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Value-at-Risk Based Risk Management: Optimal Policies and Asset Management

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Abstract

This paper analyzes optimal, dynamic portfolio and wealth/consumption policies of utility maximizing investors who must also manage market-risk exposure using a given risk-management model. We focus on the industry standard, the Value-at-Risk (VaR) based risk management, and find that VaR risk managers often optimally choose a larger exposure to risky assets than non risk managers, and consequently incur larger losses, when losses occur. We suggest an alternative risk management model, based on the expectation of a loss, to remedy the shortcomings of VaR. A general-equilibrium analysis reveals that the presence of VaR risk managers in a pure-exchange economy amplifies the stock-market volatility at times of down markets (and low output) and attenuates the volatility at times of up markets.

JEL Classifications: G11, G12, C61, D51

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1. Introduction

In recent years, we have witnessed an unprecedented surge in the usage of risk management practices, with the Value-at-Risk (VaR)-based risk management emerging as the industry standard by choice or by regulation (Dowd (1998), Jorion (1997), Saunders (1998)). VaR describes the loss that can occur over a given period, at a given confidence level, due to exposure to market risk. The wide usage of the VaR-based risk management (VaR-RM) by financial, as well as non-financial firms (Bodnar, Hayt, and Marston (1998)), stems from the fact that VaR is an easily interpretable summary measure of risk,[1] which has also an appealing rationale, as it allows its users to focus attention on "normal market conditions" in their routine operations. However, evidence abounds that, in practice, VaR estimates serve not as mere summary statistics for decision makers, but are also used as a tool to manage and control risk – where economic agents struggle to maintain the VaR of their market exposure at a prespecified level.[2] Surprisingly, the academic literature has largely overlooked this fact; at the present, we lack rigorous understanding of its economic implications, and, in particular, little is known about optimal behavior consistent with the VaR-RM.

Our objective is to undertake a comprehensive analysis of the VaR-RM while retaining the standard financial-economics paradigms of rational expectations, optimization, and market clearing. In particular, we study the implications of the VaR-RM for optimal portfolio policies, (horizon) wealth choice, and equilibrium prices. To the best of our knowledge, ours is the first attempt to embed risk management objectives into an optimizing framework. Recognizing that risk management is typically not an economic agent's primary objective, we focus on portfolio choice within the familiar (continuous time) expected utility maximizing framework, where the novel feature of our analysis is the assumption that agents may limit their risks while maximizing utility. In particular, we assume that a risk-managing agent is constrained to maintain the VaR of his horizon wealth at a prespecified level; in other words, he is constrained to maintain, below some prespecified level α, the probability of his wealth falling below some "floor." Our setting has the convenient property that it nests ($\alpha = 1$) the benchmark agent (who does not limit losses; Cox and Huang (1989), Karatzas, Lehoczky, and Shreve (1987)) and ($\alpha = 0$) the portfolio insurer (who maintains his horizon wealth above the floor in all states; Basak (1995), Grossman and Vila (1989), Grossman and Zhou (1996)).

[1]Regulators also view VaR as a useful summary measure; since 1997, the SEC has required banks and other large-capitalization registrants to quantify and report their market-risk exposure (Regulation S-K, Item 305), with VaR disclosure being one way to comply.

[2]See, for example, the lead article of *the Economist* (October 17, 1998), and Smith, Smithson, and Wilford (1995). The risk-monitoring facet of VaR is encouraged by regulators, and to that end, the Basle Committee on Banking Supervision (and the Federal Reserve, in particular) decided, effective January 1998, to allow large banks the option to use a VaR measure to set the capital reserves necessary to cover their market-risk exposure.

Our main results are as follows: First, under general security-price uncertainty and general state-independent preferences, we show that an agent, with his VaR capped, optimally chooses to insure against intermediate-loss states, while to incur losses in the worst states of the world. The somewhat surprising feature of the solution is that the uninsured states are chosen independently of preferences and endowments; they are simply the worst states up to a probability of exactly α. The intuition is that the VaR risk manager is willing to incur losses, in compliance with the VaR constraint, and it is optimal for him to incur losses in those states against which it is most expensive to insure. We exhibit a problematic feature of the derived optimal behavior, in that although the probability of a loss is fixed, when a large loss occurs, it is *larger* than when not engaging in the VaR-RM.

Second, under Constant Relative Risk Aversion (CRRA) preferences and lognormal state prices, we show the VaR risk manager's dynamic portfolio choice to deviate considerably from that of a portfolio insurer and a benchmark agent. The deviation is most pronounced in "transitional" states, where there is the highest uncertainty regarding whether losses will occur. Then, the risk manager takes on large equity positions to finance a high wealth level should economic conditions turn favorable at the horizon, while allowing for large losses in unfavorable conditions.

Third, recognizing the shortcomings of the VaR-RM to stem from its focus on the probability of a loss, regardless of the magnitude, we propose and evaluate an alternative form of risk management, which maintains limited expected losses (LEL). In contrast to the VaR-RM, under the LEL-RM, when losses occur, they are *lower* than those when not engaging in the LEL-RM. Our analysis predicts that if regulators, and hence risk managers, would be concerned with disclosing and monitoring *expected* losses (instead of VaR), then agents' optimal behavior should be consistent with reducing losses in *any* of the most adverse states of the world.

Finally, to investigate the impact of extensive usage of the VaR-RM, we move from the partial equilibrium analysis to a general equilibrium setting. We allow agents to consume continuously, while keeping the VaR of their horizon wealth at a prespecified level. For tractability and realism, we do not require the VaR horizon to coincide with the investment horizon. We work in a familiar Lucas (1978)-type pure-exchange economy and focus on the implications for stock-market volatility. We find that when the economy contains VaR risk managers, the stock-market volatility (and risk premium) increases relative to the benchmark case in down markets and decreases in up markets. The highest departure from the benchmark occurs as a response to VaR risk managers' aggressive bidding for stocks in the "transitional" states.

Our results may shed some light on the controversy surrounding the large losses incurred by some banks and hedge funds during the recent (August 1998) stock-market downturn. If indeed, as it appears, the use of VaR-based models of risk management was prevalent by these institutions (*the Economist*, October 17, 1998), then our model does, assuming deteriorating fundamentals, offer a rational explanation for their large losses. It is also interesting to note that the recent downturn was associated with high stock-market volatility, consistent with our general-equilibrium results. According to our model, when the fundamentals are deteriorating, it is then, in the transition from the good-states of the world to the bad states, that the presence of VaR risk managers in the economy should cause the stock volatility to increase relative to the benchmark.

The extant academic literature has been mostly concerned with measuring VaR (see, e.g., Duffie and Pan (1997), Linsmeier and Pearson (1996)) or with theoretically evaluating properties of VaR and other risk measures (Artzner, Delbaen, Eber, and Heath (1998), Cvitanić and Karatzas (1998), Wang (1998)). Ahn, Boudoukh, Richardson, and Whitelaw (1998) and Luciano (1998) appear to be the only related academic work that explicitly acknowledges economic agents' wish to limit the VaR of their market exposure. Ahn et al. address the one specific question of how to design a put option to minimize the VaR of a position in a stock and options, given a cost constraint on hedging. Luciano (1998), as we do, focuses on optimal portfolio policies, and also maps the VaR regulatory requirements into a constraint similar to ours. However, she does not explicitly apply the constraint to the agent's optimization problem, but only analyzes deviations from the constraint having solved the unconstrained optimization problem. Such an analysis allows one to examine whether an optimizing agent would automatically comply with the VaR regulation or not. In contrast, we apply the VaR constraint *directly* to the optimization problem, which allows us to analyze the impact of the VaR-RM on endogenously-determined economic quantities.

The paper is organized as follows. Section 2 describes the economy. Section 3 solves the individual's optimization problem under the VaR-RM, and Section 4 analyzes the optimization under the LEL-RM. Section 5 provides the equilibrium analysis. Section 6 concludes the paper. The appendix contains the proofs.

2. The Economic Setting

2.1. The Economy

We consider a finite-horizon, $[0, T]$, economy with a single consumption good (the numeraire). Uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, on which is defined an N-dimensional Brownian motion $w(t) = (w_1(t), \ldots, w_N(t))^\top$, $t \in [0, T]$. All stochastic processes are assumed adapted to $\{\mathcal{F}_t; t \in [0,T]\}$, the augmented filtration generated by w. All stated (in)equalities involving random variables hold P-almost surely. In what follows, given our focus is on characterization, we assume all stated processes to be well-defined, without explicitly stating the regularity conditions ensuring this.[3]

Investment opportunities are represented by $N+1$ securities; an instantaneously riskless bond in zero net supply, and N risky stocks, each in constant net supply of 1 and paying dividends at rate δ_j, $j = 1, \ldots, N$. The bond-price, B, and stock prices, S_j, are assumed to follow

$$
\begin{aligned}
dB(t) &= B(t)r(t)dt \ , && (1)\\
dS_j(t) + \delta_j(t)dt &= S_j(t)[\mu_j(t)dt + \sigma_j(t)dw(t)], \quad j = 1, \ldots, N \ , && (2)
\end{aligned}
$$

where the interest rate r, the drift coefficients $\mu \equiv (\mu_1, \ldots, \mu_N)^\top$, and the volatility matrix $\sigma \equiv \{\sigma_{jk},\ j = 1, \ldots, N;\ k = 1, \ldots, N\}$ are possibly path-dependent.

Dynamic market completeness (under no-arbitrage) implies the existence of a unique state price density process, ξ, given by

$$
d\xi(t) = -\xi(t)[r(t)dt + \kappa(t)^\top dw(t)] \ , \tag{3}
$$

where $\kappa(t) \equiv \sigma(t)^{-1}(\mu(t)) - r(t)\bar{1})$ is the market price of risk (or the Sharpe ratio) process, and $\bar{1} \equiv (1, \ldots, 1)^\top$. The quantity $\xi(T,\omega)$ is interpreted as the Arrow-Debreu price per unit probability P of one unit of consumption good in state $\omega \in \Omega$ at time T.

The economy is populated by 2 types of agents. Agent i is endowed at time 0 with e_{ij} shares of the risky security j, providing him with an initial wealth of $W_i(0) = e_i^\top S(0)$. (Since our focus until Section 5 is on the optimal behavior of a single risk-managing agent, we drop, for now, the subscript i.) Each agent chooses a nonnegative, terminal-horizon wealth $W(T)$ and a portfolio process θ, where $\theta(t) \equiv (\theta_1(t), \ldots, \theta_N(t))^\top$ denotes the vector of fractions of wealth invested in each stock. The agent's pre-horizon wealth process W then follows

$$
dW(t) = W(t)\left[r(t) + \theta(t)^\top(\mu(t) - r(t)\bar{1})\right] dt + W(t)\theta(t)^\top\sigma(t)dw(t) \ . \tag{4}
$$

Each agent is assumed to derive state-independent utility $u(W(T))$ over terminal wealth. The function $u(\cdot)$ is assumed twice continuously differentiable, strictly increasing, strictly concave, and to satisfy $\lim_{x\to 0} u'(x) = \infty$ and $\lim_{x\to\infty} u'(x) = 0$.

[3] Anticipating the quantities to be introduced in this section and in Section 5, see, for example, Karatzas and Shreve (1998) for the required integrability conditions on consumption policies, prices, and portfolio holdings, as well as the associated Novikov's condition. In the equilibrium constructed in Section 5, these conditions (which, in particular, guarantee nonsingularity of σ in (2)) can be shown to be satisfied.

2.2. Modeling the VaR-RM

The financial industry has standardized on the following definition of $VaR(\alpha)$ (see, e.g., Duffie and Pan (1997), Jorion (1997)): *It is the loss, which is exceeded with some given probability, α, over a given horizon.* Assuming the VaR horizon to coincide with the investment horizon, this definition translates into our setting as

$$P(W(0) - W(T) \leq VaR(\alpha)) \equiv 1 - \alpha \; , \qquad \alpha \in [0,1] \; . \tag{5}$$

Note that VaR can be interpreted as the worst loss over a given time interval, under "normal market conditions."

