Learning To Be Overconfident

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Abstract

We develop a multi-period market model describing both the process by which traders learn about their ability and how a bias in this learning can create overconfident traders. A trader in our model initially does not know his own ability, that is, the probability that he will receive a valid signal in each period. He infers this ability from his successes and failures. In assessing his ability the trader takes too much credit for his successes, i.e. he weighs his successes more heavily than would a true Bayesian agent. This leads him to become overconfident. A trader's expected level of overconfidence increases in the early stages of his career. Then, with more experience, he comes to better recognize his own ability. An overconfident trader trades too aggressively, thereby increasing trading volume and market volatility while lowering his own expected profits. Though a greater number of past successes indicates greater probable ability, a more successful trader may actually have lower expected profits in the next period than a less successful trader due to his greater overconfidence. Since overconfidence is generated by success, overconfident traders are not the poorest traders. Their survival in the market is not threatened. Overconfidence does not make traders wealthier, but the process of becoming wealthy can make traders overconfident.

1 Introduction

An old Wall Street adage advises "Don't confuse brains with a bull market." The need for such wisdom stems from traders' willingness to attribute too much of their success to their own abilities and not enough to their good fortune. Successful traders are, thus, prone to become overconfident in their abilities.

It is a common feature of human existence that we constantly learn about our own abilities by observing the consequences of our actions. For most people there is an attribution bias to this learning: we tend to overestimate the degree to which we are responsible for our own successes (Wolosin, Sherman, and Till, 1973; Miller and Ross, 1975; Langer and Roth, 1975). As Hastorf, Schneider, and Polifka (1970) write, "we are prone to attribute success to our own dispositions and failure to external forces." People recall their successes more easily than their failures and "even misremember their own predictions so as to exaggerate in hindsight what they knew in foresight" (Fischhof, 1982).

In this paper, we develop a multi-period market model describing both the process by which traders learn about their ability and how a bias in this learning can create overconfident traders. Traders in our model initially do not know their ability. They learn about their ability through experience. Traders who successfully forecast next period dividends improperly update their beliefs; they weigh too heavily the possibility that their success was due to superior ability. In so doing they become overconfident.[1]

In our model, a trader's level of overconfidence changes dynamically with his successes and failures. A trader is not overconfident when he begins to trade. Ex ante, his expected overconfidence increases over his first several trading periods and then declines.[2] Thus the greatest expected overconfidence in a trader's lifespan comes early in his career. After this he tends to develop a progressively more realistic assessment of his abilities as he ages.

One criticism of models of non-rational behavior is that non-rational traders will underperform rational traders and eventually be driven to the margins of markets if not out of them altogether.[3] This is, however, not always the case. De Long, Schleifer, Summers, and Waldmann (1990) present a model where non-rational traders in an overlapping generations model earn higher expected profits than rational traders by bearing a disproporitionate amount of the risk that they themselves create.

[1] We do not explicitly model traders' tendency to attribute failure to outside forces since, as Fiske and Taylor (1991) write, "self-enhancing attributions for success are more common than self-protective attributions for failure."

[2] More precisely, this will happen for insiders with learning biases that are not too large, where "not too large" is precisely defined in section 4.

[3] Early proponents of this view include Alchian (1950), and Friedman (1953). More recently, Blume and Easley (1982, 1992) have reinforced these ideas analytically.

Rational traders are unwilling to take long-term arbitrage positions to eliminate these higher profits because of the risk that they may die before the arbitrage pays off.

In our model, the most overconfident and non-rational traders are not the poorest traders. In fact, for any given level of learning bias and trading experience, it is successful traders, though not necessarily the most successful traders, who are the most overconfident. This is because our traders can only become overconfident when they are successful, and so overconfident traders tend to be wealthy. In sum, overconfidence does not make traders wealthy, but the process of becoming wealthy *can* make traders overconfident.

An interesting feature of our model is that success affects traders' conditional future expected profits in two ways. First, success is indicative of higher ability and, therefore, greater expected future profits. Second, success can increase overconfidence and thereby lower expected future profits. It is tempting to conclude that more successful traders generally have greater expected future profits due to their greater ability. However, the detrimental effect of the more successful traders' greater overconfidence on their future expected profits may more than offsets their greater probable ability.

Most models of financial market microstructure assume that all trader characteristics are common knowledge; in particular, traders' risk aversion, their wealth, and the distribution of their information are known by all market participants. Exceptions include Blume, Easley and O'Hara (1994), Gervais (1996), and Subramanyam (1996). In these papers, the precision of the traders' information is random. Each traders' precision is known to himself but is uncertain to other market participants who must infer it from his actions. Our model builds on these works by extending this uncertainty to the trader himself. He initially does not know the precision of his own information, and must infer it by observing his signals and subsequent outcomes.

A large literature demonstrates that people are usually overconfident and that, in particular, they are overconfident about the precision of their knowledge.[4] Benos (1996), Kyle and Wang (1996), Odean (1996), and Wang (1997) examine models with statically overconfident traders. In these models greater overconfidence leads to greater expected trading volume and greater price volatility.[5] In our model, a greater learning bias causes greater expected overconfidence, which leads to greater expected trading volume and greater price volatility.

Daniel, Hirshleifer and Subrahmanyam (1997) look at trader overconfidence in a dynamic model. Our paper differs from theirs in that we concentrate our agents' updating on the informed trader's

[4]See for example Alpert and Raiffa (1982), and Lichtenstien, Fischhoff and Phillips (1982). Odean (1996) provides an overview of this literature.

[5]In one exception, Odean shows that an overconfident, risk-averse market-maker may reduce market volatility.

ability, and not on the joint distribution of trader ability and the risky security's final payoff.[6] Our approach has the advantage of being analytically tractable, and thus does not require the use of numerical results. The scope of our analysis also differs from that of Daniel, Hirshleifer and Subrahmanyam: we are interested in the effects of the dynamic learning process on trading volume, price volatility and trading profits, whereas their main objective is to show that overconfidence, whether dynamic or not, implies different price correlation patterns in the short run and the long run.

Overconfidence is determined, in our model, endogenously and changes dynamically over a trader's life. This enables us to make predictions about which traders will be most overconfident (young, successful traders) and how overconfidence will change during a trader's life (it will, on average, rise and then fall). Our model also has implications for changing market conditions. For example, most equity market participants have long positions and benefit from upward price movements. We would therefore expect aggregate oveconfidence to be higher after market gains and lower after market losses. Since, as we show, greater overconfidence leads to greater trading volume, this suggests that trading volume will be greater after market gains and lower after market losses. Indeed, Statman and Thorley (1997) find that this is the case. We would expect aggregate overconfidence to be particularly high in a market with many young traders who have experienced only a long bull run. Thus, our model predicts that trading volume and volatility should be higher in the late stages of a bull market than in the late stages of a bear market.

The rest of this paper is organized as follows. In section 2, we introduce a one-security multi-period economy with one insider, one liquidity trader and one market maker. We show in section 3 that there is a unique linear equilibrium in our economy. This linear equilibrium is used in section 4 to analyze the effects of the insider's learning bias on his overconfidence and profits, as well as on the market's trading volume and volatility. Section 5 concludes and discusses some potential topics for future research on trader overconfidence. All the proofs are contained in the appendices.

2 The Economy

We study a multi-period economy in which only one risky asset is traded among three market participants: an informed trader, a liquidity trader, and a market maker. At the end of period t, the risky asset pays off a dividend $\hat{v}_t$, unknown to all the market participants at the beginning of the period.[7]

At the beginning of each period t, the informed trader (also called *insider*) observes a signal $\hat{\theta}_t$

[6] This is because the risky security's dividend is announced at the end of every period in our model.

[7] Throughout the whole paper, we use a 'hat' over a variable to denote the fact that it is a random variable.

which is correlated with $\hat{v}_t$. In particular, the signal $\hat{\theta}_t$ is given by

$$\hat{\theta}_t = \hat{\delta}_t \hat{v}_t + (1 - \hat{\delta}_t)\hat{\varepsilon}_t, \tag{1}$$

where $\hat{\varepsilon}_t$ has the same distribution as $\hat{v}_t$, but is independent from it. The variable $\hat{\delta}_t$ takes the values 0 or 1. Since $\hat{\varepsilon}_t$ is independent from $\hat{v}_t$, the insider's information will only be useful when $\hat{\delta}_t$ is equal to one. We assume that this will happen with probability $\hat{a}$, i.e.

$$\hat{\delta}_t \mid \hat{a} = \begin{cases} 1, & \text{prob. } \hat{a} \\ 0, & \text{prob. } 1 - \hat{a}. \end{cases} \tag{2}$$

This last equation shows that, the higher $\hat{a}$ is, the more likely that $\hat{\delta}_t$ will be equal to one. For this reason, we call $\hat{a}$ the insider's *ability*.[8] We assume that nobody (including the insider himself) knows precisely the insider's ability $\hat{a}$ at the outset (i.e. at time zero). Instead, we assume that, a priori, the insider's ability is drawn from the following distribution:

$$\hat{a} = \begin{cases} H, & \text{prob. } \phi_0 \\ L, & \text{prob. } 1 - \phi_0, \end{cases} \tag{3}$$

where $0 < L < H < 1$, and $0 < \phi_0 < 1$. Of course, since the security dividend $\hat{v}_t$ is announced at the end of every period t, the insider will know at the end of every period whether his information for that period was real ($\hat{\delta}_t = 1$) or was just pure noise ($\hat{\delta}_t = 0$).[9] For tractability reasons, we also assume that the market maker observes $\hat{\theta}_t$ at the end of period t, so that his information at the end of every period is the same as the insider's.[10] This information will be useful to both the insider and the market maker in assessing the insider's ability $\hat{a}$.

In making this last assumption, we are essentially saying that the insider's informational advantage over the market maker is one of market timing. Indeed, the insider's information set at the beginning of every period is always exactly one period ahead of the market maker's. Since our goal is not to explain the differences between these two information sets, we reduce the information gap between the insider and the market maker to only (and exactly) one period. Our analysis is then simplified in that both the insider and the market maker perform the same one-period updating at the end of every period, except that the insider's updating will be biased. Preventing the market maker from observing $\hat{\theta}_t$ at the end of period t would simply result in a more complex (non Markov) updating process for the market maker, but would not affect the insider's updating process, which is what we are utimately interested in. It would, however, increase his expected profits since the market maker's informational disadvantage would then be greater.

[8] Equivalently, we could call $\hat{a}$ the insider's information precision.

[9] This will be the case since $\hat{\varepsilon}_t = \hat{v}_t$ happens with zero probability.

[10] As will become clear below, we could have equivalently assumed that every trader's order and identity (insider vs liquidity trader) are revealed at the end of every period.

As mentioned above, our model seeks to describe the behavior of an informed trader with a learning bias. In particular we want to model the phenomenon that traders usually think "too much of their ability" when they have been successful at predicting the market in the past. In statistical terms, this will mean that traders update their ability beliefs too much when they are right. Before we formally include this behavior into our model, let us describe how a "rational" insider would react to the information he gathers from past trading rounds.

Let $\hat{s}_t$ denote the number of times that the insider's information was real in the first t periods, that is

$$\hat{s}_t = \sum_{u=1}^{t} \hat{\delta}_u. \tag{4}$$

It can be shown, using Bayes' rule, that, at the end of t periods, a rational insider's updated beliefs about his own ability will be given by

$$\phi_t(s) \equiv \Pr\{\hat{a} = H \mid \hat{s}_t = s\} = \frac{H^s(1-H)^{t-s}\phi_0}{H^s(1-H)^{t-s}\phi_0 + L^s(1-L)^{t-s}(1-\phi_0)}. \tag{5}$$

We define his updated expected ability by

$$\begin{aligned} \mu_t(s) &\equiv \mathrm{E}[\hat{a} \mid \hat{s}_t = s] \\ &= H\phi_t(s) + L[1-\phi_t(s)]. \end{aligned} \tag{6}$$

In fact, since we do not assume any kind of irrational behavior on the part of the market maker, this will be the market maker's updated belief at the end of period t also.

