt{1-\rho^2}dW_{X,t}, \tag{46}$$

where $\mu_{\widehat{V}} \equiv -\alpha\left(\mu_X - 1/2\sigma_X^2\right) + \left(1-\beta\right)\left(\mu_V - 1/2\sigma_V^2\right) + \frac{1}{2}\sigma_{\widehat{V}}^2$ denotes the expected growth rate of the scaled fundamental exchange rate and $\sigma_{\widehat{V}} \equiv \sqrt{\left(\left(1-\beta\right)\sigma_V - \alpha\sigma_X\rho\right)^2 + (\alpha\sigma_X\sqrt{1-\rho^2})^2}$.

We are now ready to solve the (scaled) equilibrium exchange rate $\widehat{S}(\widehat{V}_t)$. It is straightforward to prove that $\widehat{S}(\widehat{V}_t)$ is monotonically increasing. Define $\widehat{V}_*$ and $\widehat{V}^*$ such that $\widehat{S}(\widehat{V}_*) = 1-b$ and $\widehat{S}(\widehat{V}^*) = 1+b$. As the length of the period Δt converges to zero, with probability one $\widehat{V}_{t+\Delta t} > \widehat{V}^*$ (or $\widehat{V}_{t+\Delta t} < \widehat{V}_*$) if $\widehat{V}_t > \widehat{V}^*$ (or $\widehat{V}_t < \widehat{V}_*$). Therefore, from equation (40), it must be true that: $\widehat{S}(\widehat{V}_t) = 1-b$ if $\widehat{V} < \widehat{V}_*$, and $1+b$ if $\widehat{V} > \widehat{V}^*$.

If $\widehat{V} \in (\widehat{V}_*, \widehat{V}^*)$, it is straightforward to show that $\widehat{S}(\widehat{V}_t)$ must satisfy the following equation based on equation (40):

$$\widehat{S}'\left(\widehat{V}_t\right)\widehat{V}_t\mu_{\widehat{V}} + \frac{1}{2}\widehat{S}''\left(\widehat{V}_t\right)\widehat{V}_t^2\sigma_{\widehat{V}}^2 + \left(\mu_{CP} - \mu_V - \lambda\right)\widehat{S}\left(\widehat{V}_t\right) + \lambda\widehat{V}_t = 0. \tag{47}$$

The solution to this ordinary differential equation is: $\widehat{S}(\widehat{V}_t) = C_0\widehat{V} + C_1\widehat{V}^{\eta_1} + C_2\widehat{V}^{\eta_2}$, where η_1 and η_2 are the two roots of the quadratic equation:

$$\frac{1}{2}\sigma_{\widehat{V}}^2\eta^2 + \left(\mu_{\widehat{V}} - \frac{1}{2}\sigma_{\widehat{V}}^2\right)\eta + \left(\mu_{CP} - \mu_V - \lambda\right) = 0, \tag{48}$$

and the coefficient C_0 is given by

$$C_0 = \frac{\lambda}{\lambda + \mu_V - \mu_{\widehat{V}} - \mu_{CP}}, \tag{49}$$

and the coefficients C_1, C_2 and the thresholds $\widehat{V}_*$, $\widehat{V}^*$ are determined from the value-matching and smooth-pasting conditions:

$$\begin{aligned}
C_0\widehat{V}_* + C_1(\widehat{V}_*)^{\eta_1} + C_2(\widehat{V}_*)^{\eta_2} &= 1-b, && (50)\\
C_0\widehat{V}^* + C_1(\widehat{V}^*)^{\eta_1} + C_2(\widehat{V}^*)^{\eta_2} &= 1+b, && (51)\\
C_0 + \eta_1 C_1(\widehat{V}_*)^{\eta_1-1} + \eta_2 C_2(\widehat{V}_*)^{\eta_2-1} &= 0, && (52)\\
C_0 + \eta_1 C_1(\widehat{V}^*)^{\eta_1-1} + \eta_2 C_2(\widehat{V}^*)^{\eta_2-1} &= 0. && (53)
\end{aligned}$$

■

Appendix B2: The Offshore CNH Market

Define $\widehat{T}_t \equiv \frac{\widetilde{T}_t}{S_t^{CL}}$ and $\widehat{U}_t \equiv \frac{V_t}{S_t^{CL}} = \frac{V_t}{S_t^{CP}} \frac{S_t^{CP}}{S_t^{CL}} = \frac{\widehat{V}_t}{H_b\left(\widehat{S}\left(\widehat{V}_t\right)\right)}$, where $H_b\left(\widehat{S}\left(\widehat{V}_t\right)\right) \equiv H\left(\widehat{S}\left(\widehat{V}_t\right), 1; b\right)$. The equation for $\widetilde{T}_t$ can be written as

$$
\begin{aligned}
\widehat{T}\left(\widehat{U}_t\right) &= \frac{1+r^{\$}}{1+r^{offshore}} p E_t^Q \left[\frac{S_{t+1}^{CL}}{S_t^{CL}} H_c\left(\widehat{T}\left(\widehat{U}_{t+1}\right)\right)\right] + (1-p)\,\widehat{U}_t \\
&= \frac{1+r^{\$}}{1+r^{offshore}} p E_t^Q \left[\frac{S_{t+1}^{CP}}{S_t^{CP}} \frac{\widehat{S}_{t+1}}{\widehat{S}_t} H_c\left(\widehat{T}\left(\widehat{U}_{t+1}\right)\right)\right] + (1-p)\,\widehat{U}_t,
\end{aligned}
$$

where $\widehat{S}_t^{CL} \equiv S_t^{CL}/S_t^{CP}$, and $H_c\left(\widehat{T}\left(\widehat{U}_{t+1}\right)\right) \equiv H\left(\widehat{T}\left(\widehat{U}_{t+1}\right), 1; c\right)$.

We solve the model in continuous time. Denote $1 + r^{offshore} \equiv \exp\left(r_{CNH}\Delta t\right)$. Note that $\widehat{U}_t$ is a monotonically increasing function of $\widehat{V}_t$, given by

$$
\widehat{U}_t = \frac{V_t}{S_t^{CL}} = \frac{\widehat{V}_t}{\widehat{S}\left(\widehat{V}_t\right)} = \begin{cases} \frac{1}{1-b}\widehat{V}_t, \text{ if } \widehat{V}_t < \widehat{V}_* \\ \frac{\widehat{V}_t}{C_0\widehat{V}_t + C_1\widehat{V}_t^{\eta_1} + C_2\widehat{V}_t^{\eta_2}}, \text{ if } \widehat{V}_t \in \left[\widehat{V}_*, \widehat{V}^*\right] \\ \frac{1}{1+b}\widehat{V}_t, \text{ if } \widehat{V} > \widehat{V}^* \end{cases}.
$$

By the same argument as in the special case with $b = 0$, there exist $\widehat{V}_{**}$ and $\widehat{V}^{**}$ such that $\widehat{T}\left(\widehat{V}_t\right) = 1 - c$ for $\widehat{V}_t < \widehat{V}_{**}$, and $1 + c$ for $\widehat{V}_t > \widehat{V}^{**}$. To complete the derivation, we just need to solve for $\widehat{T}\left(\widehat{V}_t\right)$ for $\widehat{V}_t \in \left[\widehat{V}_{**}, \widehat{V}^{**}\right]$. Below we derive the ODE for $\widehat{T}(\widehat{V}_t)$.

First, recall that

$$
\frac{d\widehat{V}_t}{\widehat{V}_t} = \mu_{\widehat{V}} dt + \left(\left(1-\beta\right)\sigma_V - \alpha\sigma_X\rho\right) dW_{V,t} - \alpha\sigma_X\sqrt{1-\rho^2}\, dW_{X,t} \equiv \mu_{\widehat{V}} dt + \sigma_{\widehat{V}} dW_{\widehat{V},t},
$$

where $dW_{\widehat{V},t} \equiv \frac{1}{\sigma_{\widehat{V}}}\left[\left(\left(1-\beta\right)\sigma_V - \alpha\sigma_X\rho\right) dW_{V,t} - \alpha\sigma_X\sqrt{1-\rho^2}\, dW_{X,t}\right]$. By Ito's lemma, we have

$$
d\widehat{S}_t = \left[\widehat{S}'\widehat{V}_t\mu_{\widehat{V}} + \frac{1}{2}\widehat{S}''\sigma_{\widehat{V}}^2\widehat{V}_t^2\right] dt + \widehat{S}'\widehat{V}_t\sigma_{\widehat{V}} dW_{\widehat{V},t} \equiv \widehat{S}_t\left[\mu_{\widehat{S}} + \sigma_{\widehat{S}} dW_{\widehat{V},t}\right],
$$

where $\mu_{\widehat{S}}$ and $\sigma_{\widehat{S}}$ are zero for $\widehat{V}_t \notin \left[\widehat{V}_*, \widehat{V}^*\right]$, and for $\widehat{V}_t \in \left[\widehat{V}_*, \widehat{V}^*\right]$, $\mu_{\widehat{S}} = -\frac{1}{\widehat{S}_t}\left[\left(\mu_{CP} - \mu_V - \lambda\right)\widehat{S}_t + \lambda\widehat{V}_t\right]$ and $\sigma_{\widehat{S}} = \frac{1}{\widehat{S}_t}\widehat{S}'\widehat{V}\sigma_{\widehat{V}}$. Therefore,

$$
\begin{aligned}
d\widehat{U}_t &= d\left(\frac{\widehat{V}_t}{\widehat{S}_t}\right) = \widehat{U}_t\left(\mu_{\widehat{V}} dt + \sigma_{\widehat{V}} dW_{\widehat{V},t}\right) - \widehat{U}_t\left(\mu_{\widehat{S}} dt + \sigma_{\widehat{S}} dW_{\widehat{V},t}\right) + \widehat{U}_t\sigma_{\widehat{S}}^2 dt - \widehat{U}_t\sigma_{\widehat{V}}\sigma_{\widehat{S}} dt \\
&= \widehat{U}_t\left[\left(\mu_{\widehat{V}} - \mu_{\widehat{S}} + \sigma_{\widehat{S}}\left(\sigma_{\widehat{S}} - \sigma_{\widehat{V}}\right)\right) dt - \left(\sigma_{\widehat{S}} - \sigma_{\widehat{V}}\right) dW_{\widehat{V},t}\right] \\
&\equiv \widehat{U}_t\left(\mu_{\widehat{U}} dt + \sigma_{\widehat{U}} dW_{\widehat{V},t}\right).
\end{aligned}
$$