Our objective is to embed the VaR-RM strategy into a utility maximizing framework. This could be interpreted either as an agent himself managing risk, or as an intermediary managing risk, using the VaR approach, on an agent's behalf. The most convenient and natural way to embed the VaR-RM is to assume that an additional constraint is imposed on the agent's optimization problem, requiring the $VaR(\alpha)$ to be maintained below some prespecified level, i.e.,

$$VaR(\alpha) \leq W(0) - \underline{W} \; , \tag{6}$$

where the "floor" $\underline{W}$ is specified exogenously. Equations (5)-(6) can be combined to yield the "VaR constraint:"

$$P(W(T) \geq \underline{W}) \geq 1 - \alpha \; . \tag{7}$$

Constraint (7) requires of an agent that only with probability α, or less, will he lose more than $W(0) - \underline{W}$. Clearly, if $P(W^B(T) \geq \underline{W}) > 1 - \alpha$ for the wealth in the benchmark (B) case of no constraints, then the VaR constraint never binds, $VaR(\alpha) < W(0) - \underline{W}$; otherwise, $VaR(\alpha) = W(0) - \underline{W}$.

Note that the formulation in (7) nests the B-case; specifically, when $\alpha = 1$ the VaR constraint is never binding. More interestingly, when $\alpha = 0$, our formulation reduces to the case of portfolio insurance (PI), which constrains the horizon wealth to be above the floor $\underline{W}$ in all states (see, e.g., Basak (1995), Grossman and Vila (1989), Grossman and Zhou (1996)). One can thus view the VaR constraint as a "softer" portfolio-insurance constraint, permitting the portfolio value to deteriorate below the floor of $\underline{W}$ with a prespecified probability.[4]

[4]In the context of developing a model of international portfolio choice, Das and Uppal (1998) constrain the distribution of an agent's portfolio return, imposing an upper bound on the portfolio's excess kurtosis. The authors interpret this constraint as an implicit limit the agent imposes on the portfolio's VaR

3. Optimization under the VaR-RM

In this section, we solve the optimization problem of a VaR risk manager, and then analyze the properties of the solution.

3.1. Agent's Optimization

We solve the dynamic optimization problem of the VaR agent using the martingale representation approach (Cox and Huang (1989), Karatzas, Lehoczky, and Shreve (1987)), which allows the problem to be restated as the following static variational problem:

$$\begin{aligned} &\max_{W(T)} E[u(W(T))] \\ \textit{subject to} \quad & E[\xi(T)W(T)] \leq \xi(0)W(0) \ , \\ & P(W(T) \geq \underline{W}) \geq 1 - \alpha \ . \end{aligned} \tag{8}$$

We note that the VaR constraint complicates the maximization by introducing nonconcavity into the problem. Proposition 1 characterizes the optimal solution, assuming it exists.[5]

Proposition 1. *The time-T optimal wealth of the VaR agent is*

$$W^{VaR}(T) = \begin{cases} I(y\xi(T)) & \textit{if} \ \ \xi(T) < \underline{\xi} \ , \\ \underline{W} & \textit{if} \ \ \underline{\xi} \leq \xi(T) < \overline{\xi} \ , \\ I(y\xi(T)) & \textit{if} \ \ \overline{\xi} \leq \xi(T) \ , \end{cases} \tag{9}$$

where $I(\cdot)$ is the inverse function of $u'(\cdot)$, $\underline{\xi} \equiv u'(\underline{W})/y$, $\overline{\xi}$ is such that $P(\xi(T) > \overline{\xi}) \equiv \alpha$, and $y \geq 0$ solves $E[\xi(T)W^{VaR}(T;y)] = \xi(0)W(0)$. The VaR constraint in (7) is binding if, and only if, $\underline{\xi} < \overline{\xi}$. Moreover, the Lagrange multiplier y is decreasing in α, so that $y \in [y^B, y^{PI}]$.

Figure 1 depicts the optimal terminal wealth of a VaR agent ($\alpha \in (0,1)$), a benchmark agent ($\alpha = 1$) and a portfolio insurer ($\alpha = 0$). Here, $\underline{\underline{W}}$ is defined by

$$\underline{\underline{W}} \equiv \begin{cases} I(y\overline{\xi}) & \text{if} \ \ \underline{\xi} < \overline{\xi}, \\ \underline{W} & \text{otherwise.} \end{cases}$$

[5] Existence is not discussed here, but we will provide explicit numerical solutions for a variety of parameters' values. From (9), a feasibility bound on $\underline{W}$ for a solution is $\underline{W} \leq W(0)\xi(0)/E[\xi(T)1_{\{\xi(T)<\overline{\xi}\}}]$.

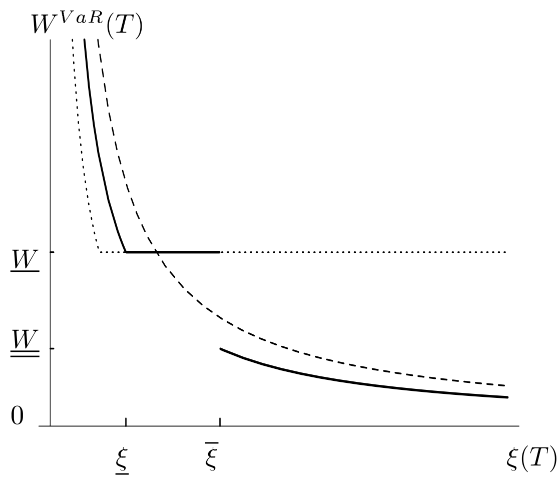


Figure 1: The time-T optimal wealth for the VaR-RM case (solid plot), the PI case (dotted plot), and the B case (dashed plot).

In "good states" (low $\xi(T)$), the portfolio insurer behaves like a B-agent, but then he must insure against all unfavorable (high $\xi(T)$) states. In contrast, Figure 1 reveals the VaR agent to endogenously classify unfavorable states into two subsets: the "bad states" ($\xi(T) \geq \overline{\xi}$), which he leaves fully uninsured, and the "intermediate states" ($\underline{\xi} \leq \xi(T) < \overline{\xi}$), which he fully insures against.[6] Since he is only concerned with the probability (and not the magnitude) of a loss, the VaR agent chooses to leave the worst states uninsured because they are the most expensive ones to insure against. The measure of these bad states is chosen to exactly comply with the VaR constraint. Consequently, $\overline{\xi}$ depends solely on α and the distribution of $\xi(T)$, and is independent of the agent's preferences and endowment. The agent can be thought of as "ignoring" losses in this upper tail of the $\xi(T)$ distribution, where consumption is the most costly.

Inspection of Figure 1 allows us to summarize the dependence of the solution on the parameters $\underline{W}$ and α. As the floor is increased, more states need to be insured against, and the intermediate region grows at the expense of the good-states region. Accordingly, the wealth in the good and bad regions must be decreased to be able to meet the higher floor in the intermediate region. As α increases, i.e., the agent is allowed to make a loss with higher probability, the intermediate, insured region can shrink, while the good and bad regions both grow. The agent's horizon wealth can increase in both the good and bad states since he is not required to insure against as large a state space. Consequently, in the bad-states region $W^{VaR}(T) < W^{B}(T) < W^{PI}(T)$. This may be a source of concern for regulators and real-world risk managers. The VaR-RM is viewed by many as a tool to shield economic agents from large losses, which, when they occur, could

[6]Later, in equilibrium, we will verify that a low $\xi(T)$ is associated with a high equity market value and vice versa.

cause credit and solvency problems. But our solution reveals that when a large loss occurs, it is a yet larger loss under the VaR-RM and hence more likely to lead to credit problems, defeating the very purpose of using the VaR-RM. Proposition 2 later amplifies upon this point.

Figure 2 depicts the shape of the probability density function of terminal wealth in the B, PI, and VaR solutions:

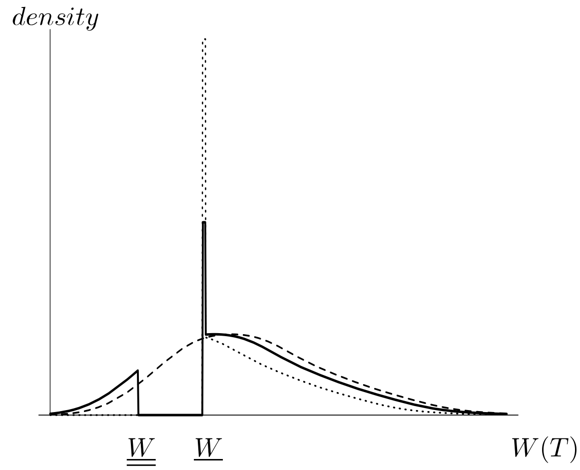


Figure 2: The probability density function of the time-T optimal wealth for the VaR-RM case (solid plot), the PI case (dotted plot), and the B case (dashed plot).

There is a probability mass build up in the VaR agent's horizon wealth, at the floor $\underline{W}$, as for the portfolio insurer. The VaR agent then has a discontinuity, with no states having wealth between $\underline{W}$ and $\underline{\underline{W}}$, while a probability α of states have wealth below $\underline{\underline{W}}$. Note that relative to the benchmark, the distribution in these bad states is shifted to the left, meaning more loss.

It has been commonly observed (e.g., Basak (1995), Grossman and Zhou (1996)) that the optimal PI horizon wealth can be expressed as the B wealth plus a put option thereon, i.e., $W^{PI}(T; y^{PI}) = W^B(T; y^{PI}) + \max[\underline{W} - W^B(T; y^{PI}), 0]$. Analogously, the VaR optimal-wealth plan in (9) can be expressed as

$$\begin{aligned} W^{VaR}(T; y(W(0))) &= W^{PI}(T; y^B(W_*)) - (\underline{W} - W^B(T; y^B(W_*)))1_{\{\bar{\xi} \leq \xi(T)\}} \\ &= W^B(T; y^B(W_*)) + (\underline{W} - W^B(T; y^B(W_*)))1_{\{\underline{\xi} \leq \xi(T) < \bar{\xi}\}}, \end{aligned}$$

where W_* is set so that $y^B(W_*) = y(W(0))$. In other words, adjusting for the initial endowment, W^{VaR} is equivalent to a PI-solution plus a short position in "binary" options, or to a B-solution plus an appropriate position in "corridor" options.[7] More precisely, since

$$W_* = W(0) - E\left[\frac{\xi(T)}{\xi(0)} \max(\underline{W} - W^B(T; y^B(W_*)), 0)\right] + E\left[\frac{\xi(T)}{\xi(0)} (\underline{W} - W^B(T; y^B(W_*)))1_{\{\bar{\xi} \leq \xi(T)\}}\right],$$

[7]For details on binary and corridor options see, for example, Briys, Bellalah, Mai, and Varenne (1998).

$W^B(T; y^B(W_*)))$ is the optimal policy of an unconstrained agent, whose initial endowment is simply $W(0)$ decreased by the price of a put (needed to implement the PI component) and increased by the proceeds of short selling the binary options.

3.2. Properties of the VaR-RM Strategy

To perform a detailed analysis of the optimal behavior under the VaR-RM strategy, we specialize the setting to CRRA preferences, $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, and to log-normal state prices with constant interest rate and market price of risk. Figures 1 and 2 appear to indicate higher losses in the bad-states region under the VaR-RM than without risk management. However, since the bad-states region itself shifts, the figures do not directly imply lower expected losses. Proposition 2 shows explicitly that under the VaR-RM the expected extreme losses are indeed higher than those incurred by an agent who does not concern himself with (7).

Proposition 2. *Assume $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, and r and κ are constant. For a given terminal-wealth plan $W(T)$, define the following two measures of loss:*
$L_1(W) = E[(\underline{\underline{W}} - W(T))1_{\{W(T)\leq \underline{\underline{W}}\}}]$ *and* $L_2(W) = E[\frac{\xi(T)}{\xi(0)}(\underline{\underline{W}} - W(T))1_{\{W(T)\leq \underline{\underline{W}}\}}]$. *Then,*
(i) $L_1(W^{VaR}) \geq L_1(W^B)$, *and* *(ii)* $L_2(W^{VaR}) \geq L_2(W^B)$.

In Proposition 2, we focus on the bad states, that is on the states where large losses occur. $L_1(W)$ measures the expected future value of a loss, when there is a large loss, while $L_2(W)$ measures its present value. Proposition 2 highlights further the undesirable features of VaR-RM, when viewed from a regulator's perspective. A regulatory requirement to manage risk using the VaR approach is designed to prevent large, frequent losses that may drive economic agents out of business. True, under the VaR-RM losses are not frequent. But the largest losses are more severe than without the VaR-RM.