Now, let us assume that the insider adjusts his beliefs too much every time he correctly predicts the security's dividend (i.e. every time that $\hat{\delta}_t = 1$). For example, at the end of the first period, if the insider finds that $\hat{\theta}_1 = \hat{v}_1$ (i.e. $\hat{\delta}_1 = 1$), the insider will adjust his beliefs to

$$\bar{\phi}_1(1) \equiv \Pr_b\{\hat{a} = H \mid \hat{s}_1 = 1\} = \frac{\gamma H\phi_0}{\gamma H\phi_0 + L(1-\phi_0)}, \tag{7}$$

where $\gamma \geq 1$ is the insider's learning bias parameter ($\gamma = 1$ representing a rational insider), and the subscript to 'Pr' denotes the fact that the probability is calculated by a biased insider. As can be seen from (7), $\bar{\phi}_1(1)$ will be higher the larger γ is, and $\bar{\phi}_1(1) \to 1$ as $\gamma \to \infty$; in other words, the learning bias dictates by how much the insider adjusts his beliefs towards being a high ability insider. It is easily shown that, in this case,

$$\bar{\phi}_t(s) \equiv \Pr_b\{\hat{a} = H \mid \hat{s}_t = s\} = \frac{(\gamma H)^s(1-H)^{t-s}\phi_0}{(\gamma H)^s(1-H)^{t-s}\phi_0 + L^s(1-L)^{t-s}(1-\phi_0)}, \tag{8}$$

and the (biased) insider's updated expected ability is given by

$$\begin{aligned}\bar{\mu}_t(s) &\equiv \mathrm{E}_b[\hat{a} \mid \hat{s}_t = s] \\ &= H\bar{\phi}_t(s) + L[1 - \bar{\phi}_t(s)]. \end{aligned} \tag{9}$$

At the beginning of every period t, the insider observes his signal $\hat{\theta}_t$; he then chooses his demand for the risky security in order to maximize his expected period t profits,[11] $\hat{\pi}_t$, conditional on both his signal and his ability beliefs $\bar{\mu}_{t-1}(\hat{s}_{t-1})$ (which only depend on $\hat{s}_{t-1}$). We denote this demand by

$$\hat{x}_t = X_t(\hat{\theta}_t, \hat{s}_{t-1}). \tag{10}$$

The other trader in the economy is a trader who trades for liquidity purposes in every period. This *liquidity trader*'s demand in period t is given exogenously by the random variable $\hat{z}_t$. Both orders, $\hat{x}_t$ and $\hat{z}_t$, are sent to a market maker who fills the orders. As in Kyle (1985), we assume that the market maker is risk-neutral and competitive, and will therefore set prices so as to make zero expected profits. So, if we denote the total order flow coming to the market maker in period t by

$$\hat{\omega}_t = \hat{x}_t + \hat{z}_t, \tag{11}$$

the market maker will set the security's price equal to

$$\hat{p}_t = P_t(\hat{\omega}_t, \hat{s}_{t-1}) \equiv \mathrm{E}[\hat{v}_t \mid \hat{\omega}_t, \hat{s}_{t-1}] \tag{12}$$

in period t. An equilibrium to our model is defined as a sequence of pairs of functions (X_t, P_t), $t = 1, 2, \ldots$, such that the insider's demand in period t, $X_t(\hat{\theta}_t, \hat{s}_{t-1})$ maximizes his expected profits (according to his own beliefs) for that period given that he faces a price curve P_t, while the market maker is expecting zero profit in that period.

3 A Linear Equilibrium

In this section, we show that, when $\hat{v}_t$, $\hat{\varepsilon}_t$, and $\hat{z}_t$ are jointly and independently normal, there is a linear equilibrium to our economy. We use that linear equilibrium in Section 4 to illustrate the properties of the model. More precisely, for this and the next section, we assume that

$$\begin{bmatrix} \hat{v}_t \\ \hat{\varepsilon}_t \\ \hat{z}_t \end{bmatrix} \sim \mathrm{N}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma & 0 & 0 \\ 0 & \Sigma & 0 \\ 0 & 0 & \Omega \end{bmatrix} \right), \; t = 1, 2, \ldots, \tag{13}$$

[11] As we mentioned above, both the risky dividend and the insider's signal are announced at the end of every period, so that the market maker is always only one period behind the insider in terms of information. This implies that the insider never finds it optimal to suboptimally choose his one period demand in order to maximize longer-term profits.

and that each such vector is independent of each other. Note that it is crucial that $\text{Var}(\hat{v}_t) = \text{Var}(\hat{\varepsilon}_t)$, since we do not want the size of $\hat{\theta}_t$ to reveal anything about the likelihood that $\hat{\delta}_t = 1$ until $\hat{v}_t$ is announced.

Let us conjecture that, in equilibrium, the function $X_t(\theta, s)$ is linear in θ, and that the function $P_t(\omega, s)$ is linear in ω:

$$X_t(\theta, s) = \beta_t(s)\,\theta, \tag{14a}$$

$$P_t(\omega, s) = \lambda_t(s)\,\omega. \tag{14b}$$

Our objective is to find $\beta_t(s)$ and $\lambda_t(s)$ which are consistent with this conjecture. This means that the insider's expected period t profits, when sending a market order of x_t to the market maker, are given by

$$\begin{aligned}
\text{E}_b[\hat{\pi}_t \mid \hat{\theta}_t, \hat{s}_{t-1}] &= \text{E}_b\left\{x_t[\hat{v}_t - P_t(\hat{\omega}_t, \hat{s}_{t-1})] \mid \hat{\theta}_t, \hat{s}_{t-1}\right\} \\
&= \text{E}_b\left\{x_t[\hat{v}_t - \lambda_t(\hat{s}_{t-1})\hat{\omega}_t] \mid \hat{\theta}_t, \hat{s}_{t-1}\right\} \\
&= \text{E}_b\left\{x_t[\hat{v}_t - \lambda_t(\hat{s}_{t-1})(x_t + \hat{z}_t)] \mid \hat{\theta}_t, \hat{s}_{t-1}\right\} \\
&= x_t\left[\text{E}_b(\hat{v}_t \mid \hat{\theta}_t, \hat{s}_{t-1}) - \lambda_t(\hat{s}_{t-1})x_t\right],
\end{aligned} \tag{15}$$

where the last equality follows from the fact that $\hat{z}_t$ is independent from both $\hat{\theta}_t$ and $\hat{s}_{t-1}$. Differentiating this last expression with respect to x_t and setting the result equal to zero yields

$$\begin{aligned}
\hat{x}_t &= \underset{x_t}{\text{argmax}}\ \text{E}_b[\hat{\pi}_t \mid \hat{\theta}_t, \hat{s}_{t-1}] \\
&= \frac{\text{E}_b(\hat{v}_t \mid \hat{\theta}_t, \hat{s}_{t-1})}{2\lambda_t(\hat{s}_{t-1})}.
\end{aligned} \tag{16}$$

Also, a simple use of iterated expectations and the projection theorem for normal variables[12] shows that

$$\begin{aligned}
\text{E}_b(\hat{v}_t \mid \hat{\theta}_t, \hat{s}_{t-1}) &= \text{E}_b\left[\text{E}_b(\hat{v}_t | \hat{\theta}_t, \hat{s}_{t-1}, \hat{\delta}_t) \mid \hat{\theta}_t, \hat{s}_{t-1}\right] \\
&= \text{E}_b\left[\hat{\delta}_t\hat{\theta}_t + (1 - \hat{\delta}_t)\cdot 0 \mid \hat{\theta}_t, \hat{s}_{t-1}\right] \\
&= \text{E}_b\left[\hat{\delta}_t \mid \hat{s}_{t-1}\right]\hat{\theta}_t \\
&= \text{E}_b\left[\hat{a} \mid \hat{s}_{t-1}\right]\hat{\theta}_t \\
&= \bar{\mu}_{t-1}(\hat{s}_{t-1})\hat{\theta}_t,
\end{aligned} \tag{17}$$

[12] The projection theorem for normal variables is as follows: suppose that

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} \sim \text{N}\left(\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{bmatrix}\right).$$

Then $\text{E}[\hat{x}|\hat{y}] = \mu_x + \frac{\Sigma_{12}}{\Sigma_{22}}(\hat{y} - \mu_y)$.

where the next to last equality results from the fact that $\hat{\theta}_t$ does not contain any information about $\hat{\delta}_t$ (or, equivalently, about $\hat{a}$).[13] So, using (16) and (17), we can indeed write $\hat{x}_t = \beta_t(\hat{s}_{t-1})\hat{\theta}_t$ with

$$\beta_t(s) = \frac{\bar{\mu}_{t-1}(s)}{2\lambda_t(s)}. \tag{18}$$

Next, we solve for the market maker's price schedule. As discussed in section 2, the market maker's price is a function of the information he gathers from the order flow that is sent to him. More precisely,

$$\begin{aligned}
\hat{p}_t &= \mathrm{E}[\hat{v}_t \mid \hat{\omega}_t, \hat{s}_{t-1}] \\
&= \mathrm{E}\left[\mathrm{E}(\hat{v}_t \mid \hat{\omega}_t, \hat{s}_{t-1}, \hat{\delta}_t) \mid \hat{\omega}_t, \hat{s}_{t-1}\right] \\
&= \mathrm{E}\left\{\hat{\delta}_t \mathrm{E}\left[\hat{v}_t \mid \hat{\omega}_t = \beta_t(\hat{s}_{t-1})\hat{\theta}_t + \hat{z}_t, \hat{s}_{t-1}\right] + (1-\hat{\delta}_t)\cdot 0 \mid \hat{\omega}_t, \hat{s}_t\right\}.
\end{aligned} \tag{19}$$

Use of the projection theorem for normal variables shows that

$$\mathrm{E}\left[\hat{v}_t \mid \hat{\omega}_t = \beta_t(\hat{s}_{t-1})\hat{\theta}_t + \hat{z}_t, \hat{s}_{t-1}\right] = \frac{\beta_t(\hat{s}_{t-1})\Sigma}{\beta_t^2(\hat{s}_{t-1})\Sigma + \Omega}\hat{\omega}_t,$$

so that we can rewrite (19) as

$$\begin{aligned}
\hat{p}_t &= \mathrm{E}\left[\hat{\delta}_t \frac{\beta_t(\hat{s}_{t-1})\Sigma}{\beta_t^2(\hat{s}_{t-1})\Sigma + \Omega}\hat{\omega}_t \;\Big|\; \hat{\omega}_t, \hat{s}_{t-1}\right] \\
&= \mathrm{E}[\hat{\delta}_t \mid \hat{s}_{t-1}]\frac{\beta_t(\hat{s}_{t-1})\Sigma}{\beta_t^2(\hat{s}_{t-1})\Sigma + \Omega}\hat{\omega}_t \\
&= \mathrm{E}[\hat{a} \mid \hat{s}_{t-1}]\frac{\beta_t(\hat{s}_{t-1})\Sigma}{\beta_t^2(\hat{s}_{t-1})\Sigma + \Omega}\hat{\omega}_t \\
&= \frac{\mu_{t-1}(\hat{s}_{t-1})\beta_t(\hat{s}_{t-1})\Sigma}{\beta_t^2(\hat{s}_{t-1})\Sigma + \Omega}\hat{\omega}_t.
\end{aligned} \tag{20}$$

From this last expression, it is clear that, as conjectured, we have $\hat{p}_t = P_t(\hat{\omega}_t, \hat{s}_{t-1}) = \lambda_t(\hat{s}_{t-1})\hat{\omega}_t$, where

$$\lambda_t(s) = \frac{\mu_{t-1}(s)\beta_t(s)\Sigma}{\beta_t^2(s)\Sigma + \Omega}. \tag{21}$$

Solving for $\beta_t(s)$ and $\lambda_t(s)$ in (18) and (21) should yield the conjectured linear equilibrium. However, as we show next, this linear equilibrium will only exist when $\bar{\mu}_{t-1}(s) \leq 2\mu_{t-1}(s)$.

