Dynamic Capital Structure and Related Models

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Leland Models

- Leland (1994): A workhorse model in modern structural corporate finance
  - If you want to combine model with data, this is the typical setting
- A dynamic version of traditional trade-off model, but capital structure decision is static
  - Trade-off model: a firm's leverage decision trades off the tax benefit with bankruptcy cost
- Relative to the previous literature (say Merton's 1974 model), Leland setting emphasizes equity holders can decide default timing ex post
  - So-called "endogenous default," an useful building block for more complicated models
  - Merton 1974 setting: given V_T distribution, default if $\widetilde{V}_T < F_T$. No default before T and the path of V_t does not matter

Firm and Its Cash Flows

- A firm's asset-in-place generates cash flows at a rate of δ_t
  - Over interval $[t, t+dt]$ cash flows is $\delta_t dt$
  - Leland '94, state variable unlevered asset value $V_t = \frac{\delta_t}{r-\mu}$ (just relabeling)
- Cash flow rate follows a Geometric Brownian Motion (with drift μ and volatility σ)

$$\frac{d\delta_t}{\delta_t} = \mu dt + \sigma dZ_t$$

  - $\{Z_t\}$ is a standard Brownian motion (Wiener process): $Z_t \sim \mathcal{N}(0, t)$, $Z_t - Z_s$ is independent of $\mathcal{F}(\{Z_{u<s}\})$
  - Given δ_0, $\delta_t = \delta_0 \exp\left(\left(\mu - 0.5\sigma^2\right) t + \sigma Z_t\right) > 0$
  - Arithmetic Brownian Motion: $d\delta_t = \mu dt + \sigma dZ_t$ so $\delta_t = \delta_0 + \mu t + \sigma Z_t$
- Persistent shocks, i.i.d. return. Today's shock dZ_t affects future level of δ_s for $s > t$
- One interpretation: firm produces one unit of good per unit of time, with market price fluctuating according to a GBM
- In this model, everything is observable, i.e. no private information

Debt as Perpetual Coupon

- Firm is servicing its debt holders by paying coupon at the rate of C
  - Debt holders are receiving cash flows Cdt over time interval $[t, t+dt]$
- Debt tax shield, with tax rate τ
- Debt is deducted before calculating taxable income implies that debt can create DTS
- The previous cash flows are after-tax cash flows, so before-tax cash flows are $\delta_t / (1-\tau)$
  - So-called Earnings Before Interest and Taxes (EBIT)
- By paying coupon C, taxable earning is $\delta_t / (1-\tau) - C$, so equity holders' cash flows are

$$\left(\frac{\delta_t}{1-\tau} - C\right)(1-\tau) = \delta_t - (1-\tau)\,C$$

- The firm investors in total get (Modigliani-Miller idea)

$$\underbrace{\delta_t - (1-\tau)\,C}_{\text{Equity}} + \underbrace{C}_{\text{Debt}} = \underbrace{\delta_t}_{\text{Firm's Asset}} + \underbrace{\tau C}_{\text{DTS}}$$

Endogenous Default Boundary

- Equity holders receiving δ_t which might become really low, but is paying constant $(1-\tau)\,C$
- When $\delta_t \to 0$, holding the firm almost has zero value—then why pay those debt holders?
- Equity holders default at $\delta_B > 0$ where equity value at δ_B has $E\left(\delta_B\right) = 0$ and $E'\left(\delta_B\right) = 0$
  - Value matching $E\left(\delta_B\right) = 0$, just says that at default equity holders recover nothing
  - Smooth pasting $E'\left(\delta_B\right) = 0$, optimality: equity can decide to wait and default at $\delta_B - \epsilon$, but no benefit of doing so
- At bankruptcy, some deadweight cost, debt holders recover a fraction $1-\alpha$ of first-best firm value $(1-\alpha)\,\delta_B/\left(r-\mu\right)$
  - First-best unlevered firm value $\delta_B/\left(r-\mu\right)$, Gordon growth formula
- Two steps:
  1. Derive debt $D\left(\delta\right)$ and equity $E\left(\delta\right)$, given default boundary δ_B
  2. Using smooth pasting condition to solve for δ_B

Valuation or Halmilton-Jacobi-Bellman (HJB) Equation (1)

- $V(y) = \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)} f(y_s)\, ds \,|y_t = y\right]$ s.t.
  $dy_t = \mu(y_t)\, dt + \sigma(y_t)\, dZ_t$
- Discrete-time Bellman equation

$$V(y) = \frac{1}{1+r}\left(f(y) + \mathbb{E}\left[V(y')\,|y\right]\right) \text{ s.t. } y' = y + \mu(y) + \sigma(y)\,\varepsilon$$