Proposition 3 presents explicit expressions for (and properties of) the VaR agent's optimal wealth and portfolio strategies before the horizon.

Proposition 3. *Assume $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, and r and κ are constant. Then:*
(i) The time-t optimal wealth is given by

$$\begin{aligned} W^{VaR}(t) = \frac{e^{\Gamma(t)}}{(y\xi(t))^{\frac{1}{\gamma}}} \quad &- \quad \left[\frac{e^{\Gamma(t)}}{(y\xi(t))^{\frac{1}{\gamma}}} \mathcal{N}(-d_1(\underline{\xi})) - \underline{W} e^{-r(T-t)} \mathcal{N}(-d_2(\underline{\xi})) \right] \\ &+ \quad \left[\frac{e^{\Gamma(t)}}{(y\xi(t))^{\frac{1}{\gamma}}} \mathcal{N}(-d_1(\overline{\xi})) - \underline{W} e^{-r(T-t)} \mathcal{N}(-d_2(\overline{\xi})) \right], \end{aligned} \tag{10}$$

where $\mathcal{N}(\cdot)$ *is the standard-normal cumulative distribution function,* y *is as in Proposition 1, and*

$$
\begin{aligned}
\underline{\xi} &= \frac{1}{y\underline{W}^{\gamma}} \ , \\
\Gamma(t) &\equiv \frac{1-\gamma}{\gamma}\left(r+\frac{||\kappa||^2}{2}\right)(T-t)+\left(\frac{1-\gamma}{\gamma}\right)^2\frac{||\kappa||^2}{2}(T-t) \ , \\
d_2(x) &\equiv \frac{\ln\frac{x}{\xi(t)}+(r-\frac{||\kappa||^2}{2})(T-t)}{||\kappa||\sqrt{T-t}} \ , \\
d_1(x) &\equiv d_2(x)+\frac{1}{\gamma}||\kappa||\sqrt{T-t} \ .
\end{aligned}
$$

(ii) The fraction of wealth invested in stocks is

$$\theta^{VaR}(t) = q^{VaR}(t)\theta^{B}(t) \ ,$$

where the benchmark value, θ^B*, and the exposure to risky assets relative to the benchmark,* q^{VaR}*, are*

$$
\begin{aligned}
\theta^B(t) &= \frac{1}{\gamma}[\sigma(t)^T]^{-1}\kappa, \\
q^{VaR}(t) &= 1-\frac{\underline{W}e^{-r(T-t)}(\mathcal{N}(-d_2(\underline{\xi}))-\mathcal{N}(-d_2(\overline{\xi})))}{W^{VaR}(t)}+\frac{\gamma(\underline{W}-\underline{\underline{W}})e^{-r(T-t)}\phi(d_2(\overline{\xi}))}{W^{VaR}(t)||\kappa||\sqrt{T-t}}, \quad (11)
\end{aligned}
$$

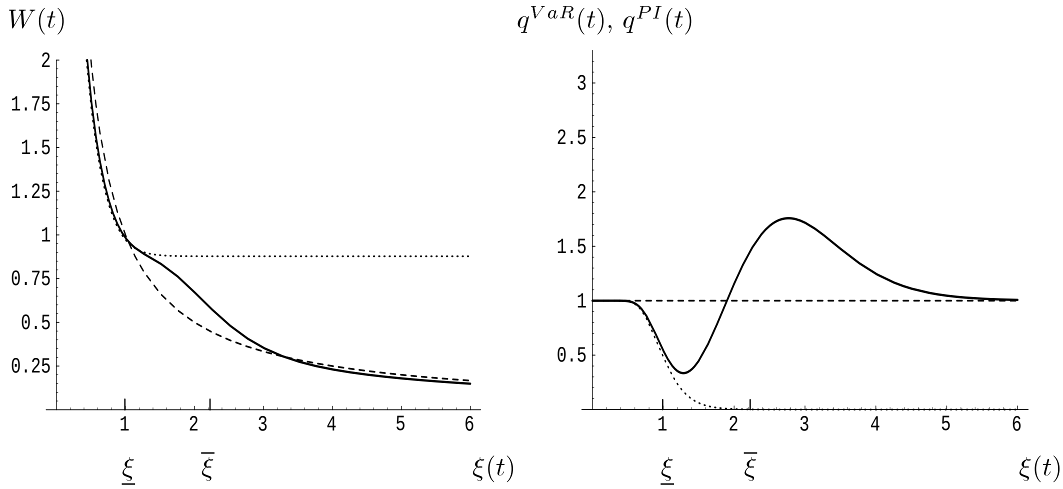
respectively, and $\phi(\cdot)$ *is the standard-normal probability distribution function.*
(iii) The exposure to risky assets relative to the benchmark is bounded below: $q^{VaR}(t) \geq 0$*, and*

$$\lim_{\xi(t)\to 0} q^{VaR}(t) = \lim_{\xi(t)\to\infty} q^{VaR}(t) = 1.$$

(iv) When the VaR constraint is binding ($\underline{\xi} < \overline{\xi}$)*, then* $q^{VaR}(t) > 1$ *if, and only if,* $\xi(t) > \xi^*(t)$*, where* $\xi^*(t)$ *is deterministic and bounded:*

$$\sqrt{\overline{\xi}\underline{\xi}}e^{(r-||\kappa||^2/2)(T-t)} \leq \xi^*(t) \ \leq \ \overline{\xi}e^{(r-||\kappa||^2/2)(T-t)}e^{(||\kappa||^2/\gamma)(T-t)} \ .$$

Figure 3 compares graphically the optimal time-t wealth and the relative stock exposure in the B, PI, and VaR cases:



(a) Wealth for PI, VaR-RM, B

(b) $q(t)$ for PI and VaR-RM, vs B

Figure 3: The (a) time-t wealth and (b) time-t exposure to risky assets relative to the benchmark, for the VaR (solid plot), PI (dotted plot), and B (dashed plot) agents. The parameters used are: $\gamma = 1$, $\alpha = 0.01$, $W(0) = 1$, $\underline{W} = 0.9$, $r = 0.05$, $||\kappa|| = 0.4$, $T = 1$, $t = 0.5$, $\xi(0) = 1$. Then, $\underline{\xi} = 0.99$, $\overline{\xi} = 2.23$.

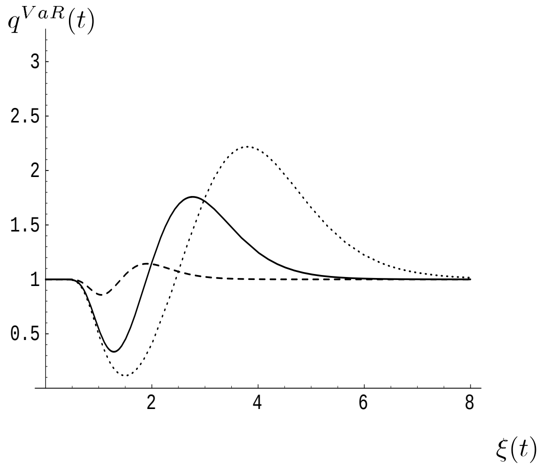
Figure 3(a) reveals that the pre-horizon wealth of the VaR agent behaves similarly to that of a portfolio insurer in the good states, while in the upper tail of the $\xi(t)$ distribution he behaves similarly to the B-case. In the intermediate region, the VaR agent's wealth exhibits concavity in $\xi(t)$ and it is easy to visualize how this concavity will increase as time approaches the horizon, and tend to the discontinuous shape in Figure 1. In these intermediate states, the VaR agent is beginning to insure himself.

Figure 3(b) illustrates the typical shape of the VaR agent's optimal asset allocation, exhibiting some surprising features. We may characterize 5 segments in the $\xi(t)$ space. In the two extremes, the benchmark behavior prevails. But in between, there are three distinct patterns: First, in the relatively cheap states, the VaR agent acts similarly to a portfolio insurer investing a higher fraction of his wealth in the bond. Second, as $\xi(t)$ rises, instead of moving further out of the equity market the VaR agent begins to increase his equity exposure, tending back towards his B-policy, then surpassing it considerably so that in the relatively expensive consumption states he invests a *higher* fraction of his wealth in stocks compared to the B-case.[8] The third segment

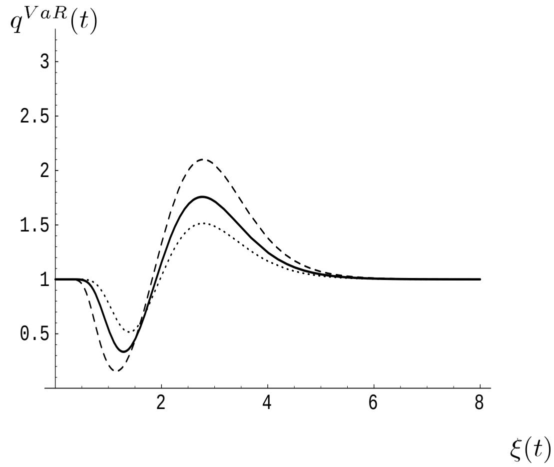
[8]For the parameters used in Figure 3, using the bounds in item (iv) of Proposition 3, $q^{VaR}(t)$, as a function of $\xi(t)$, must rise above 1 while $\xi(t)$ takes values in the (1.46, 2.38) interval. The bounds in (iv) identify, analytically, a transition from an underexposure to overexposure, relative to the B-case, for all parameters' values, and Figure 3(b) confirms this for the chosen parameters. In addition, Figure 3(b) illustrates that the VaR agent deviates considerably from the B and the PI cases when $\xi(t)$ takes values within these bounds.

occurs when $\xi(t)$ is high enough to deter the agent from further risk taking, and he converges to his benchmark policy. Formally, this nonmonotonic behavior across the state-space is linked to the replication of a portfolio of binary options. Intuitively, the asset allocation is driven by the agent's desire to insure the intermediate-states region. When $\xi(t)$ is already very high, then it is very likely that the agent will end-up in the bad-states region and it is too costly for him to bet on a favorable realization of a large equity investment. Hence, the VaR agent behaves similarly to the B-case. On the other hand, when $\xi(t)$ is in the proximity of $\overline{\xi}$, not all hope is lost, and the agent attempts, via a relatively large exposure to equity, to reach the $\underline{W}$ level of wealth, under favorable time-T economic conditions.

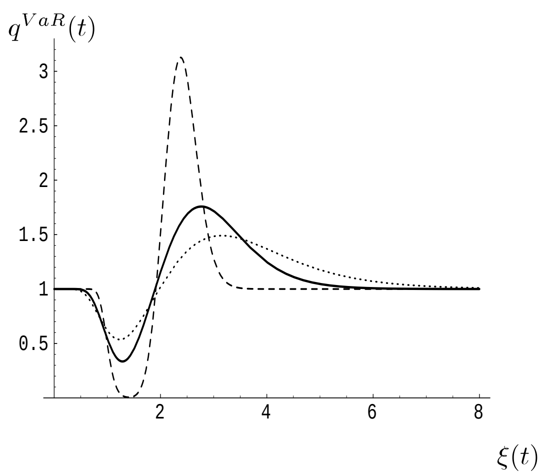
Figure 4 displays a sensitivity analysis of $q^{VaR}(t)$ to α, $\underline{W}$, and time:



(a) the effect of α



(b) the effect of $\underline{W}$

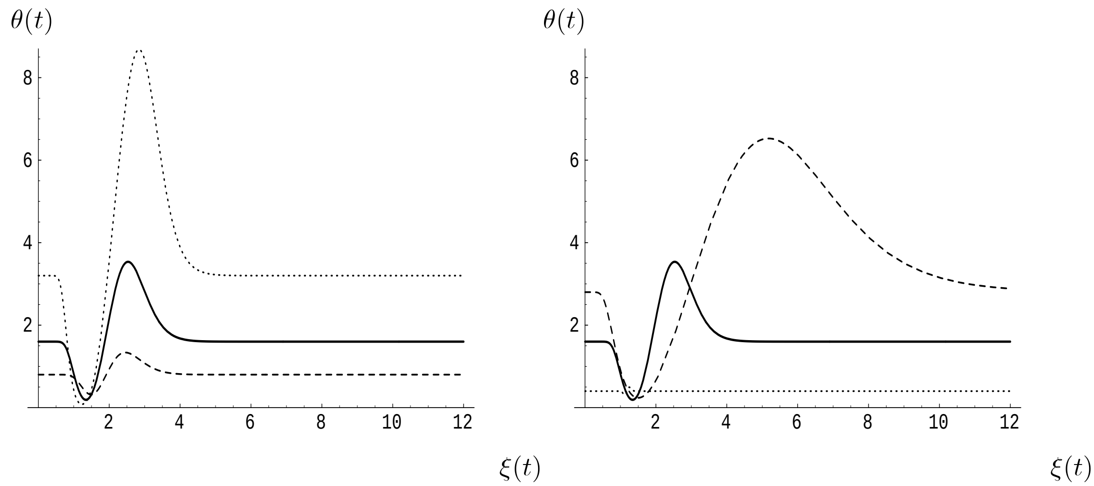


(c) the effect of t

Figure 4: The effect of exogenous parameters (α, $\underline{W}$) that define the VaR constraint, and the effect of time, on the exposure to risky assets relative to the benchmark case. The solid line in all four charts represents the following case: $\gamma = 1$, $\alpha = 0.01$, $W(0) = 1$, $\underline{W} = 0.9$, $r = 0.05$, $||\kappa|| = 0.4$, $T = 1$, $\xi(0) = 1$. Then $\underline{\xi} = 0.99$, $\overline{\xi} = 2.23$. (a) The dotted plot is for $\alpha = 0.001$, the dashed for $\alpha = 0.1$. (b) The dotted plot is for $\underline{W} = 0.8$, the dashed for $\underline{W} = 1$. (c) The dotted plot is for $t = 0.1$, the dashed for $t = 0.9$.

