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Abstract

We study the dynamic equilibrium behavior of security prices in an economy where nonfundamental risk arises from agents' heterogeneous beliefs about extraneous processes. We provide a complete characterization of equilibrium in terms of the primitives of the economy, via construction of a representative agent with stochastic weights. Besides the standard pricing of fundamental risk, an agent now also prices the nonfundamental risk with a market price which is a risk-tolerance weighted average of his extraneous disagreement with all remaining agents. Consequently, for given risk tolerances, agents' perceived state prices and consumption streams are more volatile in the presence of extraneous risk. The interest rate inherits additional terms arising from agents' misperceptions about consumption growth, and from precautionary savings motives against the nonfundamental uncertainty.

JEL classification: C60; D51; D90; G12

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1. Introduction

Within the financial community there is continual analysis of subjects, such as market psychology or consumer sentiment, that arguably have little to do with the fundamentals of the economy. The implication is that investors should take account of such extraneous factors in their decision process. Considerable evidence also suggests that belief in such nonfundamentals may in fact be self-fulfilling; fluctuations in the financial markets appear to occur even when there is no news about the market fundamentals, and are commonly attributed to extraneous variables.¹ The origin of academic interest in this viewpoint dates back to Keynes (1936) who attributed market fluctuations in part to “waves of pessimism/optimism” in the economy. More recently, the seminal theoretical work of Cass and Shell (1983) and Azariadis (1981), and subsequent developments, have expounded the viewpoint that the standard equilibrium notion in which agents form their expectations based only on fundamentals, is too restricted, and so the set of rational expectations equilibria should be expanded to include ones where nonfundamentals (or so-called sunspots) also matter. The main focus of much of this literature has been to establish the existence, and pervasiveness, of equilibria where sunspots matter, and to attempt to understand the conditions required for existence.²

The objective of this paper is to study the consequences for the dynamic behavior of security prices in equilibria where nonfundamental uncertainty matters, while working in a most familiar, well-understood environment in financial economics. Part of this objective is for the formulation to be fully consistent with rational expectations, market clearing and no arbitrage. To this end, we develop an extension of a continuous-time pure-exchange, Radner (1972) equilibrium economy, a standard workhorse model for dynamic asset pricing theory. The advantage of a continuous-time environment is the prevalence of much recent literature for comparison, and the provision of exact formulae where a discrete time model would yield approximations. (For closely related models, see Duffie and Huang, 1985; Duffie, 1986; Huang, 1987; Duffie and Zame, 1989; Karatzas, Lehoczky, and Shreve, 1990). While retaining the standard assumptions of dynamic market completeness (relative to agents’ information) and frictionless markets, we allow incomplete but symmetric information, thus allowing for heterogeneous beliefs across agents.

For extraneous uncertainty to matter in equilibrium requires some imperfection to either negate

¹The “fundamentals” of an economy are understood to include quantities such as the aggregate consumption supply, and the endowments and preferences of the agents; “nonfundamentals” or “extraneous” quantities are other quantities which only influence decisions because they are believed to affect the future.

²This body of literature (see Chiappori and Guesnerie, 1991; Guesnerie and Woodford, 1993, for two recent surveys) has built up a fairly unified understanding of when sunspots cannot exist, and hence the set of conditions one must break down to allow sunspots. However, there is a less unified understanding of the models in which sunspots do arise. Most of the literature follows two strands; following Cass and Shell (1983) in mostly finite horizon models involving incompleteness before the sunspots are introduced, or following Azariadis (1981) in infinite horizon, overlapping generations models where the sunspots are uninsurable. Our model follows the former strand more closely, having a finite horizon and insurable sunspots.

the first welfare theorem or allow Pareto optimal allocations in which extraneous uncertainty matters. We choose perhaps the most natural imperfection (mentioned by Cass and Shell, 1983): incomplete information about the extraneous process leading to heterogeneous beliefs across agents. Our motivation for this choice is two-fold. Firstly, the presence of heterogeneous beliefs is now well-recognized in the financial economics literature, and attracting a growing interest. There is little reason in general to believe that agents know (and hence agree on) the true probability distribution of any observables, but perhaps an even stronger case for disagreement can be made for the less tangible types of processes considered as extraneous. Secondly, we prefer not to deviate too much from the standard model, hence ruling out any market incompleteness (beyond the most elementary form required to sustain heterogeneous beliefs).

Our analysis both complements and extends the growing literature on (non-sunspot) models with differences of opinion amongst agents regarding the fundamentals of an economy (see Abel, 1990; De Long, Shleifer, Summers, and Waldmann, 1990; Detemple and Murthy, 1994, 1997; Harris and Raviv, 1993; Shefrin and Statman, 1994; Varian, 1985, 1989; Wang, 1994; Williams, 1977; Zapatero, 1997).³ The most closely related work to ours is the continuous-time dynamic equilibrium models of Detemple and Murthy (production) and Zapatero (pure-exchange) with logarithmic utility agents having heterogeneous beliefs about, respectively, the unobservable growth of the production process and the unobservable growth of the aggregate consumption process. Although our focus is on nonfundamentals, an important by-product of our analysis is to show (Remark 5, Section 4) how to extend the literature on heterogeneous beliefs about fundamentals, to general complete market economies with arbitrary (von Neumann-Morgenstern) utility functions.

We consider an economy in which two or more agents, with heterogeneous arbitrary (von Neumann-Morgenstern) utility functions, observe two exogenous processes: the consumption supply (aggregate endowment) process, and an extraneous process which affects none of the fundamentals. Agents have full information on the endowment process, but incomplete information on the extraneous process from which they make inferences heterogeneously about its growth. We assume that all agents believe the extraneous process may affect the real economic quantities and form their expectations taking this into account. The standard equilibria where agents ignore the extraneous process are also equilibria of this economy, but the generic equilibria are ones in which the extraneous process does influence real quantities. Hence the benchmark equilibrium notion is too restricted and its uniqueness is not robust to a perturbation in agents' beliefs, pointing to a shortcoming of the standard asset pricing models.

Equilibrium is determined in terms of a representative agent's utility function, assigning stochas-

³Other related continuous-time work on incomplete information includes Detemple (1986, 1991), Dothan and Feldman (1986), Feldman (1989, 1992), Genotte (1986), Karatzas and Xue (1991).

tic weights to each agent in the economy (Cuoco and He, 1994, incomplete markets; Serrat, 1995, liquidity constraints; Basak and Cuoco, 1997, restricted market participation). The current paper is unique as far as the stochastic weighting (capturing agents' differing individual-specific state prices) is identified in terms of exogenous quantities, the extraneous uncertainty and agents' disagreement about the extraneous growth. This "extraneous disagreement" process in a sense quantifies how close and how "lucky" an agent has been in his past estimation of the unobservable, and so modulates the weight assigned to him at a state and time. All equilibrium quantities (state prices, security prices and consumption allocations) are now driven by this stochastic extraneous disagreement process in addition to the endowment. Hence, excess uncertainty is created beyond that generated by fundamentals alone.

We provide a complete characterization of equilibrium via explicit expressions for all the pertinent quantities in terms of the economic primitives (preferences, endowments, beliefs). This allows us to readily explore the properties of equilibrium asset risk premia and the interest rate in the presence of extraneous risk. In addition to the standard expression for the market price of fundamental risk, an agent also prices the nonfundamental risk with a perceived market price that is a risk-tolerance-weighted average of his disagreement with the remaining agents in the economy. A security's perceived risk premium, then, is not purely explained by the covariance of its return with the aggregate consumption, but also by the covariance with the extraneous uncertainty. Our analysis with heterogeneous beliefs about nonfundamentals has an appealing simplicity in continuous-time, in that fundamental and nonfundamental risk are disentangled in the pricing relationships. The nonfundamental risk is also absorbed into agents' consumption allocations. Due to these additional extraneous risk components, for given agents' risk tolerances, the total volatility of the state prices and consumption allocations is higher than would be predicted by a model with no extraneous risk.

We also show that in the presence of extraneous uncertainty two additional contributions need to be taken into account in the determination of the interest rate, and hence agents' consumption growths. Firstly, there is an adjustment due to agents' misperceptions of their own consumption growths, which they will attempt to compensate for in their savings policy. In a two-agent economy, this term leads to an increase in the interest rate, but in a multi-agent economy may lead to an increase or decrease. Secondly, the interest rate decreases so as to counteract agents' additional precautionary savings against the extra nonfundamental risk in their future consumption. In the special cases of agents exhibiting constant absolute risk aversion (CARA) preferences or identical constant relative risk aversion (CRRA) preferences less risk averse than logarithmic, the net effect is an increased interest rate over the standard model. For identical CRRA preferences more risk averse than logarithmic, the standard model overestimates the interest rate.

In Section 2 we outline the economy and in Section 3 present the methodology for determination of equilibrium. Section 4 characterizes equilibrium and provides explicit representations for the state price and consumption volatilities, interest rate and consumption growths. Section 5 specializes to CRRA preferences. Section 6 extends the analysis to multiple agents and multiple extraneous processes. Section 7 concludes and the Appendix provides all proofs.

2. Economy with Extraneous Uncertainty

We present an extension of a continuous-time, pure-exchange, Radner (1972) equilibrium economy, to allow for the possibility of a so-called “nonfundamental” uncertainty influencing the real quantities in equilibrium (such as agents’ consumption allocations). By a nonfundamental (or extraneous) uncertainty we mean uncertainty which does not affect any of the fundamentals of the economy, i.e., the aggregate consumption supply or the agents’ endowments or preferences.