Next, we derive the dynamics for $S_t^{CL} = S_t^{CP}\widehat{S}_t$. Because

$$
\begin{aligned}
\frac{dS_t^{CP}}{S_t^{CP}} &\equiv \mu_{CP}dt + (\alpha\rho\sigma_X + \beta\sigma_V)\, dW_{V,t} + \alpha\sqrt{1-\rho^2}\sigma_X dW_{X,t} \\
\frac{d\widehat{S}}{\widehat{S}} &= \mu_{\widehat{S}}dt + \frac{\sigma_{\widehat{S}}}{\sigma_{\widehat{V}}}\left[\left((1-\beta)\,\sigma_V - \alpha\sigma_X\rho\right) dW_{V,t} - \alpha\sigma_X\sqrt{1-\rho^2}dW_{X,t}\right]
\end{aligned}
$$

we have

$$
\begin{aligned}
\frac{d\left(S_t^{CP}\widehat{S}\right)}{S_t^{CP}\widehat{S}} &= \mu_{CP}dt + (\alpha\rho\sigma_X + \beta\sigma_V)\, dW_{V,t} + \alpha\sqrt{1-\rho^2}\sigma_X dW_{X,t} \\
&\quad + \mu_{\widehat{S}}dt + \sigma_{\widehat{S}}\frac{1}{\sigma_{\widehat{V}}}\left[\left((1-\beta)\,\sigma_V - \alpha\sigma_X\rho\right) dW_{V,t} - \alpha\sigma_X\sqrt{1-\rho^2}dW_{X,t}\right] \\
&\quad + \sigma_{\widehat{S}}\frac{1}{\sigma_{\widehat{V}}}\left[(\alpha\rho\sigma_X + \beta\sigma_V)\left((1-\beta)\,\sigma_V - \alpha\sigma_X\rho\right) - \sigma_X\alpha^2\left(1-\rho^2\right)\right] dt \\
&\equiv \mu_{CL}dt + \left[(\alpha\rho\sigma_X + \beta\sigma_V) + \left(\sigma_{\widehat{S}}/\sigma_{\widehat{V}}\right)\left((1-\beta)\,\sigma_V - \alpha\sigma_X\rho\right)\right] dW_{V,t} \\
&\quad + \left(1 - \sigma_{\widehat{S}}/\sigma_{\widehat{V}}\right)\alpha\sqrt{1-\rho^2}\sigma_X dW_{X,t},
\end{aligned}
$$

where $\mu_{CL} \equiv \mu_{CP} + \mu_{\widehat{S}} + \frac{\sigma_{\widehat{S}}}{\sigma_{\widehat{V}}}\left[(\alpha\rho\sigma_X + \beta\sigma_V)\left((1-\beta)\,\sigma_V - \alpha\sigma_X\rho\right) - \sigma_X\alpha^2\left(1-\rho^2\right)\right]$.

Lastly, in continuous time, equation (17) implies the following ODE for $\widehat{T}\left(\widehat{U}\right)$ as a function of $\widehat{U}$:

$$
0 = \widehat{T}'\left(\widehat{U}\right)\widehat{U}\mu_{\widehat{U}} + \frac{1}{2}\widehat{T}''\left(\widehat{U}\right)\widehat{U}^2\sigma_{\widehat{U}}^2 + (r_{USD} - r_{CNH} - \lambda + \mu_{CL})\,\widehat{T}\left(\widehat{U}\right) + \lambda\widehat{U}_t.
$$

It is easier to solve for $\widehat{T}\left(\widehat{V}\right)$ as a function of $\widehat{V}$ instead. Using $\widehat{T}'\left(\widehat{V}\right) = \widehat{T}'\left(\widehat{U}\right)\widehat{U}'\left(\widehat{V}\right)$ and $\widehat{T}''\left(\widehat{V}\right) = \widehat{T}''\left(\widehat{U}\right)\widehat{U}'\left(\widehat{V}\right)^2 + \widehat{T}'\left(\widehat{U}\right)\widehat{U}''\left(\widehat{V}\right)$, the above ODE can be rewritten as one for $\widehat{T}\left(\widehat{V}\right)$:

$$
\begin{aligned}
0 &= \frac{1}{2}\frac{\widehat{U}^2\sigma_{\widehat{U}}^2}{\widehat{U}'\left(\widehat{V}\right)^2}\widehat{T}''\left(\widehat{V}\right) + \left[\frac{\widehat{U}\mu_{\widehat{U}}}{\widehat{U}'\left(\widehat{V}\right)} - \frac{1}{2}\widehat{U}^2\sigma_{\widehat{U}}^2\frac{\widehat{U}''\left(\widehat{V}\right)}{\widehat{U}'\left(\widehat{V}\right)^3}\right]\widehat{T}'\left(\widehat{V}\right) \\
&\quad + (r_{USD} - r_{CNH} - \lambda + \mu_{CL})\,\widehat{T}\left(\widehat{V}\right) + \lambda\widehat{U}\left(\widehat{V}\right) \\
&\equiv \theta_2\widehat{T}''\left(\widehat{V}\right) + \theta_1\widehat{T}'\left(\widehat{V}\right) + \theta_0\widehat{T}\left(\widehat{V}\right) + \lambda\widehat{U}\left(\widehat{V}\right),
\end{aligned} \tag{54}
$$

where $\theta_0, \theta_1, \theta_2$ denote the coefficients of $\widehat{T}\left(\widehat{V}\right)$, $\widehat{T}'\left(\widehat{V}\right)$, and $\widehat{T}''\left(\widehat{V}\right)$, respectively. The boundary conditions are $\widehat{T}\left(\widehat{V}_{**}\right) = 1-c$, $\widehat{T}\left(\widehat{V}^{**}\right) = 1+c$, $\widehat{T}'\left(\widehat{V}_{**}\right) = \widehat{T}'\left(\widehat{V}^{**}\right) = 0$. So it is a free-boundary problem. Because of no closed-form solution, we solve it numerically using the finite-difference method (FDM). We provide the detail about our numerical method shortly below.

Numerical Method

Because of no closed-form solution for nonzero arbitraging costs, we numerically solve the free-boundary problem in (54) with free boundaries $\widehat{V}_{**}$, $\widehat{V}^{**}$ using the finite-difference method (FDM). The algorithm of solving for the optimal boundaries is based on Muthuraman and Kumar (2006).

We use the following two-step procedure to solve the free-boundary problem. We begin by choosing arbitrary boundaries $\widehat{V}_{**}^{(0)}$, $\widehat{V}^{**(0)}$, and denote the closed interval between these two boundaries as $\Omega^0 = \left[\widehat{V}_{**}^{(0)}, \widehat{V}^{**(0)}\right]$. Initially, we choose the interval large enough to embed the optimal interval Ω^* within it. In the next step, we use a boundary update procedure to obtain a new interval Ω^1, which is a subset of Ω^0 (i.e., $\Omega^1 \subseteq \Omega^0$). We will repeat the procedure to get a sequence of the intervals Ω^0, Ω^1, $\cdots$ until it converges to the optimal interval Ω^* associated with the optimal boundaries $\widehat{V}_{**}$, $\widehat{V}^{**}$.

- In step 1, given $\Omega^m = \left[\widehat{V}_{**}^{(m)}, \widehat{V}^{**(m)}\right]$, we solve the ODE in (54) with the boundary conditions that $\widehat{T}\left(\widehat{V}_{**}^{(m)}\right) = 1 - c$ and $\widehat{T}\left(\widehat{V}^{**(m)}\right) = 1 + c$. It can be easily solved using a FDM scheme. Specifically, we construct a equally spaced grid $\left\{\widehat{V}_n^{(m)}\right\}_{n=1}^{N}$ within the interval Ω^m such that $\widehat{V}_1^{(m)} = \widehat{V}_{**}^{(m)}$ and $\widehat{V}_N^{(m)} = \widehat{V}^{**(m)}$. Let $\Delta\widehat{V}$ denote the distance between two consecutive grid points. To simplify notation, we denote $\widehat{T}\left(\widehat{V}_n^{(m)}\right)$ simply by $\widehat{T}_n^{(m)}$. For $n \in \{2, \cdots, N-1\}$, we use FDM to evaluate the derivatives of $\widehat{T}\left(\widehat{V}\right)$ at the grid point $\widehat{V}_n^{(m)}$:

$$\widehat{T}'\left(\widehat{V}_n\right) = \frac{\widehat{T}\left(\widehat{V}_{n+1}\right) - \widehat{T}\left(\widehat{V}_n\right)}{\Delta\widehat{V}} = \frac{\widehat{T}_{n+1} - \widehat{T}_n}{\Delta\widehat{V}} \text{ and } \widehat{T}''\left(\widehat{V}_n\right) = \frac{\widehat{T}_{n+1} - 2\widehat{T}_n + \widehat{T}_{n-1}}{\Delta\widehat{V}^2}.$$