In general terms, (a) decreasing α, (b) increasing $\underline{W}$, or (c) decreasing the time-to-horizon all cause the agent to deviate more from the B-behavior, as the VaR constraint exerts more influence. As α decreases, the deviation from the benchmark also spreads to a larger region of $\xi(t)$, while as the time-to-horizon decreases, the deviation shrinks to a smaller region of $\xi(t)$.

Figure 5 displays the sensitivity, to γ and κ of the risky asset holdings of the VaR agent, for a market with one risky stock:



(a) the effect of γ (b) the effect of κ

Figure 5: The effect of (a) risk aversion (γ) and (b) market price of risk (κ) on the fraction of wealth that the VaR agent allocates to a stock investment. The solid line in both charts represents the following case: $\gamma = 1$, $\alpha = 0.01$, $W(0) = 1$, $\underline{W} = 0.9$, $r = 0.05$, $\kappa = 0.4$, $\sigma = 0.25$, $T = 1$, $\xi(0) = 1$. Then $\theta^B(t) = 1.6$. (a) The dotted plot is for $\gamma = 0.5$, for which $\theta^B(t) = 3.2$. The dashed plot is for $\gamma = 2$, for which $\theta^B(t) = 0.8$. (b) The dotted plot is for $\kappa = 0.1$, for which $\theta^B(t) = 0.4$. The dashed plot is for $\kappa = 0.7$, for which $\theta^B(t) = 2.8$.

The deviation from the benchmark holdings becomes more pronounced for both lower γ (less risk averse agent) and higher κ (higher market price of risk). This behavior is fairly intuitive; as an agent becomes less risk averse, or as the stock's Sharpe ratio increases, he responds more aggressively to changes in the state variable ξ that affect his likelihood to end-up with $W^{VaR}(T) \geq \underline{W}$, as opposed to $W^{VaR}(T) \leq \underline{\underline{W}}$. Note that, contrary to the B-case (but similarly to the PI-case), the more risk averse agent takes on more risk than the less risk averse in the "better" intermediate states; the more risk averse agent invests more in the stock, preparing to end-up with $W^{VaR}(T) > \underline{W}$, as opposed to $W^{VaR}(T) = \underline{W}$. Somewhat more surprising is that, contrary to the B-case (and the PI case), in the "worse" intermediate states a higher Sharpe ratio does not necessarily cause the VaR agent to allocate more wealth to the stock. To understand why, note that a change in κ affects the dynamics of $\xi(t)$; in particular, the boundary into the bad-states region, $\overline{\xi}$, is increasing in κ. Hence, at some given $\xi(t)$, such as 2 (in this example), the lower the κ, the closer the agent is to the transition into the bad states region so the more heavily he invests in the stock, targeting to finance $W^{VaR}(T) = \underline{W}$ should the bad states not occur.

4. Optimization under the LEL-RM

In this section, we introduce the LEL-RM strategy as an alternative to the VaR-RM strategy. We then solve the optimization problem of a LEL risk manager, and analyze the properties of the solution.

4.1. The LEL-RM

The shortcomings of the VaR-RM, highlighted in the previous section, stem from the fact that the VaR agent is concerned with controlling the probability of a loss, rather than its magnitude. It turns out that the expected losses, in the states where there are large losses, are higher than those the agent would have incurred if he had not engaged in the VaR-RM in the first place. Ideally, to control the magnitude of losses, one ought to control all moments of the loss distribution. As a first step, in this section, we focus on controlling the first moment, and examine how one can remedy the shortcomings of the VaR-RM. We leave the analysis of higher moments for future work.

We define a LEL-RM strategy as one under which the present value of the agent's losses, as measured by the $L_2(\cdot)$ measure, are constrained:

$$E[\xi(T)(\underline{W} - W(T))1_{\{W(T)\leq\underline{W}\}}] \leq \epsilon, \tag{12}$$

where $\epsilon \geq 0$ is a constant. Observe that, since $E[\xi(T)(\underline{W} - W(T))1_{\{W(T)\leq\underline{W}\}}] = E[\xi(T)(\underline{W} - W(T))|W(T) \leq \underline{W}]P(W(T) \leq \underline{W})$, this constraint penalizes both a high probability of a loss, and a high expected loss given there is a loss. We may note that the constrained quantity in (12) bears some similarity to the "tail conditional expectation," introduced by Artzner, Delbaen, Eber, and Heath (1998). But their objection to the VaR measure of risk is for a different reason: VaR fails to display a sub-additivity when combining the risk of two or more portfolios (the probability of a loss of the whole may be greater than the probability of a loss of the individual parts).

Analogously to the treatment of (7), we impose (12) as a constraint on the agent's optimization problem, thereby incorporating the LEL-RM directly into the optimization. The formulation again nests the B-case ($\epsilon = \infty$) and the PI-case ($\epsilon = 0$). As we show next, when $0 < \epsilon < \infty$, the LEL strategy has the appealing property that it indeed yields results, which are consistent with the stated goal of "managing risk" in the following sense: The LEL risk manager optimally chooses a wealth level, which in the low-wealth states is above the benchmark wealth.

4.2. Agent's Optimization

Using the martingale representation approach, the dynamic optimization problem of the LEL risk manager (henceforth, the LEL agent) is restated as the following variational problem:

$$\max_{W(T)} E[u(W(T))] \quad s.t. \quad E[\xi(T)W(T)] \leq \xi(0)W(0)\ , \ E[\xi(T)(\underline{W} - W(T))1_{\{W(T)\leq \underline{W}\}}] \leq \epsilon\ . \quad (13)$$

Proposition 4 characterizes the optimal solution, assuming it exists.[9]

Proposition 4. *The time-T optimal wealth of the LEL agent is*

$$W^{LEL}(T) = \begin{cases} I(z_1\xi(T)) & \text{if } \ \xi(T) < \underline{\xi}_\epsilon\ , \\ \underline{W} & \text{if } \ \underline{\xi}_\epsilon \leq \xi(T) < \overline{\xi}_\epsilon\ , \\ I((z_1 - z_2)\xi(T)) & \text{if } \ \overline{\xi}_\epsilon \leq \xi(T)\ , \end{cases} \quad (14)$$

where $\underline{\xi}_\epsilon \equiv \frac{u'(\underline{W})}{z_1}$, $\overline{\xi}_\epsilon \equiv \frac{u'(\underline{W})}{(z_1 - z_2)}$, and $(z_1 \geq 0,\ z_2 \geq 0)$ solve the following system:

$$\begin{cases} E[\xi(T)W^{LEL}(T; z_1, z_2)] = \xi(0)W(0)\ , \\ E[\xi(T)(\underline{W} - W^{LEL}(T; z_1, z_2))1_{\{W^{LEL}(T;z_1,z_2)\leq \underline{W}\}}] = \epsilon \quad or \quad z_2 = 0. \end{cases}$$

The LEL constraint in (12) is binding if, and only if, $\underline{\xi}_\epsilon < \overline{\xi}_\epsilon$. Moreover, the Lagrange multiplier z_1 is decreasing in ϵ, so that $z_1 \in [z_1^B, z_1^{PI}]$. Also, $z_1 - z_2 \leq z_1^B$.

Figure 6 depicts the optimal terminal wealth of a LEL agent ($\epsilon \in (0, \infty)$), a benchmark agent ($\epsilon = \infty$) and a portfolio insurer ($\epsilon = 0$):

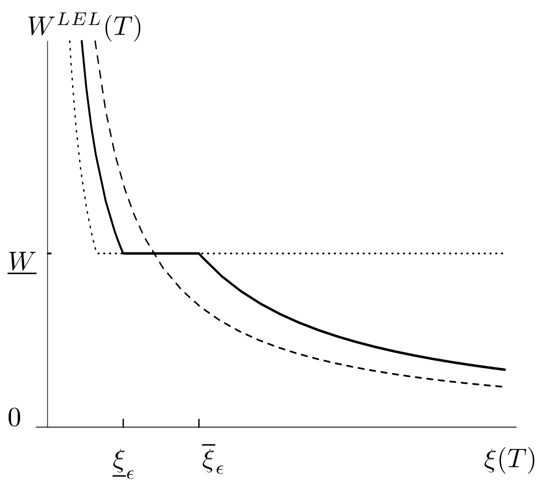


Figure 6: The time-T optimal wealth for the LEL-RM case (solid plot), the PI case (dotted plot), and the B case (dashed plot).

[9] From (14), the feasibility bound on $\underline{W}$ for a solution is $\underline{W} \leq (W(0)\xi(0) + \epsilon)/E[\xi(T)]$.

Figure 6 reveals that in contrast to the findings in the VaR case, now in the bad-states region $W^B(T) < W^{LEL}(T) < W^{PI}(T)$. This highlights the most surprising, but also encouraging, feature of the optimal behavior of the LEL agent; although in some states he is willing to settle for a wealth lower than $\underline{W}$, he does so while endogenously choosing a higher $W^{LEL}(T)$ than $W^B(T)$. The LEL agent endogenously decides to classify unfavorable states into two subsets: the bad states, against which he partially insures, and the intermediate states, against which he fully insures. Again, he chooses the worst states in which to maintain a loss, because these are the most expensive states to insure against, but maintains some level of insurance. Insuring a terminal wealth at the $\underline{W}$ level is too costly, so he settles for less, but enough to comply with the LEL constraint. Note that, the LEL agent not only chooses $\underline{\xi}_\epsilon$ endogenously, but also endogenously determines the value of $\overline{\xi}_\epsilon$; unlike $\overline{\xi}$, $\overline{\xi}_\epsilon$ does depend on the agent's preferences and endowment. A further distinction with the VaR-RM is that the terminal wealth policy under the LEL-RM is continuous across the states of the world.

Figure 7 depicts the shape of the probability density function of terminal wealth in the B, PI, and LEL solutions:

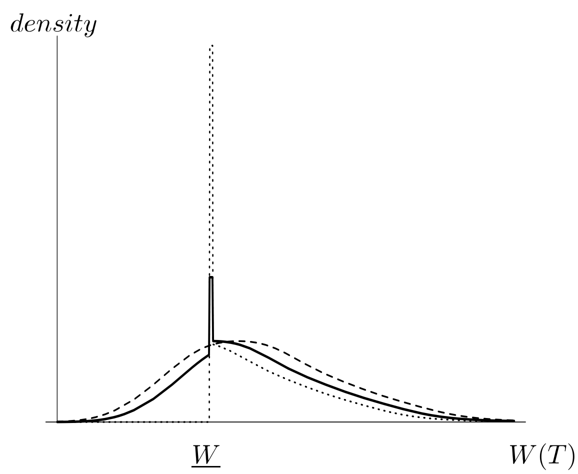


Figure 7: The probability density function of the time-T optimal wealth for the LEL-RM case (solid plot), the PI case (dotted plot), and the B case (dashed plot).