- Continuous-time, $V(y)$ can be written as

$$\begin{aligned}
V(y) &= \mathbb{E}_t\left[f(y_t)\, dt + \int_{t+dt}^{\infty} e^{-r(s-t)} f(y_s)\, ds \,|y_{t+dt} = y_t + \mu(y_t)\, dt + \sigma(y_t)\, dZ_t\right] \\
&= f(y)\, dt + e^{-r\cdot dt}\mathbb{E}_t\left[\int_{t+dt}^{\infty} e^{-r(s-t-dt)} f(y_s)\, ds \,|y_{t+dt} = y_t + \mu(y_t)\, dt + \sigma(y_t)\, dZ_t\right] \\
&= f(y)\, dt + e^{-r\cdot dt}\mathbb{E}_t\left[\mathbb{E}_{t+dt}\left(\int_{t+dt}^{\infty} e^{-r(s-t-dt)} f(y_s)\, ds \,|y_{t+dt} = y_t + \mu(y_t)\, dt + \sigma(y_t)\, dZ_t\right)\right] \\
&= f(y)\, dt + (1 - rdt)\,\mathbb{E}_t\left[V(y_t + \mu(y)\, dt + \sigma(y_t)\, dZ_t)\right] \\
&= f(y)\, dt + (1 - rdt)\,\mathbb{E}_t\left[V(y_t) + V'(y_t)\,\mu(y_t)\, dt + V'(y_t)\,\sigma(y_t)\, dZ_t + \frac{1}{2}V''(y_t)\,\sigma^2(y_t)\, dt\right] \\
&= f(y)\, dt + (1 - rdt)\left[V(y) + V'(y)\,\mu(y)\, dt + \frac{1}{2}V''(y)\,\sigma^2(y)\, dt\right]
\end{aligned}$$

Valuation or Halmilton-Jacobi-Bellman (HJB) Equation (2)

- Expansion of RHS:

$$\begin{aligned} V(y) &= f(y)\,dt + (1 - rdt)\left[V(y) + V'(y)\,\mu(y)\,dt + \frac{1}{2}V''(y)\,\sigma^2(y)\,dt\right] \\ &= f(y)\,dt + V(y) + V'(y)\,\mu(y)\,dt + \frac{1}{2}V''(y)\,\sigma^2(y)\,dt \\ &\quad -rV(y)\,dt - rV'(y)\,\mu(y)\,(dt)^2 - r\frac{1}{2}V''(y)\,\sigma^2(y)\,(dt)^2 \end{aligned}$$

- From higher to lower orders, until non-trivial identity
  - At order $O(1)$, $V(y) = V(y)$, trivial identity
  - At order $O(dt)$, non–trivial identity

$$0 = \left[f(y) + V'(y)\,\mu(y) + \frac{1}{2}V''(y)\,\sigma^2(y) - rV(y)\right]dt$$

- As a result, we have

$$\underbrace{rV(y)}_{\text{required return}} = \underbrace{f(y)}_{\text{flow (dividend) payoff}} + \underbrace{V'(y)\,\mu(y) + \frac{1}{2}\sigma^2(y)\,V''(y)}_{\text{local change of value function (capital gain, long-term payoffs)}}$$

- That is how I write down value functions for any process (later I will introduce jumps)

General Solution for GBM process with Linear Flow Payoffs

- In the Leland setting, the model is special because

$$f\left(y\right)=a+by,\mu\left(y\right)=\mu y,\text{ and }\sigma\left(y\right)=\sigma y$$

- It is well known that the general solution to $V\left(y\right)$ is

$$V\left(y\right)=\frac{a}{r}+\frac{b}{r-\mu}y+K_{\gamma}y^{-\gamma}+K_{\eta}y^{\eta}$$

where the "power" parameters are given by

$$\begin{aligned}
-\gamma &= -\frac{\mu-\frac{1}{2}\sigma^2+\sqrt{\left(\frac{1}{2}\sigma^2-\mu\right)^2+2\sigma^2 r}}{\sigma^2}<0,\\
\eta &= -\frac{\mu-\frac{1}{2}\sigma^2-\sqrt{\left(\frac{1}{2}\sigma^2-\mu\right)^2+2\sigma^2 r}}{\sigma^2}>1
\end{aligned}$$

- The constants K_{γ} and K_{η} are determined by boundary conditions

Side Note: How Do You Get Those Two Power Parameters

- Those two power parameters $-\gamma$ and η are roots to the fundamental quadratic equations
- Consider the homogenous ODE:

$$rV(y) = \mu y V'(y) + \frac{1}{2}\sigma^2 y^2 V''(y)$$

- Guess the $V(y) = y^x$, then $V'(y) = xy^{x-1}$ and $V''(y) = x(x-1)y^{x-2}$

$$\begin{aligned} ry^x &= \mu x y^x + \frac{1}{2}\sigma^2 x(x-1) y^x \\ r &= \mu x + \frac{1}{2}\sigma^2 x(x-1) \\ 0 &= \frac{1}{2}\sigma^2 x^2 + \left(\mu - \frac{1}{2}\sigma^2\right) x - r \end{aligned}$$