2.1. Agents’ Endowments, Information Structure and Perceptions

We consider a finite horizon $[0, T]$ economy in which there is a single consumption good (the numeraire). The uncertainty is represented by the filtered probability space $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}, \mathcal{P})$ on which is defined a two-dimensional Brownian motion (W_ϵ, W_z) . Throughout this paper, we use the notation $\{\mathcal{F}_t^{x,y,z}\}$ to denote the augmented filtration generated by the processes x, y, z . Letting \mathcal{H} denote a σ -field independent of $\mathcal{F}_T^{W_\epsilon, W_z}$, the complete information filtration $\{\mathcal{G}_t\}$ is the augmentation of $\mathcal{H} \times \{\mathcal{F}_t^{W_\epsilon, W_z}\}$. W_ϵ represents the *fundamental uncertainty*, and W_z the independent *nonfundamental* (or *extraneous*) *uncertainty*.

There are two agents in the economy. Each agent ($n = 1, 2$) is endowed with an exogenously specified $\{\mathcal{F}_t^{W_\epsilon}\}$ -progressively measurable endowment process, ϵ_n , strictly positive and bounded, such that the aggregate endowment $\epsilon(t) \equiv \epsilon_1(t) + \epsilon_2(t)$ follows an Itô process

$$d\epsilon(t) = \mu_\epsilon(t) dt + \sigma_\epsilon(t) dW_\epsilon(t) . \quad (1)$$

The mean growth μ_ϵ and volatility σ_ϵ are assumed bounded, $\{\mathcal{F}_t^{W_\epsilon}\}$ -progressively measurable processes, so that the extraneous uncertainty does not affect the fundamentals of the economy. We refer to σ_ϵ as the *fundamental aggregate risk* in the economy. We assume there exists an extraneous process, z , that agents believe may affect some of the real quantities in the economy:

$$dz(t) = \mu_z(t) dt + \sigma_z(t) dW_z(t) . \quad (2)$$

We restrict σ_z to be nonzero, bounded and of the form $\sigma_z(z(\cdot), t)$, a progressively measurable functional of the trajectory of z up to time t , and to be such that a strong solution to (2) exists.

We refer to σ_z as the *nonfundamental risk* in the economy. The growth μ_z is assumed bounded and adapted to $\mathcal{H} \times \{\mathcal{F}_t^z\}$; in particular, $\mu_z(0)$ is \mathcal{H} -measurable.⁴

Remark 1. It is without loss of generality that we assume the process z to only have a dW_z component, with dW_z independent of dW_ϵ . We could assume more generally, an extraneous process with dynamics $d\bar{z}(t) = \mu_z(t) dt + \sigma_{z\epsilon}(t) dW_\epsilon(t) + \sigma_{zz}(t) dW_z(t)$, but due to the information structure of the agents (described below), the process $\sigma_{z\epsilon}$ would turn out to have no effect on the equilibrium, in which case we may simply redefine z by $dz(t) = d\bar{z}(t) - \sigma_{z\epsilon}(t) dW_\epsilon(t)$. Although the nonfundamentals by definition do not affect the fundamentals, they are allowed to be partially driven by the fundamentals.

The fundamental uncertainty W_ϵ is assumed observable by all agents, as are ϵ , μ_ϵ and σ_ϵ . In addition, agents also observe the process z , but do not have complete information about its dynamics. The agents' common observation filtration $\{\mathcal{F}_t\}$ is then $\{\mathcal{F}_t^{W_\epsilon, z}\}$, so they have *incomplete information* since $\mathcal{F}_t \subset \mathcal{G}_t$, $t \in [0, T]$. The agents may deduce σ_z from the quadratic variation of z , but can only draw inferences from their observations of z about the mean growth μ_z . Agents have equivalent probability measures \mathcal{P}^n , $n = 1, 2$, also equivalent to \mathcal{P} , but which may disagree on \mathcal{H} , so that agents have heterogeneous prior beliefs. Agents update their beliefs about μ_z in a Bayesian fashion, via $\mu_z^n(t) = E^n[\mu_z(t) \mid \mathcal{F}_t^{W_\epsilon, z}]$, where $E^n[\cdot]$ denotes the expectation operator relative to probability measure \mathcal{P}^n . We will not provide an explicit analysis of the inferencing problem, but note that due to their heterogeneous priors, agents may draw heterogeneous inferences about $\mu_z(t)$ at all finite times. We allow for heterogeneous inferences about μ_z in anticipation of Proposition 2.

We define, in the classical filtering sense, the innovation processes W_z^n induced by the agents' beliefs \mathcal{P}^n and filtration $\{\mathcal{F}_t\}$, by $dW_z^n(t) = [dz(t) - \mu_z^n(t) dt] / \sigma_z(t)$, $n = 1, 2$. The innovation process of each agent is such that given his perceived growth of the extraneous process, μ_z^n , he believes the observed extraneous process to be described by the dynamics

$$dz(t) = \mu_z^n(t) dt + \sigma_z(t) dW_z^n(t), \quad n = 1, 2, \quad (3)$$

and indeed "agrees" with the process he observes. Furthermore, the agents' information filtration and the innovation filtrations coincide, i.e., $\{\mathcal{F}_t^{W_\epsilon, z}\} = \{\mathcal{F}_t^{W_\epsilon, W_z^n}\}$, assuming (3) has a strong solution. We denote $\{\mathcal{F}_t^{W_\epsilon, W_z^n}\}$ by $\{\mathcal{F}_t^n\}$. Effectively we may view each agent as being endowed with the filtered probability space $(\Omega, \mathcal{F}^n, \{\mathcal{F}_t^n\}, \mathcal{P}^n)$ where, by Girsanov's theorem, the innovation process W_z^n is a Brownian motion on that space. Hence, we may interpret the innovation process of agent n , W_z^n , as the nonfundamental uncertainty perceived by agent n .

⁴As in Detemple and Murthy (1994), we put the structure on σ_z to keep the filtering problem tractable. The only role of the σ -field \mathcal{H} will be to allow for heterogeneity in agents' priors; \mathcal{G}_0 must be non-trivial (unlike $\mathcal{F}_0^{W_\epsilon, W_z}$).

The innovations of the two agents are related by

$$dW_z^2(t) = dW_z^1(t) + \bar{\mu}(t) dt; \quad \bar{\mu}(t) \equiv \frac{\mu_z^1(t) - \mu_z^2(t)}{\sigma_z(t)}, \quad (4)$$

with $\bar{\mu}$ a bounded, $\{\mathcal{F}_t\}$ -progressively measurable process. The two agents disagree on the news conveyed by the observed extraneous process z , and $\bar{\mu}$ parameterizes their disagreement. This disagreement is driven by the relative “optimism/pessimism” of the two agents about the growth of the extraneous process, $\mu_z^1 - \mu_z^2$, and also by the nonfundamental risk σ_z .⁵ We will interpret a positive change $dW_z^n > 0$ as agent n perceiving “good” news in the extraneous process z , and $dW_z^n < 0$ as perceived “bad” news. We will see that, besides the inferencing problem, equilibrium itself does not place any further restrictions on the process $\bar{\mu}$, and so from our point-of-view we shall consider it an exogenous parameter.

2.2. Securities Market

In the spirit of the Arrow-Debreu tradition, in this paper we always assume there exists a sufficient number of zero net supply securities to allow dynamic completion of the markets relative to agents’ observation filtrations.⁶ Of course, due to our introduction of the unobserved and hence unverifiable random variable H , the market must be incomplete under the full information filtration in the

⁵As an example of a specific process for $\bar{\mu}$, consider the case of agents both knowing that μ_z is a constant, but having distinct initial priors as to the value of that constant. They both have normally distributed priors with variance $\nu(0)$, but with different means $\mu_z^1(0) \neq \mu_z^2(0)$. (Of course, this example is inconsistent with our boundedness assumption on $\mu_z(0)$, but we could modify our discussion to work on \mathcal{L}^2 instead.) Also, assume that σ_z is a constant. Then, agents’ beliefs over time for the growth have the dynamics (Liptser and Shirayev, 1977)

$$d\mu_z^n(t) = \frac{\nu(t)}{\sigma_z} dW_z^n(t),$$

where $\nu(t) = \nu(0) \sigma_z^2 / (\nu(0)t + \sigma_z^2)$, implying

$$d\bar{\mu}(t) = \frac{\nu(t)}{\sigma_z} [dW_z^1(t) - dW_z^2(t)] = -\frac{\nu(t)}{\sigma_z} \bar{\mu}(t) dt,$$

(using (4)). In this particular case, $\bar{\mu}$ is deterministic with solution

$$\bar{\mu}(t) = \bar{\mu}(0) \exp \left\{ -\frac{1}{\sigma_z} \int_0^t \nu(s) ds \right\} = \bar{\mu}(0) \left(\frac{\sigma_z^2}{\nu(0)t + \sigma_z^2} \right)^{\sigma_z}.$$

If on the other hand, we allow $\nu^1(0) \neq \nu^2(0)$, we deduce $\bar{\mu}$ to be stochastic, solving

$$d\bar{\mu}(t) = -\frac{\nu^2(t)}{\sigma_z^2} \bar{\mu}(t) + \frac{(\nu^1(t) - \nu^2(t))}{\sigma_z^2} dW_z^1(t).$$

⁶The viewpoint, that the introduction of extrinsic uncertainty should be accompanied by the introduction of claims contingent on that uncertainty is common in the finite horizon sunspot literature (Guesnerie and Woodford, 1993). The infinite horizon sunspot literature generally does not assume sunspots are insurable, one viewpoint being that insurability may wipe out or at least attenuate extraneous risk. We assume insurability to maintain the tractability of a complete market; based on the infinite horizon literature viewpoint, this assumption would if anything lead us to underestimate the impact of extraneous risk.