Similarly to Figure 2, there is a probability mass build up in the LEL agent's horizon wealth, at the floor $\underline{W}$. However, the LEL has no discontinuities across states. Also, relative to the benchmark, the distribution in the bad states is shifted to the right, meaning less loss.

The optimal-wealth plan in (14) can be expressed as

$$\begin{aligned} W^{LEL}(T; z_1(W(0)), z_2(W(0))) &= \min[W^{PI}(T; y^B(W_\epsilon)), W^B(T; y^B(W_\epsilon) - z_2(W(0)))] \\ &= \max[W^B(T; y^B(W_\epsilon)), \min[\underline{W}, W^B(T; y^B(W_\epsilon) - z_2(W(0))], \end{aligned}$$

where we set W_ϵ so that $y^B(W_\epsilon) = z_1(W(0))$. Hence, adjusting for the initial endowment, W^{LEL} is equivalent to an option on a minimum of two "securities" (one being riskless), where the nonstandard feature of the option is that the strike price is stochastic.[10] The wealth adjustment, which equates the strike price to the wealth of a fictitious unconstrained agent, is obtained by valuing this non-standard option at the initial date:

$$W_\epsilon = W(0) - E\left[\frac{\xi(T)}{\xi(0)} \max\left[\min[\underline{W}, W^B(T; y^B(W_\epsilon) - z_2(W(0))] - W^B(T; y^B(W_\epsilon)), 0\right]\right] .$$

4.3. Properties of the LEL-RM Strategy

We now specialize the setting to CRRA preferences, $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, and to log-normal state prices with constant interest rate and market price of risk, analogous to the VaR analysis in Section 3. Using the notation defined in Proposition 3, Proposition 5 summarizes the wealth dynamics and the portfolio choice of the LEL agent.

Proposition 5. *Assume $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, and r and κ are constant. Then:*
(i) The time-t optimal wealth is given by

$$\begin{aligned} W^{LEL}(t) = \frac{e^{\Gamma(t)}}{(z_1\xi(t))^{\frac{1}{\gamma}}} \quad &- \quad \left[\frac{e^{\Gamma(t)}}{(z_1\xi(t))^{\frac{1}{\gamma}}}\mathcal{N}(-d_1(\underline{\xi}_\epsilon)) - \underline{W}e^{-r(T-t)}\mathcal{N}(-d_2(\underline{\xi}_\epsilon))\right] \\ &+ \quad \left[\frac{e^{\Gamma(t)}}{(z_1 - z_2)^{\frac{1}{\gamma}}\xi(t)^{\frac{1}{\gamma}}}\mathcal{N}(-d_1(\overline{\xi}_\epsilon)) - \underline{W}e^{-r(T-t)}\mathcal{N}(-d_2(\overline{\xi}_\epsilon))\right], \end{aligned} \quad (15)$$

where $\Gamma(t)$, $d_1(x)$, $d_2(x)$ are as given in Proposition 3, (z_1, z_2) are as given in Proposition 4,

$$\underline{\xi}_\epsilon = \frac{1}{z_1\underline{W}^\gamma} \quad \text{and} \quad \overline{\xi}_\epsilon = \frac{1}{(z_1 - z_2)\underline{W}^\gamma} .$$

(ii) The fraction of wealth invested in stocks is

$$\theta^{LEL}(t) = q^{LEL}(t)\theta^B(t),$$

where the exposure to risky assets relative to the benchmark, $q^{LEL}(t)$ is

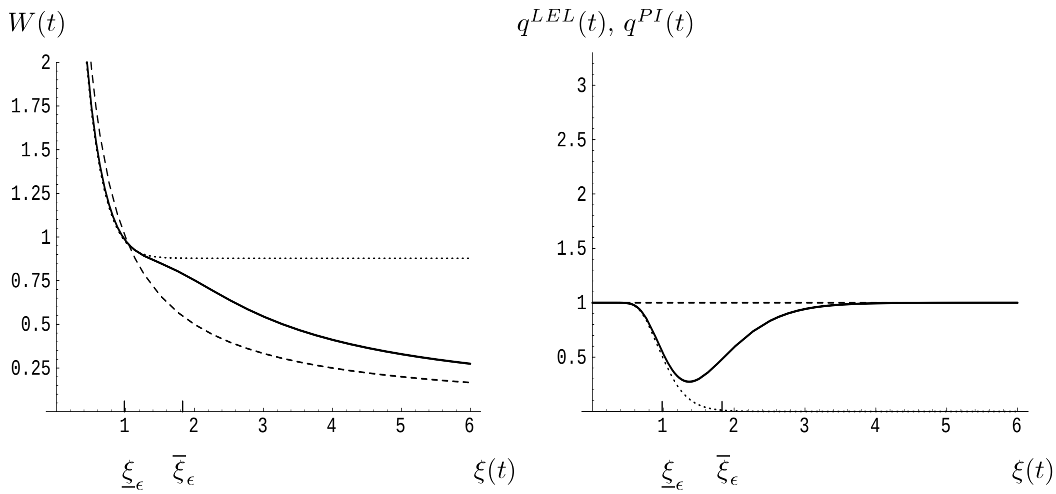
$$q^{LEL}(t) = 1 - \frac{\underline{W}e^{-r(T-t)}(\mathcal{N}(-d_2(\underline{\xi}_\epsilon)) - \mathcal{N}(-d_2(\overline{\xi}_\epsilon)))}{W^{LEL}(t)}.$$

(iii) The exposure to risky assets relative to the benchmark is bounded below and above: $0 \leq q^{LEL}(t) \leq 1$, and

$$\lim_{\xi(t)\to 0} q^{LEL}(t) = \lim_{\xi(t)\to\infty} q^{LEL}(t) = 1.$$

[10]See Stulz (1982) for the analysis and applications of an option on a minimum of two assets (both risky), where the strike price is fixed.

Figure 8 compares graphically the optimal wealth and the stock exposure in the B, PI, and LEL cases:



(a) Wealth for PI, LEL-RM, B

(b) $q(t)$ for PI and LEL-RM, vs B

Figure 8: The (a) time-t wealth and (b) time-t exposure to risky assets relative to the benchmark, for the LEL (solid plot), PI (dotted plot), and B (dashed plot) agents. The parameters used are: $\gamma = 1$, $\epsilon = 0.01$, $W(0) = 1$, $\underline{W} = 0.9$, $r = 0.05$, $||\kappa|| = 0.4$, $T = 1$, $t = 0.5$, $\xi(0) = 1$. Then $\underline{\xi}_\epsilon = 0.98$, $\overline{\xi}_\epsilon = 1.83$.

Figure 8(a) illustrates that, as in the VaR case, for low and intermediate values of $\xi(t)$ the agent's pre-horizon wealth behaves similarly to a portfolio insurer's wealth. In the intermediate range, the LEL agent attempts to insure as many states as he can afford, but in the higher tail of the $\xi(t)$ distribution, he reverts to a behavior similar to the B-behavior. However, unlike in the VaR case, in this upper tail of the distribution the LEL agent maintains a higher wealth than in the B-case. Again, one can easily visualize how the wealth in the intermediate states approaches the shape in Figure 6, as the time approaches the horizon.

In Figure 8(b), we clearly see the properties of $q^{LEL}(t)$ stated in item (iii) of Proposition 5. The LEL agent manoeuvres between his behavior in the B and the PI cases, never investing a *higher* fraction of his wealth in stocks compared to the B-case. The agent's asset allocation has four distinct patterns over the $\xi(t)$ space. In the two extremes, the benchmark behavior prevails. But in between, there are now only two distinct patterns: First, the LEL agent acts as a portfolio insurer, and then, as $\xi(t)$ rises, instead of moving further into the riskless asset the agent increases his equity exposure, tending back towards his B-policy, but never surpassing it in terms of the exposure to equity. Intuitively, the asset allocation of the LEL agent differs from that of the VaR agent because W^{LEL} is continuous across states. In the VaR case, if $\xi(t)$ is close to $\overline{\xi}$ as he approaches the horizon, the VaR agent must allow for the need to finance highly distinct wealths: $\underline{W}$ or $\underline{\underline{W}}$. For the LEL agent, however, a slight change in $\xi(t)$ as t approaches T does not necessitate the financing of a very different level of wealth. Therefore, the LEL-RM never leads risk managers to take extreme leveraged positions compared to the positions they would have taken as non risk managers.

As an aside, we may calculate the probability, $\alpha(\underline{W})$, of making a loss lower than $W(0) - \underline{W}$ for the benchmark- and the LEL-agents. We have

$$\alpha^{LEL}(\underline{W}) = P\left(\xi(T) > \frac{1}{(z_1 - z_2)\underline{W}^{\gamma}}\right) \leq P\left(\xi(T) > \frac{1}{z_1^B \underline{W}^{\gamma}}\right) = \alpha^B(\underline{W}) \ .$$

Hence, the probability of a loss is also lowered by the LEL-RM strategy; to some extent, the LEL agent also manages his VaR.

5. Equilibrium Implications of the VaR-RM

Given that the VaR-RM is becoming an industry standard, it is of interest to evaluate the impact of the presence of VaR risk managers on market prices. In this section, to examine price effects of the VaR-RM, we develop a pure-exchange general equilibrium model of an economy containing VaR risk managers. Since much attention has been directed towards understanding the impact of portfolio insurance on equilibrium prices (Brennan and Schwartz (1989), Donaldson and Uhlig (1993), Basak (1995, 1998), Grossman and Zhou (1996)), given the relationship between VaR risk managers and portfolio insurers, a comparison of equilibrium effects is warranted.

5.1. The Equilibrium Setting

A problem with extending the economic setting in Section 2 to a standard pure-exchange general equilibrium model is that the VaR constraint is imposed directly on the agent's terminal wealth, and hence on his terminal consumption. In equilibrium, this imposes restrictions on the exogenous source that supplies the goods for the terminal consumption. Specifically, Proposition 1 (and Figure 1) revealed the VaR agent's wealth to be discontinuous, never taking values between $\underline{\underline{W}}$ and $\underline{W}$. Therefore, good-market clearing would require a discontinuity in the exogenous terminal consumption source, which seems too contrived a primitive. To circumvent this problem, we instead assume that the VaR horizon, T, is shorter than the agent's lifetime, T', so that the VaR-horizon wealth, $W(T)$, (rather than equating to a lump-sum consumption) represents the value of future consumption. As a result, the VaR constraint is imposed on a quantity, which need not be directly provided by an exogenous consumption supply. A side benefit of this assumption is that it probably renders our model a more realistic description of the VaR-RM, because in reality the VaR horizon would rarely coincide with the consumption horizon. To distinguish the setting here from that of Section 2, we refer to the VaR risk manager as the long-lived VaR agent. We will see that the basic optimal (partial equilibrium) behavior presented in Sections 3-4, survives under this modified setting.

We assume the two types of agents in the economy, the normal agent (n) and the long-lived VaR agent (v), to derive utility from intertemporal (continuous) consumption over their lifetime $[0, T']$. As opposed to the normal agent, the long-lived VaR agent is subject to the additional VaR constraint (7) over time-T wealth, where $T < T'$. For simplicity, we specialize to both agents having logarithmic utility of consumption, and assume the (exogenously) given aggregate consumption process $\delta(t) \equiv \sum_{j=1}^{N} \delta_j(t)$ to follow a geometric Brownian motion process:

$$d\delta(t) = \delta(t) \left[\mu_\delta dt + \sum_{j=1}^{N} \sigma_{\delta_j} dw_j(t) \right] , \quad t \in [0, T'] ,$$

with μ_δ, σ_{δ_j} constants, and $\delta(0) > 0$.

We can anticipate (in light of Basak (1995)) that the constraint applied at the VaR horizon T may result in jumps in the equilibrium security and state prices. Hence, we need to modify accordingly our posited price dynamics in (1)-(2). We posit that the price dynamics in $[0, T)$ and $(T, T']$ are still given by (1)-(2), but at time T we allow for an additional jump component, $\eta dA(t)$, in the changes of security prices. Here, $A(t)$ is a (right-continuous) step function defined by $A(t) \equiv 1_{\{t \geq T\}}$, so that $dA(t)$ is a measure assigning unit mass to time T, and the jump coefficient, η, is an $\mathcal{F}_T$-measurable random variable related to the price jumps by

$$\eta = \ln(B(T)/B(T-)) = \ln(S_j(T)/S_j(T-)) = \ln(\xi(T)/\xi(T-)) , \quad j = 1, \ldots, N, \tag{16}$$

where $S_j(T-)$ is the left limit of $S_j(\cdot)$ at T. Notice that, since $\mathcal{F}_{T-} = \mathcal{F}_T$, to prevent arbitrage on these jumps, the jump coefficient η in all security prices must be the same so that the deflated prices and wealth, $\xi(t)B(t)$, $\xi(t)S_j(t)$, and $\xi(t)W_i(t)$, remain continuous at all times.