To see this, recall from (15)-(17) that the insider chooses x_t so as to maximize his expected profits in period t, which can be written as

$$\mathrm{E}_b[\hat{\pi}_t \mid \hat{\theta}_t, \hat{s}_{t-1}] = x_t\left[\bar{\mu}_{t-1}(\hat{s}_{t-1})\hat{\theta}_t - \lambda_t(\hat{s}_{t-1})x_t\right]. \tag{22}$$

[13] Again, this is the case since both $\hat{v}_t$ and $\hat{\varepsilon}_t$ have the same distribution.

We let $\hat{x}_t$ denote the insider's optimal demand, keeping in mind that a rational insider would have chosen a different demand.[14] From (22), we have

$$\hat{x}_t = \frac{\bar{\mu}_{t-1}(\hat{s}_{t-1})\hat{\theta}_t}{2\lambda_t(\hat{s}_{t-1})}. \tag{23}$$

In this last expression, $\lambda_t(\hat{s}_{t-1})$ is the slope of the market maker's linear price schedule. We expect that slope to be positive in equilibrium.

Now, the insider's real (unbiased) profits, as they would be calculated by the market maker (if he knew $\hat{\theta}_t$), are then given by

$$\mathrm{E}[\pi_t \mid \hat{\theta}_t, \hat{s}_{t-1}] = \hat{x}_t \left[\bar{\mu}_{t-1}(\hat{s}_{t-1})\hat{\theta}_t - \lambda_t(\hat{s}_{t-1})\hat{x}_t\right]. \tag{24}$$

As long as this number is positive, the market maker loses money to the insider on average. To offset these expected losses, the market maker will indeed quote an upward sloping price schedule ($\lambda_t(\hat{s}_{t-1}) > 0$) that enables him to make some profits off the liquidity trader. Since the market maker is competitive, these expected profits will exactly match his expected losses in equilibrium.

However, when (24) is negative, the competitive market maker cannot keep quoting an upward sloping price schedule, since he would then be making profits off both the insider and the liquidity trader. Also, quoting a decreasing price schedule ($\lambda_t(\hat{s}_{t-1}) \leq 0$) does not solve the problem since, for any $\lambda_t(\hat{s}_{t-1}) \leq 0$, the insider then wants to trade an infinite number of shares, which would push the market maker to decrease the slope of his price schedule even more. So, in this situation, no equilibrium exists.

When will such a situation arise? The answer is when (24) is negative which, after replacing x_t by (23), reduces to $\bar{\mu}_{t-1}(s) \geq 2\mu_{t-1}(s)$. A necessary and sufficient condition for this situation to be prevented for all integers s, t satisfying $0 \leq s < t$ is that $H \leq 2L$. This is why we will assume that this condition holds in the rest of the paper. So, in essence, this condition, insures that the insider will make positive profits even though these profits may not be optimized. We now have the following result.

Lemma 3.1 *As long as $H \leq 2L$, there is a unique linear equilibrium to the economy described in Section 2. In this equilibrium, the insider's demand and the market maker's price schedule are given by (14a) and (14b) with*

$$\beta_t(s) = \sqrt{\frac{\Omega}{\Sigma}\frac{\bar{\mu}_{t-1}(s)}{2\mu_{t-1}(s) - \bar{\mu}_{t-1}(s)}}, \quad \textit{and} \tag{25a}$$

$$\lambda_t(s) = \frac{1}{2}\sqrt{\frac{\Sigma}{\Omega}\bar{\mu}_{t-1}(s)\left[2\mu_{t-1}(s) - \bar{\mu}_{t-1}(s)\right]}. \tag{25b}$$

[14] We will discuss the suboptimality of $\hat{x}_t$ in section 4.6.

Proof : See Appendix A.

In what follows, we use this equilibrium to study the effect of the insider's learning bias on the economy.

4 Properties of the Model

In this section, we analyze the effects of the insider's learning bias on the properties of the economy in equilibrium. In particular, we are interested in how the size of the learning bias, γ, will affect volume, price volatility, and insider profits. Before we do this, however, let us first look at the behavioral patterns of the insider in equilibrium.

4.1 Convergence

If this financial market game is played to infinity, we would expect both the insider and the market maker to eventually learn the exact ability $\hat{a}$ of the insider. This in fact would be true for a rational insider ($\gamma = 1$). However, since our insider learns his ability with a personal bias, this result is not immediate; in fact, as we shall see, this result is not true for a highly biased insider.

When $\hat{a} = H$, we expect the insider to correctly guess the one-period dividend a fraction H of the time. So, as we play the game more and more often (as t tends to ∞), we expect his updated posteriors

$$\begin{aligned}\bar{\phi}_t(s) &= \frac{(\gamma H)^s(1-H)^{t-s}\phi_0}{(\gamma H)^s(1-H)^{t-s}\phi_0 + L^s(1-L)^{t-s}(1-\phi_0)} \\ &= \frac{1}{1+\left(\frac{L}{\gamma H}\right)^s\left(\frac{1-L}{1-H}\right)^{t-s}\frac{1-\phi_0}{\phi_0}}\end{aligned}$$

to behave like

$$\frac{1}{1+\left(\frac{L}{\gamma H}\right)^{Ht}\left(\frac{1-L}{1-H}\right)^{t-Ht}\frac{1-\phi_0}{\phi_0}} = \frac{1}{1+\left[\left(\frac{L}{\gamma H}\right)^{H}\left(\frac{1-L}{1-H}\right)^{1-H}\right]^t\frac{1-\phi_0}{\phi_0}}.$$

This last quantity will converge to 1 as desired[15] if

$$\left[\left(\frac{L}{\gamma H}\right)^{H}\left(\frac{1-L}{1-H}\right)^{1-H}\right]^t \to 0 \quad \text{as } t \to \infty,$$

or equivalently if

$$\left(\frac{L}{\gamma H}\right)^{H}\left(\frac{1-L}{1-H}\right)^{1-H} < 1.$$

[15] When $\hat{a} = H$, we want the conditional probability that the insider has high ability to converge to 1.

The following lemma shows that this is indeed the case.

Lemma 4.1 *When $\hat{a} = H$, the updated posteriors of the insider $\bar{\phi}_t(\hat{s}_t)$ will converge to 1 almost surely as $t \longrightarrow \infty$.*

Proof: See Appendix B.

So both the insider and the market maker will eventually learn the insider's ability precisely when it is high (when $\hat{a} = H$). Let us now turn to the case where $\hat{a} = L$. In this case, we expect the insider to correctly guess the one-period dividend a fraction L of the time. So, as we play the game more and more often (as t tends to ∞), we expect his updated posteriors $\bar{\phi}_t(s)$ to behave like

$$\frac{1}{1 + \left(\frac{L}{\gamma H}\right)^{Lt} \left(\frac{1-L}{1-H}\right)^{t-Lt} \frac{1-\phi_0}{\phi_0}} = \frac{1}{1 + \left[\left(\frac{L}{\gamma H}\right)^{L} \left(\frac{1-L}{1-H}\right)^{1-L}\right]^t \frac{1-\phi_0}{\phi_0}}.$$

As the following lemma shows, this quantity only converges to zero when γ is close enough to 1. This means that the market maker will always learn the insider's ability when it is low (when $\hat{s} = L$),[16] but the insider will only do so if his learning bias is not too large.

Lemma 4.2 *When $\hat{a} = L$, the updated posteriors of the insider $\bar{\phi}_t(\hat{s}_t)$ will converge as follows:*

$$\bar{\phi}_t(\hat{s}_t) \xrightarrow{a.s.} \begin{cases} 0, & \text{if } \gamma < \gamma^* \\ \phi_0, & \text{if } \gamma = \gamma^* \\ 1, & \text{if } \gamma > \gamma^*, \end{cases}$$

where $\gamma^* = \frac{L}{H}\left(\frac{1-L}{1-H}\right)^{(1-L)/L}$.

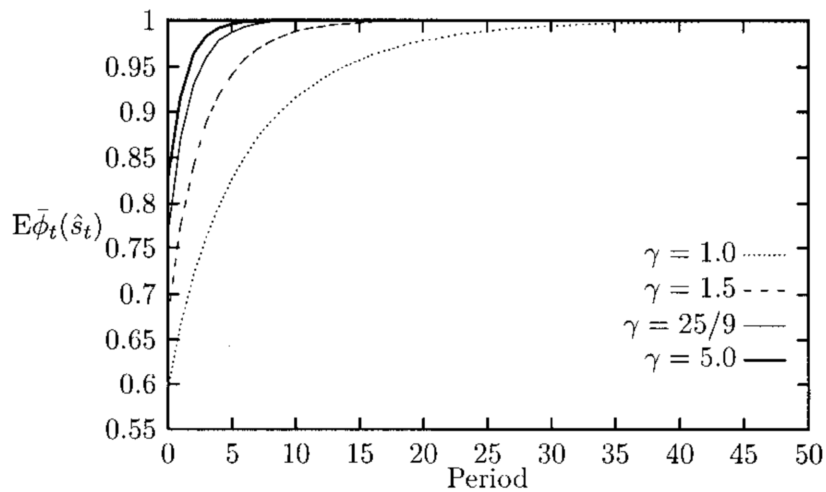
Proof: See Appendix B.

A disturbing implication of this lemma is that, if his learning bias is large enough, an insider who has been successful about $L \times 1{,}000{,}000$ times in the first 1,000,000 periods could still believe that he is a high ability insider. In other words, when an insider is highly biased, a few successful periods are enough to sustain his beliefs that he is a high ability insider.

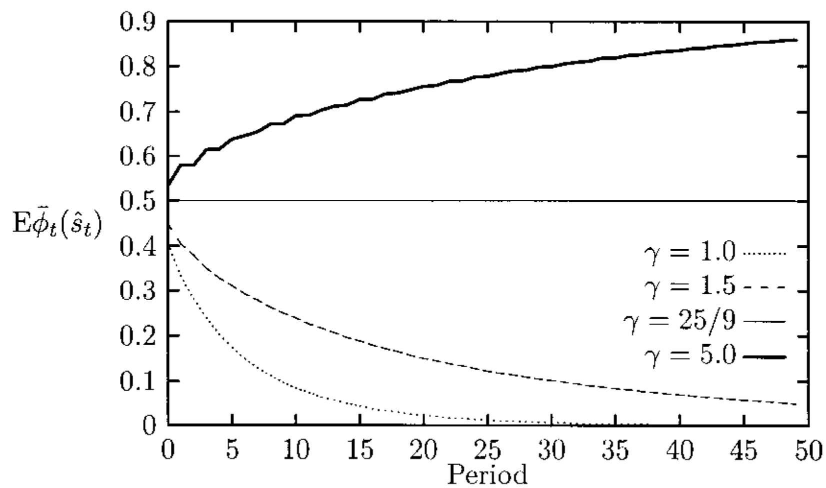
To illustrate this, we show in Figure 1 how we expect an insider to adjust his beliefs about his own ability ($\bar{\phi}_t$) when his actual ability is high (Panel 1(a)) and when it is low (Panel 1(b)). As seen in that figure, the biased insider's beliefs are always on average larger than an unbiased insider's beliefs. Since unbiased insiders always eventually learn their ability, it is therefore not surprising to find that high ability insiders also always learn their own ability:[17] they naturally tend to update towards that high ability. However, as shown in Panel 1(b), a biased insider may not always give in to his observations: more precisely, if $\gamma > \gamma^*$, he will never find out if he is a low ability insider.

[16] This is due to the fact that we assumed that the market maker's learning is unbiased.

[17] In fact, they will do so faster the more biased they are.



(a) Convergence when $\hat{a} = H$.



(b) Convergence when $\hat{a} = L$.