Substituting the above expressions into the ODE (54) and manipulating it, we obtain

$$U_n\widehat{T}_{n+1}^{(m)} + M_n\widehat{T}_n^{(m)} + D_n\widehat{T}_{n-1}^{(m)} = -\lambda\widehat{U}\left(\widehat{V}_n^{(m)}\right),$$

where $U_n \equiv \frac{\theta_1}{\Delta\widehat{V}} + \frac{\theta_2}{\Delta\widehat{V}^2}$, $M_n \equiv -\frac{\theta_1}{\Delta\widehat{V}} - \frac{2\theta_2}{\Delta\widehat{V}^2} + \theta_0$, and $D_n \equiv \frac{\theta_2}{\Delta\widehat{V}^2}$. The value of the function $\widehat{T}\left(\widehat{V}\right)$ at the grid can then be solved by the linear system below:

$$\begin{bmatrix} 1 & & & & & & \\ D_2 & M_2 & U_2 & & & & \\ & & \ddots & & & & \\ & & D_n & M_n & U_n & & \\ & & & & \ddots & & \\ & & & & D_{N-1} & M_{N-1} & U_{N-1} \\ & & & & & & 1 \end{bmatrix} \begin{bmatrix} \widehat{T}_1^{(m)} \\ \widehat{T}_2^{(m)} \\ \vdots \\ \widehat{T}_n^{(m)} \\ \vdots \\ \widehat{T}_{N-1}^{(m)} \\ \widehat{T}_N^{(m)} \end{bmatrix} = \begin{bmatrix} 1 - c \\ -\lambda\widehat{U}\left(\widehat{V}_2^{(m)}\right) \\ \vdots \\ -\lambda\widehat{U}\left(\widehat{V}_n^{(m)}\right) \\ \vdots \\ -\lambda\widehat{U}\left(\widehat{V}_{N-1}^{(m)}\right) \\ 1 + c \end{bmatrix}.$$

- In step 2, we update the boundaries from Ω^m to Ω^{m+1}. Specifically, given the solution $\left\{\widehat{T}_n^{(m)}\right\}_{n=1}^{N}$ from the first step, we choose the updated boundary $\widehat{V}_{**}^{(m+1)}$ (or $\widehat{V}^{**(m+1)}$) as the grid point at which the function $\widehat{T}^{(m)}$ achieves its minimum (or maximum). Note that upon convergence to the optimal boundaries $\widehat{V}_{**}$ and $\widehat{V}^{**}$, the first derivatives are both zero, satisfying the additional boundary conditions: $\widehat{T}'\left(\widehat{V}_{**}\right) = \widehat{T}'\left(\widehat{V}^{**}\right) = 0$. As argued in Muthuraman and Kumar (2006), the boundary update procedure in this step guarantees that the generated sequence of Ω's are nested.

Appendix C: International Trades and the Balance of Payments

There are totally $(N+1)$ countries, including the home country (indexed by N), the numeraire country (indexed by 0), and countries in the rest of the world or ROW (indexed by 1 through $N-1$). In the context of this paper, the home country refers to China and the numeraire country refers to the US. We lump countries 1 through $N-1$ into the rest of the world (RoW). So from now on we focus on three countries/regions: China (home country), the US (numeraire country), and the RoW.

Suppose China (the home country, or country N) trades with N other countries, indexed by $i \in \{0, \cdots, N-1\}$. Let $R_t^{(i)}$ denote the price of currency i in dollars for $i = 0, \cdots, N$. Note that (i) $R_t^{(0)}$ is always equal to unity; (ii) $R_t^{(N)}$ is the inverse of the central parity of the RMB against the USD; that is, $R_t^{(N)} = 1/S_t^{CP,CNY/USD}$ in the previous section. Assume the price of each country's product is 1 in terms of its currency. Hence, country i's product costs $R_t^{(i)}$ dollar.

The surplus in the home country's balance of trade in terms of dollars, TB_t, is defined as exports minus imports:

$$TB_t \equiv R_t^{(N)} \sum_{j=0}^{N-1} D_t^{j,N} - \sum_{j=0}^{N-1} R_t^{(j)} D_t^{N,j},$$

where $D_t^{j,N}$ denotes country j's demand for the home country's output (i.e., export from China to country j), and $D_t^{N,j}$ denotes the home country's demand for country j's output (i.e., import to China from country j). The loglinear approximation of the trade-balance growth is given below:

$$\Delta \log TB_t \approx \sum_{j=0}^{N} \gamma_{t-1}^{(j)} \Delta \log R_t^{(j)},$$

where the derivation and the expressions of $\gamma_{t-1}^{(j)}$ in Appendix A.

Let $c_t \equiv \log S_t^{CP,CNY/USD}$ denote the logarithm of the central parity $S^{CP,CNY/USD}$. Note that $R_t^{(N)} = 1/S_t^{CP,CNY/USD}$ and hence $\Delta \log R_t^{(j)} = -(c_t - c_{t-1})$. Conditional on the observations of $\left\{\Delta \log R_t^{(j)}\right\}_{j=0}^{N-1}$, minimizing the variability in the trade-balance growth leads to a basket peg. In fact,

$$\begin{aligned} & \min_{c_t} \left(\Delta \log TB_t\right)^2 \\ = & \min_{c_t} \left(\sum_{j=1}^{N-1} \gamma_{t-1}^{(j)} \Delta \log R_t^{(j)} - \gamma_{t-1}^{(N)} \left(c_t - c_{t-1}\right)\right)^2 . \end{aligned}$$

The solution to the above optimization problem is:

$$c_t = c_{t-1} - \sum_{j=1}^{N-1} \left(-\gamma_{t-1}^{(j)}/\gamma_{t-1}^{(N)}\right) \Delta \log R_t^{(j)}. \tag{55}$$

The optimal exchange rate policy that minimizes the variability in the trade-balance growth is thus a basket peg. To see this, consider a RMB index or basket of currencies of countries $j = 0, \cdots, N-1$. For currency $j \in \{1, \cdots, N-1\}$, its weight in the basket, denoted by ω_j, is given by

$$\omega_j = -\gamma_{t-1}^{(j)}/\gamma_{t-1}^{(N)}, \ j = 1, \cdots, N-1$$

Under additional simplifying assumptions (see Flanders and Helpman (1979) and also the appendix in the end of the paper), we can show that the optimal weight for currency $j \in \{0, 1, \cdots, N-1\}$ is equal to share of country j in the home country's exports. Given that the export shares are relatively stable, we can roughly consider the weights $\{\omega_j\}_{j=0}^{N-1}$ largely time invariant. The weight for the USD in the basket $\omega_0 = 1 - \sum_{j=1}^{N-1} \omega_j$ is determined as well. Note that ω_0 corresponds to ω_{USD} in the previous section.

With the basket composition $\{\omega_j\}_{j=0}^{N-1}$, the optimal central parity in (55) can be rewritten as:

$$c_t = c_{t-1} - \sum_{j=1}^{N-1} \omega_j \Delta \log R_t^{(j)}.$$

It is straightforward to show that the above central parity is actually equal to the logarithm of the stability pillar $\overline{S}_t$ corresponding to the RMB index:

$$\begin{aligned} B_t &= C_B \left(R_t^{(N)}\right)^{\omega_0} \prod_{j=1}^{N-1} \left(R_t^{(N)}/R_t^{(j)}\right)^{\omega_j} \\ &\equiv \chi R_t^{(N)} X_t^{1-\omega_0}, \end{aligned}$$

where $X_t \equiv C_X \prod_{j=1}^{N-1} \left(R_t^{(j)}\right)^{-\frac{\omega_i}{1-\omega_0}}$ and $\chi \equiv C_B/C_X^{1-\omega_0}$ are defined in the same way as in the previous section.

Let $\overline{s}_t \equiv \log \overline{S}_t$ denote the logarithm of the stability pillar $\overline{S}_t$, then it is given by

$$\begin{aligned} \overline{s}_t &= c_{t-1} - \sum_{j=1}^{N-1} \omega_j \Delta \log R_t^{(j)} = c_{t-1} + (1-\omega_0) \Delta \log X_t \\ &\equiv c_{t-1} + (1-\omega_0) \Delta x_t, \end{aligned}$$

where $x_t \equiv \log X_t$ denotes the logarithm of the index-implied USD index X_t. As a result, a basket peg is optimal if the objective is only to minimize the variability in the trade-balance growth.

Appendix D: Derivation and Proofs in the Theoretical Model

Appendix D1: The central bank's AM problem and the solution

Recall that the central bank's AM objective function is stated in (24), repeated below

$$\mathcal{U}(Y_t) = \min_{c_t} \xi_{\Delta x}\left((1-\omega_0)\Delta x_t - \Delta c_t\right)^2 + \xi_{\Delta e}\left(c_t - e_{t-1}\right)^2 + \xi_c c_t^2 \Delta t + E_t^{AM}\left[\mathcal{V}(X_t)\right].$$

Here we show that the value function $\mathcal{U}(Y_t)$ is also a quadratic function of the state vector Y_t.