5.2. Optimization of a Long-Lived VaR Agent

The long-lived VaR risk manager solves the following problem:

$$\max_{(c_v, W(T-))} E\left[\int_0^{T'} \ln(c_v(t))dt\right]$$

$$\textit{subject to} \qquad E\left[\int_0^{T} \xi(t)c_v(t)dt + \xi(T-)W_v(T-)\right] \leq \xi(0)W_v(0) , \tag{17}$$

$$E\left[\int_T^{T'} \xi(t)c_v(t)dt \mid \mathcal{F}_T\right] \leq \xi(T-)W_v(T-) \ , \tag{18}$$

$$P(W_v(T-) \geq \underline{W}) \geq 1-\alpha \ . \tag{19}$$

The static budget constraint is broken into two components, (17) and (18), to facilitate understanding of the impact of the VaR constraint (19) on the optimization problem. The VaR constraint is imposed on the left limit of time-T wealth to maintain the standard convention of right continuity of wealth processes. The optimal solutions, if they exist, for the long-lived VaR agent and the normal agent are summarized in Proposition 6.

Proposition 6. *The optimal consumption policies and time-T optimal wealth of the two agents are*

$$c_n(t) = \frac{1}{y_n\xi(t)} \ , \qquad t \in [0,T'] \ , \tag{20}$$

$$c_v(t) = \begin{cases} \frac{1}{y_{v1}\xi(t)} \ , & t \in [0,T) \ , \\ \frac{1}{y_{v2}\xi(t)} \ , & t \in [T,T'] \ , \end{cases} \tag{21}$$

$$W_n(T-) = \frac{T'-T}{y_n\xi(T-)} \ , \tag{22}$$

$$W_v(T-) = \begin{cases} \frac{T'-T}{y_{v1}\xi(T-)} & \text{if } \ \xi(T-) < \frac{T'-T}{y_{v1}\underline{W}} \ , \\ \underline{W} & \text{if } \ \frac{T'-T}{y_{v1}\underline{W}} \leq \xi(T-) < \overline{\xi} \ , \\ \frac{T'-T}{y_{v1}\xi(T-)} & \text{if } \ \overline{\xi} \leq \xi(T-) \ , \end{cases} \tag{23}$$

where the constants y_n, y_{v1}, and the $\mathcal{F}_T$-measurable random variable y_{v2} satisfy

$$\frac{T'}{y_n} = \xi(0)W_n(0) \ , \tag{24}$$

$$\frac{T'}{y_{v1}} + E\left[\left(\xi(T-)\underline{W} - \frac{T'-T}{y_{v1}}\right) 1_{\{\frac{T'-T}{y_{v1}\underline{W}} \leq \xi(T-) < \overline{\xi}\}}\right] = \xi(0)W_v(0) \ , \tag{25}$$

$$\frac{T'-T}{y_{v1}} + \left(\xi(T-)\underline{W} - \frac{T'-T}{y_{v1}}\right) 1_{\{\frac{T'-T}{y_{v1}\underline{W}} \leq \xi(T-) < \overline{\xi}\}} = \frac{T'-T}{y_{v2}} \ , \tag{26}$$

and $\overline{\xi}$ is defined by $P(\xi(T-) > \overline{\xi}) \equiv \alpha$.

The solution for the VaR-horizon wealth of the long-lived VaR agent, (23), is analogous to (9), and the intuition for the solution, discussed in Section 3.1, prevails. The only new aspect in which the long-lived VaR agent differs from the normal agent is that he is given different "weighting" before (y_{v1}) and after (y_{v2}) the VaR horizon. When the VaR constraint is binding, $y_{v1} > y_{v2}$ in states where the agent is insuring himself. This resembles the result in Basak (1995) for the portfolio insurer, the idea being that post-horizon consumption not only provides the VaR risk manager with utility but also contributes towards meeting his VaR constraint.

5.3. Equilibrium State Prices

We now define and then characterize the equilibrium in our setting.

Definition 1. *An equilibrium is a collection of* $(r,\ \mu,\ \sigma,\ \eta)$ *and optimal* $(c_n,\ c_v,\ \theta_n,\ \theta_v)$*, such that the good, stock, and bond markets clear, i.e.,* $\forall t \in [0, T']$*,*

$$c_n(t) + c_v(t) = \delta(t)\ , \tag{27}$$

$$\theta_{nj}(t) + \theta_{vj}(t) = S_j(t)\ , \quad j = 1, \ldots, N\ , \tag{28}$$

$$W_n(t) + W_v(t) = \sum_{j=1}^{N} S_j(t)\ . \tag{29}$$

Proposition 7 solves for the equilibrium state price density and its dynamics.

Proposition 7. *The equilibrium state price density is given by*

$$\xi(t) = \begin{cases} (y_n^{-1} + y_{v1}^{-1})\delta(t)^{-1}\ , & t \in [0, T) \\ (y_n^{-1} + y_{v2}^{-1})\delta(t)^{-1}\ , & t \in [T, T'], \end{cases} \tag{30}$$

where y_n*,* y_{v1}*,* y_{v2} *satisfy (24)-(26), with (30) substituted in. Moreover, the equilibrium interest rate and market price of risk are constants, at all* $t \in [0, T']$*, given by* $r = \mu_\delta - ||\sigma_\delta||^2$*, and* $\kappa_j = \sigma_{\delta_j}$*,* $j = 1, \ldots, N$*, and the jump-size parameter is* $\eta = ln((y_n^{-1} + y_{v1}^{-1})/(y_n^{-1} + y_{v2}^{-1})) \leq 0$*.*

Proposition 7 reveals the anticipated (upward) jump in ξ at time T; the price of consumption, ξ, jumps up to counteract the upward jump in aggregate consumption demand at time T, where the jump in demand is due to the VaR risk manager no longer postponing consumption to meet the VaR constraint.

5.4. Equilibrium Market Price, Volatility, and Risk Premium

The price of the *equity market* portfolio, W_{em}, is defined as the aggregate optimally-invested wealth in the risky securities. In equilibrium, W_{em} is also equal to both the aggregate optimally invested wealth and the sum of the risky asset prices:

$$W_{em}(t) \equiv \sum_{j=1}^{N}(\theta_{nj}(t)W_n(t) + \theta_{vj}(t)W_v(t)) = W_n(t) + W_v(t) = \sum_{j=1}^{N} S_j(t)\ .$$

The equilibrium market dynamics can be represented by

$$dW_{em}(t) + \delta(t)dt = W_{em}(t)\left[\mu_{em}(t)dt + \sum_{j=1}^{N}\sigma_{em,j}(t)dw_j(t) + \eta dA(t)\right]\ ,$$

where μ_{em} is the equity market drift and $||\sigma_{em}(t)|| = \sqrt{\sum_{j=1}^N \sigma^2_{em,j}(t)}$ is the equity-market volatility. Proposition 8 presents these quantities in equilibrium and contrasts them with the benchmark (B) economy with all normal agents.[11]

Proposition 8. *The equilibrium market price, volatility, and risk premium in a logarithmic-utility normal-agent benchmark economy are given, $\forall t \in [0, T']$, by*

$$W^B_{em}(t) = (T'-t)\delta(t)\ , \qquad ||\sigma^B_{em}(t)|| = ||\sigma_\delta||\ , \qquad \mu^B_{em}(t) - r = ||\sigma_\delta||^2\ .$$

Before the VaR horizon, the corresponding quantities in the economy with one logarithmic-utility long-lived VaR agent and one logarithmic-utility normal agent are

$$\begin{aligned} W^{VaR}_{em}(t) = (T'-t)\delta(t) \quad &- \quad \left[\frac{T'-T}{y_{v1}}\delta(t)\mathcal{N}(-\hat{d}_1(\overline{\delta})) - \underline{W}e^{-(\mu_\delta - ||\sigma_\delta||^2)(T-t)}\mathcal{N}(-\hat{d}_2(\overline{\delta}))\right] \\ &+ \quad \left[\frac{T'-T}{y_{v1}}\delta(t)\mathcal{N}(-\hat{d}_1(\underline{\delta})) - \underline{W}e^{-(\mu_\delta - ||\sigma_\delta||^2)(T-t)}\mathcal{N}(-\hat{d}_2(\underline{\delta}))\right], \end{aligned} \tag{31}$$

$$\begin{aligned} ||\sigma^{VaR}_{em}(t)|| &= \hat{q}(t)||\sigma_\delta||\ , \\ \mu^{VaR}_{em}(t) - r &= \hat{q}(t)||\sigma_\delta||^2\ , \end{aligned}$$

where

$$\begin{aligned} \overline{\delta} &\equiv \frac{\underline{W}y_{v1}}{T'-T}\ , \\ \underline{\delta} &\equiv 1/\overline{\xi}\ , \\ \hat{d}_1(x) &\equiv \frac{\ln\frac{\delta(t)}{x} + (\mu_\delta - \frac{1}{2}||\sigma_\delta||^2)(T-t)}{||\sigma_\delta||\sqrt{T-t}}\ , \\ \hat{d}_2(x) &\equiv \hat{d}_1(x) - ||\sigma_\delta||\sqrt{T-t}\ , \\ \hat{q}(t) &\equiv 1 - \frac{\underline{W}e^{-(\mu_\delta - ||\sigma_\delta||^2)(T-t)}(\mathcal{N}(-\hat{d}_2(\overline{\delta})) - \mathcal{N}(-\hat{d}_2(\underline{\delta})))}{W^{VaR}_{em}(t)} \\ &\quad + \frac{(\underline{W} - \frac{1}{y_{v1}}(T'-T)\underline{\delta})e^{-(\mu_\delta - ||\sigma_\delta||^2)(T-t)}\phi(\hat{d}_2(\underline{\delta}))}{W^{VaR}_{em}(t)||\sigma_\delta||\sqrt{T-t}}\ . \end{aligned}$$

After the VaR horizon, market prices, volatility, and risk premia in both economies are identical. Consequently, before the VaR horizon,
(i) $W^{VaR}_{em}(t) > W^B_{em}(t)$,
(ii) $||\sigma^{VaR}_{em}(t)|| > ||\sigma^B_{em}(t)||$ *and* $\mu^{VaR}_{em}(t) > \mu^B_{em}(t)$ *if, and only if,* $\delta(t) < \delta^*(t)$, *where* $\delta^*(t)$ *is deterministic and bounded:*

$$\underline{\delta}e^{-(\mu_\delta - ||\sigma_\delta||^2/2)(T-t)} \le \delta^*(t) \ \le\ \sqrt{\underline{\delta}\overline{\delta}}e^{-(\mu_\delta - ||\sigma_\delta||^2/2)(T-t)}e^{||\sigma_\delta||^2(T-t)}\ .$$

[11] Although not the focus of our discussion, we note that, under appropriate restrictions on exogenous parameters, existence of equilibrium (demonstrated via existence of the y's in (24)-(26)) can be straightforwardly verified.

Item (i) reveals the pre-horizon market price in the VaR economy to be higher than in the benchmark economy. This result is as in the PI economy, and comes about because the long-lived VaR agent values post-horizon dividends more than the pre-horizon consumption, since these dividends help him to meet his constraint. The pre-horizon value of the equity market is then pushed up, because equities are claims against the post-horizon dividends.

When the VaR agent behaves like a portfolio insurer ($\alpha = 0$), it is immediate to verify that $\hat{q}(t) \in [0, 1]$, and equity-market volatility is never higher than in the B-case, as indeed was shown by Basak (1995). Otherwise, as long as the VaR constraint is binding ($\overline{\delta} > \underline{\delta}$), item (ii) reveals that there are always states of the world in which the VaR economy stock volatility is higher than in the benchmark. This is a consequence of the risky asset demands of the VaR agent, discussed in Section 3.2. Since the interest rate and the market price of risk are pinned down as constants in equilibrium, favorability of the risky equity market relative to the bond is controlled by its volatility. Whenever the presence of the VaR agent elevates the demand for risky assets, the market volatility will increase to compensate (so to clear markets), and conversely when the VaR agent depresses the demand for risky assets. The market risk premium increases (or decreases) in parallel. Furthermore, item (ii) implies that the increased volatility arises in states of low output, or down stock markets, or more specifically, in the transition from the intermediate states of the world to the bad states.