Figure 1: Convergence patterns of the insider's beliefs about his own ability when (a) $\hat{a} = H$; (b) $\hat{a} = L$. Both figures were obtained with $H = 0.9$, $L = 0.5$, $\phi_0 = 0.5$, and $\Sigma = \Omega = 1$. Each line was obtained with a different γ, shown in the legends. Note that, with these parameter values, $\gamma^* = 25/9 \approx 2.78$.

4.2 Patterns of Overconfidence

As we show in section 4.1, the insider will eventually learn his own ability, provided his learning is not too biased (i.e. provided that γ is not too large). This means that, when the insider's ability is low, the insider eventually comes to his senses, and recognizes the fact that he is a low ability insider. In this section, we are interested in the *overconfidence level* of the insider throughout his life. Indeed, even though the insider may eventually learn his own ability, his irrational behavior will make his learning pattern different from that of a rational insider.

An insider is very overconfident in a particular period if his updated expected ability at that time ($\bar{\mu}_t(\hat{s}_t)$) is large compared to the updated expected ability that a rational insider would have with the same past history of sucesses and failures ($\mu_t(\hat{s}_t)$). To measure the insider's overconfidence in period t, we therefore define the random variable

$$\hat{\kappa}_t \equiv K_t(\hat{s}_t) \equiv \frac{\bar{\mu}_t(\hat{s}_t)}{\mu_t(\hat{s}_t)}. \tag{26}$$

Of course, when the insider is rational ($\gamma = 1$), the numerator is exactly equal to the denominator of this expression, and $\hat{\kappa}_t = 1$ for all $t = 1, 2, \ldots$. On the other hand, when the insider's learning is biased ($\gamma > 1$), we have $\bar{\mu}_t(\hat{s}_t) \geq \mu_t(\hat{s}_t)$, and $\hat{\kappa}_t \geq 1$ for all t. As the next proposition shows, the insider's overconfidence in period t, $\hat{\kappa}_t$, is greater when the insider's learning bias is large. In other words, the insider's overconfidence is directly attributable to his learning bias.

Proposition 4.1 *The function $K_t(s)$ defined in (26) is increasing in γ.*

Proof : See Appendix B.

Our measure of overconfidence at any point in time is therefore increasing in the insider's learning bias, but is it also increasing in the number of his past successful predictions? Since the insider's overconfidence results from his learning bias when he is successful, it is tempting to conclude that the more successful an insider is, the more overconfident he will be. As we next show, this intuition is wrong.

First, since the insider updates his beliefs incorrectly only after successful predictions, it is always true that $\bar{\mu}_t(0) = \mu_t(0)$, and therefore $K_t(0) = 1$. However, as soon as the insider successfully predicts one risky dividend, his learning bias makes him overconfident,[18] and $\bar{\mu}_t(1) > \mu_t(1)$, so that $K_t(1) > 1$. So, it is always true that the insider's first successful prediction makes him overconfident.[19] However, it is not always the case that an additional successful prediction always makes the insider more overconfident.

To see this, suppose that we are at the end of the second period. The insider will then have been successful 0, 1 or 2 times. We already know that $K_2(1) > K_2(0) = 1$ for any value of the

[18]In our model traders are not overconfident when they begin to trade. It is through making forecasts and trading that they become overconfident. This leads market participants to be, on average, overconfident. In real markets selection bias may cause even beginning traders to be overconfident. Indeed, since not everyone trades, it is likely that people who rate their own trading abilities most highly will seek jobs as traders or will trade actively on their own account. Those with actual high ability and those with high overconfidence will rate their own ability highest. Thus, even at the entry level, we would expect to find overconfident traders.

[19]And, as section 4.1 shows, he will remain so for the rest of his life.

insider's learning bias parameter γ. Now, suppose that γ is large. This means that if the insider is successful in the first period, he will immediately (and perhaps falsely) jump to the conclusion that he is a high ability insider, i.e. $\bar{\mu}_1(1)$ is close to H. Since this one successful period has already convinced the insider that his ability is high, the second period results will not have much of an effect on his beliefs, whether he is successful or not in that period, i.e. $\bar{\mu}_2(2)$ is close to $\bar{\mu}_2(1)$. On the other hand, if the insider had been rational ($\gamma = 1$), he would have adjusted gradually his expected ability beliefs towards H (L) after a successful (an unsuccessful) period, so that $\mu_2(2)$ will be somewhat larger than $\mu_2(1)$. Therefore, since $\bar{\mu}_2(2) \approx \bar{\mu}_2(1)$ and $\mu_2(2) > \mu_2(1)$, we have

$$K_2(2) \equiv \frac{\bar{\mu}_2(2)}{\mu_2(2)} < \frac{\bar{\mu}_2(1)}{\mu_2(1)} \equiv K_2(1),$$

and we see that $K_2(s)$ decreases when s goes from 1 to 2.

In short, the biased insider adjusts his beliefs non-rationally with every successful prediction, making him overconfident. However, when the insider's past performance is sufficiently good (large number of successful predictions), it is the case that even an unbiased insider would reach the conclusion that he is a high ability insider. In other words, the significance of the insider's past performance overweighs the significance of his overconfidence. The following proposition describes this phenomenon in more details.

Proposition 4.2 *The function $K_t(s)$ defined in (26) is either*

- *increasing over all $s \in \{0, \dots, t\}$, or*
- *increasing over $s \in \{0, \dots, s_t^*\}$ and decreasing over $s \in \{s_t^*, \dots, t\}$, for some $s_t^* \in \{1, \dots, t\}$.*

The latter case will occur when γ, t/s, and/or t are large.

Proof: See Appendix B.

Intuitively, this result says that a trader who has been very successful in only a few rounds of trading or one who has been moderately successful in several rounds of trading will have a greatly inflated opinion of his ability. But a trader who has been very successful over many rounds of trading probably does have high ability. And while he may overestimate his expected ability he does not do so by as much as do moderately successful traders.

In this model, traders are rational in all respects except that they have a common learning bias: they tend to attribute their successes disproportionately to their own ability. This leads successful traders to become overconfident. Other learning biases can also lead to overconfidence. For example, it is well known that, when updating beliefs from sequential information, people tend to weigh recent information too heavily and older information too little (Anderson, 1959, 1981; Einhorn and Hogarth, 1992).[20] We do not introduce this recency effect into our model because doing so would negate the Markov property of the insider's (and the market maker's) updating process. It is clear though that, if traders weigh recent outcomes more heavily than older ones, recently successful traders will tend to become overconfident.

The last two propositions describe how the insider's overconfidence in a particular period depends on his learning bias and on his previous performance. Let us now turn to how his overconfidence is expected to behave over time. To do this, we calculate the ex ante expected period t overconfidence level of the insider, which we denote by

$$\Upsilon_t \equiv \mathrm{E}\left[\hat{\kappa}_t\right]. \tag{27}$$

It is easy to show that[21]

$$\Upsilon_t = \sum_{s=0}^{t} \binom{t}{s} \left[H^s(1-H)^{t-s}\phi_0 + L^s(1-L)^{t-s}(1-\phi_0)\right] \frac{\bar{\phi}_t(s)}{\phi_t(s)}, \tag{28}$$

where $\phi_t(s)$ and $\bar{\phi}_t(s)$ are as in (5) and (8). In Figure 2, we show the patterns in the level of overconfidence for different values of γ. When γ is relatively small ($\gamma < \gamma^*$), the insider will on average be overconfident at first but, over time, will converge to a rational behavior.

This can be explained as follows. Over a small number of trading periods a trader's success rate may greatly exceed that predicted by his ability. Very successful traders will overestimate the likelihood that success is due to ability rather than luck. But over many trading periods a trader's success rate is likely to be close to that predicted by his ability. Only those traders with extreme learning bias (or with very unlikely success patterns) will fail to recognize their true ability. Indeed, as γ increases, the insider tends to put more and more weight on his past successes, and so takes a little more time to rationally find his ability. However, if γ is too large (more precisely, if $\gamma > \gamma^*$), it is possible that the insider puts so much weight on his past successes in the stock market that he never becomes rational.

Thus our model predicts that more younger traders will be more overconfident than older traders. This is not due to any attribute of youth other than lack of experience. Young traders are more likely to have success records which are unrepresentative of their abilities. For some this will lead to overconfidence. By the law of large numbers older traders are likely to have success

[20] In experimental studies, subjects sometimes also exhibit a 'primacy effect,' weighting the earliest observations of a time series heavily (Anderson, 1981). This happens most often in situations where subjects lose interest in the data (Einhorn and Hogarth, 1992). It is unlikely that traders would lose interest in their own successes and failures, and so we would not expect to find a large primacy effect in their updating.

[21] See Appendix B.

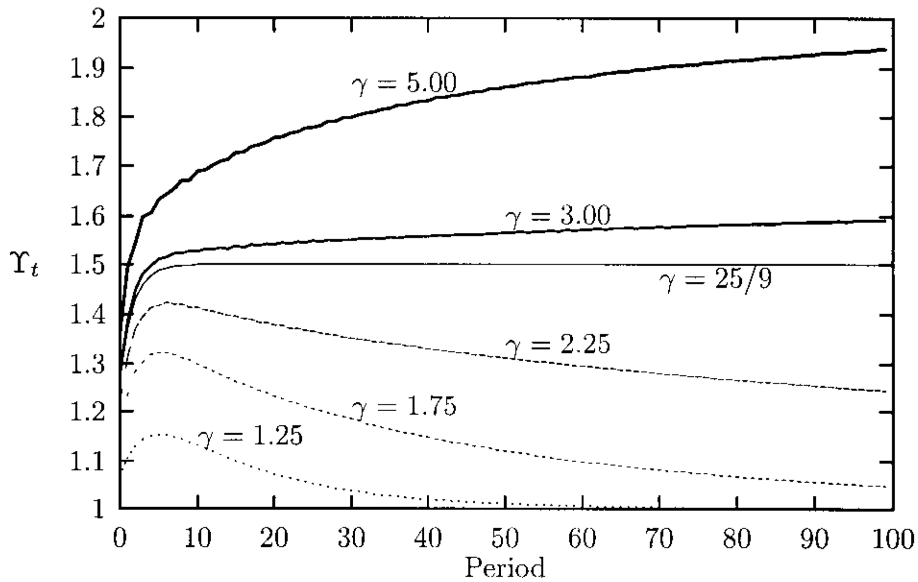


Figure 2: Ex ante expected patterns in the level of overconfidence of the insider over time. The figure was obtained with $H = 0.9$, $L = 0.5$, $\phi_0 = 0.5$, $\Sigma = \Omega = 1$. Each line was obtained with a different value of γ shown on the figure. Note that, with these parameter values, $\gamma^* = 25/9 \approx 2.78$.

records which are more representative of their abilities; they will, on average, have more realistic self assessments. In a sufficiently large group of traders, however, there will be some successful, older, low-ability traders with whom the odds have not yet caught up. These traders are likely to make large mistakes in the future.

4.3 The Updating Process

Before we proceed to analyze trading volume, volatility, and insider profits, let us look at the market maker's updating process about the insider's ability. In particular, we would like to carefully describe the sense in which the insider's number of successful predictions at any point in time is a good indicator of both his past and future performance.

From the point of view of the market maker, a successful insider at a particular point in time is an insider who has often predicted the market correctly in the past.[22] So suppose that the market maker has observed that the insider has been successful s times in the first t periods, that is suppose that $\hat{s}_t = s$. The next proposition describes the distribution of past and present insider successes conditional on this information. Before we state the proposition, recall that for any two real random variables $\hat{x}$ and $\hat{y}$ with discrete density functions f and g respectively, $\hat{x}$ is said to be

[22]This is due to the fact that the market maker observes $\hat{\theta}_t$ at the end of every period t. Without that assumption, the market maker would learn about the insider by observing the correlation between order flow and dividends, a large correlation indicating a high ability insider.

larger than $\hat{y}$ in the likelihood ratio order (which we denote by $\hat{x} \succeq_{\text{lr}} \hat{y}$) if $\frac{g(x)}{f(x)}$ is decreasing in x over the union their support.[23]

Proposition 4.3 *Let $\hat{s}_u[t,s]$ denote a random variable whose distribution is that of $\hat{s}_u$ conditional on $\hat{s}_t = s$. For any $u \neq t$,*

$$\hat{s}_u[t,t] \succeq_{\text{lr}} \hat{s}_u[t,t-1] \succeq_{\text{lr}} \hat{s}_u[t,t-2] \succeq_{\text{lr}} \cdots \succeq_{\text{lr}} \hat{s}_u[t,0].$$

Proof: See Appendix B.