We decompose the matrix V in the value function $\mathcal{V}$ accordingly. Note that because

$$\begin{aligned}
\frac{1}{\Delta t}E_t^{AM}\left[\mathcal{V}(X_t)\right] &= E_t^{AM}\left[X_t^{(1)\prime}V^{(1,1)}X_t^{(1)} + 2X_t^{(3)}V^{(3,1)}X_t^{(1)} + 2X_t^{(1)\prime}V^{(1,2)}X_t^{(2)}\right] \\
&\quad + V^{(3,3)}c_t^2 + 2c_tV^{(3,2)}X_t^{(2)} + X_t^{(2)\prime}V^{(2,2)}X_t^{(2)} + V_0 \\
&= V^{(3,3)}c_t^2 + 2c_t\left[V^{(3,1)}A^{(1,1)}X_{t-1}^{(1)} + V^{(3,2)}X_t^{(2)}\right] + X_t^{(2)\prime}V^{(2,2)}X_t^{(2)} + V_0 \\
&\quad + X_{t-1}^{(1)\prime}A^{(1,1)\prime}V^{(1,1)}A^{(1,1)}X_{t-1}^{(1)} + E\left[\epsilon_{X,t}^{(1)\prime}V^{(1,1)}\epsilon_{X,t}^{(1)}\right] + 2X_{t-1}^{(1)\prime}A^{(1,1)\prime}V^{(1,2)}X_t^{(2)}
\end{aligned}$$

Denote $V_{cc} \equiv V^{(3,3)}$. Note that $X_{t-1}^{(1)} = Y_t^{(2)}$ and $X_t^{(2)} = Y_t^{(3)}$. We can rewrite the above equation below

$$\begin{aligned}
\frac{1}{\Delta t}E_t^{AM}\left[\mathcal{V}(X_t)\right] &= V_{cc}c_t^2 + 2c_t\left[V^{(3,1)}A^{(1,1)}Y_t^{(2)} + V^{(3,2)}Y_t^{(3)}\right] + Y_t^{(3)\prime}V^{(2,2)}Y_t^{(3)} \\
&\quad + Y_t^{(2)\prime}A^{(1,1)\prime}V^{(1,1)}A^{(1,1)}Y_t^{(2)} + 2Y_t^{(2)\prime}A^{(1,1)\prime}V^{(1,2)}Y_t^{(3)} + E\left[\epsilon_{X,t}^{(1)\prime}V^{(1,1)}\epsilon_{X,t}^{(1)}\right] + V_0
\end{aligned}$$

Solving the optimization problem in (24) is equivalent to solving the problem below:

$$\min_{c_t} \frac{\xi_{\Delta x}}{\Delta t}\left((1-\omega_0)\Delta x_t - \Delta c_t\right)^2 + \frac{\xi_{\Delta e}}{\Delta t}\left(c_t - e_{t-1}\right)^2 + \left(V_{cc} + \xi_c\right)c_t^2 + 2c_t\left[V^{(3,1)}A^{(1,1)}Y_t^{(2)} + V^{(3,2)}Y_t^{(3)}\right]$$

The solution is the following generalized two-pillar policy:

$$c_t = w_1 e_{t-1} + w_2\left(c_{t-1} + (1-\omega_0)\Delta x_t\right) + h_{t-1},$$

where $w_1 \equiv \frac{\xi_{\Delta e}/\Delta t}{V_{cc}+\xi_c+(\xi_{\Delta e}+\xi_{\Delta x})/\Delta t}$, $w_2 \equiv \frac{\xi_{\Delta x}/\Delta t}{V_{cc}+\xi_c+(\xi_{\Delta e}+\xi_{\Delta x})/\Delta t}$, and $h_{t-1} \equiv -\frac{V^{(3,1)}A^{(1,1)}Y_t^{(2)}+V^{(3,2)}Y_t^{(3)}}{V_{cc}+\xi_c+(\xi_{\Delta e}+\xi_{\Delta x})/\Delta t}$.

Denote $\widehat{V}^{(3,2)} = V^{(3,2)} - \xi_{\Delta e}/\Delta t\,[0,1,0]$. After tedious algebra (see detailed derivations in the online appendix), we can show that

$$
\begin{aligned}
& \frac{1}{\Delta t}\mathcal{U}\left(Y_t\right) \\
= \; & \frac{\xi_{\Delta x}}{\Delta t}\left(\left(1-\omega_0\right)\Delta x_t-\Delta c_t\right)^2+\frac{\xi_{\Delta e}}{\Delta t}\left(c_t-e_{t-1}\right)^2+\left(V_{cc}+\xi_c\right)c_t^2+2c_t\left[V^{(3,1)}A^{(1,1)}Y_t^{(2)}+\widehat{V}^{(3,2)}Y_t^{(3)}\right] \\
& +Y_t^{(3)\prime}V^{(2,2)}Y_t^{(3)}+Y_t^{(2)\prime}A^{(1,1)\prime}V^{(1,1)}A^{(1,1)}Y_t^{(2)}+2Y_t^{(2)\prime}A^{(1,1)\prime}V^{(1,2)}Y_t^{(3)}+E\left[\epsilon_{X,t}^{(1)\prime}V^{(1,1)}\epsilon_{X,t}^{(1)}\right]+V_0 \\
= \; & Y_t^{(1)\prime}U^{(1,1)}Y_t^{(1)} \\
& -\frac{\left[Y_t^{(2)\prime}A^{(1,1)\prime}V^{(3,1)\prime}V^{(3,1)}A^{(1,1)}Y_t^{(2)}+Y_t^{(3)\prime}\widehat{V}^{(3,2)\prime}\widehat{V}^{(3,2)}Y_t^{(3)}+2Y_t^{(2)\prime}A^{(1,1)\prime}V^{(3,1)\prime}\widehat{V}^{(3,2)}Y_t^{(3)}\right]}{V_{cc}+\xi_c+\left(\xi_{\Delta e}+\xi_{\Delta x}\right)/\Delta t} \\
& +2Y_t^{(1)\prime}\begin{bmatrix} w_2\left(1-\omega_0\right) \\ w_1+w_2 \end{bmatrix}\left(V^{(3,1)}A^{(1,1)}Y_t^{(2)}+\widehat{V}^{(3,2)}Y_t^{(3)}\right)+\xi_{\Delta e}/\Delta t\left(d_{t-1}^2+2c_{t-1}d_{t-1}\right) \\
& +Y_t^{(3)\prime}V^{(2,2)}Y_t^{(3)}+Y_t^{(2)\prime}A^{(1,1)\prime}V^{(1,1)}A^{(1,1)}Y_t^{(2)}+2Y_t^{(2)\prime}A^{(1,1)\prime}V^{(1,2)}Y_t^{(3)}+E\left[\epsilon_{X,t}^{(1)\prime}V^{(1,1)}\epsilon_{X,t}^{(1)}\right]+V_0 \\
= \; & Y_t^{(1)\prime}U^{(1,1)}Y_t^{(1)}+2Y_t^{(1)\prime}\begin{bmatrix} w_2\left(1-\omega_0\right) \\ w_1+w_2 \end{bmatrix}V^{(3,1)}A^{(1,1)}Y_t^{(2)} \\
& +2Y_t^{(1)\prime}\left(\begin{bmatrix} w_2\left(1-\omega_0\right) \\ w_1+w_2 \end{bmatrix}\widehat{V}^{(3,2)}+\frac{\xi_{\Delta e}}{\Delta t}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right)Y_t^{(3)} \\
& +Y_t^{(2)\prime}\left[A^{(1,1)\prime}V^{(1,1)}A^{(1,1)}-\frac{A^{(1,1)\prime}V^{(3,1)\prime}V^{(3,1)}A^{(1,1)}}{V_{cc}+\xi_c+\left(\xi_{\Delta e}+\xi_{\Delta x}\right)/\Delta t}\right]Y_t^{(2)} \\
& +Y_t^{(3)\prime}\left[V^{(2,2)}-\frac{\widehat{V}^{(3,2)\prime}\widehat{V}^{(3,2)}}{V_{cc}+\xi_c+\left(\xi_{\Delta e}+\xi_{\Delta x}\right)/\Delta t}+\frac{\xi_{\Delta e}}{\Delta t}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]Y_t^{(3)} \\
& +2Y_t^{(2)\prime}\left[A^{(1,1)\prime}V^{(1,2)}-\frac{A^{(1,1)\prime}V^{(3,1)\prime}\widehat{V}^{(3,2)}}{V_{cc}+\xi_c+\left(\xi_{\Delta e}+\xi_{\Delta x}\right)/\Delta t}\right]Y_t^{(3)}+E\left[\epsilon_{X,t}^{(1)\prime}V^{(1,1)}\epsilon_{X,t}^{(1)}\right]+V_0 \\
\equiv \; & Y_t^{\prime}UY_t+U_0,
\end{aligned}
$$

where $U_0 \equiv E\left[\epsilon_{X,t}^{(1)\prime}V^{(1,1)}\epsilon_{X,t}^{(1)}\right]+V_0$, and $U=\left[U^{(i,j)}\right]$, $i,j=1,2,3$, and

$$
\begin{aligned}
U^{(1,1)} &= \left(V_{cc}+\xi_c+\frac{\xi_{\Delta x}+\xi_{\Delta e}}{\Delta t}\right)\begin{bmatrix} \left(1-\omega_0\right)^2 w_2\left(1-w_2\right) & \left(1-\omega_0\right)w_2\left(1-w_1-w_2\right) \\ \left(1-\omega_0\right)w_2\left(1-w_1-w_2\right) & \left(w_1+w_2\right)\left(1-w_1-w_2\right) \end{bmatrix}, \\
U^{(2,2)} &= A^{(1,1)\prime}V^{(1,1)}A^{(1,1)}-\frac{A^{(1,1)\prime}V^{(3,1)\prime}V^{(3,1)}A^{(1,1)}}{V_{cc}+\xi_c+\left(\xi_{\Delta e}+\xi_{\Delta x}\right)/\Delta t}, \\
U^{(3,3)} &= V^{(2,2)}-\frac{\widehat{V}^{(3,2)\prime}\widehat{V}^{(3,2)}}{V_{cc}+\xi_c+\left(\xi_{\Delta e}+\xi_{\Delta x}\right)/\Delta t}+\frac{\xi_{\Delta e}}{\Delta t}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
U^{(1,2)} &= \begin{bmatrix} w_2(1-\omega_0) \\ w_1+w_2 \end{bmatrix} V^{(3,1)}A^{(1,1)}, \\
U^{(1,3)} &= \begin{bmatrix} w_2(1-\omega_0) \\ w_1+w_2 \end{bmatrix} \widehat{V}^{(3,2)} + \frac{\xi_{\Delta e}}{\Delta t}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \\
U^{(2,3)} &= A^{(1,1)\prime}V^{(1,2)} - \frac{A^{(1,1)\prime}V^{(3,1)\prime}\widehat{V}^{(3,2)}}{V_{cc}+\xi_c+(\xi_{\Delta e}+\xi_{\Delta x})/\Delta t},
\end{aligned}
$$

and $U^{(2,1)} = U^{(1,2)\prime}$, $U^{(3,1)} = U^{(1,3)\prime}$, $U^{(3,2)} = U^{(2,3)\prime}$.