Note that the equilibrium analysis provides a justification for our identification of low (high) $\xi(t)$ with good (bad) states of the world. (30) reveals $\xi(t)$, the price of consumption, to be decreasing in the consumption supply $\delta(t)$, while (31) reveals the equity market value to be increasing in $\delta(t)$. Hence, what we call "good (bad)" states are those associated with high (low) aggregate output and with high (low) equity prices.

6. Conclusion

We analyze the effects of risk management on optimal wealth and consumption choices and on optimal portfolio policies. We first focus on modeling risk managers as expected utility maximizers, who derive utility from wealth at some horizon, and who must comply with a VaR constraint imposed at that horizon, requiring that the wealth may decrease below a given floor only with a prespecified probability. Having embedded VaR into an optimizing framework, we reveal several surprising effects, some of which may be viewed as undesirable by regulators. In particular, VaR risk managers incur larger losses than non risk managers in the most adverse states of the world. To address that, we next propose an alternative model of risk management, a LEL-RM, where expected losses, rather than the probability of losses, are limited. We demonstrate how this alternative model remedies the shortcomings of the VaR-RM.

Both the partial-equilibrium and the general-equilibrium analyses of the economy with VaR risk managers yield profoundly different implications compared to the extensively-studied case of portfolio insurance: VaR risk managers differ from portfolio insurers both in their endogenously chosen quantities and in their impact on equilibrium prices. In particular, in the worse states of the world, the VaR agents may take on more risk than the benchmark agents and consequently increase the stock-market volatility, which is exactly the opposite behavior and impact on volatility as compared with portfolio insurers.

While we demonstrate how to embed two particular forms of risk management into an optimizing framework, our analysis may also pave the way towards evaluation of further alternative risk management practices of interest to regulators. In particular, there is room to consider risk management models that require agents to focus on the higher moments of the distribution of a loss. For example, from an econometric perspective, volatilities can be estimated more efficiently than means, and it is therefore of interest to compare the LEL-RM framework with one that binds the second moment of a loss, which may be an easier framework to implement in practice.

Appendix: Proofs

Proof of Proposition 1: Let $\hat{W}(T) = W^{VaR}(T)$. If $P(\hat{W}(T) < \underline{W}) < \alpha$, then by their definition, $\overline{\xi} < \underline{\xi}$, and $W^{VaR}(T) = I(y\xi(T)) = W^B(T)$, which is optimal following the standard arguments as in the benchmark case. Otherwise, $P(\hat{W}(T) < \underline{W}) = \alpha$, and $\overline{\xi} \geq \underline{\xi}$. The remainder of the proof is for the latter case. We adapt the common convex-duality approach (see, e.g., Karatzas and Shreve (1998)) to incorporate the VaR constraint. The expression in Lemma 1 is the convex conjugate of u with an additional term capturing the VaR constraint.

Lemma 1. *Expression (9) solves the following pointwise problem* $\forall \xi(T)$*:*

$$u(\hat{W}(T)) - y\xi(T)\hat{W}(T) + y_2 1_{\{\hat{W}(T) \geq \underline{W}\}} = \max_W \{u(W) - y\xi(T)W + y_2 1_{\{W \geq \underline{W}\}}\} \ ,$$

where $y_2 \equiv u(I(y\overline{\xi})) - y\overline{\xi}I(y\overline{\xi}) - u(\underline{W}) + y\overline{\xi}\underline{W} \geq 0$.

Proof: The function on which $\max\{\cdot\}$ operates is not concave in W, but can only exhibit local maxima at $W = I(y\xi(T))$ and/or $W = \underline{W}$. To find the global maximum, we need to compare the value of these two local maxima. When $\xi(T) < \underline{\xi}$, we have $I(y\xi(T)) > \underline{W}$ and

$$u(I(y\xi(T))) - y\xi(T)I(y\xi(T)) + y_2 > u(\underline{W}) - y\xi(T)\underline{W} + y_2,$$

so $I(y\xi(T))$ is the global maximum. When $\underline{\xi} \leq \xi(T) < \overline{\xi}$, we have $I(y\xi(T)) \leq \underline{W}$ and

$$\begin{aligned} u(\underline{W}) - y\xi(T)\underline{W} + y_2 &= u(I(y\overline{\xi})) - y\overline{\xi}I(y\overline{\xi}) + y\underline{W}(\overline{\xi} - \xi(T)) \\ &> u(I(y\xi(T))) - y\xi(T)I(y\xi(T)) \ , \end{aligned} \qquad \text{(A1)}$$

where the inequality follows from $\xi(T) < \overline{\xi}$ and $\frac{\partial}{\partial \xi}\{u(I(y\xi)) - y\xi I(y\xi) + y\underline{W}\xi\} = -yI(y\xi) + y\underline{W} \geq 0$ whenever $\xi \geq \underline{\xi}$. So $\underline{W}$ is the global maximum. When $\xi(T) \geq \overline{\xi}$, the inequality in (A1) is reversed and so $I(y\xi(T))$ is the global maximum. Finally, to show $y_2 \geq 0$, note that

$$y_2 = [u(I(y\overline{\xi})) - y\overline{\xi}I(y\overline{\xi}) + y\underline{W}\overline{\xi}] - [u(I(y\underline{\xi})) - y\underline{\xi}I(y\underline{\xi}) + y\underline{W}\underline{\xi}] \geq 0 \ ,$$

again from $\frac{\partial}{\partial \xi}\{u(I(y\xi)) - y\xi I(y\xi) + y\underline{W}\xi\} \geq 0$ and $\overline{\xi} \geq \underline{\xi}$. ■

Now, let $W(T)$ be any candidate optimal solution, which satisfies the VaR constraint (7) and the static budget constraint (8). We have

$$\begin{aligned} & E[u(\hat{W}(T))] - E[u(W(T))] \\ = \ & E[u(\hat{W}(T))] - E[u(W(T))] - y\xi(0)W(0) + y\xi(0)W(0) + y_2(1-\alpha) - y_2(1-\alpha) \\ \geq \ & E[u(\hat{W}(T))] - E[u(W(T))] - E[y\xi(T)\hat{W}(T)] + E[y\xi(T)W(T)] \\ & + E[y_2 1_{\{\hat{W}(T) \geq \underline{W}\}}] - E[y_2 1_{\{W(T) \geq \underline{W}\}}] \geq 0, \end{aligned}$$

where the former inequality follows from the static budget constraint and the VaR constraint holding with equality for $\hat{W}(T)$, while holding with inequality for $W(T)$. The latter inequality follows from Lemma 1. Hence $\hat{W}(T)$ is optimal. Finally, since the VaR constraint must hold with equality, we deduce the definition of $\overline{\xi}$. From (9) it is clear that $\partial W^{VaR}(T;y)/\partial\alpha|_y < 0$, and in particular $W^{PI}(T;y) \geq W^{VaR}(T;y)$. Furthermore, except when equal to $\underline{W}$, all wealth policies are decreasing in y. Hence, to allow the static budget constraint hold with equality, we must have y decreasing in α and $y \in [y^B, y^{PI}]$. ■

Proof of Proposition 2:

(i) It is straightforward to verify that $L_1(W^B) = G_1(a_B, y^B)$, $L_1(W^{VaR}) = G_1(a_V, y)$, where

$$G_1(a,x) \;=\; \underline{\underline{W}}\mathcal{N}(a) - x^{-\frac{1}{\gamma}} e^{\left(\frac{m}{\gamma}+\frac{s^2}{2\gamma^2}\right)} \mathcal{N}(a - \frac{s}{\gamma}),$$

$$m = E[-ln\xi(T)], \qquad s^2 = Var[-ln\xi(T)],$$

$$a_B = \frac{(\ln \underline{\underline{W}}^{\gamma} y^B \;-\; m)}{s}, \qquad a_V = \frac{(\ln \underline{\underline{W}}^{\gamma} y \;-\; m)}{s},$$

and y solves $E[\xi(T)I(y\xi(T))] = \xi(0)W(0)$. Next, it is also straightforward to show that, for $x > 0$, $\frac{\partial}{\partial a}G_1(a,x) \geq 0$ if, and only if, $a \leq a_V$. Hence, since $a_B \leq a_V$, $G_1(a_B, y) \leq G_1(a_V, y)$. Also, since $\frac{\partial}{\partial x}G_1(a,x) \geq 0$ and $y \geq y^B$, $G_1(a, y^B) \leq G_1(a, y)$. Then,

$$\begin{aligned} L_1(W^{VaR}) - L_1(W^B) &= G_1(a_V, y) - G_1(a_B, y^B) \\ &\geq G_1(a_V, y) - G_1(a_B, y) \geq 0. \end{aligned}$$

(ii) It is straightforward to verify that $L_2(W^B) = G_2(a_B, y^B)$, $L_2(W^{VaR}) = G_2(a_V, y)$, where

$$\begin{aligned} G_2(a,x) &= (\underline{\underline{W}}e^{-m+\frac{s^2}{2}}\mathcal{N}(a+s) - x^{-\frac{1}{\gamma}}e^{\Gamma}\mathcal{N}(a - \frac{1-\gamma}{\gamma}s))/\xi(0), \\ \Gamma &= \frac{1-\gamma}{\gamma}m + \left(\frac{1-\gamma}{\gamma}\right)^2\frac{s^2}{2}, \end{aligned}$$

a_B, a_V, as in part (i). Also, for $x > 0$, $\frac{\partial}{\partial a}G_2(a,x) \geq 0$ if, and only if, $a \leq a_V$, and since $\frac{\partial}{\partial x}G_2(a,x) \geq 0$, $G_2(a, y^B) \leq G_2(a, y)$. Therefore,

$$\begin{aligned} L_2(W^{VaR}) - L_2(W^B) &= G_2(a_V, y) - G_2(a_B, y^B) \\ &\geq G_2(a_V, y) - G_2(a_B, y) \geq 0. \end{aligned}$$

■

Proof of Proposition 3:

(i) From (3) and (4), Itô's lemma implies that $\xi(t)W^{VaR}(t)$ is a martingale:

$$W^{VaR}(t) = E[\frac{\xi(T)}{\xi(t)}W^{VaR}(T)|\mathcal{F}_t] \;. \tag{A2}$$

When r and κ are constant, conditional on $\mathcal{F}_t$, $\ln \xi(T)$ is normally distributed with mean $\ln \xi(t) - (r + \frac{||\kappa||^2}{2})(T-t)$ and variance $||\kappa||^2(T-t)$. Substituting (9) into (A2), using $I(x) = x^{-\frac{1}{\gamma}}$, and evaluating the conditional expectations over each of the three regions of $\xi(T)$ yields (10).

(ii) Applying Itô's lemma to (10), using $\kappa = \sigma(t)^{-1}(\mu(t) - r\bar{1})$, we get

$$\sigma_{W^{VaR}}(t) = \frac{1}{\gamma}\frac{e^{\Gamma(t)}}{(y\xi(t))^{\frac{1}{\gamma}}}[1 - \mathcal{N}(-d_1(\underline{\xi})) + \mathcal{N}(-d_1(\overline{\xi})) + \frac{\gamma(\underline{W} - \underline{\underline{W}})e^{-r(T-t)-\Gamma(t)}\phi(d_2(\overline{\xi}))}{(y\xi(t))^{-\frac{1}{\gamma}}||\kappa||\sqrt{T-t}}]\kappa \; .$$

From (4), $\sigma_{W^{VaR}}(t)$ must equal $\sigma(t)^\top \theta^{VaR}(t) W^{VaR}(t)$. So, using the well-known value of θ^B, we get

$$q^{VaR}(t) = \frac{e^{\Gamma(t)}}{W^{VaR}(t)(y\xi(t))^{\frac{1}{\gamma}}}[1 - \mathcal{N}(-d_1(\underline{\xi})) + \mathcal{N}(-d_1(\overline{\xi})) + \frac{\gamma(\underline{W} - \underline{\underline{W}})e^{-r(T-t)-\Gamma(t)}\phi(d_2(\overline{\xi}))}{(y\xi(t))^{-\frac{1}{\gamma}}||\kappa||\sqrt{T-t}}] \tag{A3}$$

Rearranging (A3), we get (11).