- $-\gamma$ and η are the two roots of this equation

Debt Valuation (1)

- For debt, flow payoff is C so

$$D(\delta) = \frac{C}{r} + K_\gamma \delta^{-\gamma} + K_\eta \delta^\eta$$

- Two boundary conditions
  - When $\delta = \infty$, default never occurs, so $D(\delta = \infty) = \frac{C}{r}$ perpetuity. Hence $K_\eta = 0$ (otherwise, D goes to infinity)
  - When $\delta = \delta_B$, debt value is $\frac{(1-\alpha)\delta_B}{r-\mu}$. $D(\delta_B) = \frac{(1-\alpha)\delta_B}{r-\mu}$ implies that

$$\frac{C}{r} + K_\gamma \delta_B^{-\gamma} = \frac{(1-\alpha)\delta_B}{r-\mu} \Rightarrow K_\gamma = \frac{\frac{(1-\alpha)\delta_B}{r-\mu} - \frac{C}{r}}{\delta_B^{-\gamma}}$$

Debt Valuation (2)

- We obtain the closed-form solution for debt value

$$
\begin{aligned}
D(\delta) &= \frac{C}{r} + \left(\frac{\delta}{\delta_B}\right)^{-\gamma}\left(\frac{(1-\alpha)\,\delta_B}{r-\mu} - \frac{C}{r}\right) \\
&= \left(\frac{\delta}{\delta_B}\right)^{-\gamma}\frac{(1-\alpha)\,\delta_B}{r-\mu} + \left(1 - \left(\frac{\delta}{\delta_B}\right)^{-\gamma}\right)\frac{C}{r}
\end{aligned}
$$

- Present value of 1 dollar contingent on default:

$$
\mathbb{E}\left[e^{-r\tau_B}\right] = \left(\frac{\delta}{\delta_B}\right)^{-\gamma} \text{ where } \tau_B = \inf\left\{t : \delta_t < \delta_B\right\}
$$

- The debt value can also be written in the following intuitive form

$$
\begin{aligned}
D(\delta) &= \mathbb{E}\left[\int_0^{\tau_B} e^{-rs}\,C ds + e^{-r\tau_B}\frac{(1-\alpha)\,\delta_B}{r-\mu}\right] \\
&= \mathbb{E}\left[\frac{C}{r}\left(-\int_0^{\tau_B} de^{-rs}\right) + e^{-r\tau_B}\frac{(1-\alpha)\,\delta_B}{r-\mu}\right] \\
&= \mathbb{E}\left[\frac{C}{r}\left(1 - e^{-r\tau_B}\right) + e^{-r\tau_B}\frac{(1-\alpha)\,\delta_B}{r-\mu}\right]
\end{aligned}
$$

Equity Valuation (1)

- For equity, flow payoff is $\delta_t - (1-\tau)\, C$, so

$$E(\delta) = \frac{\delta}{r-\mu} - \frac{(1-\tau)\, C}{r} + K_\gamma \delta^{-\gamma} + K_\eta \delta^{\eta}$$

- When $\delta = \infty$, equity value cannot grow faster than first-best firm value which is linear in δ. So $K_\eta = 0$
- When $\delta = \delta_B$, we have

$$E(\delta_B) = \frac{\delta_B}{r-\mu} - \frac{(1-\tau)\, C}{r} + K_\gamma \delta_B^{-\gamma} = 0 \Rightarrow K_\gamma = \frac{\frac{(1-\tau)C}{r} - \frac{\delta_B}{r-\mu}}{\delta_B^{-\gamma}}$$

Thus

$$E(\delta) = \underbrace{\frac{\delta}{r-\mu} - \frac{(1-\tau)\, C}{r}}_{\text{Equity value if never defaults (pay } (1-\tau)C \text{ forever)}} + \underbrace{\left(\frac{(1-\tau)\, C}{r} - \frac{\delta_B}{r-\mu}\right)\left(\frac{\delta}{\delta_B}\right)^{-\gamma}}_{\text{Option value of default}}$$

Equity Valuation (2)