sense that payoffs contingent on H cannot be attained. Since agents believe there are two relevant dimensions of uncertainty, there is one riskless bond and two non-dividend-paying risky securities, with price dynamics perceived by each agent

$$\begin{aligned} dP_0(t) &= P_0(t) r(t) dt , \\ dP_i(t) &= P_i(t) [\mu_i(t) dt + \sigma_{i\epsilon}(t) dW_\epsilon(t) + \sigma_{iz}(t) dW_z(t)] \\ &= P_i(t) [\mu_i^n(t) dt + \sigma_{i\epsilon}(t) dW_\epsilon(t) + \sigma_{iz}(t) dW_z^n(t)] , \quad i = 1, 2 , \quad n = 1, 2 . \end{aligned}$$

The interest rate r , the vector of perceived drifts $\mu^n \equiv (\mu_1^n, \mu_2^n)^\top$ and the volatility matrix $\sigma \equiv \{\sigma_{ij}; i = 1, 2, j = \epsilon, z\}$ are posited to be (possibly path-dependent) bounded $\{\mathcal{F}_t\}$ -progressively measurable processes, while $\mu \equiv (\mu_1, \mu_2)^\top$ to be bounded and $\{\mathcal{G}_t\}$ -progressively measurable. The price system (r, μ, σ) is to be determined endogenously in equilibrium. In standard dynamic asset pricing models, for market completeness the volatility matrix is typically posited to be non-singular. However, we do not want to force the the security prices to covary with the nonfundamental uncertainty. Accordingly, we allow two cases: either the matrix $\sigma(t)$, $t \in [0, T]$, is non-singular, in which case at least one security must at all times covary with W_z , or $\sigma_{iz}(t) = 0$, $t \in [0, T]$, $i = 1, 2$ (a particular singularity). For tractability, we employ zero net supply securities paying no dividends, which can be interpreted as continuously resettled contracts between agents to reallocate wealth (e.g., futures contracts). The analysis we present is analogous for a model with two positive net supply securities paying out dividends summing to ϵ (Remark 3, Section 4).

Agents observe the security prices, but do not observe the mean returns of the two risky securities and so draw their own inferences, μ_i^n , from their Bayesian updating on the extraneous process, z . Common observation of the risky security price processes and equation (4) imply the following relationship between agents' perceptions of the security price drifts:

$$\mu_i^1(t) - \mu_i^2(t) = \sigma_{iz}(t) \bar{\mu}(t) , \quad i = 1, 2 . \quad (5)$$

As long as the security price covaries with the extraneous process, i.e., $\sigma_{iz}(t) \neq 0$, the agents disagree on the mean expected return. If, however, $\sigma_{iz}(t) = 0$, then $\mu_i^1(t) = \mu_i^2(t)$.

From the perceived price processes, provided σ is non-singular perceived dynamic market completeness allows us to construct a unique system of Arrow-Debreu securities for each agent. Accordingly, we define the perceived *state price density* process ξ^n , under each agent's probability measure as the process with dynamics

$$d\xi^n(t) = -\xi^n(t) [r(t) dt + \theta_\epsilon^n(t) dW_\epsilon(t) + \theta_z^n(t) dW_z^n(t)] , \quad n = 1, 2 .$$

By a state price density process we mean the process such that the time-0 value of any consumption stream c is given by $E^n \left[\int_0^T \xi^n(t) c(t) dt \right] / \xi^n(0)$. Furthermore, $\xi^n(t, \omega)$ is interpreted as the Arrow-

Debreu price (per unit of probability \mathcal{P}^n) of a unit of consumption in state $\omega \in \Omega$ at time t , as perceived by agent n . The processes θ_ϵ^n and θ_z^n are the unique, bounded, $\{\mathcal{F}_t^n\}$ -progressively measurable perceived *market price of fundamental risk* and *market price of nonfundamental risk* processes such that $\theta^n \equiv (\theta_\epsilon^n, \theta_z^n)^\top$ is given by $\sigma(t)\theta^n(t) = (\mu^n(t) - r(t)\mathbf{1})$, $n = 1, 2$, where $\mathbf{1} = (1, 1)^\top$. Some manipulation and the use of equation (5) implies

$$\theta_\epsilon^1(t) - \theta_\epsilon^2(t) = 0, \quad (6)$$

$$\theta_z^1(t) - \theta_z^2(t) = \bar{\mu}(t), \quad (7)$$

if $\sigma(t)$ is non-singular. If $\sigma_{iz}(t) = 0$, $i = 1, 2$, then $\theta_z^n(t) = 0$, $n = 1, 2$. Hence, under our formulation, both agents have the same perceived market price of fundamental risk, but may disagree on the market price of nonfundamental risk.

2.3. Agents' Preferences and Optimization

Each agent n chooses a nonnegative consumption process, c_n , and a portfolio process, π_n , from the set of $\{\mathcal{F}_t\}$ -progressively measurable processes satisfying $\int_0^T c_n(t) dt < \infty$ and $\int_0^T \|\pi_n(t)\|^2 dt < \infty$ a.s. Here $\pi_n \equiv (\pi_{n1}, \pi_{n2})^\top$ denotes the vector of amounts (in units of good) invested in each risky security. An *admissible* consumption-portfolio pair (c_n, π_n) is defined as one for which the associated wealth process, X_n , satisfies

$$dX_n(t) = X_n(t)r(t)dt + (\epsilon_n(t) - c_n(t))dt + \pi_n(t)^\top [\mu^n(t) - r(t)\mathbf{1}]dt + \pi_n(t)^\top \sigma(t) dW^n(t),$$

where $W^n \equiv (W_\epsilon, W_z)^\top$, and is bounded from below and obeys $X_n(T) \geq 0$ a.s. Each agent is assumed to derive time-additive, state-independent utility $u_n(c_n(t))$ from intertemporal consumption in $[0, T]$. The function $u_n(\cdot)$ is assumed to be three times continuously differentiable, strictly increasing, strictly concave, and to satisfy $\lim_{c \rightarrow 0} u'_n(c) = \infty$ and $\lim_{c \rightarrow \infty} u'_n(c) = 0$. An agent's dynamic optimization problem is to maximize $E^n \left[\int_0^T u_n(c_n(t)) dt \right]$ over all admissible (c_n, π_n) pairs for which the expected integral is well-defined, where the expectation is taken relative to his individual information structure $(\Omega, \mathcal{F}^n, \{\mathcal{F}_t^n\}, \mathcal{P}^n)$.

It is well-known (Cox and Huang, 1989; Karatzas, Lehoczky, and Shreve, 1987) that each agent's dynamic optimization problem can be converted into the following static variational problem given that agent's individual-specific Arrow-Debreu prices, ξ^n :

$$\max_{c_n(\cdot)} E^n \left[\int_0^T u_n(c_n(t)) dt \right] \quad \text{subject to} \quad E^n \left[\int_0^T \xi^n(t) c_n(t) dt \right] \leq E^n \left[\int_0^T \xi^n(t) \epsilon_n(t) dt \right].$$

The necessary and sufficient conditions for optimality of the consumption streams to the above

problem are⁷

$$c_n(t) = I_n(y_n \xi^n(t)), \quad n = 1, 2, \quad (8)$$

where I_n is the inverse of u'_n and y_n is the unique number such that agent n 's static budget constraint holds with equality at the optimum, i.e., y_n satisfies

$$E^n \left[\int_0^T \xi^n(t) I_n(y_n \xi^n(t)) dt \right] = E^n \left[\int_0^T \xi^n(t) \epsilon_n(t) dt \right], \quad n = 1, 2. \quad (9)$$

Equation (8) implies that the individual consumption processes inherit the dynamics

$$dc_n(t) = \mu_{c_n}^n(t) dt + \sigma_{c_n \epsilon}(t) dW_\epsilon(t) + \sigma_{c_n z}(t) dW_z^n(t), \quad n = 1, 2,$$

where $\mu_{c_n}^n$ denotes agent n 's perceived mean consumption growth, and the consumption volatility components, $\sigma_{c_n \epsilon}$ and $\sigma_{c_n z}$, denote the covariability of consumption with fundamental and perceived nonfundamental uncertainty, respectively.

3. Equilibrium in the Presence of Extraneous Uncertainty

We define equilibrium in the economy with extraneous risk as follows.⁸

Definition 1. An *equilibrium* is a price system (r, μ, σ) , inferred drifts of the extraneous process μ_z^n , and consumption-portfolio processes (c_n^*, π_n^*) such that: (i) both agents choose their optimal consumption-portfolio strategies at their perceived price processes in their individual filtered probability spaces $(\Omega, \mathcal{F}^n, \{\mathcal{F}_t^n\}, \mathcal{P}^n)$, $\mathcal{F}^n = \mathcal{F}^{W_\epsilon, W_z^n}$; (ii) the W_z^n are the innovation processes associated with inferences μ_z^n , and (iii) the price system is such that the good and security markets clear, i.e.,

$$c_1^*(t) + c_2^*(t) = \epsilon(t), \quad \pi_{1i}^*(t) + \pi_{2i}^*(t) = 0, \quad i = 1, 2, \quad X_1(t) + X_2(t) = 0. \quad (10)$$

Note that since we assume agents employ Bayesian updating to make their inferences, and since they are still making choices consistent with utility maximization, these agents are not acting irrationally. In particular, the equilibrium definition above requires that the price system perceived by the agents to clear the markets at a time and state, does actually clear the markets once that time has arrived and state been revealed, i.e., agents' expectations are rational and self-fulfilled in

⁷Existence of an optimal solution can be established by an appropriate growth condition on $u_n(\cdot)$ and a moment condition on ξ^n (Cox and Huang, 1991).

⁸As in Karatzas, Lehoczky, and Shreve (1990) and He and Leland (1993) we take (r, μ, σ) to be endogenously determined in equilibrium. Some authors (e.g., Cuoco and He, 1994) have taken the viewpoint that σ may be specified exogenously for zero net supply securities, since equilibrium only places a joint restriction on μ and σ . This alternative viewpoint would not affect our analysis, but only change the interpretation (Sections 3.1 and 3.2) that an economy yields multiple equilibria with extraneous risk mattering generically. Instead, each choice of σ would yield one equilibrium, with extraneous risk mattering for generic choices of σ .

equilibrium. The definition above includes both cases where the extraneous risk does affect the real quantities in equilibrium, and cases where the economy essentially collapses to a benchmark economy; we consider these two cases in Sections 3.1 and 3.2.