Appendix D2: Dynamics of the state variables

Define $Z_t = \begin{bmatrix} X_t \\ d_t \end{bmatrix}$. We now derive the dynamics of the state variables. First, note that

$$
\begin{aligned}
X_{t+1}^{(1)} &= \begin{bmatrix} q_{t+1} \\ i_{t+1}^{*} \end{bmatrix} = \begin{bmatrix} 1-\rho_q\Delta t & & 0_{1\times5} \\ & 1-\rho_{i^*}\Delta t & 0_{1\times5} \end{bmatrix} Z_t + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u_t + \begin{bmatrix} \epsilon_{q,t+1} \\ \epsilon_{i^*,t+1} \end{bmatrix} \\
&\equiv A^{(1,\cdot)}Z_t + B^{(1,\cdot)}u_t + \epsilon_{X,t+1}^{(1)}.
\end{aligned}
$$

and

$$
\begin{aligned}
X_{t+1}^{(2)} &= \begin{bmatrix} m_t \\ d_t \\ i_t \end{bmatrix} = \begin{bmatrix} m_{t-1}+u_t \\ d_t \\ \alpha^{-1}\left(d_t-(m_{t-1}+u_t)+c_t-q_t\right) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1/\alpha & 0 & -1/\alpha & 0 & 0 & 1/\alpha & 1/\alpha \end{bmatrix} Z_t + \begin{bmatrix} 1 \\ 0 \\ -1/\alpha \end{bmatrix} u_t \\
&\equiv A^{(2,\cdot)}Z_t + B^{(2,\cdot)}u_t,
\end{aligned}
$$

and

$$
\begin{aligned}
X_{t+1}^{(3)} &= c_{t+1} = w_1 d_t + (w_1+w_2)c_t + w_2(1-\omega_0)\Delta x_{t+1} - \frac{V^{(3,1)}A^{(1,1)}X_t^{(1)} + V^{(3,2)}X_{t+1}^{(2)}}{V_{cc}+\xi_c+(\xi_{\Delta e}+\xi_{\Delta x})/\Delta t} \\
&= \begin{bmatrix} -\frac{V^{(3,1)}A^{(1,1)}}{V_{cc}+\xi_c+(\xi_{\Delta e}+\xi_{\Delta x})/\Delta t} & 0_{3\times1} & w_1+w_2 & 0 \end{bmatrix} Z_t \\
&\quad -\frac{\widehat{V}^{(3,2)}\left(A^{(2,\cdot)}Z_t + B^{(2,\cdot)}u_t\right)}{V_{cc}+\xi_c+(\xi_{\Delta e}+\xi_{\Delta x})/\Delta t} + w_2(1-\omega_0)\Delta x_{t+1} \\
&\equiv A^{(3,\cdot)}Z_t + B^{(3,\cdot)}u_t + w_2(1-\omega_0)\Delta x_{t+1},
\end{aligned}
$$

and[33]

$$
\begin{aligned}
E_t d_{t+1} &= (1-w_1)\, d_t + (i_t - i_t^*)\,\Delta t + (1-w_1-w_2)\, c_t - h_t \\
&= (1-w_1)\, d_t + \left[\alpha^{-1}\left(d_t - (m_{t-1}+u_t) + c_t - q_t\right) - i_t^*\right]\Delta t + (1-w_1-w_2)\, c_t - h_t \\
&= -\frac{\Delta t}{\alpha} q_t - i_t^* \Delta t + \left(\frac{\Delta t}{\alpha} + (1-w_1-w_2)\right) c_t - \frac{\Delta t}{\alpha} m_{t-1} + \left(1 - w_1 + \frac{\Delta t}{\alpha}\right) d_t - \frac{\Delta t}{\alpha} u_t \\
&\quad + \frac{V^{(3,1)} A^{(1,1)} X_t^{(1)} + V^{(3,2)} X_{t+1}^{(2)}}{V_{cc} + \xi_c + (\xi_{\Delta e} + \xi_{\Delta x})/\Delta t} \\
&= \begin{bmatrix} -\frac{\Delta t}{\alpha} & -\Delta t & -\frac{\Delta t}{\alpha} & 0 & 0 & \frac{\Delta t}{\alpha} + (1-w_1-w_2) & 1 + \frac{\Delta t}{\alpha} \end{bmatrix} Z_t + \left(-\frac{\Delta t}{\alpha}\right) u_t \\
&\quad + \frac{V^{(3,1)} A^{(1,1)}}{V_{cc} + \xi_c + (\xi_{\Delta e} + \xi_{\Delta x})/\Delta t} X_t^{(1)} + \frac{\widehat{V}^{(3,2)}\left(A^{(2,\cdot)} Z_t + B^{(2,\cdot)} u_t\right)}{V_{cc} + \xi_c + (\xi_{\Delta e} + \xi_{\Delta x})/\Delta t} \\
&\equiv A^{(4,\cdot)} Z_t + B^{(4,\cdot)} u_t.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\begin{bmatrix} X_{t+1} \\ E_t\left[d_{t+1}\right] \end{bmatrix} &= \begin{bmatrix} X_{t+1}^{(1)} \\ X_{t+1}^{(2)} \\ X_{t+1}^{(3)} \\ E_t\left[d_{t+1}\right] \end{bmatrix} = \begin{bmatrix} A^{(1,\cdot)} \\ A^{(2,\cdot)} \\ A^{(3,\cdot)} \\ A^{(4,\cdot)} \end{bmatrix} Z_t + \begin{bmatrix} 0 \\ B^{(2,\cdot)} \\ B^{(3,\cdot)} \\ B^{(4,\cdot)} \end{bmatrix} u_t + \begin{bmatrix} \epsilon_{X,t+1}^{(1)} \\ 0 \\ w_2(1-w_0)\,\Delta x_{t+1} \\ 0 \end{bmatrix} \\
&\equiv AZ_t + Bu_t + \begin{bmatrix} \epsilon_{X,t+1}^{(1)} \\ 0 \\ w_2(1-w_0)\,\Delta x_{t+1} \\ 0 \end{bmatrix} \qquad (56)
\end{aligned}
$$

[33] This is because

$$
i_t = \alpha^{-1}\left(d_t - (m_{t-1}+u_t) + c_t - q_t\right)
$$

and

$$
\begin{aligned}
d_{t+1} &= e_{t+1} - \left[w_1 d_t + (w_1+w_2)\, c_t + w_2(1-w_0)\,\Delta x_{t+1} + h_t\right] \\
&= e_{t+1} - e_t + (1-w_1)\, d_t + (1-w_1-w_2)\, c_t - w_2(1-w_0)\,\Delta x_{t+1} - h_t
\end{aligned}
$$

and

$$
\begin{aligned}
E_t\left[d_{t+1}\right] &= E_t\left[e_{t+1} - e_t\right] + (1-w_1)\, d_t + (1-w_1-w_2)\, c_t - w_2(1-w_0)\, E_t\left[\Delta x_{t+1}\right] - h_t \\
&= (1-w_1)\, d_t + (i_t - i_t^*)\,\Delta t + (1-w_1-w_2)\, c_t - w_2(1-w_0)\, E_t\left[\Delta x_{t+1}\right] - h_t \\
&= (1-w_1)\, d_t + (i_t - i_t^*)\,\Delta t + (1-w_1-w_2)\, c_t - h_t
\end{aligned}
$$

where in deriving the last equality we have used the result $E_t\left[\Delta x_{t+1}\right] = 0$ under the assumption that $\Delta x_{t+1} = \epsilon_{\Delta x,t+1}$.