- Finally, smooth pasting condition

$$
\begin{aligned}
0 &= E'(\delta)\Big|_{\delta=\delta_B} \\
&= \frac{1}{r-\mu} + \left(\frac{(1-\tau)C}{r} - \frac{\delta_B}{r-\mu}\right)(-\gamma)\left(\frac{\delta}{\delta_B}\right)^{-\gamma-1}\frac{1}{\delta_B}\Bigg|_{\delta=\delta_B} \\
&= \frac{1}{r-\mu} + (-\gamma)\left(\frac{(1-\tau)C}{r\delta_B} - \frac{1}{r-\mu}\right)
\end{aligned}
$$

- Thus

$$\delta_B = (1-\tau)C\frac{r-\mu}{r}\frac{\gamma}{1+\gamma}$$

What if the firm can decide optimal coupon

- At $t = 0$, what is the optimal capital structure (leverage)?
- Given δ_0 and C, the total levered firm value $v(\delta_0) = E(\delta_0) + D(\delta_0)$ is

$$\underbrace{\frac{\delta_0}{r-\mu}}_{\text{Unlevered value}} + \underbrace{\frac{\tau C}{r}\left(1 - \left(\frac{\delta}{\delta_B}\right)^{-\gamma}\right)}_{\text{Tax shield}} - \underbrace{\frac{\alpha \delta_B}{r-\mu}\left(\frac{\delta}{\delta_B}\right)^{-\gamma}}_{\text{Bankruptcy cost}}$$

- Realizing that δ_B is linear in C, we can find the optimal C^* that maximizing the levered firm value to be

$$C^* = \frac{\delta_0}{r-\mu}\frac{r(1+\gamma)}{(1-\tau)\gamma}\left(1 + \gamma + \frac{\alpha\gamma(1-\tau)}{\tau}\right)^{-1/\gamma}$$

- Important observation: optimal C^* is linear in δ_0! So called scale-invariance
  - It implies that if the firm is reoptimizing, its decision is just some constant scaled by the firm size

Trade-off Theory: Economics behind Leland (1994)

- Benefit: borrowing gives debt tax shield (DTS)
- Equity holders makes default decision ex post
- The firm fundamental follows GBM, **persistent** income shocks
- After enough negative shocks, equity holders' value of keeping the firm alive can be really low
- Debt obligation is fixed, so when δ_t is sufficiently low, it is optimal to default
  - Debt-overhang—Equity holders do not care if default impose losses on debt holders
- But, at time zero when equity holders issue debt, debt holders price default in $D(\delta_0)$
  - And equity holders will receive $D(\delta_0)$!
- Hence equity holders optimize $E(\delta_0) + D(\delta_0)$, realizing that coupon C will affect DTS (positively) and bankruptcy cost (negatively)
- If equity holders can commit ex ante about ex post default behavior, what do they want to do?

Leland, Goldstein and Ju (2000, Journal of Business)

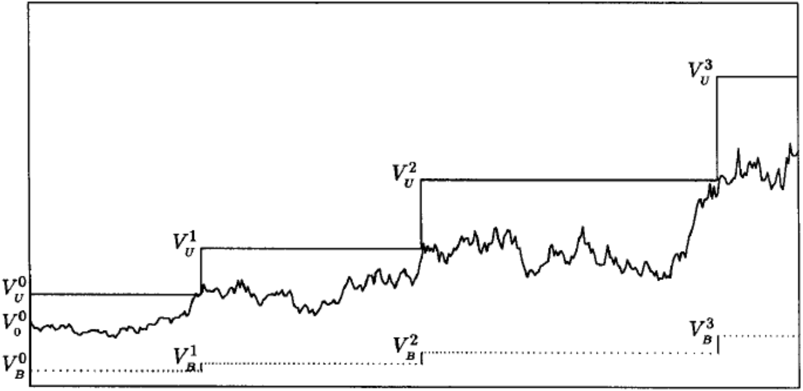
- There are two modifications relative to Leland (1994):
- First, directly modelling pre-tax cashflows – so-called EBIT, rather than after-tax cashflows
- It makes clear that there are three parties to share the cashflows: equity, debt, and government
- When we take comparative statics w.r.t. tax rate τ, in Leland (1994) you will ironically get that levered firm value $\uparrow$ when τ $\uparrow$
  - In Leland, raising τ does not change δ_t (which is after-tax cashflows)
- In LGJ, after-tax cashflows are $(1-\tau)\,\delta_t$, so raising τ lowers firm value

Leland, Goldstein and Ju (2000, Journal of Business)

- Second, more importantly, allowing for firms to upward adjust their leverage if it is optimal to do so in the future
  - When future fundamental goes up, leverage goes down, optimal to raise more debt
  - Need fix cost to do so—otherwise tend to do it too often
- Key assumption for tractability: when adjusting leverage, the firm has to buy back all existing debt
  - Say that this rule is written in debt covenants
  - As a result, there is always one kind of debt at any point of time
- After buying back, when equity holders decide how much debt to issue, they are solving the same problem again with new firm size
  - But the model is scale invariant, so the solution is the same (except a larger scale)
- F face value. A firm with (δ, F) faces the same problem as $(k\delta, kF)$