3.1. Determination of Equilibrium where Extraneous Risk Matters

Definition 2. An *equilibrium where extraneous risk matters* is one in which at least one of the agents' consumption processes depends with nonzero probability on the extraneous process z .⁹ This is equivalent to the condition that at least one of the agents' perceived state price densities depends with nonzero probability on the extraneous process z .

For analytical convenience we introduce a representative agent formulation (following, for example, Huang, 1987). The representative agent's utility function is defined by

$$U(c; \Lambda) \equiv \max_{c_1, c_2} \lambda_1 u_1(c_1) + \lambda_2 u_2(c_2)$$

subject to $c_1 + c_2 = c$, where $\Lambda \equiv (\lambda_1, \lambda_2) \in (0, \infty)^2$.

Identifying $\lambda_1 = 1/y_1$, $\lambda_2(t) = \eta(t)/y_2$ where $\eta(t) \equiv \xi^1(t)/\xi^2(t)$, yields the equilibrium conditions in Proposition 1.¹⁰

Proposition 1. Assume $\sigma(t)$, $t \in [0, T]$, is non-singular. If equilibrium exists, the equilibrium state price density processes of the two agents are given by

$$\xi^1(t) = U'(\epsilon(t); 1/y_1, \eta(t)/y_2), \quad \xi^2(t) = U'(\epsilon(t); 1/y_1, \eta(t)/y_2)/\eta(t), \quad (11)$$

where the ratio y_1/y_2 satisfies either agent's budget constraint, i.e.,

$$\begin{aligned} E^1 \left[\int_0^T U'(\epsilon(t); 1/y_1, \eta(t)/y_2) I_1 \left(y_1 U'(\epsilon(t); 1/y_1, \eta(t)/y_2) \right) dt \right] \\ = E^1 \left[\int_0^T U'(\epsilon(t); 1/y_1, \eta(t)/y_2) \epsilon_1(t) dt \right], \end{aligned} \quad (12)$$

and the stochastic weighting $\eta(t) \equiv \xi^1(t)/\xi^2(t)$ follows

$$\frac{d\eta(t)}{\eta(t)} = -\bar{\mu}(t) dW_z^1(t) = \bar{\mu}(t)^2 dt - \bar{\mu}(t) dW_z^2(t) = \bar{\mu}(t) \frac{(\mu_z^1(t) - \mu_z(t))}{\sigma_z(t)} dt - \bar{\mu}(t) dW_z(t). \quad (13)$$

⁹This definition is analogous to the one offered by Cass and Shell (1983) for "sunspots mattering" in a two-period model.

¹⁰Establishing the existence of equilibrium in this context would involve showing: (i) there exists a solution y_1/y_2 to (12); (ii) given ξ in (11)–(12), there exists an optimal (c_n^*, π_n^*) which satisfies the assumed integrability conditions and clears all markets; and (iii) the r , μ and σ of the posited price dynamics exist, whose associated ξ satisfies (11)–(12) and which satisfy the assumed boundedness conditions. The latter existence and boundedness are easily verified from the explicit formulae for θ and r provided in Propositions 3 and 5 of Section 4.

The equilibrium consumption allocations are given by

$$c_1^*(t) = I_1\left(y_1 U'(\epsilon(t); 1/y_1, \eta(t)/y_2)\right), \quad c_2^*(t) = I_2\left(y_2 U'(\epsilon(t); 1/y_1, \eta(t)/y_2)/\eta(t)\right). \quad (14)$$

If there exists ξ satisfying (11)–(12), then the market clearing conditions (10) are satisfied by the associated optimal policies.

Equations (11)–(14) fully characterize the equilibrium and determine the state price densities (up to a multiplicative constant), the consumption allocations and their dynamics, in terms of the exogenously specified primitives (preferences, endowments, beliefs). The representative agent in this economy with extraneous uncertainty has stochastic weights (as opposed to constant weights in the benchmark economy) due to the agents' facing differing state price densities. However, since $\bar{\mu}$ is essentially exogenously given by agents' perceptions on the (exogenously specified) z process, so is the process η from (13). We may set $\eta(0) = 1$ without loss of generality. Then (12) determines the ratio y_1/y_2 uniquely for given ϵ and η processes; the budget constraints do not put further conditions on η itself. We note that the stated equilibrium conditions do not separately determine both the mean returns μ , and the volatility matrix σ , of zero net supply securities paying out no dividends. The state price density dynamics (market prices of risk and interest rate) merely imply joint restrictions on μ and σ . This indeterminacy is also well-recognized in a benchmark economy with perfect information and homogeneous beliefs (Karatzas, Lehoczky, and Shreve, 1990; He and Leland, 1993).

Since the volatility matrix is not separately restricted in equilibrium, Proposition 1 describes equilibria where the real quantities depend on the extraneous uncertainty, since the consumption processes in (14) are driven by the extraneous process z via the stochastic weighting η . Hence, equilibria where extraneous risk matters are generic.

Now we turn our discussion to the process η to gain further insight into the stochastic weighting. The random variable $\eta(T)$ is the Radon-Nikodym derivative of agent 2's beliefs, \mathcal{P}^2 , with respect to agent 1's beliefs, \mathcal{P}^1 , i.e., $\eta(T) = d\mathcal{P}^2/d\mathcal{P}^1$, with the property $E^1[\eta(T)] = 1$. Then the process $\eta(t) \equiv E^1[\eta(T) | \mathcal{F}_t]$, $\eta(0) = 1$, is a strictly positive \mathcal{P}^1 -martingale. According to (13), all the uncertainty in η is driven by the extraneous uncertainty, with its volatility being driven by $\bar{\mu}$, the disagreement between the agents about μ_z , normalized by the nonfundamental risk, σ_z . Accordingly, from now on, we shall refer to η as the *extraneous disagreement process*, and $\bar{\mu}$ as the *extraneous disagreement risk*. The more heterogeneous the two agents in their perceptions, the higher the volatility (and drift in agent 2's world) of η ; the more nonfundamental risk present, the

lower the volatility. The solution to (13) may be expressed as

$$\eta(t) = \exp \left\{ \int_0^t \left(\bar{\mu}(s)^2 + \frac{(\mu_z^1(s) - \mu_z(s))^2}{\sigma_z(s)^2} - \frac{(\mu_z^2(s) - \mu_z(s))(\mu_z^1(s) - \mu_z(s))}{\sigma_z(s)^2} \right) ds - \int_0^t \bar{\mu}(s) dW_z(s) \right\}.$$

The $(\mu_z^1(s) - \mu_z(s))^2$ term reveals that $\eta(t)$ tends to be higher the further away agent 1's estimation of $\mu_z(s)$ is from its real value, due to the term $(\mu_z^1(s) - \mu_z(s))^2$. However, the term $-(\mu_z^2(s) - \mu_z(s))(\mu_z^1(s) - \mu_z(s))$ mitigates this effect if agent 2's estimation is biased in the same direction as agent 1's while amplified if agent 2's estimation has the opposite bias. $\eta(t)$ is also relatively higher if, over the past, agent 1 has tended either to be optimistic relative to the other agent ($\bar{\mu}(s) > 0$) whenever the extraneous uncertainty underwent a random decrease ($dW_z(s) < 0$), or to be relatively pessimistic whenever the extraneous uncertainty underwent a random increase. This last condition can be interpreted as agent 1 having tended to be relatively unlucky in his predictions of the drift process.

Remark 2 (The Benchmark Radner Economy). In the following sections of the paper, to highlight our results, we often make statements relative to a benchmark economy with complete information, homogeneous beliefs and no extraneous uncertainty. This model consists of two agents, who receive a fraction of the aggregate endowment process as specified in (1), but do not believe any other extraneous process to affect the real quantities in equilibrium. The agents live in the complete, symmetric information structure $(\Omega, \mathcal{F}^{W_\epsilon}, \{\mathcal{F}_t^{W_\epsilon}\}, \mathcal{P})$. To dynamically complete the market, in addition to the bond, the agents only need to introduce one risky security, with the fully observed price dynamics $dP_1(t) = P_1(t)[\mu_1(t) dt + \sigma_{1\epsilon}(t) dW_\epsilon(t)]$, and hence the associated state price density process is symmetrical across agents and driven only by the fundamental uncertainty, i.e., $d\xi(t) = -\xi(t)[r(t) dt + \theta_\epsilon(t) dW_\epsilon(t)]$, $\theta_\epsilon(t) \equiv (\mu_1(t) - r(t))/\sigma_{1\epsilon}(t)$. The market price of nonfundamental risk is not defined in this economy. Agents' optimization problems can be solved for as above and yield agents' consumption processes with dynamics $dc_n(t) = \mu_{c_n}(t) dt + \sigma_{c_n\epsilon}(t) dW_\epsilon(t)$. If the standard Radner general equilibrium is defined, we obtain equilibrium conditions identical to (11)–(14), but with $\eta(t)$ replaced by 1. The formulae for the quantities of interest in Section 4 for this benchmark economy are either well-known (for example, Duffie and Zame, 1989; Karatzas, Lehoczky, and Shreve, 1990) or straightforward to derive, and will be quoted without proof.

3.2. Equilibria where Extraneous Risk Does Not Matter

For completeness of our discussion, Proposition 2 points out that a homogeneous-beliefs economy only supports equilibria where extraneous risk does not matter. This is simply an application of the Cass and Shell (1983) “Ineffectivity Theorem”.