This result will prove useful in the rest of section 4 in showing how the past successes of the insider at one point in time indicate (predict) past (future) variables in the economy. One such variable is trading volume, and it is the object of the next section.

4.4 Trading Volume

As shown in section 4.2, the insider, because of his learning bias, will always be overconfident about his own ability over his lifetime.[24] In other words, the insider always thinks that his signal $\hat{\theta}_t$ in period t is more informative than it really is. Since this is the case, the insider will always use his information more aggressively than he should, that is he will trade more than a rational insider would. This, of course, leads to higher expected trading volume in the risky security.

Let $\hat{\psi}_t$ denote the trading volume in period t. Since this trading volume comes from both the insider and the liquidity trader, it is given by

$$\hat{\psi}_t = \frac{1}{2}\left(|\hat{x}_t| + |\hat{z}_t|\right). \tag{29}$$

The following lemma shows how the expected one-period trading volume is calculated, conditional on the insider having been successful s times in the first t periods.

Lemma 4.3 *The conditional expected volume in any given period u given that $\hat{s}_t = s$ is equal to*

$$E\left[\hat{\psi}_u \mid \hat{s}_t = s\right] = \frac{1}{\sqrt{2\pi}}\left\{\sqrt{\Sigma} E\left[\beta_u(\hat{s}_{u-1}) \mid \hat{s}_t = s\right] + \sqrt{\Omega}\right\}. \tag{30}$$

Proof: See Appendix B.

[23]See for example Shaked and Shanthikumar (1994), page 27. Note that, in this definition, $a/0$ is taken to be equal to ∞ whenever $a > 0$. Finally, note that the likelihood ratio order is stronger than the more common first-order stochastic dominance order, i.e. if $\hat{x} \succeq_{\text{lr}} \hat{y}$, then $\hat{x} \succeq_{\text{st}} \hat{y}$.

[24]More precisely, the insider will be at least as confident as a rational trader. He becomes overconfident as soon as he correctly predicts a one-period dividend.

First, recall from equation (14a) in section 3 that the insider will multiply his period t signal, $\hat{\theta}_t$, by $\beta_t(s)$ to obtain his demand for the risky asset in that period. In other words, $\beta_t(s)$ represents the insider's trading intensity in period t after having been successful s times in the first $t-1$ periods. The above lemma then shows that greater average insider intensity in a particular period leads to larger expected volume in that period. Also, note that this lemma applies to future periods $(u > t)$ as well as past periods $(u \leq t)$. Finally, when $t = s = 0$, the expected volume given in (30) represents the ex ante expected volume in period u. Our first result shows how expected volume varies with the learning bias parameter γ.

Proposition 4.4 *Given that $\hat{s}_t = s$, the expected volume in period u is increasing in the insider's learning bias parameter γ.*

Proof: See Appendix B.

This result applies to both past $(u \leq t)$ and future $(u > t)$ volume. Although the likelihood of the event that $\hat{s}_t = s$ does not depend on the insider's learning bias parameter γ, it is still true that $\hat{s}_t = s$ will on average be obtained with more volume when γ is large. It is also true that $\hat{s}_t = s$ announces more future volume.

Another way to interpret Proposition 4.4 is as follows. First, as we mention in the paragraph following Lemma 4.3, expected volume in a particular period will be larger the larger the expected insider trading intensity is for that period. Now, notice that we can rewrite $\beta_t(s)$ given in (25a) as

$$\beta_t(s) = \sqrt{\frac{\Omega}{\Sigma}\left[\frac{2}{K_{t-1}(s)} - 1\right]^{-1}}, \tag{31}$$

where $K_{t-1}(s)$ is as defined in equation (26). This tells us that the trading intensity $\beta_t(s)$ of the insider in period t given that he has been successful s times in the first $t-1$ periods is a monotonically increasing function of the insider's overconfidence $K_{t-1}(s)$ after $t-1$ periods. Since we showed in Proposition 4.1 that the insider's overconfidence in any period, given any number of past successes, is increasing in γ, it is not surprising to find that expected volume in a particular period will also be increasing in γ.

This monotonic relationship between $\beta_t(s)$ and $K_{t-1}(s)$ also helps us characterize the conditional expected volume in a particular period u, given different past histories at the end of t periods. Indeed, given this relationship, it is not surprising to find that the expected one-period volume given s insider successes in the first t periods will have the same shape as $K_t(s)$ as a function of s, which we described in Proposition 4.2.

Proposition 4.5 *The expected volume in period u, conditional on the insider having been successful s times in the first t periods (given $\hat{s}_t = s$), is either*

- *increasing over all $s \in \{0, \ldots, t\}$, or*
- *increasing over $s \in \{0, \ldots, s_t^\circ\}$ and decreasing over $s \in \{s_t^\circ, \ldots, t\}$, for some $s_t^\circ \in \{1, \ldots, t\}$.*

The latter case will occur when γ, t/s, and/or t are large.

Proof: See Appendix B.

This result also makes intuitive sense. The insider will tend to trade aggressively in a given period u when he thinks highly of himself, i.e. when $\bar{\mu}_{u-1}(\hat{s}_{u-1})$ is large. When the market maker does not agree with the insider about the insider's ability (i.e. when $\mu_{u-1}(\hat{s}_{u-1})$ is much smaller than $\bar{\mu}_{u-1}(\hat{s}_{u-1})$), he will not let this insider's unwarranted intensity affect his price schedule, and so expected volume for that period will be high. However, when the market maker also thinks highly of the insider (i.e. when $\mu_{u-1}(\hat{s}_{u-1})$ is also high), he suspects the insider to often have useful information about the one-period risky dividend, and will therefore react more abruptly to a particular order flow; in other words, the market maker will quote a steeper price schedule, i.e. $\lambda_u(\hat{s}_{u-1})$ will be large. This steeper price schedule will in turn dampen the insider's trading intensity and, at the same time, reduce expected volume for the period.

4.5 Volatility

In this section, we show that price volatility in any period is larger in the presence of a highly biased insider. Also, at any point in time, expected future price volatility will be larger the more successful the insider has been in predicting the market in previous periods.

In this economy, the security price in one period reflects only the dividend paid at the end of that one period. Since the dividend's unconditional mean is zero, the price's unconditional mean is also zero, and the expected volatility in period u given that $\hat{s}_t = s$ can be measured by

$$\mathrm{Var}(\hat{p}_u \mid \hat{s}_t = s) = \mathrm{E}\left(\hat{p}_u^2 \mid \hat{s}_t = s\right). \tag{32}$$

As we show in Appendix B, this is equal to

$$\mathrm{E}\left(\hat{p}_u^2 \mid \hat{s}_t = s\right) = \frac{\Sigma}{2}\mathrm{E}\big[\bar{\mu}_{u-1}(\hat{s}_{u-1})\,\mu_{u-1}(\hat{s}_{u-1}) \bigm| \hat{s}_t = s\big]. \tag{33}$$

The following proposition shows how this quantity moves with the insider's learning bias paramater γ.

Proposition 4.6 *Given that $\hat{s}_t = s$, the expected volatility in period u is increasing in the insider's learning bias parameter γ.*

Proof: See Appendix B.

This result implies that a learning bias on the part on the insider will cause prices to have excess volatility. As we show next, this excess volatility will be more extreme the more successful the insider is.

Proposition 4.7 *At the end of period t, the conditional expected volatility in period u is increasing in the number of past successful predictions by the insider in the first t periods.*

Proof: See Appendix B.

Interestingly, expected volatility does not exhibit the same patterns as overconfidence and volume in terms how it is affected by the insider's past successes. Indeed, although both expected overconfidence and trading volume can both be non-monotonic in the number of past insider successes, expected volatility is always increased by one more insider success. More precisely, large posteriors by the biased insider ($\bar{\mu}_t(s)$) and the rational market maker ($\mu_t(s)$) both contribute to more expected volatility: the former by his unwarranted aggressiveness, and the latter by his steeper price schedule.

4.6 Insider Profits

Let us now look at the effect of the learning bias on the insider's profits. Section 4.4 showed that the biased insider trades too aggressively on his information; in other words, the insider's learning bias makes him act suboptimally. It is therefore not surprising that we find in this section that the insider's expected profits in any given period are decreasing in his learning bias parameter γ. To show this, we calculate the insider's expected profits in period u after he has been successful s times in the first t periods to be[25]

$$\mathrm{E}\left[\hat{\pi}_u \mid \hat{s}_t = s\right] = \frac{1}{2}\sqrt{\Sigma\Omega}\,\mathrm{E}\left[\sqrt{\bar{\mu}_{u-1}(\hat{s}_{u-1})\left[2\mu_{u-1}(\hat{s}_{u-1}) - \bar{\mu}_{u-1}(\hat{s}_{u-1})\right]}\,\middle|\, \hat{s}_t = s\right]. \tag{34}$$

The following proposition shows that this quantity is indeed decreasing in γ.

Proposition 4.8 *Given that $\hat{s}_t = s$, the insider expected profits in period u are decreasing in the insider's learning bias parameter γ.*

[25] See Appendix B.

Proof: See Appendix B.

The fact that the insider's profits are affected negatively by his learning bias is not surprising. However, as we shall see next, this learning bias can have a perverse effect on the insider's future expected profits as a function of his past successes. In particular, it is possible for a successful insider's expected future profits to be smaller than a less successful insider's. The is due to the fact that two forces affect an insider's expected future profits: his overconfidence and his expected ability.

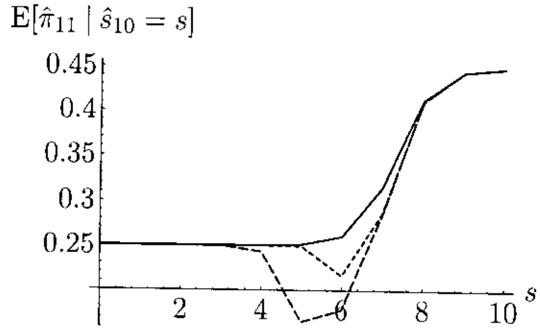
To disentangle these two forces, let us describe the insider's expected profits in period $t+1$ after he has been successful s times in the first t periods. We know from section 4.2 that the insider's overconfidence at the end of t periods is at a minimum of 1 at $s=0$. We also know from Proposition 4.2 that this measure of overconfidence increases with the number s of past insider successful dividend predictions (up to s_t^*). This means that the insider's decision in period $t+1$ will be more and more distorted as s increases.[26]

At the same time, as s is increased, it becomes more and more likely that the insider's ability is high,[27] even if it is not as likely as the insider thinks. Of course, an insider without a learning bias would take advantage of this revelation; that is his expected profits for the next period would get higher with his number of past successes. However, a biased insider, as discussed above, becomes more overconfident, and may act so suboptimally after many successes that he more than offsets the potential increase in expected profits coming from his likely higher ability. Obviously, as the insider's overconfidence comes back down ($s > \hat{s}_t^*$), both forces affect expected insider profits positively and, from then on, any additional past success results in additional expected future profits.