Optimal Policies in LGJ

- $\frac{\delta_B}{\delta_0} = \psi$: default factor, $\frac{\delta_U}{\delta_0} = \gamma$: leverage adjustment factor



- LGJ: can precommit to γ. No precommitment in Fischer-Heinkel-Zechner (1989)

How Do We Model Finite Maturity

- Perpetual debt in Leland (1994). In practice debt has finite maturity
- Debt maturity is very hard to model in a dynamic model
- You can do exponentially decaying debt (Leland, 1994b, 1998)
- Rough idea: what if your debt randomly matures in a Poisson fashion with intensity $1/m$?
- Exponential distribution, the expected maturity is $\int_0^\infty x \frac{1}{m} e^{-x/m} dx = m$
- It is memoriless—if the debt has not expired, looking forward the debt price is always the same!
- Actually, you do not need random maturing. Exponential decaying coupon payment also works!
- So, debt value is $D(\delta)$, not $D(\delta, t)$ where t is remaining maturity
- If all debt maturity is i.i.d, large law of numbers say that at $[t, t+dt]$, $\frac{1}{m}dt$ fraction of debt mature

Leland (1998)

- Using exponentially decaying finite maturity debt
- Equity holders can ex post choose risk

$$\sigma \in \{\sigma_H, \sigma_L\}$$

- Research question: how does **asset substitution** work in this dynamic framework? How does it depend on debt leverage and debt maturity?
- Typically with default option, asset substitution occurs optimally (default option gets more value if volatility is higher)
- With asset substitution, the optimal maturity is shorter, consistent with the idea that short-term debt helps curb agency problems (numerical result, not sure robust)
- Quantitatively, agency cost due to asset substitution is small

Leland (1998) (2)

- Assume threshold strategy that there exists δ_S s.t.

$$\sigma = \sigma_H \text{ for } \delta < \delta_S \text{ and } \sigma = \sigma_L \text{ for } \delta \geq \delta_S$$

- Solve for equity, debt, DTS, BC the same way as before, with one important change
- Need to piece solutions on $[\delta_B, \delta_S)$ and $[\delta_S, \infty)$ together
- $-\gamma_H, \eta_H, -\gamma_L, \eta_L$: solutions to fundamental quadratic equations

$$\begin{aligned} D^H(\delta) &= \frac{C}{r} + K_\gamma^H \delta^{-\gamma_H} + K_\eta^H \delta^{\eta_H} \text{ for } [\delta_B, \delta_S) \\ D^L(\delta) &= \frac{C}{r} + K_\gamma^L \delta^{-\gamma_L} + K_\eta^L \delta^{\eta_L} \text{ for } [\delta_S, \infty) \end{aligned}$$

- Four boundary conditions to get $K_\gamma^H, K_\eta^H, K_\gamma^L, K_\eta^L$
- $K_\eta^L = 0$ because $D(\delta = \infty) < \frac{C}{r}$. The other three: $D^H(\delta_S) = D^L(\delta_S)$ (value matching), $D^{H\prime}(\delta_S) = D^{L\prime}(\delta_S)$ (smooth pasting), $D^H(\delta_B) = \frac{(1-\alpha)\delta_B}{r-\mu}$ (value matching)
    - Here, smooth pasting at δ_S always holds, because Brownian crosses δ_S "super" fast. The process does not stop there (like at δ_B)

Leland and Toft (1996)

- Deterministic maturity, but keep uniform distribution of debt maturity structure
- Say we have debts with a total measure of 1, maturity is uniformly distributed $U[0, T]$, same principal P, same coupon C
- Tough: now debt price is $D(\delta, t)$, need to solve a PDE
- Equity promises to keep the same maturity structure in the future
- Equity holders' cashflows are

$$\delta_t dt - (1-\tau) Cdt - \frac{1}{T} dt (P - D(\delta, T))$$

  - Cashflows $\delta_t dt$; Coupon Cdt; and Rollover losses/gains
- Over $[t, t+dt]$, there is $\frac{1}{T} dt$ measure of debt matures, equity holders need to pay

$$\frac{1}{T} dt (P - D(\delta, T))$$

as equity holders get $D(\delta, T) \frac{1}{T} dt$ by issuing new debt

Leland and Toft (1996)

- First step: solve the PDE

$$rD(\delta, t) = C + D_t(\delta, t) + \mu\delta D_\delta(\delta, t) + \frac{1}{2}\sigma^2\delta^2 D_{\delta\delta}(\delta, t)$$