Proposition 2. *If agents have homogeneous beliefs, almost surely for all $t \in [0, T]$, about the extraneous process, then there exists no equilibrium in which extraneous risk matters. In such an economy, $\theta_z^1(t) = \theta_z^2(t) = 0$, $t \in [0, T]$.*

The equilibria in the homogeneous-beliefs economy are identical to those in the benchmark economy.

Even with heterogeneous beliefs, if an equilibrium exists in the benchmark economy for a given endowment process and preferences, those individual consumption processes and state prices would be expected to also be an equilibrium in any economy with extraneous uncertainty and the same endowment process and preferences. One way to construct these equilibria is to have neither risky security covary with the extraneous uncertainty, i.e., $\sigma_{iz}(t) = 0$, $i = 1, 2$. Then both risky securities must be identical and must be perceived identically by both agents ($\sigma_{1\epsilon}(t) = \sigma_{2\epsilon}(t)$, $\mu_1^1(t) = \mu_2^1(t) = \mu_1^2(t) = \mu_2^2(t)$), even though the agents disagree on μ_z^n . These are degenerate examples of equilibria in which extraneous risk does not matter, having $\theta_z^1(t) = \theta_z^2(t) = 0$, $t \in [0, T]$. In fact, whenever extraneous risk does not matter, the equilibrium is effectively equivalent to one in the benchmark model, since $\theta_z^1(t) = \theta_z^2(t) = 0$ implies $\eta(t) = 1$, $t \in [0, T]$.

A natural question to ask is which type of equilibrium Pareto dominates. Due to the incomplete market for hedging against the random variable H , neither type of equilibrium is necessarily Pareto optimal under the full uncertainty. However, since its market is dynamically complete under the agents' observation filtrations, an equilibrium where extraneous risk matters is Pareto optimal relative to all other $\{\mathcal{F}_t^n\}$ -progressively measurable allocations. In an equilibrium where extraneous risk does not matter, there is no means to hedge against the extraneous uncertainty, and so such an allocation is only Pareto optimal relative to a reduced space, all other $\{\mathcal{F}_t^{W\epsilon}\}$ -progressively measurable allocations. Intuitively, the latter will be dominated by an allocation dependent on the extraneous uncertainty, in which agents give up consumption in states they believe to be less likely in return for consumption in states they believe to be more likely. Hence, informally, we may argue that an equilibrium in which extraneous risk matters Pareto dominates one in which extraneous risk does not matter.

4. Characterization of Equilibrium with Extraneous Risk

This section characterizes the generic equilibria when the extraneous risk does matter, arising whenever $\sigma(t)$, $t \in [0, T]$, is non-singular.

4.1. Agents' Equilibrium Consumption Allocations

Comparing the two agents' equilibrium consumption, from (14), we deduce that $c_1^*(t) > c_2^*(t)$ if and only if $I_1(y_1 U'(\epsilon(t); 1/y_1, \eta(t)/y_2)) > I_2(y_2 U'(\epsilon(t); 1/y_1, \eta(t)/y_2)/\eta(t))$. In the case of homogeneous preferences, $u_1(\cdot) = u_2(\cdot)$, we have $c_1^*(t) > c_2^*(t)$ if and only if $y_2/(y_1 \eta(t)) > 1$. In the benchmark economy, this comparison is driven purely by the constant y_2/y_1 and hence agents' initial wealths; if agent 1 is initially more wealthy than agent 2, his weight $1/y_1$ is higher and he consumes more in all states at all times. In the extraneous risk economy, however, the comparison is state-dependent, being also driven by the extraneous disagreement process. Even if agent 1 is initially wealthier than agent 2, there will be states and times characterized by a high $\eta(t)$, when he consumes less than agent 2. Agent 1's consumption is unambiguously decreasing in the extraneous disagreement process:

$$\frac{\partial c_1^*(t, w)}{\partial \eta(t, w)} = -\frac{\partial c_2^*(t, w)}{\partial \eta(t, w)} = -\frac{\eta(t)}{a_1(t) + a_2(t)} < 0, \quad \omega \in \Omega,$$

where $a_n(t) \equiv -u_n''(c_n^*(t))/u_n'(c_n^*(t))$ denotes the absolute risk aversion of agent n . We recall that a relatively high $\eta(t)$ arises when agent 1's prediction of μ_z has tended to be poor in the past, when agent 1's estimation error has opposed that of agent 2, and when agent 1 has tended to be relatively unlucky in his prediction. These conditions are all unfavorable to agent 1 and so cause his consumption to decrease relative to agent 2. In the limits, we have $\lim_{\eta(t) \rightarrow 0} c_1^*(t) = \epsilon(t)$ and $\lim_{\eta(t) \rightarrow \infty} c_1^*(t) = 0$. In a given state and time, as the extraneous disagreement process becomes very small, agent 1 dominates the economy in that state, even if he were initially (possibly much) less wealthy than agent 2.

4.2. State Price and Consumption Volatilities

Proposition 3 summarizes the main results concerning the behavior of the market prices of risk perceived by the two agents.¹¹

¹¹An alternative, less illuminating, representation for the perceived market prices of risk is solely in terms of the representative agent's utility function:

$$\begin{aligned} \theta_\epsilon^1(t) = \theta_\epsilon^2(t) &= -\frac{U''(\epsilon(t); 1/y_1, \eta(t)/y_2)}{U'(\epsilon(t); 1/y_1, \eta(t)/y_2)} \sigma_\epsilon(t), \\ \theta_z^1(t) &= \frac{U'_\eta(\epsilon(t); 1/y_1, \eta(t)/y_2)}{U'(\epsilon(t); 1/y_1, \eta(t)/y_2)} \eta(t) \bar{\mu}(t), \quad \theta_z^2(t) = \left[\frac{U'_\eta(\epsilon(t); 1/y_1, \eta(t)/y_2)}{U'(\epsilon(t); 1/y_1, \eta(t)/y_2)} \eta(t) - 1 \right] \bar{\mu}(t). \end{aligned}$$

Proposition 3. *In an equilibrium where extraneous risk matters, the perceived market prices of fundamental and nonfundamental risk and the total volatility of state prices are given by*

$$\theta_\epsilon^1(t) = \theta_\epsilon^2(t) = A(t) \sigma_\epsilon(t), \quad (15)$$

$$\theta_z^1(t) = \frac{A(t)}{a_2(t)} \bar{\mu}(t), \quad \theta_z^2(t) = -\frac{A(t)}{a_1(t)} \bar{\mu}(t), \quad (16)$$

$$\|\theta^n(t)\| = A(t) \sqrt{\sigma_\epsilon(t)^2 + (\bar{\mu}(t)/a_m(t))^2}, \quad (n, m) = (1, 2), (2, 1), \quad (17)$$

where

$$A(t) \equiv -\frac{U''(\epsilon(t); 1/y_1, \eta(t)/y_2)}{U'(\epsilon(t); 1/y_1, \eta(t)/y_2)} = \frac{1}{1/a_1(t) + 1/a_2(t)}, \quad a_n(t) \equiv -\frac{u_n''(c_n^*(t))}{u_n'(c_n^*(t))},$$

and $c_n^*(t)$, $n = 1, 2$, satisfy (14). Consequently, $|\theta_z^1(t)| > |\theta_z^2(t)|$ and $\|\theta^1(t)\| > \|\theta^2(t)\|$ if and only if $a_1(t) > a_2(t)$.

An agent's perceived market price of fundamental risk has an expression identical to the benchmark economy, being given by the aggregate fundamental risk (σ_ϵ) normalized by the representative risk tolerance in the economy ($1/A$). The more risk averse the economy, the higher the market price of fundamental risk. The extraneous risk only appears via the representative risk aversion A . An agent's perceived market price of nonfundamental risk, nonexistent in the benchmark economy, is proportional to the extent of his disagreement (about the growth of the extraneous process) with the other agent, $\pm\bar{\mu}(t)$, normalized by the other agent's risk tolerance. The more risk tolerant agent has the higher market price of nonfundamental risk, with both agents' market prices of nonfundamental risk increasing in the agents' heterogeneity of beliefs. The market price of fundamental risk is increasing in either agent's risk aversion, while an agent's perceived market price of nonfundamental risk is increasing in his own risk aversion, but decreasing in the other agent's. The formulae suggest that, whereas the *priced fundamental risk* is $\sigma_\epsilon(t)$, homogeneous across all agents, we can think of the *priced nonfundamental risk* as $\pm\bar{\mu}(t)/a_m(t)$ (where m denotes the other agent), which is heterogeneous across all agents. An important implication of the presence of the priced extraneous risk is that the volatility of the state prices, $\|\theta^n(t)\|$, is higher, for given $A(t)$, than in the benchmark economy.

For reassurance, we note that in the limit where one agent's, say agent 2's, risk tolerance tends to zero, the market prices of risk essentially collapse to their benchmark expressions. We obtain $\lim_{a_2(t) \rightarrow \infty} \theta_\epsilon^n(t) = a_1(t) \sigma_\epsilon(t)$, $\lim_{a_2(t) \rightarrow \infty} \theta_z^1(t) = 0$, $\lim_{a_2(t) \rightarrow \infty} \theta_z^2(t) = \bar{\mu}(t)$. The market price of fundamental risk tends to its value in a benchmark economy containing only agent 1. Agent 1's market price of nonfundamental risk tends to zero as it should in a benchmark economy; agent 2's does not tend to zero, but since the agent is essentially insignificant, his market price of risk is

meaningless.