Appendix D3: The central bank's PM problem and the solution

Recall that $\frac{1}{\Delta t}\mathcal{V}(X_t) = \min_{u_t} \xi_d d_t^2 + \xi_i i_t^2 + \beta E_t^{PM}\left[\frac{1}{\Delta t}\mathcal{U}(Y_{t+1})\right]$. Note that:

$$\begin{aligned}
& \xi_d d_t^2 + \xi_i i_t^2 \\
= \; & \xi_d d_t^2 + \frac{\xi_i}{\alpha^2}\left(d_t - m_{t-1} - u_t + c_t - q_t\right)^2 \\
= \; & \frac{\xi_i}{\alpha^2} q_t^2 + 2q_t\left(\frac{\xi_i}{\alpha^2} m_{t-1} - \frac{\xi_i}{\alpha^2}\left(c_t + d_t\right)\right) + \frac{\xi_i}{\alpha^2} m_{t-1}^2 + 2m_{t-1}\left(-\frac{\xi_i}{\alpha^2}\left(c_t + d_t\right)\right) + \left(\xi_d + \frac{\xi_i}{\alpha^2}\right) d_t^2 \\
& + \frac{\xi_i}{\alpha^2} c_t^2 + 2\frac{\xi_i}{\alpha^2} c_t d_t + \left(\frac{\xi_u}{\Delta t} + \frac{\xi_i}{\alpha^2}\right) u_t^2 + 2\left[\frac{\xi_i}{\alpha^2} q_t + \frac{\xi_i}{\alpha^2} m_{t-1} - \frac{\xi_i}{\alpha^2} d_t - \frac{\xi_i}{\alpha^2} c_t\right] u_t \\
\equiv \; & Z_t' Q Z_t + Z_t' W u_t + u_t' W' Z_t + u_t' R u_t,
\end{aligned}$$

where $\Xi \equiv \xi_i/\alpha^2$, $R = \Xi$, and

$$Z_t = \begin{pmatrix} q_t \\ i_t^* \\ m_{t-1} \\ d_{t-1} \\ i_{t-1} \\ c_t \\ d_t \end{pmatrix}, W = \begin{bmatrix} \Xi \\ 0 \\ \Xi \\ 0 \\ 0 \\ -\Xi \\ -\Xi \end{bmatrix}, Q = \begin{bmatrix} \Xi & 0 & \Xi & 0 & 0 & -\Xi & -\Xi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Xi & 0 & \Xi & 0 & 0 & -\Xi & -\Xi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\Xi & 0 & -\Xi & 0 & 0 & \Xi & \Xi \\ -\Xi & 0 & -\Xi & 0 & 0 & \Xi & \xi_d + \Xi \end{bmatrix}.$$

First, note that

$$\begin{aligned}
& E_t^{PM}\left[Y_{t+1}' U Y_{t+1}\right] \\
= \; & U_{11} Var(\Delta x) + U_{22} c_t^2 + 2\left(U_{23} c_t q_t + U_{24} c_t i_t^* + U_{25} c_t m_t + U_{26} c_t d_t + U_{27} c_t i_t\right) + U_{33} q_t^2 \\
& + 2\left(U_{34} q_t i_t^* + U_{35} q_t m_t + U_{36} q_t d_t + U_{37} q_t i_t\right) + U_{44}\left(i_t^*\right)^2 + 2\left(U_{45} i_t^* m_t + U_{46} i_t^* d_t + U_{47} i_t^* i_t\right) \\
& + U_{55} m_t^2 + 2\left(U_{56} m_t d_t + U_{57} m_t i_t\right) + U_{66} d_t^2 + U_{67} d_t i_t + U_{77} i_t^2,
\end{aligned}$$

where we have used the results $E_t^{PM}\left[\frac{1}{\Delta t}\mathcal{U}(Y_{t+1})\right] = E_t^{PM}\left[Y_{t+1}' U Y_{t+1}\right] + U_0$, $E_t^{PM}\left[\left(\Delta x_{t+1}\right)^2\right] = Var(\Delta x) = \sigma_{\Delta x}^2 \Delta t$, and $E_t^{PM}\left[Y_{t+1}\right] = \begin{bmatrix} 0 & c_t & q_t & i_t^* & m_t & d_t & i_t \end{bmatrix}'$.

Then after tedious algebra (see detailed derivations in the online appendix), we can show that

$$
\begin{aligned}
& E_t^{PM}\left[Y_{t+1}' U Y_{t+1}\right] \\
= \; & \left(U_{33} - \frac{2}{\alpha}U_{37} + \frac{U_{77}}{\alpha^2}\right) q_t^2 + 2q_t \begin{pmatrix} \left(U_{34} - \frac{U_{47}}{\alpha}\right) i_t^* + \left(U_{35} - \frac{U_{57}+U_{37}}{\alpha} + \frac{U_{77}}{\alpha^2}\right) m_{t-1} \\ + \left(U_{23} - \frac{U_{27}-U_{37}}{\alpha} - \frac{U_{77}}{\alpha^2}\right) c_t + \left(U_{36} - \frac{U_{67}-U_{37}}{\alpha} - \frac{U_{77}}{\alpha^2}\right) d_t \end{pmatrix} \\
& + U_{44}\left(i_t^*\right)^2 + 2i_t^*\left(\left(U_{45} - \frac{1}{\alpha}U_{47}\right) m_{t-1} + \left(U_{24} + \frac{1}{\alpha}U_{47}\right) c_t + \left(U_{46} + \frac{1}{\alpha}U_{47}\right) d_t\right) \\
& + \left(U_{55} - \frac{2}{\alpha}U_{57} + \frac{1}{\alpha^2}U_{77}\right) m_{t-1}^2 + 2m_{t-1} \begin{pmatrix} \left(U_{25} - \frac{1}{\alpha}\left(U_{27} - U_{57}\right) - \frac{1}{\alpha^2}U_{77}\right) c_t \\ + \left(U_{56} - \frac{1}{\alpha}\left(U_{67} - U_{57}\right) - \frac{1}{\alpha^2}U_{77}\right) d_t \end{pmatrix} \\
& + \left(U_{22} + \frac{2}{\alpha}U_{27} + \frac{U_{77}}{\alpha^2}\right) c_t^2 + 2\left(U_{26} + \frac{U_{27}+U_{67}}{\alpha} + \frac{U_{77}}{\alpha^2}\right) c_t d_t + \left(U_{66} + \frac{2}{\alpha}U_{67} + \frac{U_{77}}{\alpha^2}\right) d_t^2 \\
& + 2u_t \begin{pmatrix} \left(U_{35} - \frac{U_{57}+U_{37}}{\alpha} + \frac{U_{77}}{\alpha^2}\right) q_t + \left(U_{45} - \frac{U_{47}}{\alpha}\right) i_t^* + \left(U_{55} - \frac{2}{\alpha}U_{57} + \frac{U_{77}}{\alpha^2}\right) m_{t-1} \\ + \left(U_{25} + \frac{U_{57}-U_{27}}{\alpha} - \frac{U_{77}}{\alpha^2}\right) c_t + \left(U_{56} + \frac{U_{57}-U_{67}}{\alpha} - \frac{U_{77}}{\alpha^2}\right) d_t \end{pmatrix} \\
& + \left(U_{55} - \frac{2}{\alpha}U_{57} + \frac{1}{\alpha^2}U_{77}\right) u_t^2 + U_{11} Var\left(\Delta x\right) \\
\equiv \; & Z_t' \widetilde{Q} Z_t + Z_t' \widetilde{W} u_t + u_t' \widetilde{W}' Z_t + u_t' \widetilde{R} u_t + U_{11} Var\left(\Delta x\right),
\end{aligned}
$$

where $\widetilde{R} = U_{55} - \frac{2}{\alpha}U_{57} + \frac{1}{\alpha^2}U_{77}$, and

$$
\widetilde{W} = \begin{bmatrix} U_{35} - \frac{1}{\alpha}U_{57} - \frac{1}{\alpha}U_{37} + \frac{1}{\alpha^2}U_{77} \\ U_{45} - \frac{1}{\alpha}U_{47} \\ U_{55} - \frac{2}{\alpha}U_{57} + \frac{1}{\alpha^2}U_{77} \\ 0 \\ 0 \\ U_{25} + \frac{1}{\alpha}U_{57} - \frac{1}{\alpha}U_{27} - \frac{1}{\alpha^2}U_{77} \\ U_{56} + \frac{1}{\alpha}U_{57} - \frac{1}{\alpha}U_{67} - \frac{1}{\alpha^2}U_{77} \end{bmatrix}, \widetilde{Q} = \begin{bmatrix} \widetilde{Q}^{(1,1)} & \widetilde{Q}^{(1,2)} & \widetilde{Q}^{(1,3)} \\ \widetilde{Q}^{(2,1)} & \widetilde{Q}^{(2,2)} & \widetilde{Q}^{(2,3)} \\ \widetilde{Q}^{(3,1)} & \widetilde{Q}^{(3,2)} & \widetilde{Q}^{(3,3)} \end{bmatrix},
$$

and

$$
\begin{aligned}
\widetilde{Q}^{(1,1)} &= \begin{bmatrix} U_{33} + \alpha^{-2}U_{77} - 2\alpha^{-1}U_{37} & U_{34} - \alpha^{-1}U_{47} \\ U_{34} - \alpha^{-1}U_{47} & U_{44} \end{bmatrix}, \\
\widetilde{Q}^{(1,2)} &= \begin{bmatrix} U_{35} + \alpha^{-2}U_{77} - \alpha^{-1}\left(U_{57} + U_{37}\right) & 0 \\ U_{45} - \alpha^{-1}U_{47} & 0 \end{bmatrix}, \\
\widetilde{Q}^{(1,3)} &= \begin{bmatrix} 0 & U_{23} - \alpha^{-2}U_{77} - \alpha^{-1}\left(U_{27} - U_{37}\right) & U_{36} - \alpha^{-2}U_{77} - \alpha^{-1}\left(U_{67} - U_{37}\right) \\ 0 & U_{24} + \alpha^{-1}U_{47} & U_{46} + \alpha^{-1}U_{47} \end{bmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{Q}^{(2,2)} &= \begin{bmatrix} U_{55}+\alpha^{-2}U_{77}-2\alpha^{-1}U_{57} & 0 \\ 0 & 0 \end{bmatrix}, \\
\widetilde{Q}^{(2,3)} &= \begin{bmatrix} 0 & U_{25}-\alpha^{-2}U_{77}-\alpha^{-1}\left(U_{27}-U_{57}\right) & U_{56}-\alpha^{-2}U_{77}-\alpha^{-1}\left(U_{67}-U_{57}\right) \\ 0 & 0 & 0 \end{bmatrix}, \\
\widetilde{Q}^{(3,3)} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & U_{22}+\alpha^{-2}U_{77}+2\alpha^{-1}U_{27} & U_{26}+\alpha^{-2}U_{77}+\alpha^{-1}\left(U_{27}+U_{67}\right) \\ 0 & U_{26}+\alpha^{-2}U_{77}+\alpha^{-1}\left(U_{27}+U_{67}\right) & U_{66}+\alpha^{-2}U_{77}+2\alpha^{-1}U_{67} \end{bmatrix},
\end{aligned}
$$

and $\widetilde{Q}^{(2,1)} = \widetilde{Q}^{(1,2)\prime}$, $\widetilde{Q}^{(3,1)} = \widetilde{Q}^{(1,3)\prime}$, $\widetilde{Q}^{(2,3)} = \widetilde{Q}^{(3,2)\prime}$.