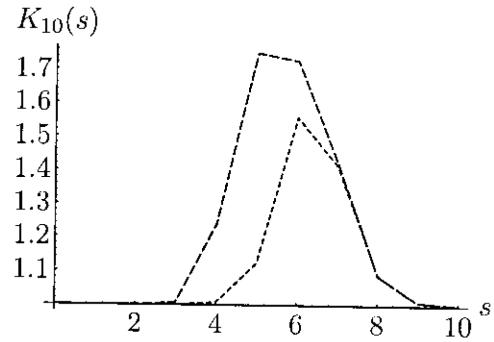
Figure 3 illustrates how the insider's overconfidence and expected ability counterbalance each other. In Panel 3(a) of that figure, we look at the insider's expected profits in period 11, as a function of the number of successes he has had in the first 10 periods; we do this for three different values of γ (1, 2, and 5), and otherwise use the same parameters as in Figures 1 and 2. It is clear from that figure that an unbiased insider always benefits from an additional past success, since his expected ability is higher. However, when the insider's learning is biased, it is possible that his overconfidence (which we plot in Panel 3(b)) prevents him from benefiting from the boost in expected ability that results from an additional success. In fact, for this example, we can see that an insider with $\gamma = 2$ or $\gamma = 5$ who has had six successes in the first 10 periods does worse than an

[26]This was seen to be true in equation (31), where we show that $\beta_t(s)$ is monotonically increasing in $K_t(s)$.

[27]Result C.1 of Appendix C shows that more past successes increase the likelihood $\bar{\phi}_t(s)$ that the insider's ability is high.



(a) Expected insider profits.



(b) Overconfidence.

Figure 3: Insider expected profits and overconfidence in period 11 as functions of his success in the first 10 periods. The figure was obtained with $H = 0.9$, $L = 0.5$, $\phi_0 = 0.5$, $\Sigma = \Omega = 1$. The continuous line was obtained with $\gamma = 1$ (unbiased insider), the small-dashed line with $\gamma = 2$, and the large-dashed line with $\gamma = 5$.

insider who has yet to predict one dividend correctly!

The next proposition shows that the intuition derived for the expected insider profits in period $t + 1$ after observing that $\hat{s}_t = s$ generalizes to any period u.

Proposition 4.9 *Given that $\hat{s}_t = s$, the expected insider profits in period u are increasing over $s \in \{0, \ldots, s'_t\}$ and $s \in \{s''_t, \ldots, t\}$, but are decreasing over $s \in \{s'_t, \ldots, s''_t\}$ for some $(s'_t, s''_t) \in \{1, \ldots, t\}^2$ such that $s'_t \leq s''_t$.*

Proof: See Appendix B.

In our model traders trade on their own account. We do not model the agency issues associated with money managers investing for others. Nevertheless, this last proposition along with Figure 3 may provide some guidance about the choice of managers. We show that a trader with more past successes may have lower expected future profits than a trader with fewer successes. This is because the more successful trader, though objectively more likely to possess high ability, will not make maximum use of his ability due to his overconfidence. An investor choosing a money manager cannot usually observe that manager's level of overconfidence. If the investor has personal contact with a manager he may try to assess that manager's overconfidence through social cues, but when such cues are not available, our model suggests that a manager's success record will be indicative of his overconfidence.

Using a manager's success record as a measure of his overconfidence creates a dilemma for the investor since the investor uses the same success record to assess the manager's ability. While a trader would always prefer to have as good a success record as possible it is not clear that, when choosing a money manager, an investor will always prefer the one with the best past record. A very successful trader may be too overconfident and therefore trade too aggressively. The effects of overconfidence on trading are likely to be exacerbated when risk aversion and agency issues are introduced to the picture. An overconfident money manager may take risks with his client's money which the client would not endorse. Investors can try to protect themselves from choosing the most overconfident managers by avoiding managers who are successful but underexperienced. They should also judge managers on their long term performance, rather than their most recent successes.

Another interesting aspect of our model is that all insiders, even those with low ability, have positive expected profits.[28] These profits are derived from the liquidity trader's willingness to trade at any price and the market maker's willingness to break even. Low abilitiy traders earn less than those with high ability, but they earn enough that even after they realize they have low ability they continue to trade, albeit not as aggressively as before. In real markets traders who have experienced repeated failures are likely to lose their jobs, their money, or their confidence, and quit trading. The traders who remain will be those with the greatest ability and the greatest overconfidence. This survivorship bias, like the selection bias mentioned in footnote 18, will make the overconfidence level of those active in the marketplace higher than that of the general population. This contrasts the results of the natural selection literature,[29] which argue that overconfident and irrational traders will be driven out of financial markets over time. This phenomenon does not happen here, since trading profits is what makes insiders overconfident.

[28] This is because of our assumption that $H \leq 2L$, without which an equilibrium may not exist in some periods.
[29] See, for example, Blume and Easley (1982, 1992).

5 Conclusion

We go through life learning about ourselves as well as the world around us. We assess our own abilities not so much through introspection as by observing our successes and failures. Most of us tend to take too much credit for our own successes. This leads to overconfidence. It is in this way that overconfidence develops in our model. When a trader is successful, he attributes too much of his success to his own ability and revises his beliefs about his ability upward too much. In our model overconfidence is dynamic, changing with successes and failures. Average levels of overconfidence are greatest in those who have been trading for a short time. With more experience, people develop better self assessments.

Since it is through success that traders become overconfident, successful traders, though not necessarily the most successful traders, are most overconfident. These traders are also, as a result of success, wealthy. Overconfidence does not make traders wealthy, but the process of becoming wealthy can make them overconfident. Thus, as opposed to other models of trader irrationality, this model suggests that overconfident traders can play an important long-term role in financial markets. So the assumption that a trader's overconfidence endogenously results from his learning bias, although apparently equivalent to an assumption of static trader overconfidence, leads to very different market dynamics. It is therefore our view that behavioral finance should not only be concerned with the potential presence of overconfidence in financial markets, but also by the origin of such overconfidence.

As shown in our model, an overconfident trader trades too aggressively, and this increases expected market volume. Volatility is increasing in a trader's number of past successes (for a given number of periods). Both volume and volatility increase with the degree of a trader's learning bias. Overconfident traders behave suboptimally, thereby lowering their expected profits. A more successful trader is likely to have more information gathering ability but he may not use his information well. Thus the expected future profits of a more successful trader may actually be lower than those of a less successful trader. Successful traders do tend to be good, but not as good as they think they are. As we point out, this may result in perverse effects for the selection of money managers, whose ability *and* overconfidence can only be assessed by their past performance. However, to better assess the significance of these effects, one would need to take into account both the agency problems between a money manager and his clients, as well as the potential risk aversion of these agents.

Appendix A

Proof of Lemma 3.1

By using (21) in (18) and rearranging, we obtain

$$2\mu_{t-1}(s)\Sigma\beta_t^2(s) = \bar{\mu}_{t-1}(s)\Sigma\beta_t^2(s) + \bar{\mu}_{t-1}(s)\Omega,$$

which is quadratic in $\beta_t(s)$. As long as $2\mu_{t-1}(s) \geq \bar{\mu}_{t-1}(s)$, we can solve for $\beta_t(s)$ and obtain (25a), as desired.[30] Also, a necessary and sufficient condition for this inequality to be satisfied for any integers s and t such that $0 \leq s < t$ is that $H \leq 2L$. Finally, using (25a) for $\beta_t(s)$ in (21) yields (25b). ∎

[30]The other root is rejected, since it represents a minimum, not a maximum.

Appendix B

Proof of Lemma 4.1

As discussed in tha paragraph preceding the lemma, all we need to show is that

$$\left(\frac{L}{\gamma H}\right)^{H}\left(\frac{1-L}{1-H}\right)^{1-H} < 1.$$

By taking log's of both sides and rearranging, this can be shown to be equivalent to showing that

$$f_{\gamma,L}(H) \equiv (1-H)\log(1-L) - (1-H)\log(1-H) - H\log\gamma - H\log H + H\log L < 0 \qquad \text{(B.1)}$$

for all $H \in (L, 1]$. First, note that $f_{\gamma,L}(L) = -L\log\gamma \leq 0$. So, if we can show that $f'_{\gamma,L}(H) < 0$ for all $H \in (L, 1]$, we will have the desired result. Indeed,

$$\begin{aligned} f'_{\gamma,L}(H) &= -\log(1-L) + \log(1-H) - log\gamma - \log H + \log L \\ &= -\log\gamma - \log\left(\frac{H(1-L)}{L(1-H)}\right) < 0, \end{aligned}$$

since $\gamma \geq 1$ and $\frac{H(1-L)}{L(1-H)} > 1$. ∎

Proof of Lemma 4.2

As $t \to \infty$, since $\hat{a} = L$, we expect the insider to correctly guess the one-period dividend a fraction L of the times. So, as we play the game more and more often (t tends to ∞), we expect his updated posteriors $\bar{\phi}_t(s)$ to behave like

$$\frac{1}{1+\left(\frac{L}{\gamma H}\right)^{Lt}\left(\frac{1-L}{1-H}\right)^{t-Lt}\frac{1-\phi_0}{\phi_0}} = \frac{1}{1+\left[\left(\frac{L}{\gamma H}\right)^{L}\left(\frac{1-L}{1-H}\right)^{1-L}\right]^{t}\frac{1-\phi_0}{\phi_0}}.$$

So $\bar{\phi}_t(s)$ will converge to 0, ϕ_0, or 1 according to whether the expression in square brackets is greater than, equal to, or smaller than 1. By taking log's, this is equivalent to finding whether

$$g_{\gamma,H}(L) \equiv (1-L)\log(1-L) - (1-L)\log(1-H) - L\log\gamma - L\log H + L\log L$$

is greater than, equal to, or smaller than 0. Let us first check that $g_{1,H}(L) > 0$ for all $L \in [0, H)$, which we do by checking that $g_{1,H}(H) = 0$, and $g'_{1,H}(L) < 0$ for all $L \in [0, H)$. The first part is easily verified, and

$$\begin{aligned} g'_{1,H}(L) &= -\log(1-L) + \log(1-H) - \log H + \log L \\ &= -\log\left(\frac{H(1-L)}{L(1-H)}\right) < 0, \end{aligned}$$

since $H(1-L) > L(1-H)$. Now, observe that $\frac{\partial}{\partial\gamma} g_{\gamma,H}(L) = -\frac{L}{\gamma} < 0$, so that $g_{1,H}(L) > g_{\gamma,H}(L)$ for all $\gamma \in (1,\infty)$ and $\lim_{\gamma\to\infty} g_{\gamma,H}(L) = -\infty$. Since $g_{1,H}(L) > 0$, this means that there will always be a value γ^* such that

$$g_{\gamma,H}(L) \begin{cases} > 0, & \text{if } \gamma < \gamma^* \\ = 0, & \text{if } \gamma = \gamma^* \\ < 0, & \text{if } \gamma > \gamma^*. \end{cases}$$

This value γ^* solves $g_{\gamma^*,H}(L) = 0$, and it is easily shown to be given by

$$\gamma^* = \frac{L}{H}\left(\frac{1-L}{1-H}\right)^{(1-L)/L}.$$

This completes the proof. ▮

Proof of Proposition 4.1

Since the denominator of $K_t(s)$ in (26) is not a function of γ, $\frac{\partial K_t(s)}{\partial\gamma}$ will have the same sign as $\frac{\partial\bar{\mu}_t(s)}{\partial\gamma}$. We show in Result C.2 of Appendix C that $\frac{\partial\bar{\mu}_t(s)}{\partial\gamma} > 0$. ▮

Proof of Proposition 4.2

In our model, the number of successes in the first t periods is obviously an integer in $\{0, 1, \ldots, t\}$, but the function $K_t(s)$ is well defined for any $s \in [0, t]$. We first show that this function is increasing for $s \in [0, s_0]$ and decreasing for $s \in [s_0, t]$ for some $s_0 \in [0, t]$.