  Boundary conditions

$$\begin{aligned}
D(\delta = \infty, t) &= \frac{C}{r}\left(1 - e^{-rt}\right) + Pe^{-rt}: \text{ defaultless bond} \\
D(\delta = \delta_B, t) &= (1 - \alpha)\frac{\delta_B}{r - \mu}: \text{ defaulted bond} \\
D(\delta, 0) &= P \text{ for } \delta \geq \delta_B: \text{ paid back in full when it matures}
\end{aligned}$$

- Leland-Toft (1996) get closed-form solutions for debt values; have a look
  - Better know the counterpart of Feyman-Kac formula. The point is to know it admits closed-form solution

Leland and Toft (1996)

- Equity value satisfies the ODE

$$rE(\delta) = \delta - (1-\tau)C + \frac{1}{T}\left[D(\delta, T) - P\right] + \mu\delta E_\delta(\delta) + \frac{1}{2}\sigma^2\delta^2 E_{\delta\delta}(\delta)$$

- This is also very tough, given the complicated form of $D(\delta, T)$!
- Leland and Toft have a trick (Modigliani-Miller idea): $E(\delta) =$

$$v(\delta) - \frac{1}{T}\int_0^T D(\delta, t)\,dt = \frac{\delta}{r-\mu} + DTS(\delta) - BC(\delta) - \frac{1}{T}\int_0^T D(\delta, t)\,dt$$

- $DTS(\delta)$ and $BC(\delta)$ are much easier to price
  - $DTS(\delta)$ is the value for constant flow payoff τC till default occurs
  - $BC(\delta)$ is the value of bankruptcy cost incurred on default
  - We have derived them given δ_B
- After getting $E(\delta; \delta_B)$, δ_B is determined by smooth pasting $E'(\delta_B; \delta_B) = 0$
- In He-Xiong (2012), we introduce market trading frictions for corporate bonds
  - Some deadweight loss during trading, the above trick does not work

Calculation of Debt Tax Shield

- Let us price $DTS(\delta)$ which is the value for constant flow payoff τC till default occurs
- We can have

$$\begin{aligned} DTS(\delta) &= \mathbb{E}\left[\int_0^{\tau_B} e^{-rs}\tau C ds\right] \\ &= \mathbb{E}\left[\frac{\tau C}{r}\left(1-e^{-r\tau_B}\right)\right] = \frac{\tau C}{r}\left(1-\left(\frac{\delta}{\delta_B}\right)^{-\gamma}\right) \end{aligned}$$

- Or, $F(\delta) = DTS(\delta)$

$$\begin{aligned} rF(\delta) &= \tau C + \mu\delta F_\delta(\delta) + \frac{1}{2}\sigma^2\delta^2 F_\delta(\delta) \\ F(\delta) &= \frac{\tau C}{r} + K_\gamma \delta^{-\gamma} + K_\eta \delta^\eta \end{aligned}$$

plugging $F(\delta_B) = 0$ and $F(\infty) = \frac{\tau C}{r}$ (so $K_\eta = 0$) we have

$$F(\delta) = \frac{\tau C}{r}\left(1-\left(\frac{\delta}{\delta_B}\right)^{-\gamma}\right)$$

MELLA-BARRAL and PERRAUDIN (1997) (1)

- How to model negotiation and strategic debt service?
- Consider a firm producing one widget per unit of time, random widget price
$$dp_t / p_t = \mu dt + \sigma dZ_t$$
- Constant production cost $w > 0$ so cash flows are $p_t - w$
- If debt holders come in to manage the firm, cash flows are $\xi_1 p_t - \xi_0 w$ with $\xi_1 < 1$ and $\xi_0 > 1$
- Even without debt, p_t can be so low that shutting down the firm is optimal
- This is so called "**operating leverage**"
  - One explanation for why Leland models predict too high leverage relative to data: Leland model includes operating leverage
- For debt holders, if they take over, value is $X(p)$ (need to figure out their hypothetical optimal stopping time by using smooth-pasting condition)

MELLA-BARRAL and PERRAUDIN (1997) (2)

- Now imagine the original coupon is $b > 0$
- When p_t goes down, what if equity holders can make a take-it-or-leave-it offer to debt holders?
- Denote the equilibrium coupon service $s(p)$, and resulting debt value $L(p)$
- In equilibrium there exist two thresholds $p_c < p_s$
  - When $p_t \geq p_s$, $s(p) = b$, nothing happens
  - When $p_t \in (p_c, p_s)$, we have $s(p) < b$ and $L(p) = X(p)$. As long as debt service is less than the contracted coupon, the value of debt equals that of debtholders' outside option $X(p)$
  - When p_t hits p_c, liquidating the firm
- When $s(p) < b$ we have $s(p) = \xi_1 p_t - \xi_0 w$ which is as if debt holders take the firm.
  - In the paper, there is some complication of $\gamma > 0$ which is the firm's scrap value