(iii) Inspection of (A3) clearly reveals that it is nonnegative. The limits are immediate to verify.

(iv) For a given t, to save notation, we suppress the dependence of ξ, q^{VaR}, and W^{VaR} on t. The proof first establishes the existence of ξ^*, for a given t, by explicitly computing a region (in the ξ-space) within which q^{VaR} rises, as a function of ξ, from below to above 1. Then, uniqueness of ξ^* is established. As stated in the proposition, the above region is defined in terms of two sufficient conditions: the first is that $q^{VaR} < 1$ if $\xi < \sqrt{\overline{\xi}\underline{\xi}}e^{(r-||\kappa||^2/2)(T-t)}$, the second is that $q^{VaR} > 1$ if $\xi > \overline{\xi}e^{(r-||\kappa||^2/2)(T-t)}e^{(||\kappa||^2/\gamma)(T-t)}$. For brevity, we only present the proof of the former, as the proof of the latter follows similar steps. For $X \in [\underline{\underline{W}}, \underline{W}]$, let

$$F(X, \xi) \equiv \frac{\gamma(X - \underline{\underline{W}})\phi(d_2(\overline{\xi}))}{||\kappa||\sqrt{T-t}} - X(\mathcal{N}(-d_2(X^{-\gamma}/y)) - \mathcal{N}(-d_2(\overline{\xi}))) \; .$$

Note that $d_2(X^{-\gamma}/y)$ and $d_2(\overline{\xi})$ are functions of ξ, and that $q^{VaR} = 1 + F(\underline{W}, \xi)/W^{VaR}$. Hence, for a given t and ξ, $q^{VaR} < 1$ if, and only if, $F(\underline{W}, \xi) < 0$. For analytical tractability, we only derive a sufficient condition for $F(\underline{W}, \xi) < 0$. Noting that $F(\underline{\underline{W}}, \xi) = 0$, a sufficient condition for $F(\underline{W}, \xi) < 0$ is that $\frac{\partial}{\partial X}F(X, \xi) < 0$, $\forall X \in [\underline{\underline{W}}, \underline{W}]$. It is straightforward to verify that a sufficient condition for $\frac{\partial}{\partial X}F(X, \xi) < 0$, $\forall X \in [\underline{\underline{W}}, \underline{W}]$, is that $\xi < \sqrt{\frac{\overline{\xi}}{yX^\gamma}}e^{(r-||\kappa||^2/2)(T-t)}$. But, because $\underline{\xi} = \frac{1}{y\underline{W}^\gamma} \leq \frac{1}{yX^\gamma}$, $\forall X \in [\underline{\underline{W}}, \underline{W}]$, the latter inequality holds when $\xi < \sqrt{\overline{\xi}\underline{\xi}}e^{(r-||\kappa||^2/2)(T-t)}$. To summarize: $\xi < \sqrt{\overline{\xi}\underline{\xi}}e^{(r-||\kappa||^2/2)(T-t)} \Rightarrow \xi < \sqrt{\frac{\overline{\xi}}{yX^\gamma}}e^{(r-||\kappa||^2/2)(T-t)}$, $\forall X \in [\underline{\underline{W}}, \underline{W}] \Rightarrow \frac{\partial}{\partial X}F(X, \xi) < 0$, $\forall X \in [\underline{\underline{W}}, \underline{W}] \Rightarrow F(\underline{W}, \xi) < 0 \Rightarrow q^{VaR} < 1$. This, combined with the condition for $q^{VaR} > 1$, using the limits in (iii), and the fact that $F(\underline{W}, \xi)$ is differentiable with respect to ξ imply that there exists a ξ^* for which $F(\underline{W}, \xi^*) = 0$, and there exist ξ_L, ξ_H satisfying $\xi_L < \xi^* < \xi_H$ for which $F(\underline{W}, \xi_L) < 0 < F(\underline{W}, \xi_H)$ and $\frac{\partial}{\partial \xi}F(\underline{W}, \xi_L) = \frac{\partial}{\partial \xi}F(\underline{W}, \xi_H) = 0$. To complete the proof, we need to show that ξ^* is unique. To prove this, it is enough, in our setting, to verify that $\frac{\partial}{\partial \xi}F(\underline{W},\xi)=0$ has at most two distinct roots. To verify the latter, note that $\frac{\partial}{\partial \xi}F(\underline{W},\xi)=0$ if, and only if, $f(\underline{W},d_2(\overline{\xi}))=0$, where $f(\underline{W},h)=a_1h-a_2e^{a_3h}+\underline{W}$, and $a_1=\frac{\gamma(\underline{W}-\underline{\underline{W}})}{||\kappa||\sqrt{T-t}}$, $a_2=\underline{W}e^{-\frac{a_3^2}{2}}$, $a_3=\frac{\gamma\ln(\underline{W}/\underline{\underline{W}})}{||\kappa||\sqrt{T-t}}$, are all positive and independent of ξ. But $\frac{\partial^2}{\partial h^2}f(\underline{W},h)=-a_2a_3^2e^{a_3h}<0$, for all h, and hence $f(\underline{W},h)=0$ has no more than two distinct roots: h_1, h_2. Assume, without loss of generality, $h_1>h_2$. Since there is a one-to-one mapping between $d_2(\overline{\xi})$ and ξ, $\xi_L=\overline{\xi}e^{(r-\frac{||\kappa||^2}{2})(T-t)-h_1||\kappa||\sqrt{T-t}}$, and $\xi_H=\overline{\xi}e^{(r-\frac{||\kappa||^2}{2})(T-t)-h_2||\kappa||\sqrt{T-t}}$ are the unique, global minimizer and maximizer, respectively, of F. ∎

Proof of Proposition 4: This is the direct analog of Proposition 1. Let $\hat{W}(T)=W^{LEL}(T)$. If $E[\xi(T)(\underline{W}-\hat{W}(T))1_{\{\hat{W}(T)\le\underline{W}\}}]<\epsilon$, then $z_2=0$ and $\overline{\xi}_\epsilon=\underline{\xi}_\epsilon$, and $W^{LEL}(T)=I(z_1\xi(T))=W^B(T)$, which is optimal following the standard arguments.
Otherwise, $E[\xi(T)(\underline{W}-\hat{W}(T))1_{\{\hat{W}(T)\le\underline{W}\}}]=\epsilon$, and $\overline{\xi}\ge\underline{\xi}$. The remainder of the proof is for the latter case.

Lemma 2. *Expression (14) solves the following pointwise problem $\forall\xi(T)$:*

$$
\begin{aligned}
&u(\hat{W}(T))-z_1\xi(T)\hat{W}(T)-z_2(\underline{W}-\hat{W}(T))1_{\{\hat{W}(T)<\underline{W}\}}\\
&=\max_W\{u(W)-z_1\xi(T)W-z_2(\underline{W}-W)1_{\{W<\underline{W}\}}\}\ .
\end{aligned}
$$

Proof: The function on which $\max\{\cdot\}$ operates is not concave in W, but can only exhibit local maxima at $W=I(z_1\xi(T))$ if $I(z_1\xi(T))\ge\underline{W}$, or $W=I((z_1-z_2)\xi(T))$ if $I((z_1-z_2)\xi(T))<\underline{W}$, or $W=\underline{W}$. When $\xi(T)<\underline{\xi}_\epsilon$, $I(z_1\xi(T))>\underline{W}$, $I((z_1-z_2)\xi(T))>\underline{W}$, and

$$
u(I(z_1\xi(T)))-z_1\xi(T)I(z_1\xi(T))>u(\underline{W})-z_1\xi(T)\underline{W},
$$

so $I(z_1\xi(T))$ is the global maximum. When $\xi(T)\ge\overline{\xi}_\epsilon$, $I((z_1-z_2)\xi(T))<\underline{W}$, $I(z_1\xi(T))<\underline{W}$ and

$$
u(I((z_1-z_2)\xi(T)))-(z_1-z_2)\xi(T)I((z_1-z_2)\xi(T))\ge u(\underline{W})-(z_1-z_2)\xi(T)\underline{W},
$$

so $I((z_1-z_2)\xi(T))$ is the global maximum. When $\underline{\xi}_\epsilon\le\xi(T)<\overline{\xi}_\epsilon$, $I(z_1\xi(T))\le\underline{W}$,
$I((z_1-z_2)\xi(T))>\underline{W}$, so $W=\underline{W}$ is the only local maximum and hence the solution. ∎

Now, let $W(T)$ be any candidate optimal solution, which satisfies the static budget constraint and the LEL constraint in (12). We have

$$
\begin{aligned}
&E[u(\hat{W}(T))]-E[u(W(T))]\\
=\ &E[u(\hat{W}(T))]-E[u(W(T))]-z_1\xi(0)W(0)+z_1\xi(0)W(0)-z_2\epsilon+z_2\epsilon\\
\ge\ &E[u(\hat{W}(T))]-E[u(W(T))]-E[z_1\xi(T)\hat{W}(T)]+E[z_1\xi(T)W(T)]\\
&-E[z_2(\underline{W}-\hat{W}(T))1_{\{\hat{W}(T)\le\underline{W}\}}]+E[z_2(\underline{W}-W(T))1_{\{W(T)\le\underline{W}\}}]\ge 0,
\end{aligned}
$$

where the former inequality follows from the static budget constraint and the LEL constraint holding with equality for $\hat{W}(T)$, while holding with inequality for $W(T)$. The latter inequality follows from Lemma 1. Hence $\hat{W}(T)$ is optimal. Suppose $z_1 > z_1^{PI}$. Then $W^{PI}(T) > W^{LEL}(T)$ in all states, contradicting the budget constraint holding with equality for both. Hence, by contradiction, $z_1 \leq z_1^{PI}$. Suppose $z_1 - z_2 \leq z_1 < z_1^B$. Then $W^{LEL}(T) > W^B(T)$ in all states. Similarly, if $z_1^B < z_1 - z_2 < z_1$, then $W^{LEL}(T) < W^B(T)$ in all states. Either case contradicts the budget constraint holding with equality for both, so we must have $z_1 - z_2 \leq z_1^B \leq z_1$. ∎

Proof of Proposition 5: The proof is as of Proposition 3, except with $\underline{\xi}$ and $\overline{\xi}$ replaced appropriately by $\underline{\xi}_\epsilon$ and $\overline{\xi}_\epsilon$. ∎

Proof of Proposition 6: (20), (22), and (24) are well-known to solve the unconstrained optimization. To show that (21), (23), (25), and (26) are the optimal solution to the optimization problem of the long-lived VaR agent is a straightforward extension of the proof of Proposition 1, and is therefore omitted. ∎

Proof of Proposition 7: (30) follows from the clearing of the consumption good market. Then, r and κ are determined by applying Itô's lemma to (30) and equating terms with (3), and η follows by substituting (30) into (16). ∎

Proof of Proposition 8: In equations (24)-(26), the y's are only determined up to a multiplicative constant, and we therefore, without loss of generality, set $y_n^{-1} + y_{v1}^{-1} = 1$. The expression for $W_{em}^{VaR}(t)$ follows by substituting $(T' - T)\delta(t)/y_{v1}$, $||\sigma_\delta||$, $\mu_\delta - ||\sigma_\delta||^2$, for $1/y\xi(t)$, $||\kappa||$, r, respectively, in the time-t wealth equation (10) of Proposition 3, and adding the $(T' - t)\delta(t)$ term to account for intermediate consumption. Applying Itô's lemma to $W_{em}^{VaR}(t)$ yields the expressions for $||\sigma_{em}^{VaR}(t)||$, $\mu_{em}^{VaR}(t)$. To show property (i), use (22)-(23) and (30) to note that when $\hat{W}_v(T-) \neq \underline{W}$ then $\hat{W}_n(T-) + \hat{W}_v(T-) = (T' - T)\delta(T-)$, and when $\hat{W}_v(T-) = \underline{W}$, then $\hat{W}_n(T-) + \hat{W}_v(T-) > (T' - T)\delta(T-)$. Hence, $W_{em}^{VaR}(T-) > W_{em}^B(T-)$, which implies (i). Property (ii) follows by substituting the appropriate equilibrium quantities in part (iv) of Proposition 3.

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