To show this, recall that

$$K_t(s) = \frac{\bar{\mu}_t(s)}{\mu_t(s)} = \frac{L + (H-L)\bar{\phi}_t(s)}{L + (K-L)\phi_t(s)}.$$

If we define an "iso-confidence" curve by $K_t(s) = K_i$ for some constant $K_i \geq 1$, each of these curves can then be written as a straight line in a $\bar{\phi}_t(s)$-$\phi_t(s)$ diagram. More precisely, each iso-confidence curve can be expressed as

$$\phi_t(s) = \frac{1}{K_i}\left[\bar{\phi}_t(s) - \frac{(K_i - 1)L}{H-L}\right].$$

These lines are shown as thin solid line in Figure 4.

In that figure, $1 = K_1 < K_2 < \cdots < K_6$. From Result C.1 in Appendix C, we know that the parametric curve $\{\bar{\phi}_t(s), \phi_t(s)\}_{s=0}^t$ is increasing. As can be seen from Figure 4, if we can also show that this curve's concavity is increasing, we will have the desired result (that $K_t(s)$ is increasing and then decreasing as a function of s).

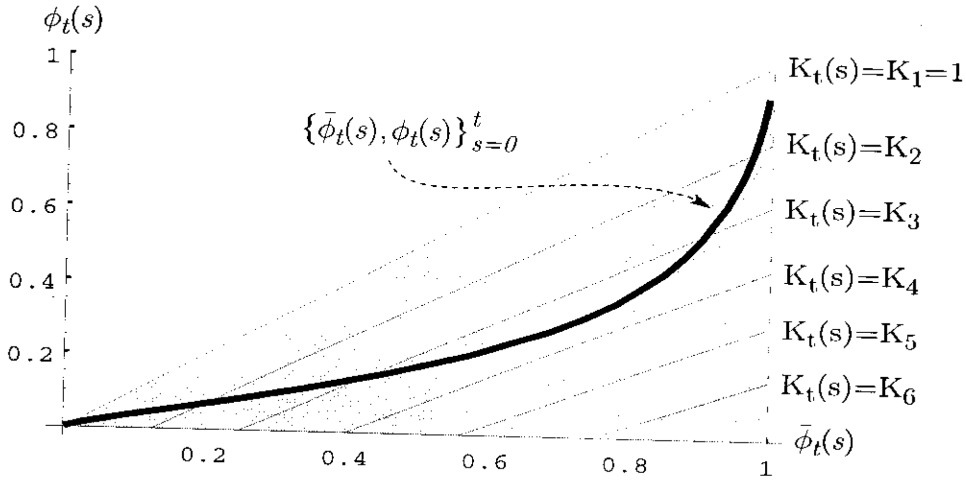


Figure 4: This figure shows $\phi_t(s)$ as a function of $\bar{\phi}_t(s)$. For any $s \in [0, t]$, we have $\bar{\phi}_t(s) \geq \phi_t(s)$, so that all the points $\{\bar{\phi}_t(s), \phi_t(s)\}_{s=0}^{t}$ must lie in the grey area. The thin solid lines represent the "iso-confidence" curves $K_t(s) = K_i$, $i = 1, \ldots, 6$ for $1 = K_1 < K_2 < \cdots < K_6$. The thick solid line represents the parametric curve $\{\bar{\phi}_t(s), \phi_t(s)\}_{s=0}^{t}$, where $\phi_t(s)$ and $\bar{\phi}_t(s)$ are given in (6) and (9) respectively.

To show this, we use Result C.1 in Appendix C to obtain

$$\frac{\partial \phi_t(s)}{\partial \bar{\phi}_t(s)} = \frac{\partial \phi_t(s)/\partial s}{\partial \bar{\phi}_t(s)/\partial s} = \frac{\phi_t(s)[1-\phi_t(s)] \log\left(\frac{H}{L}\frac{1-L}{1-H}\right)}{\bar{\phi}_t(s)[1-\bar{\phi}_t(s)] \log\left(\frac{\gamma H}{L}\frac{1-L}{1-H}\right)},$$

and

$$\frac{\partial}{\partial s}\left(\frac{\partial \phi_t(s)}{\partial \bar{\phi}_t(s)}\right) = \frac{\log\left(\frac{H}{L}\frac{1-L}{1-H}\right)}{\log\left(\frac{\gamma H}{L}\frac{1-L}{1-H}\right)} \frac{\phi_t(s)[1-\phi_t(s)]}{\bar{\phi}_t(s)[1-\bar{\phi}_t(s)]} \left\{[1-2\phi_t(s)] \log\left(\frac{H}{L}\frac{1-L}{1-H}\right) - [1-2\bar{\phi}_t(s)] \log\left(\frac{\gamma H}{L}\frac{1-L}{1-H}\right)\right\}. \tag{B.2}$$

Using standard calculus results along with (B.2) and Result C.1 in Appendix C, we have

$$\frac{\partial^2 \phi_t(s)}{\partial \bar{\phi}_t(s)^2} = \frac{\frac{\partial}{\partial s}\left(\frac{\partial \phi_t(s)}{\partial \bar{\phi}_t(s)}\right)}{\partial \bar{\phi}_t(s)/\partial s} = \frac{\phi_t(s)[1-\phi_t(s)]}{\left\{\bar{\phi}_t(s)[1-\bar{\phi}_t(s)]\right\}^2} \frac{[1-2\phi_t(s)] \log\left(\frac{H}{L}\frac{1-L}{1-H}\right) - [1-2\bar{\phi}_t(s)] \log\left(\frac{\gamma H}{L}\frac{1-L}{1-H}\right)}{\log\left(\frac{H}{L}\frac{1-L}{1-H}\right)}.$$

This last expression will always have the same sign as

$$[1-2\phi_t(s)]\log\left(\frac{H}{L}\frac{1-L}{1-H}\right)-[1-2\bar{\phi}_t(s)]\log\left(\frac{\gamma H}{L}\frac{1-L}{1-H}\right).$$

Finally, it can be shown that

$$\frac{\partial}{\partial s}\left\{[1-2\phi_t(s)]\log\left(\frac{H}{L}\frac{1-L}{1-H}\right)-[1-2\bar{\phi}_t(s)]\log\left(\frac{\gamma H}{L}\frac{1-L}{1-H}\right)\right\}=$$
$$2\left\{\bar{\phi}_t(s)[1-\bar{\phi}_t(s)]\log^2\left(\frac{\gamma H}{L}\frac{1-L}{1-H}\right)-\phi_t(s)[1-\phi_t(s)]\log^2\left(\frac{H}{L}\frac{1-L}{1-H}\right)\right\}>0.$$

This shows that $K_t(s)$ is first increasing and then decreasing as a function of s. In fact, it is easy to show that this last expression is greater the larger γ, t, and s/t are.

To complete the proof, we must deal with the fact that, in our problem, we only care about $K_t(s)$ for $s \in \{0, 1, \ldots, t\}$. However, since the conditions we derive are sufficient to make the slope of $K_t(s)$ negative for s close to t, they will also be sufficient to make $K_t(s) - K_t(s-1) < 0$ for large enough integers $s \in \{0, 1, \ldots, t\}$. ∎

Derivation of Υ_t in Equation (28)

Equation (28) follows from the fact that

$$\begin{aligned}
\mathrm{E}\left[\frac{\bar{\phi}_t(\hat{s}_t)}{\phi_t(\hat{s}_t)}\right] &= \sum_{s=0}^{t}\Pr\{\hat{s}_t=s\}\frac{\bar{\phi}_t(s)}{\phi_t(s)}\\
&= \sum_{s=0}^{t}[\Pr\{\hat{s}_t=s\mid\hat{a}=H\}\Pr\{\hat{a}=H\}+\Pr\{\hat{s}_t=s\mid\hat{a}=L\}\Pr\{\hat{a}=H\}]\frac{\bar{\phi}_t(s)}{\phi_t(s)}\\
&= \sum_{s=0}^{t}\left[\binom{t}{s}H^s(1-H)^{t-s}\phi_0+\binom{t}{s}L^s(1-L)^{t-s}(1-\phi_0)\right]\frac{\bar{\phi}_t(s)}{\phi_t(s)}.
\end{aligned}$$

Proof of Proposition 4.3

Since the likelihood ratio order is transitive, it is sufficient to show that $\hat{s}_u[t, s+1] \succeq_{\mathrm{lr}} \hat{s}_u[t, s]$ for any $s < t$. It is straightforward to show that

$$\Pr\{\hat{s}_u=v\mid\hat{s}_t=s\}=\frac{\binom{u}{v}\binom{t-u}{s-v}}{\binom{t}{s}},$$

The desired density ratio can thus be calculated to be equal to

$$\frac{\Pr\{\hat{s}_u=v\mid\hat{s}_t=s\}}{\Pr\{\hat{s}_u=v\mid\hat{s}_t=s+1\}}=\frac{(t-s)(s+1-v)}{(s+1)(t-u-s+v)},$$

which is easily shown to be decreasing in v. ∎

Proof of Lemma 4.3

First, a standard result for normal variables is that, if $\hat{y} \sim \mathrm{N}(0, \sigma^2)$, then

$$\mathrm{E}\,|\hat{y}| = \sqrt{\frac{2\sigma^2}{\pi}}.$$

We can therefore write

$$\begin{aligned}
\mathrm{E}\left[\hat{\psi}_u \,\middle|\, \hat{s}_t = s\right] &= \frac{1}{2}\mathrm{E}\left[|\hat{x}_u| + |\hat{z}_u| \,\middle|\, \hat{s}_t = s\right] \\
&= \frac{1}{2}\mathrm{E}\left[|\hat{x}_u| \,\middle|\, \right] + \frac{1}{2}\sqrt{\frac{2\Omega}{\pi}} \\
&= \frac{1}{2}\mathrm{E}\left[\left|\beta_u(\hat{s}_{u-1})\hat{\theta}_u\right| \,\middle|\, \hat{s}_t = s\right] + \sqrt{\frac{\Omega}{2\pi}} \\
&= \frac{1}{2}\mathrm{E}\left[\beta_u(\hat{s}_{u-1})\,|\hat{\theta}_u| \,\middle|\, \hat{s}_t = s\right] + \sqrt{\frac{\Omega}{2\pi}} \\
&= \frac{1}{2}\mathrm{E}\left\{\mathrm{E}\left[\beta_u(\hat{s}_{u-1})\,|\hat{\theta}_u| \,\middle|\, \hat{s}_{u-1}, \hat{s}_t = s\right] \,\middle|\, \hat{s}_t = s\right\} + \sqrt{\frac{\Omega}{2\pi}} \\
&= \frac{1}{2}\mathrm{E}\left\{\beta_u(\hat{s}_{u-1})\,\mathrm{E}\left[|\hat{\theta}_u| \,\middle|\, \hat{s}_{u-1}, \hat{s}_t = s\right] \,\middle|\, \hat{s}_t = s\right\} + \sqrt{\frac{\Omega}{2\pi}} \\
&= \frac{1}{2}\mathrm{E}\left[\beta_u(\hat{s}_{u-1})\sqrt{\frac{2\Sigma}{\pi}} \,\middle|\, \hat{s}_t = s\right] + \sqrt{\frac{\Omega}{2\pi}},
\end{aligned}$$

and this last expression is equal to (30). ∎

Proof of Proposition 4.4

Given the expression for the conditional expected volume in (30), it is sufficient to prove that

$$\frac{\partial \beta_t(s)}{\partial \gamma} > 0.$$

Straighforward differentiation of the expression for $\beta_t(s)$ in equation (25a) of Lemma 3.1 results in

$$\frac{\partial \beta_t(s)}{\partial \gamma} = \sqrt{\frac{\Omega}{\Sigma}\frac{2\mu_{t-1}(s) - \bar{\mu}_{t-1}(s)}{\bar{\mu}_{t-1}(s)}}\frac{\mu_{t-1}(s)}{[2\mu_{t-1}(s) - \bar{\mu}_{t-1}(s)]^2}\frac{\partial \bar{\mu}_{t-1}(s)}{\partial \gamma},$$

which in turn shows that it is sufficient to show that

$$\frac{\partial \bar{\mu}_{t-1}(s)}{\partial \gamma} > 0.$$

This is shown to be true in Result C.2 of Appendix C. ∎

Proof of Proposition 4.5

As shown in Lemma 4.3, the expected volume in period u is proportional to the expected trading intensity of the insider in that period. Since $\beta_t(s)$ is monotonically increasing in $K_t(s)$, the results of Propositions 4.2 and 4.3 immediately imply this result. ∎