Miao, Hackbarth, Morellec (2006)

- Firm EBIT is $y_t \delta_t$, y_t aggregate business cycle condition

$$\begin{aligned} d\delta_t/\delta_t &= \mu dt + \sigma dZ_t \\ y_t &\in \{y_G, y_B\}: \text{ Markov Chain} \end{aligned}$$

- Exponentially decaying debt, etc, same as Leland (1998)
- Default boundary depends on the current macro state: δ_B^G and δ_B^B. Same smooth-pasting condition
- $\delta_B^G < \delta_B^B$, default more in B. Help explain credit spread puzzle
  - Bond seems too cheap in the data. If bond payoff is lower in recession, then it requires a higher return
- Lots of papers about credit spread puzzle use this framework

$$d\delta_t/\delta_t = \mu_s dt + \sigma_s dZ_t$$

where $s \in \{G, B\}$ or more

- ODE in vector: $x = \ln(\delta)$, $\mathbf{D}(x) = \left[D^G(x), D^B(x)\right]'$

$$r\mathbf{D}(x) = c\mathbf{1}_{2\times 1} + \boldsymbol{\mu}_{2\times 2}\mathbf{D}'(x) + \frac{1}{2}\Sigma_{2\times 2}\mathbf{D}''(x)$$

see my recent Chen, Cui, He, Milbradt (2014) if you are interested

Debt Overhang Framework

- Investment decisions are made by shareholders to maximize the value of equity
- No renegotiation of debt contracts
- Debt holders cannot do real investment themselves (investments lost if not done by owners). No other distress costs.
- Question: Does the firm want to invest?
- Answer: The firm will forgo investment projects with NPV below the wealth transfer to debt holders plus any loss from inefficient decisions implied by the debt structure

Diamond-He (2014): Will Short-term Debt Impose Stronger Overhang?

- What is the maturity effect on debt overhang?
- Consider two otherwise identical firms, one with long-term 10 year debt and the other with short-term 5 year debt. They have the same initial leverage
- Note, short-term debt is very different from debt that has matured
  - Empirically, short-term debt means 3- or 5- year cutoff
- Say equity holders are investing right now
  - Which firm suffers more debt overhang?
- Say equity holders are facing dynamic investment opportunities
  - Which firm suffers more debt overhang?

Immediate Investment, Black-Scholes-Merton Setting (1)

- Say a firm with asset value

$$\frac{dV_t}{V_t} = rdt + \sigma dZ_t$$

- The firm has a debt outstanding, with face value F_m and maturity m. At time m, the debt payoff is $\min(F_m, V_m)$ and equity payoff is $\max(V_m - F_m, 0)$
- The equity value is $E(V_0, m)$ and debt value $D(V_0, m) = V_0 - E(V_0, m)$
  - Remind you of European call option? That is how Black-Scholes paper gets published (they apply their stuff to corporate debt)
- Suppose that investment raises V_0 by ε. How much equity/debt gain?
- It is $Delta = E_V(V_0, m)$! Debt delta is $D_V(V_0, m) = 1 - E_V(V_0, m)$
- The higher the $D_V(V_0, m)$ the greater the debt overhang

Immediate Investment, Black-Scholes-Merton Setting (2)

- Benchmark result. Yes, short-term debt always has lower overhang!
- Proposition: Suppose $m_1 < m_2$. If we choose F_m so that $D(V_0, m_1) = D(V_0, m_2)$, then

$$D_V(V_0, m_1) < D_V(V_0, m_2)$$

- This result depends on constant volatility assumption
- Two period model, and suppose period-2 volatility depends on period-1 shock

$$\sigma = \sigma_L \text{ if } Z_1 > Q \text{ and } \sigma = \sigma_H \text{ otherwise}$$

- Keep debt value constant. If $\sigma_L = \sigma_H = \varepsilon$, stronger long-term overhang; If $\sigma_L = 0$ and $\sigma_H = \varepsilon$, stronger short-term overhang
  - Use the fact that when $\varepsilon \to 0$, long-term and short-term are the same
  - Often theorists can only rigorously show limit results, but they are important (qualitatively)!
- Intuition: if volatility is higher after interim bad news, short-term debt kills the firm but long-term debt allows equity to recover a lot

Future Investment, Leland Setting

- Given investment $\widetilde{i}_t$, firm's cash flows are

$$d\delta_t / \delta_t = \widetilde{i}_t dt + \sigma dZ_t$$

- Binary investment choice, cost $\lambda\delta\widetilde{i}_t$, optimal threshold strategy (verified later)

$$i(\delta) = i \text{ if } \delta > \delta_i \text{ and } i(\delta) = 0 \text{ otherwise}$$