Next, we now solve the above problem of the central bank at the PM of period t. This solution is derived as follows. First, the matrices A, Q, and B are decomposed according to the decomposition of $Z_t = (X_t, d_t)'$:

$$
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}
$$

Second, given $d_t = DX_t$, from (56), we have

$$
\begin{aligned}
E_t\left[d_{t+1}\right] &= A_{21}X_t + A_{22}d_t + B_2u_t = D_{t+1}E_t\left[X_{t+1}\right] = D_{t+1}\left(A_{11}X_t + A_{12}d_t + B_1u_t\right), \\
d_t &= \left(A_{22}-D_{t+1}A_{12}\right)^{-1}\left[\left(D_{t+1}A_{11}-A_{21}\right)X_t + \left(D_{t+1}B_1-B_2\right)u_t\right] \equiv H_tX_t + G_tu_t,
\end{aligned}
$$

where $H_t \equiv \left(A_{22}-D_{t+1}A_{12}\right)^{-1}\left(D_{t+1}A_{11}-A_{21}\right)$ and $G_t \equiv \left(A_{22}-D_{t+1}A_{12}\right)^{-1}\left(D_{t+1}B_1-B_2\right)$.

Third, substitution and identification of the terms in (23) results in

$$
\begin{aligned}
\frac{1}{\Delta t}\mathcal{V}\left(X_t\right) &= X_t'V_tX_t + V_{0,t} \\
&= \min_{u_t} \xi_d d_t^2 + \xi_i i_t^2 + \xi_{\Delta d}\left(\Delta d_t\right)^2/\Delta t + \xi_{\Delta i}\left(\Delta i_t\right)^2/\Delta t + \xi_u u_t^2/\Delta t + \beta E_t^{PM}\left[\frac{1}{\Delta t}\mathcal{U}\left(Y_{t+1}\right)\right] \\
&= \min_{u_t}\left(X_t'Q_t^*X_t + X_t'W^*u_t + u_t'W_t^{*\prime}X_t + u_t'R_t^*u_t\right) \\
&\quad +\beta\left(X_t'\widetilde{Q}_t^*X_t + X_t'\widetilde{W}^*u_t + u_t'\widetilde{W}_t^{*\prime}X_t + u_t'\widetilde{R}_t^*u_t + U_{11}Var\left(\Delta x\right)\right),
\end{aligned}
$$

where $Q_t^* \equiv Q_{11} + Q_{12}H_t + H_t'Q_{21} + H_t'Q_{22}H_t$, $W_t^* \equiv W_1 + H_t'W_2 + Q_{12}G_t + H_t'Q_{22}G_t$, $R_t^* \equiv R + G_t'W_2 + W_2'G_t + G_t'Q_{22}G_t$. Similarly defined are $\widetilde{Q}_t^*$, $\widetilde{U}_t^*$, and $\widetilde{R}_t^*$.

Fourth, optimization gives the standard result that the optimal choice of u_t can be expressed as a feedback on X_t

$$
u_t = -\left(R_t^* + \beta\widetilde{R}_t^*\right)^{-1}\left(W_t^* + \beta\widetilde{W}_t^*\right)' X_t \equiv -F_t^*X_t,
$$

where $F_t^* = \left(R_t^* + \beta\widetilde{R}_t^*\right)^{-1}\left(W_t^* + \beta\widetilde{W}_t^*\right)'$ and $V_t = Q_t^* + \beta\widetilde{Q}_t^* - \left(W_t^* + \beta\widetilde{W}_t^*\right)F_t^* - F_t^{*\prime}\left(W_t^* + \beta\widetilde{W}_t^*\right)' + F_t^{*\prime}\left(R_t^* + \beta\widetilde{R}_t^*\right)F_t^*$.

Finally,

$$d_t = H_t X_t + G_t u_t = \left(H_t - G_t F_t^*\right) X_t \equiv D_t X_t,$$

where $D_t = H_t - G_t F_t^*$. Therefore, the stationary solution is given as in Proposition (2).

Appendix D4: Characterization of the Steady Equilibrium

In this appendix, we further characterize the steady equilibrium. In particular, we conjecture the matrix V in the value function $\mathcal{V}$ to be zero, except its elements V_{22}, V_{26} $(= V_{62})$, and V_{66}, where V_{ij} denotes the (i, j) element of the matrix V. In the online appendix, we verify that the matrix V indeed takes the conjectured form.

One immediate implication of the conjecture is that

$$\begin{aligned} h_{t-1} &= -\frac{V^{(3,1)}A^{(1,1)}Y_t^{(2)} + V^{(3,2)}Y_t^{(3)}}{V_{cc} + \xi_c + \left(\xi_{\Delta e} + \xi_{\Delta x}\right)/\Delta t} = -\frac{V_{62}\left(1 - \rho_{i^*}\Delta t\right)}{V_{cc} + \xi_c + \left(\xi_{\Delta e} + \xi_{\Delta x}\right)/\Delta t} i_{t-1}^* \\ &\equiv h i_{t-1}^*, \end{aligned}$$

where $h \equiv -\frac{V_{62}(1-\rho_{i^*}\Delta t)}{V_{cc}+\xi_c+(\xi_{\Delta e}+\xi_{\Delta x})/\Delta t}$. Note that when the US interest rate is independent over time (i.e., $(1 - \rho_{i^*}\Delta t) = 0$), then $h = 0$.

As derived in the online appendix, we can further show that $D = \begin{bmatrix} 0 & D_2 & 0 & 0 & 0 & D_6 \end{bmatrix}$, $F^* = \begin{bmatrix} 1 & F_2 & 1 & 0 & 0 & F_6 \end{bmatrix}$, $H = \begin{bmatrix} G & H_2 & G & 0 & 0 & H_6 \end{bmatrix}$, $G = \frac{\Delta t/\alpha}{1+\Delta t/\alpha - w_1(1+D_6)}$, where D_2 and D_6 are the second and sixth elements of D, respectively, and F_2, F_6, H_2, H_6 are similarly defined. These elements are determined endogenously together with the matrix V. After tedious algebra (see the online appendix for detailed derivations), we can verify indeed the matrix V takes the conjectured form.

Appendix E: Extended model with intraday government intervention

In the extended model, the government's AM problem of choosing central parity is the same as in the baseline model. Under the simplifying assumptions of i.i.d. processes of q_t and i_t^* as well as $\xi_{\Delta e} = 0$, the central parity has simplified rule as follows

$$c_{t+1} = w_1 e_t + w_2 \left(c_t + \left(1 - \omega_0\right)\Delta x_{t+1}\right),$$

where $w_1 = \frac{\xi_{\Delta e}/\Delta t}{V_{cc}+\xi_c+(\xi_{\Delta e}+\xi_{\Delta x})/\Delta t}$, $w_2 = \frac{\xi_{\Delta x}/\Delta t}{V_{cc}+\xi_c+(\xi_{\Delta e}+\xi_{\Delta x})/\Delta t}$, and $V_{cc} \equiv V^{(2,2)}$. Furthermore, the government's value function from its AM optimization problem is given by $\mathcal{U}(Y_t) = (Y_t' U Y_t + U_0)\Delta t$ where $Y_t = (\Delta x_t, c_{t-1})'$ denotes the state variable, and

$$U = \left(V_{cc} + \xi_c + \frac{\xi_{\Delta e} + \xi_{\Delta x}}{\Delta t}\right)\begin{bmatrix} (1-\omega_0)^2 w_2 (1-w_2) & (1-\omega_0) w_2 (1-w_1-w_2) \\ (1-\omega_0) w_2 (1-w_1-w_2) & (w_1+w_2)(1-w_1-w_2) \end{bmatrix}.$$

We now turn to the government's PM problem of choosing optimal monetary policy in (23):

$$\mathcal{V}(X_t) = \min_{u_t} \left[\xi_d d_t^2 + \xi_i i_t^2\right]\Delta t + \beta E_t^{PM}\left[\mathcal{U}(Y_{t+1})\right].$$