Derivation of $\mathbf{E}(\hat{p}_u^2 \mid \hat{s}_t = s)$ in Equation (33)

Using the law of iterated expectations, we can write

$$\begin{aligned}
\mathrm{E}\left(\hat{p}_u^2 \mid \hat{s}_t = s\right) &= \mathrm{E}\left[\lambda_u^2(\hat{s}_{u-1})\,\hat{\omega}_u^2 \mid \hat{s}_t = s\right] \\
&= \mathrm{E}\left[\lambda_u^2(\hat{s}_{u-1})(\hat{x}_u + \hat{z}_u)^2 \mid \hat{s}_t = s\right] \\
&= \mathrm{E}\left\{\lambda_u^2(\hat{s}_{u-1})\left[\beta_u(\hat{s}_{u-1})\hat{\theta}_u + \hat{z}_u\right]^2 \,\Big|\, \hat{s}_t = s\right\} \\
&= \mathrm{E}\left(\mathrm{E}\left\{\lambda_u^2(\hat{s}_{u-1})\left[\beta_u(\hat{s}_{u-1})\hat{\theta}_u + \hat{z}_u\right]^2 \,\Big|\, \hat{s}_{u-1}, \hat{s}_t = s\right\} \,\Big|\, \hat{s}_t = s\right) \\
&= \mathrm{E}\Big\{\lambda_u^2(\hat{s}_{u-1})\Big[\beta_u^2(\hat{s}_{u-1})\mathrm{E}(\hat{\theta}_u^2 \mid \hat{s}_{u-1}, \hat{s}_t = s) + 2\beta_u(\hat{s}_{u-1})\mathrm{E}(\hat{\theta}_u\hat{z}_u \mid \hat{s}_{u-1}, \hat{s}_t = s) \\
&\qquad + \mathrm{E}(\hat{z}_u^2 \mid \hat{s}_{u-1}, \hat{s}_t = s)\Big] \,\Big|\, \hat{s}_t = s\Big\} \\
&= \mathrm{E}\left\{\lambda_u^2(\hat{s}_{u-1})\left[\beta_u^2(\hat{s}_{u-1})\Sigma + \Omega\right] \,\Big|\, \hat{s}_t = s\right\}.
\end{aligned}$$

Now, using the expressions derived for $\beta_t(s)$ and $\lambda_t(s)$ in Lemma 3.1, it is easy to show that

$$\lambda_t^2(s)\left[\beta_t^2(s)\Sigma + \Omega\right] = \frac{\Sigma}{2}\bar{\mu}_{t-1}(s)\mu_{t-1}(s).$$

Proof of Proposition 4.6

The result easily follows from the fact that $\frac{\partial}{\partial\gamma}\bar{\mu}_{t-1}(s) > 0$, which is shown to be true in Result C.2 of Appendix C. ∎

Proof of Proposition 4.7

This amounts to show that the product $\bar{\mu}_{t-1}(s)\mu_{t-1}(s)$ is increasing in s. However, since both these quantities are increasig in s, the result follows easily. ∎

Derivation of $\mathrm{E}[\hat{\pi}_u \mid \hat{s}_t = s]$ in Equation (34)

Using the law of iterated expectations, we can write

$$
\begin{aligned}
&\mathrm{E}\left[\hat{\pi}_u \mid \hat{s}_t = s\right] \\
&= \mathrm{E}\left[\hat{x}_u(\hat{v}_u - \hat{p}_u) \mid \hat{s}_t = s\right] \\
&= \mathrm{E}\left(\beta_u(\hat{s}_{u-1})\hat{\theta}_u \left\{\hat{v}_u - \lambda_u(\hat{s}_{u-1}) \left[\beta_u(\hat{s}_{u-1})\hat{\theta}_u + \hat{z}_u\right]\right\} \,\Big|\, \hat{s}_t = s\right) \\
&= \mathrm{E}\left[\mathrm{E}\left(\beta_u(\hat{s}_{u-1})\hat{\theta}_u \left\{\hat{v}_u - \lambda_u(\hat{s}_{u-1}) \left[\beta_u(\hat{s}_{u-1})\hat{\theta}_u + \hat{z}_u\right]\right\} \,\Big|\, \hat{\theta}_u, \hat{s}_{u-1}, \hat{s}_t = s\right) \,\Big|\, \hat{s}_t = s\right] \\
&= \mathrm{E}\left\{\beta_u(\hat{s}_{u-1})\hat{\theta}_u \left[\mu_{u-1}(\hat{s}_{u-1})\hat{\theta}_u - \lambda_u(\hat{s}_{u-1})\beta_u(\hat{s}_{u-1})\hat{\theta}_u\right] \,\Big|\, \hat{s}_t = s\right\} \\
&= \mathrm{E}\left(\mathrm{E}\left\{\beta_u(\hat{s}_{u-1})\hat{\theta}_u \left[\mu_{u-1}(\hat{s}_{u-1})\hat{\theta}_u - \lambda_u(\hat{s}_{u-1})\beta_u(\hat{s}_{u-1})\hat{\theta}_u\right] \,\Big|\, \hat{s}_{u-1}, \hat{s}_t = s\right\} \,\Big|\, \hat{s}_t = s\right) \\
&= \mathrm{E}\left\{\beta_u(\hat{s}_{u-1}) \left[\mu_{u-1}(\hat{s}_{u-1}) - \lambda_u(\hat{s}_{u-1})\beta_u(\hat{s}_{u-1})\right] \mathrm{E}\left(\hat{\theta}_u^2 \mid \hat{s}_{u-1}, \hat{s}_t = s\right) \,\Big|\, \hat{s}_t = s\right\} \\
&= \mathrm{E}\left\{\beta_u(\hat{s}_{u-1}) \left[\mu_{u-1}(\hat{s}_{u-1}) - \lambda_u(\hat{s}_{u-1})\beta_u(\hat{s}_{u-1})\right] \Sigma \,\Big|\, \hat{s}_t = s\right\}.
\end{aligned}
$$

Finally, using the expressions derived for $\beta_t(s)$ and $\lambda_t(s)$ in Lemma 3.1, it is easy to show that

$$\beta_t(s)\left[\mu_{t-1}(s) - \lambda_t(s)\beta_t(s)\right] = \frac{1}{2}\sqrt{\frac{\Omega}{\Sigma}\bar{\mu}_{t-1}(s)\left[2\mu_{t-1}(s) - \bar{\mu}_{t-1}(s)\right]}.$$

Proof of Proposition 4.8

To show the desired result, we only need to show that $\bar{\mu}_{t-1}(s)\left[2\mu_{t-1}(s) - \bar{\mu}_{t-1}(s)\right]$ is decreasing in γ. This is straightforward to show since

$$\frac{\partial}{\partial\gamma}\left\{\bar{\mu}_{t-1}(s)\left[2\mu_{t-1}(s) - \bar{\mu}_{t-1}(s)\right]\right\} = -2\left[\bar{\mu}_{t-1}(s) - \mu_{t-1}(s)\right]\frac{\partial}{\partial\gamma}\bar{\mu}_{t-1}(s),$$

and $\frac{\partial}{\partial\gamma}\bar{\mu}_{t-1}(s)$ is shown to be positive in Result C.2 of Appendix C. ▮

Proof of Proposition 4.9

This result is shown in essentially the same way that Proposition 4.2 was shown earlier, except that the "iso-profit" curves are now quadratic. ▮

Appendix C

This appendix contains a few results that are used in the proofs of some propositions in section 4.

Result C.1 *The partial derivatives of $\bar{\phi}_t(s)$ in (8) with respect to γ and s are respectively equal to*

$$\frac{\partial \bar{\phi}_t(s)}{\partial \gamma} = \frac{s}{\gamma}\bar{\phi}_t(s)\left[1-\bar{\phi}_t(s)\right] \geq 0, \tag{C.1}$$

and

$$\frac{\partial \bar{\phi}_t(s)}{\partial s} = \bar{\phi}_t(s)\left[1-\bar{\phi}_t(s)\right]\log\left(\frac{\gamma H}{L}\frac{1-L}{1-H}\right) \geq 0. \tag{C.2}$$

Proof: Staightforward differentiation of $\bar{\phi}_t(s)$ yields

$$\begin{aligned}
\frac{\partial \bar{\phi}_t(s)}{\partial \gamma} &= \Big\{ s\gamma^{s-1}H^s(1-H)^{t-s}\phi_0\left[(\gamma H)^s(1-H)^{t-s}\phi_0 + L^s(1-L)^{t-s}(1-\phi_0)\right] - \\
&\quad (\gamma H)^s(1-H)^{t-s}\phi_0 s\gamma^{s-1}H^s(1-H)^{t-s}\phi_0 \Big\} \\
&\qquad \div \left[(\gamma H)^s(1-H)^{t-s}\phi_0 + L^s(1-L)^{t-s}(1-\phi_0)\right]^2 \\
&= \frac{\frac{s}{\gamma}(\gamma H)^s(1-H)^{t-s}\phi_0 L^s(1-L)^{t-s}(1-\phi_0)}{\left[(\gamma H)^s(1-H)^{t-s}\phi_0 + L^s(1-L)^{t-s}(1-\phi_0)\right]^2} \\
&= \frac{s}{\gamma}\bar{\phi}_t(s)\left[1-\bar{\phi}_t(s)\right],
\end{aligned}$$

which is obviously greater than or equal to zero. Now, since we can write

$$\bar{\phi}_t(s) = \frac{1}{1+\left(\frac{L}{\gamma H}\frac{1-H}{1-L}\right)^s\left(\frac{1-L}{1-H}\right)^t\frac{1-\phi_0}{\phi_0}},$$

we have

$$\begin{aligned}
\frac{\partial \bar{\phi}_t(s)}{\partial s} &= \frac{(-1)\left(\frac{L}{\gamma H}\frac{1-H}{1-L}\right)^s\log\left(\frac{L}{\gamma H}\frac{1-H}{1-L}\right)\left(\frac{1-L}{1-H}\right)^t\frac{1-\phi_0}{\phi_0}}{\left[1+\left(\frac{L}{\gamma H}\frac{1-H}{1-L}\right)^s\left(\frac{1-L}{1-H}\right)^t\frac{1-\phi_0}{\phi_0}\right]^2} \\
&= \bar{\phi}_t(s)\left[1-\bar{\phi}_t(s)\right]\log\left(\frac{\gamma H}{L}\frac{1-L}{1-H}\right).
\end{aligned}$$

Since $\gamma H > L$ and $1-L > 1-H$, this last quantity is obviously greater than or equal to zero. ∎

Result C.2 *The partial derivatives of $\bar{\mu}_t(s)$ in (9) with respect to γ and s are respectively equal to*

$$\frac{\partial \bar{\mu}_t(s)}{\partial \gamma} = (H-L)\frac{\partial \bar{\phi}_t(s)}{\partial \gamma} = (H-L)\frac{s}{\gamma}\bar{\phi}_t(s)\left[1-\bar{\phi}_t(s)\right] \geq 0, \tag{C.3}$$

and

$$\frac{\partial \bar{\mu}_t(s)}{\partial s} = (H-L)\frac{\partial \bar{\phi}_t(s)}{\partial s} = (H-L)\bar{\phi}_t(s)\left[1-\bar{\phi}_t(s)\right]\log\left(\frac{\gamma H}{L}\frac{1-L}{1-H}\right) \geq 0. \tag{C.4}$$

Proof : Since we have

$$\bar{\mu}_t(s) = H\bar{\phi}_t(s) + L\left[1 - \bar{\phi}_t(s)\right] = L + (H - L)\bar{\phi}_t(s)$$

and $H > L$, this result follows immediately from Result C.1 above. ∎

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