We may derive the risk premium for security $i = 1, 2$, as perceived by the two agents as

$$\begin{aligned}\mu_i^1(t) - r(t) &= A(t) \sigma_\epsilon(t) \sigma_{i\epsilon}(t) + \frac{A(t)}{a_2(t)} \bar{\mu}(t) \sigma_{iz}(t) = A(t) \operatorname{cov} \left(\frac{dP_i(t)}{P_i(t)}, d\epsilon(t) \right) - \frac{A(t)}{a_2(t)} \operatorname{cov} \left(\frac{dP_i(t)}{P_i(t)}, d\eta(t) \right), \\ \mu_i^2(t) - r(t) &= A(t) \sigma_\epsilon(t) \sigma_{i\epsilon}(t) - \frac{A(t)}{a_1(t)} \bar{\mu}(t) \sigma_{iz}(t) = A(t) \operatorname{cov} \left(\frac{dP_i(t)}{P_i(t)}, d\epsilon(t) \right) + \frac{A(t)}{a_1(t)} \operatorname{cov} \left(\frac{dP_i(t)}{P_i(t)}, d\eta(t) \right).\end{aligned}$$

As in the standard consumption-based CAPM (Duffie and Zame, 1989), a risky security's risk premium is positively related to the covariance of its return with the change in aggregate consumption. Now, however, the risk premia cannot be purely explained by the covariance with aggregate consumption; they are also driven by the covariance of the security's return with the change in the extraneous disagreement process. According to agent 1, risk premia are decreasing in this covariance, since to him, a high $\eta(t)$ is unfavorable. According to agent 2, risk premia are increasing in this covariance. In the benchmark economy, a security positively related to the aggregate consumption always demands a positive risk premium. In the economy with extraneous risk, however, if the disagreement is sufficiently large, one of the agents will be content with even a negative risk premium for that security.

Further manipulating the risk premia perceived by each agent yields

$$\frac{A(t)}{a_1(t)} \mu_i^1(t) + \frac{A(t)}{a_2(t)} \mu_i^2(t) - r(t) = A(t) \operatorname{cov} \left(\frac{dP_i(t)}{P_i(t)}, d\epsilon(t) \right), \quad i = 1, 2. \quad (18)$$

Hence, we obtain an expression resembling the standard consumption CAPM, with the risk premium of a security by a risk-tolerance-weighted average of each agent's perceived risk premium. Such a representation has also been offered by Varian (1989) and a similar one by Williams (1977) for the case of agents having heterogeneous beliefs about a fundamental process.¹²

Proposition 4 summarizes the expressions and properties of the volatilities of consumption for the two agents.

Proposition 4. *In an equilibrium where extraneous risk matters, the volatilities of individual consumption allocations are given by*

$$\sigma_{c_n^* \epsilon}(t) = \frac{A(t)}{a_n(t)} \sigma_\epsilon(t), \quad n = 1, 2, \quad (19)$$

$$\sigma_{c_1^* z}(t) = -\sigma_{c_2^* z}(t) = \frac{A(t)}{a_1(t) a_2(t)} \bar{\mu}(t), \quad (20)$$

$$\|\sigma_{c_n^*}(t)\| = \frac{A(t)}{a_n(t)} \sqrt{\sigma_\epsilon(t)^2 + (\bar{\mu}(t)/a_m(t))^2}, \quad (n, m) = (1, 2), (2, 1). \quad (21)$$

¹²Contrary to Varian's (1989, p. 32–33) assertion, (18) does not imply that the pricing of securities is independent of the dispersion of opinion about their expected values. Our market prices of risk and interest rate are all dependent on dispersion of opinion, $\bar{\mu}$ (Propositions 3 and 5), as they are in Varian's framework.

Consequently, $|\sigma_{c_1^* \epsilon}(t)| > |\sigma_{c_2^* \epsilon}(t)|$ and $\|\sigma_{c_1^*}(t)\| > \|\sigma_{c_2^*}(t)\|$ if and only if $a_1(t) < a_2(t)$.

As in the benchmark economy, the covariability of an agent's consumption with the fundamental uncertainty is proportional to the aggregate fundamental risk in the economy, weighted by his fraction of the aggregate risk tolerance (A/a_n). The agents share the fundamental risk between them, in proportion to their absolute risk tolerances ($1/a_n$), so that markets clear, $\sigma_{c_1^* \epsilon}(t) + \sigma_{c_2^* \epsilon}(t) = \sigma_\epsilon(t)$. The more risk tolerant agent absorbs more of the fundamental risk. However, additional extraneous risk is now created in the individual agents' consumption, with agents taking on equal and opposite extraneous risk, so that the extraneous risk cancels in aggregate, $\sigma_{c_1^* z}(t) + \sigma_{c_2^* z}(t) = 0$. As with the fundamental component, the nonfundamental risk either agent takes on is given by his priced nonfundamental risk weighted by his risk tolerance. Hence, the more heterogeneous the agents, the more extraneous consumption risk is created. An agent's extraneous consumption risk is decreasing in either agent's absolute risk aversion, in contrast to the fundamental component which decreases in that agent's risk aversion, but increases in the other agent's. An important implication of the presence of the additional risk, is that both agents' consumption volatilities, $\|\sigma_{c_n^*}(t)\|$, are higher, for given $a_1(t)$ and $a_2(t)$, than in the benchmark economy.

In the limit of zero risk tolerance for agent 2, we obtain $\lim_{a_2(t) \rightarrow \infty} \sigma_{c_1^* \epsilon}(t) = \sigma_\epsilon(t)$, $\lim_{a_2(t) \rightarrow \infty} \sigma_{c_2^* \epsilon}(t) = 0$, $\lim_{a_2(t) \rightarrow \infty} \sigma_{c_n^* z}(t) = 0$, $n = 1, 2$, showing again how agent 1 dominates the economy and the equilibrium converges to the benchmark case.

Remark 3 (The Model with Dividend-Paying Assets). Consider replacing the two risky zero net supply securities and endowment by two positive net supply assets paying exogenously-given dividends, $\delta_1 + \delta_2 = \epsilon$, with endogenous price processes

$$dP_i(t) + \delta_i(t)dt = P_i(t) [\mu_i(t) dt + \sigma_{i\epsilon}(t) dW_\epsilon(t) + \sigma_{iz}(t) dW_z(t)], \quad i = 1, 2.$$

All our conclusions (Propositions 1–6) regarding equilibria in which extraneous risk matters, are still valid (except in degenerate cases where sunspot equilibria do not arise). The main difference is that the following additional no-arbitrage pricing restrictions hold:

$$P_i(t) = \frac{1}{U'(\epsilon(t); 1/y_1, \eta(t)/y_2)} E^1 \left[\int_t^T U'(\epsilon(s); 1/y_1, \eta(s)/y_2) \delta_i(s) ds \mid \mathcal{F}_t^{W_\epsilon, z} \right], \quad i = 1, 2, \quad (22)$$

restricting the asset prices in terms of their dividends. Hence in this model, the mean returns μ and volatilities σ of the assets are individually determined in equilibrium, but only via non-explicit formulae, even in the benchmark Radner economy of Remark 2 (except in a few special cases). The presence of the η terms in (22) suggests that the prices will in general covary with the extraneous uncertainty, meaning that extraneous risk matters in equilibrium.

Remark 4 (The Case of No Fundamental Risk). Economies with no fundamental risk have been the main focus of the existing sunspots literature (see Guesnerie and Woodford, 1993). By a model with no fundamental risk, we mean that the aggregate consumption, ϵ , is a deterministic function of time (with no W_ϵ and σ_ϵ processes existing). In such a benchmark economy, there would be no market prices of risk, no consumption volatility and only a deterministic bond price. In such an economy with extraneous risk, the market price of fundamental risk and the fundamental component of consumption volatility do not exist, while a version of our analysis can show that the nonfundamental market price of risk and consumption volatility component are given by the same expressions as in Propositions 3 and 4. This illustrates that the situation can arise where no fundamental news is arriving, yet all markets (security prices and consumption processes) still move stochastically. Hence, if agents believe the extraneous process to affect some of the real quantities of the economy, and if agents disagree on the growth of this extraneous process, the equilibrium quantities exhibit risk where there was none in the fundamentals of the economy.

Remark 5 (Fundamental-Disagreement Model). Given that heterogeneous beliefs play such a critical role, it is of interest to contrast our implications with a model of heterogeneous beliefs about fundamentals, the focus of much related literature (such as Detemple and Murthy, 1994; Zapatero, 1997). In particular, in a model with no extraneous processes, where agents disagree on the mean growth in the aggregate endowment process, μ_ϵ , agent 1's perceived market price of fundamental risk, would be given by

$$\theta_\epsilon^1(t) = A(t) \sigma_\epsilon(t) + \frac{A(t)}{a_2(t)} \bar{\mu}(t); \quad \bar{\mu}(t) \equiv \frac{\mu_\epsilon^1(t) - \mu_\epsilon^2(t)}{\sigma_\epsilon(t)}.$$

Unlike in our case, the expression for the market price of fundamental risk is itself directly affected by the heterogeneous beliefs, and no extra dimensions of risk are priced. Our disentanglement of the disagreement risk from the fundamental risk allows us to make clearer statements about the dependencies of pertinent quantities on risk tolerances, fundamental risk and disagreement risk. For example, our result concerning the more volatile perceived state prices and consumption streams does not necessarily hold in the fundamental-disagreement model; whether a volatility is increased or decreased depends on an agent's optimism or pessimism relative to the remainder of the economy. Furthermore, in the fundamental-disagreement model, unlike in our model, a security's risk premium can still be explained by the covariance of its return with the single-factor aggregate consumption, since the fundamental-disagreement risk, $\bar{\mu}$, is itself only driven by the aggregate consumption, and so the two terms can be collapsed together; only the proportionality factor is affected compared with a homogeneous-beliefs model.