- Zero-coupon debt with principal P. Equity holders refinance $1/m$ fraction, so net cashflow $(D(\delta) - P)/m$ every period.
- Equity's cash flow:

$$\delta_t dt - \lambda\delta_t \widetilde{i}_t dt + (D(\delta_t) - P)/m \cdot dt$$

- Equity defaults when δ_t hits δ_B

Debt and Equity Valuations

- For debt

$$rD(\delta) = i(\delta) D'(\delta) + \frac{\delta^2\sigma^2}{2} D''(\delta) + \frac{1}{m}(P - D(\delta))$$

with solution $\left(p = \frac{P}{1+mr}\right)$

$$D(\delta) = \begin{cases} p + A_1\delta^{-\gamma_1} & \text{if } \delta > \delta_i \\ p + A_2\delta^{-\gamma_2} + A_3\delta^{-\gamma_3} & \text{if } \delta \in [\delta_B, \delta_i] \end{cases}$$

- $A_1 < 0, A_2, A_3$ determined by value-matching at δ_i and δ_B and smooth-pasting at δ_i
  - Why smooth-pasting at δ_i?
- Equity:

$$rE(\delta) = \max_i \delta(1 - \lambda i(\delta)) + i(\delta)\,\delta E'(\delta) + \frac{\delta^2\sigma^2}{2} E''(\delta) - \frac{1}{m}(P - D(\delta))$$

- Optimal thresholds $E'(\delta_i) = \lambda$ and $E'(\delta_B) = 0$
  - It is easier to solve for levered firm value $V(\delta)$ first and then $E(\delta) = V(\delta) - D(\delta)$

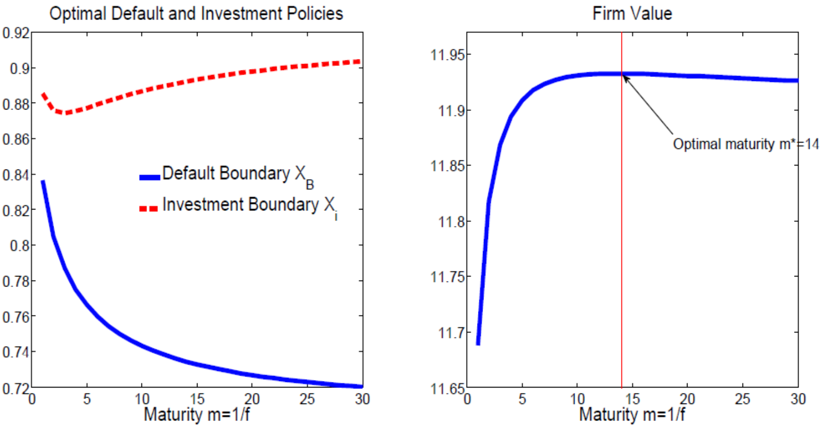
Proof of Unique Investment Threshold

- Useful technique in other situations. This also proves optimality of threshold strategy for investment
- $E'(\delta_B) = 0$, and $E'(\delta = \infty) = \frac{1-\lambda i}{r-i} > \lambda$
  - $E(\delta = \infty) = \frac{1-\lambda i}{r-i}\delta > \frac{1}{r}\delta$, i.e., $\lambda r < 1$ for investment being efficient
- Say there are potentially multiple points that $E'(\delta_i) = \lambda$. Take the smallest and construct equity valuation
- Say $\delta_2 > \delta_1 > \delta_i$, $E'(\delta_1) = E'(\delta_2) = \lambda$ but $E''(\delta_1) < 0$ and $E''(\delta_2) > 0$
- Find some middle point δ_3 with $E'(\delta_3) < \lambda$, $E''(\delta_3) = 0$ and $E'''(\delta_3) > 0$
- Taking derivative of equity equation again and evaluate at $\delta_3 > \delta_i$

$$(r-i)E'(\delta_3) - 1 + \lambda i = \underbrace{\left(i + \sigma^2\right)\delta_3 E''(\delta_3)}_{0} + \underbrace{\frac{\sigma^2\delta_3^2}{2}E'''(\delta_3)}_{>0} + \underbrace{\frac{1}{m}D'(\delta_3)}_{>0} > 0$$

- But $(r-i)E'(\delta_3) - 1 + \lambda i < (r-i)\lambda - 1 + \lambda i = \lambda r - 1 < 0$, contradiction!

Optimal debt maturity



- Without investment, long-term debt $m = 0$ is optimal (Leland-Toft)
- Two ways to make long-term debt inferior: 1) investment, so debt overhang 2) investor liquidity shocks with early consumption needs