Because Δx_t follows an i.i.d. process, $E_t^{PM}\left[\mathcal{U}(Y_{t+1})\right]$, derived below, depends on only c_t, not money supply m_t:

$$\begin{aligned} & \frac{1}{\Delta t} E_t^{PM}\left[\mathcal{U}(Y_{t+1})\right] \\ = & \frac{(V_{cc}+\xi_c)(\xi_{\Delta e}/\Delta t + \xi_{\Delta x}/\Delta t)}{V_{cc}+\xi_c+\xi_{\Delta e}/\Delta t+\xi_{\Delta x}/\Delta t} c_t^2 + \frac{(V_{cc}+\xi_c+\xi_{\Delta e}/\Delta t)(\xi_{\Delta x}/\Delta t)}{V_{cc}+\xi_c+\xi_{\Delta e}/\Delta t+\xi_{\Delta x}/\Delta t}(1-\omega_0)^2 Var(\Delta x) + U_0 \\ \equiv & Z_t'\widetilde{Q}Z_t + Z_t'\widetilde{W}m_t + m_t'\widetilde{W}'Z_t + u_t'\widetilde{R}m_t + U_{11}Var(\Delta x) + U_0, \end{aligned}$$

where $\widetilde{R} = 0$, $\widetilde{W} = 0_{5\times 1}$, and $\widetilde{Q} = diag([0,0,U_{22},0,0])$. As a result, the government's problem at the PM can be simplified as choosing the level of money supply m_t directly. That is,

$$\begin{aligned} & \min_{m_t} \xi_d d_t^2 + \xi_i i_t^2 \\ = & \min_{m_t} \xi_d d_t^2 + \frac{\xi_i}{\alpha^2}(d_t - m_t + c_t - q_t)^2 \\ \equiv & \min_{m_t} Z_t' Q Z_t + Z_t' W m_t + m_t' W' Z_t + m_t' R m_t, \end{aligned}$$

where $\Xi = \xi_i/\alpha^2$, and $R = \Xi$,

$$X_t = \begin{pmatrix} q_t \\ i_t^* \\ c_t \\ v_t \end{pmatrix}, Z_t = \begin{pmatrix} q_t \\ i_t^* \\ c_t \\ v_t \\ d_t \end{pmatrix}, Q = \begin{bmatrix} \Xi & 0 & -\Xi & 0 & -\Xi \\ 0 & 0 & 0 & 0 & 0 \\ -\Xi & 0 & \Xi & 0 & \Xi \\ 0 & 0 & 0 & 0 & 0 \\ -\Xi & 0 & \Xi & 0 & \xi_d + \Xi \end{bmatrix}, W = \Xi \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}.$$

Similarly as in the baseline model, the transition equation can be written as

$$\begin{aligned} X_{t+1} &= A_{11}X_t + A_{12}d_t + B_1 m_t + \epsilon_{X,t+1}, \\ E_t[d_{t+1}] &= A_{21}X_t + A_{22}d_t + B_2 m_t, \end{aligned}$$

where $\epsilon_{X,t+1} \equiv (\epsilon_{q,t+1}, \epsilon_{i^*,t+1}, w_2(1-\omega_0)\Delta x_{t+1}, v_{t+1})'$ and

$$
\begin{aligned}
A_{11} &= \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & w_1+w_2 & \\ & & & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 \\ 0 \\ w_1 \\ 0 \end{bmatrix}, \\
A_{21} &= \begin{bmatrix} -\Delta t/\alpha & -\Delta t & 1-w_1-w_2+\Delta t/\alpha & -\gamma Var_t\left(\rho_{t+1}\right) \end{bmatrix}, A_{22} = 1-w_1+\Delta t/\alpha, \\
B_1 &= 0_{4\times 1}, B_2 = -\Delta t/\alpha.
\end{aligned}
$$

We can solve the model similarly as before (see Appendix B1 for detailed derivation). The basic idea is that under discretion private agents' expectations incorporate the following restriction in a rational expectations equilibrium: $d_t = DX_t$. The restriction together with the dynamics of the state vector X_t and the expectation $E_t(d_{t+1})$ imply that in equilibrium $d_t = HX_t + Gm_t$ determines the exchange rate deviation as a linear function of the state vector and money supply. Substituting the expression for d_t into the government's objective function yields

$$
\begin{aligned}
& \min_{m_t} \xi_d d_t^2 + \xi_i i_t^2 \\
= & \min_{m_t} X_t' Q_{11} X_t + X_t' Q_{12} d_t + d_t' Q_{21} X_t + d_t' Q_{22} d_t \\
& + X_t' W_1 m_t + d_t' W_2 m_t + m_t' W_1' X_t + m_t' W_2' d_t + m_t' R m_t \\
= & \min_{m_t} X_t' Q_{11} X_t + X_t' Q_{12}\left(HX_t + Gm_t\right) + \left(HX_t + Gm_t\right)' Q_{21} X_t \\
& + \left(HX_t + Gm_t\right)' Q_{22}\left(HX_t + Gm_t\right) + X_t' W_1 m_t \\
& + \left(HX_t + Gm_t\right)' W_2 m_t + m_t' W_1' X_t + m_t' W_2'\left(HX_t + Gm_t\right) + m_t' R m_t \\
= & \min_{m_t} X_t' Q^* X_t + X_t' W^* m_t + m_t' W^{*\prime} X_t + m_t' R^* m_t,
\end{aligned}
$$

where $Q^* \equiv Q_{11} + Q_{12}H + H'Q_{21} + H'Q_{22}H$, $W^* \equiv W_1 + H'W_2 + Q_{12}G + H'Q_{22}G$, and $R^* \equiv R + G'W_2 + W_2'G + G'Q_{22}G$. Therefore, the first-order condition implies the optimal money supply $m_t = -\left(R^*\right)^{-1} W^{*\prime} X_t \equiv -F^* X_t$ with $F^* \equiv \left(R^*\right)^{-1} W^{*\prime}$.

Special Case with $\xi_d = 0$

We now focus on a special case with $\xi_d = 0$. We conjecture that $V = diag\left([0, 0, V_{cc}, 0]\right)$ is a 4×4 matrix of zeros, except for the last diagonal element being $V_{cc} > 0$, and that

$$
d_t = i_t^* \Delta t - c_t + \gamma Var_t\left(\rho_{t+1}\right) v_t = \begin{bmatrix} 0 & \Delta t & -1 & \gamma Var_t\left(\rho_{t+1}\right) \end{bmatrix} X_t \equiv DX_t.
$$

Under this conjecture, we have

$$\begin{aligned} H &= (A_{22} - DA_{12})^{-1}(DA_{11} - A_{21}) = \frac{1}{1+\Delta t/\alpha}\begin{bmatrix}\frac{\Delta t}{\alpha} & \Delta t & -\left(1+\frac{\Delta t}{\alpha}\right) & \gamma Var_t\left(\rho_{t+1}\right)\end{bmatrix}, \\ G &= -(A_{22} - DA_{12})^{-1}(DB_1 - B_2) = \frac{\Delta t/\alpha}{1+\Delta t/\alpha}. \end{aligned}$$

Furthermore, we can show that $R^* = \frac{\Xi}{(1+\Delta t/\alpha)^2}$, $W^* = \frac{\Xi}{(1+\Delta t/\alpha)^2}\begin{bmatrix}1 & -\Delta t & 0 & -\gamma Var_t\left(\rho_{t+1}\right)\end{bmatrix}'$, and

$$Q^* = \frac{\Xi}{(1+\Delta t/\alpha)^2}\begin{bmatrix}1 & -\Delta t & 0 & -\gamma Var_t\left(\rho_{t+1}\right) \\ -\Delta t & (\Delta t)^2 & 0 & \gamma Var_t\left(\rho_{t+1}\right)\Delta t \\ 0 & 0 & 0 & 0 \\ -\gamma Var_t\left(\rho_{t+1}\right) & \gamma Var_t\left(\rho_{t+1}\right)\Delta t & 0 & \left(\gamma Var_t\left(\rho_{t+1}\right)\right)^2\end{bmatrix}.$$

Therefore, $F^* = -(R^*)^{-1}W^{*\prime} = \begin{bmatrix}1 & -\Delta t & 0 & -\gamma Var_t\left(\rho_{t+1}\right)\end{bmatrix}$. The above result thus verifies that our conjecture for D indeed holds. We can also show that $Q^* - W^*F^* - F^{*\prime}W^{*\prime} + F^{*\prime}R^*F^* = 0$.

Next, it is also straightforward to show that $\widetilde{R}^* = 0$, $\widetilde{W}^* = 0_{4\times 1}$, and $\widetilde{Q}^* = diag\left([0, 0, U_{22}, 0]\right)$. Therefore,

$$V^* = Q^* + \beta\widetilde{Q}^* - \left(W_t^* + \beta\widetilde{W}_t^*\right)F_t^* - F_t^{*\prime}\left(W_t^* + \beta\widetilde{W}_t^*\right)' + F_t^{*\prime}\left(R_t^* + \beta\widetilde{R}_t^*\right)F_t^* = \beta\widetilde{Q}^*.$$

It thus follows that $V_{cc} = \beta U_{22} = \frac{\beta(V_{cc}+\xi_c)(\xi_{\Delta e}/\Delta t + \xi_{\Delta x}/\Delta t)}{V_{cc}+\xi_c+\xi_{\Delta e}/\Delta t+\xi_{\Delta x}/\Delta t}$, which has a unique positive root.

Because the excess return is given by $\rho_{t+1} = (i_t - i_t^*)\Delta t - (e_{t+1} - e_t)$, we have

$$\rho_{t+1} - E_t\left[\rho_{t+1}\right] = -\left(e_{t+1} - E_t\left[e_{t+1}\right]\right) = -E\epsilon_{X,t+1},$$

implying that the conditional variance of the excess return ρ_{t+1} is given by

$$Var_t\left(\rho_{t+1}\right) = EVar\left(\epsilon_{X,t+1}\right)E' = Var\left(i_t^*\right)(\Delta t)^2 + \left(\gamma Var_t\left(\rho_{t+1}\right)\right)^2 Var\left(v_t\right). \tag{57}$$

The above equation implies the conditional variance of the excess return $Var_t\left(\rho_{t+1}\right)$ is time invariant. We can thus simply write it as $Var\left(\rho_{t+1}\right)$.

Lastly, we can determine the spot exchange rate e_t as the following

$$e_t = d_t + c_t = i_t^*\Delta t + \gamma Var\left(\rho_{t+1}\right)v_t \equiv EX_t,$$

where $E \equiv \left(0, \Delta t, 0, \gamma Var\left(\rho_{t+1}\right)\right)$. Thus $Var\left(e_t\right) = Var\left(\rho_{t+1}\right)$.