4.3. Interest Rate and Consumption Growths

Proposition 5 reports the equilibrium interest rate.¹³

Proposition 5. *In an equilibrium where extraneous risk matters, the interest rate is given by:*

$$r(t) = A(t) \mu_\epsilon(t) - \frac{1}{2} A(t) B(t) \sigma_\epsilon(t)^2 + \frac{A(t)^2}{a_1(t) a_2(t)} \bar{\mu}(t)^2 - \frac{1}{2} \frac{A(t)^3 [b_1(t) + b_2(t)]}{a_1(t)^2 a_2(t)^2} \bar{\mu}(t)^2, \quad (23)$$

where

$$B(t) \equiv -\frac{U'''(\epsilon(t); 1/y_1, \eta(t)/y_2)}{U''(\epsilon(t); 1/y_1, \eta(t)/y_2)} = \left(\frac{A(t)}{a_1(t)}\right)^2 b_1(t) + \left(\frac{A(t)}{a_2(t)}\right)^2 b_2(t), \quad b_n(t) \equiv -\frac{u_n'''(c_n^*(t))}{u_n''(c_n^*(t))},$$

and $c_n^*(t)$, $n = 1, 2$, satisfy (14).

As in the benchmark economy (first two terms), the equilibrium interest rate in the presence of extraneous risk is positively related to the growth in aggregate consumption and negatively related (for agents with decreasing absolute risk aversion (DARA)) to the aggregate fundamental risk, in proportion to the representative risk aversion and prudence.¹⁴ The latter negative term arises to compensate for agents' precautionary savings motive in the face of future risky endowment. Relative to the benchmark economy, as a result of the nonfundamental risk being priced, the interest rate is driven by two extra terms, directly dependent on the extraneous disagreement risk $\bar{\mu}$. The third term arises from discrepancy in agents' perceptions about their mean consumption growths. In this two-agent case, it turns out that the aggregate perceived consumption growth, $\mu_{c_1}^1 + \mu_{c_2}^2$ ($= \mu_\epsilon + \bar{\mu}^2/(a_1 + a_2)$), is unambiguously higher than the real aggregate consumption growth, μ_ϵ , so the interest rate must increase to counteract agents' excessive saving tendency. The last term in (23) decreases (for DARA) the interest rate to compensate for agents' extra precautionary savings against the extraneous risk introduced into their consumption streams. Unlike for the fundamental risk, since agents price this nonfundamental risk heterogeneously, the individual agents' savings tendencies against nonfundamental risk do not "aggregate" to yield the representative agent's absolute prudence coefficient.

Because of the presence of the two opposing additional terms, the net effect of the extraneous risk on the interest rate is, in general, ambiguous. However, if agents are highly prudent the last term will dominate and the interest rate will be lower than predicted by the benchmark model (for

¹³An alternative representation for the interest rate purely in terms of the representative agent's utility, is

$$r(t) = -\frac{U''(\epsilon(t); 1/y_1, \eta(t)/y_2)}{U'(\epsilon(t); 1/y_1, \eta(t)/y_2)} \mu_\epsilon(t) - \frac{1}{2} \frac{U'_{\eta\eta}(\epsilon(t); 1/y_1, \eta(t)/y_2)}{U'(\epsilon(t); 1/y_1, \eta(t)/y_2)} \eta(t)^2 \bar{\mu}(t)^2.$$

¹⁴First introduced by Kimball (1990), prudence represents the sensitivity of a decision variable to risk or "the propensity to prepare and forearm oneself in the face of uncertainty." As Kimball discusses, when the utility function is additively separable, $-U'''/U''$ is the appropriate measure of absolute prudence.

given $A(t)$, $B(t)$). If agents' prudences are very low, the third term will dominate and the interest rate will be higher than predicted by the benchmark model. If agents have negative prudences ($u_n'''(\cdot) < 0$), then the interest rate will again be higher. In Section 5, we show the interest rate to unambiguously increase for CRRA preferences with $\gamma > 0$, and unambiguously decrease for CRRA with $\gamma < 0$. The interest rate also increases unambiguously for CARA preferences.

Proposition 6 summarizes the behavior of agents' equilibrium consumption growths.

Proposition 6. *In an equilibrium where extraneous risk matters, the mean perceived growths in the individual agents' consumption allocations are given by:*

$$\begin{aligned} \mu_{c_n^n}^n(t) = & \frac{A(t)}{a_n(t)} \mu_\epsilon(t) + \frac{1}{2} \frac{A(t)^2}{a_n(t)} \left[\frac{b_n(t)}{a_n(t)} - \frac{B(t)}{A(t)} \right] \sigma_\epsilon(t)^2 + \left(\frac{A(t)}{a_n(t)} \right)^2 \frac{1}{a_m(t)} \bar{\mu}(t)^2 \\ & + \frac{1}{2} \left(\frac{A(t)}{a_n(t) a_m(t)} \right)^3 [a_n(t) b_n(t) - a_m(t) b_m(t)] \bar{\mu}(t)^2, \quad (n, m) = (1, 2), (2, 1). \end{aligned} \quad (24)$$

As in the benchmark economy, the mean growth in an individual agent's consumption is positively related to the aggregate consumption growth, where the proportionality coefficient is that agent's fraction of the total risk tolerance (A/a_n), and is related to the fundamental risk, with the proportionality factor being the difference between the two agents' products of prudence and risk tolerance coefficients (b_n/a_n). The more prudent an agent is, the more he saves, but this is counterbalanced by the decrease in the interest rate, so an agent's net saving is driven by how prudent he is relative to the whole economy. Relative to the benchmark economy, when extraneous risk matters, the extraneous disagreement risk also appears directly as a driving factor in agents' consumption growths through the last two terms in (24). The third term is due to the interest rate increase caused by agents' misperceptions of their consumption growth, causing both agents to save more. The fourth term arises from precautionary savings motives against the nonfundamental risk, again driven by how prudent the agent is relative to the whole economy.

5. An Example: The Case of Constant Relative Risk Aversion

We assume here that agents have utility function $u_n(c) = c^\gamma/\gamma$, $\gamma < 1$, $\gamma \neq 0$, or $u_n(c) = \log(c)$ (which corresponds to $\gamma = 0$), for $n = 1, 2$. For this family of utility functions it is natural to express aggregate and individual consumption growths and volatilities in percentage terms as we will see below. For this special case, the previous expressions for the state price and consumption dynamics collapse to explicit formulae, and some additional unambiguous results arise.

Proposition 7. *Assume agents exhibit identical CRRA preferences. Then, in an equilibrium where extraneous risk matters, the state prices and consumption volatilities are given by*

$$\xi^1(t) = \left[y_1^{-1} \left(1 + (y_1 \eta(t)/y_2)^{1/1-\gamma} \right)^{1-\gamma} \right] \epsilon(t)^{\gamma-1}; \quad \xi^2(t) = \left[(y_1 \eta(t))^{-1} \left(1 + (y_1 \eta(t)/y_2)^{1/1-\gamma} \right)^{1-\gamma} \right] \epsilon(t)^{\gamma-1}$$

$$c_1^*(t) = \frac{1}{1 + (y_1 \eta(t)/y_2)^{1/1-\gamma}} \epsilon(t); \quad c_2^*(t) = \frac{(y_1 \eta(t)/y_2)^{1/1-\gamma}}{1 + (y_1 \eta(t)/y_2)^{1/1-\gamma}} \epsilon(t),$$

where y_1/y_2 satisfies

$$E^1 \left[\int_0^T \left[1 + (y_1 \eta(t)/y_2)^{1/1-\gamma} \right]^{-\gamma} \epsilon(t)^\gamma dt \right] = E^1 \left[\int_0^T \left[1 + (y_1 \eta(t)/y_2)^{1/1-\gamma} \right]^{1-\gamma} \epsilon(t)^{\gamma-1} \epsilon_1(t) dt \right].$$

Hence the volatilities (expressed in percentage terms) of these two processes are given by

$$\theta_\epsilon^n(t) = (1-\gamma) \frac{\sigma_\epsilon(t)}{\epsilon(t)}; \quad \theta_z^1(t) = \frac{c_2^*(t)}{\epsilon(t)} \bar{\mu}(t); \quad \theta_z^2(t) = -\frac{c_1^*(t)}{\epsilon(t)} \bar{\mu}(t),$$

$$\|\theta^n(t)\| = \sqrt{\left((1-\gamma) \sigma_\epsilon(t)/\epsilon(t) \right)^2 + \left((1-c_n^*(t)/\epsilon(t)) \bar{\mu}(t) \right)^2}, \quad n = 1, 2,$$

$$\frac{\sigma_{c_n^* \epsilon}(t)}{c_n^*(t)} = \frac{\sigma_\epsilon(t)}{\epsilon(t)}; \quad \frac{\sigma_{c_1^* z}(t)}{c_1^*(t)} = \frac{c_2^*(t)}{(1-\gamma)\epsilon(t)} \bar{\mu}(t); \quad \frac{\sigma_{c_2^* z}(t)}{c_2^*(t)} = -\frac{c_1^*(t)}{(1-\gamma)\epsilon(t)} \bar{\mu}(t),$$

$$\left\| \frac{\sigma_{c_n^*}(t)}{c_n^*(t)} \right\| = \sqrt{\left(\sigma_\epsilon(t)/\epsilon(t) \right)^2 + \left((1-c_n^*(t)/\epsilon(t)) \bar{\mu}(t)/(1-\gamma) \right)^2}, \quad n = 1, 2.$$

The interest rate and the individual consumption growths (in percentage terms) are given by

$$r(t) = (1-\gamma) \frac{\mu_\epsilon(t)}{\epsilon(t)} - \frac{1}{2} (1-\gamma)(2-\gamma) \left(\frac{\sigma_\epsilon(t)}{\epsilon(t)} \right)^2 + \frac{1}{2} \gamma(3-\gamma) \frac{c_1^*(t) c_2^*(t)}{\epsilon(t)} \bar{\mu}(t)^2,$$

$$\frac{\mu_{c_n^*}^n(t)}{c_n^*(t)} = \frac{\mu_\epsilon(t)}{\epsilon(t)} + \frac{(\epsilon(t) - c_n^*(t)) \left[(1-\gamma/2) \epsilon(t) - c_n^*(t) \right]}{(1-\gamma)^2 \epsilon(t)^2} \bar{\mu}(t)^2, \quad n = 1, 2.$$

Unlike in the benchmark economy, agents' individual consumptions are no longer simply constant multiples of the aggregate consumption, being also driven by the extraneous disagreement process. The volatilities of the state price and individual consumption processes, $\|\theta^n(t)\|$ and $\|\sigma_{c_n^*}(t)/c_n^*(t)\|$ are unambiguously increased by the presence of extraneous uncertainty.¹⁵ "Excess volatility" is created over what one would expect from the fundamentals alone. The interest rate is unambiguously increased by the presence of the extraneous uncertainty, if $0 < \gamma < 1$, i.e., the agents are less risk averse than log utility ($0 < \gamma < 1$). If agents have log utility ($\gamma = 0$), the interest rate is unchanged, and if the agents are more risk averse than log utility ($\gamma < 0$), the interest rate is unambiguously decreased. The perceived consumption growths of the two agents are both unambiguously higher in the presence of the extraneous uncertainty if $\gamma \leq 0$. If $\gamma > 0$,

¹⁵For CARA preferences, the state price volatility $\|\theta^n(t)\|$ and the consumption volatility for absolute changes $\|\sigma_{c_n^*}(t)\|$ are again unambiguously increased by the presence of extraneous uncertainty.

the perceived consumption growth of agent n is higher than in the benchmark economy if and only if his consumption is less than the fraction $(1 - \gamma/2)$ of the aggregate consumption. This condition is more likely to be met for lower γ , lower ϵ_n , or “worse” past beliefs about the extraneous process.

A further point to note is that, in the benchmark economy with CRRA agents, if the aggregate endowment process follows a geometric Brownian motion, i.e., μ_ϵ/ϵ and σ_ϵ/ϵ are constants, then all the endogenous equilibrium parameters, τ , θ , $\sigma_{c_n^*}/c_n^*$ and $\mu_{c_n^*}/c_n^*$, are also constant, implying that the state price and consumption processes also follow a geometric Brownian motion. In the extraneous uncertainty case, this is not the case. Even if $\bar{\mu}$ is a constant, i.e., the extraneous disagreement process follows a geometric Brownian motion, the state price process and individual agents’ consumption do not follow a geometric Brownian motion, and all the endogenous equilibrium diffusion parameters are stochastic.

6. Extensions

6.1. More than two agents

The analysis of Sections 2–4 extends readily to the case of $N > 2$ agents. The uncertainty-information structure is as in Section 2, where each agent is effectively endowed with the probability space $(\Omega, \mathcal{F}^n, \{\mathcal{F}_t^n\}, \mathcal{P}^n)$ and $\mathcal{F}^n = \mathcal{F}^{W_\epsilon, W_z^n} = \mathcal{F}^{W_\epsilon, z}$, $n = 1, \dots, N$. We express each agent’s innovation process relative to agent 1’s as

$$dW_z^n(t) = dW_z^1(t) + \bar{\mu}^n(t) dt, \quad \bar{\mu}^n(t) \equiv \frac{\mu_z^1(t) - \mu_z^n(t)}{\sigma_z(t)}, \quad n = 1, \dots, N,$$

where, as before, $\bar{\mu}^n$ is essentially exogenous. The remaining analysis, results and notation of Sections 2–3 carry through, with appropriate modifications. Propositions 8 and 9 report the equilibrium market prices of risk, interest rate and consumption volatilities and mean growths, revealing that the essential structure of the equilibrium in Section 4 is maintained.

Proposition 8. *In an equilibrium with N agents where extraneous risk matters, the market prices of fundamental and nonfundamental risk perceived by the agents are given by*

$$\begin{aligned} \theta_\epsilon^n(t) &= A(t) \sigma_\epsilon(t), \\ \theta_z^n(t) &= A(t) \sum_{m \neq n} \frac{(\bar{\mu}^m(t) - \bar{\mu}^n(t))}{a_m(t)}, \\ \|\theta^n(t)\| &= A(t) \sqrt{\sigma_\epsilon(t)^2 + \sum_{m \neq n} \left((\bar{\mu}^m(t) - \bar{\mu}^n(t)) / a_m(t) \right)^2}, \quad n = 1, \dots, N, \end{aligned}$$

and the agents' consumption volatilities given by

$$\begin{aligned}\sigma_{c_n^* \epsilon}(t) &= \frac{A(t)}{a_n(t)} \sigma_\epsilon(t), \\ \sigma_{c_n^* z}(t) &= \frac{A(t)}{a_n(t)} \sum_{m \neq n} \frac{(\bar{\mu}^m(t) - \bar{\mu}^n(t))}{a_m(t)}, \\ \|\sigma_{c_n^*}(t)\| &= \frac{A(t)}{a_n(t)} \sqrt{\sigma_\epsilon(t)^2 + \sum_{m \neq n} \left((\bar{\mu}^m(t) - \bar{\mu}^n(t)) / a_m(t) \right)^2}, \quad n = 1, \dots, N.\end{aligned}$$

An agent's perceived market price of fundamental risk is still proportional to the aggregate fundamental risk, with the proportionality factor being the representative risk aversion in the economy. Consequently, an agent's consumption covariance with fundamental uncertainty has the same form as in the two-agent case. However, each agent n 's perceived market price of nonfundamental risk is proportional to a weighted average of his extraneous disagreement relative to all other agents m , $(\bar{\mu}^m(t) - \bar{\mu}^n(t))$, with the weight being proportional to agent m 's risk tolerance. In a sense, this term is agent n 's pessimism/optimism relative to the average remaining agents'. In this multi-agent case, we may identify the priced nonfundamental risk as being $\sum_{m \neq n} (\bar{\mu}^m(t) - \bar{\mu}^n(t)) / a_m(t)$, again being heterogeneous across agents. Then, an agent's consumption covariability with nonfundamental risk is, as usual, proportional to this priced risk.

Proposition 9. *In an equilibrium where extraneous risk matters with N agents, the interest rate is given by*

$$\begin{aligned}r(t) &= A(t) \mu_\epsilon(t) - \frac{1}{2} A(t) B(t) \sigma_\epsilon(t)^2 + A(t)^2 \sum_{n=1}^N \frac{\bar{\mu}^n(t)}{a_n(t)} \sum_{m \neq n} (\bar{\mu}^n(t) - \bar{\mu}^m(t)) / a_m(t) \\ &\quad - \frac{1}{2} A(t)^3 \sum_{n=1}^N \frac{b_n(t)}{a_n(t)^2} \sum_{m \neq n} \left((\bar{\mu}^m(t) - \bar{\mu}^n(t)) / a_m(t) \right)^2,\end{aligned}$$

and the agents' perceived consumption growth given by

$$\begin{aligned}\mu_{c_n^*}^n(t) &= \frac{A(t)}{a_n(t)} \mu_\epsilon(t) + \frac{1}{2} \frac{A(t)^2}{a_n(t)} \left(\frac{b_n(t)}{a_n(t)} - \frac{B(t)}{A(t)} \right) \sigma_\epsilon(t)^2 + \frac{A(t)^2}{a_n(t)} \sum_{\ell=1}^N \frac{\bar{\mu}^\ell(t)}{a_\ell(t)} \sum_{m \neq \ell} (\bar{\mu}^\ell(t) - \bar{\mu}^m(t)) / a_\ell(t) \\ &\quad + \frac{1}{2} \frac{A(t)^3}{a_n(t)} \left\{ \frac{b_n(t)}{a_n(t)} \left(\sum_{\ell \neq n} 1/a_\ell(t) \right) \sum_{m \neq n} \left((\bar{\mu}^m(t) - \bar{\mu}^n(t)) / a_m(t) \right)^2 \right. \\ &\quad \left. - \sum_{\ell \neq n} \frac{b_\ell(t)}{a_\ell(t)^2} \sum_{m \neq n} \left((\bar{\mu}^m(t) - \bar{\mu}^n(t)) / a_m(t) \right)^2 \right\}, \quad n = 1, \dots, N.\end{aligned}$$

The additional (third and fourth) terms in the interest rate have similar interpretations to the case of two agents. The third term arises because agents live under differing probability spaces.

It is a risk-tolerance weighted average of each agent n 's extraneous disagreement relative to agent 1 times that agent n 's disagreement relative to the average remaining agents'. Unlike in the two-agent economy, this third term now has an ambiguous sign. The fourth term is a weighted average of each agent n 's squared disagreement relative to the remaining average agent, with the weight driven by the product of that agent n 's prudence and risk aversion parameters. Again, this term arises from the precautionary savings motives of all the agents against nonfundamental risk, but since they price the nonfundamental risk heterogeneously, these terms may not be aggregated as simply as the precautionary savings against fundamental risk.

In an agent's mean consumption growth, the additional third term is due to the effect on the interest rate of agents' misperceptions about their own consumption growth, and has ambiguous sign, unlike the two-agent case. The fourth term is driven by the agent's precautionary savings motive against nonfundamental risk minus the average precautionary savings motive against nonfundamental risk. As an agent wants to save more, his perceived consumption growth will increase, but if all agents want to save more, the interest rate must go down to counteract these tendencies.

6.2. More than one extraneous process

We return to the two-agent framework but extend the analysis to the case of $L > 1$, possibly correlated, extraneous processes driven by an L -dimensional Brownian motion process $W_z \equiv (W_1, \dots, W_L)^\top$. The extraneous process dynamics are as in (2) but with the notation $z(t)$, $\mu_z(t)$ and $\sigma_z(t)$ denoting L -dimensional vectors and an $L \times L$ matrix, respectively. The uncertainty-information structure is as in Section 2, but with the processes z and W_z replaced by their L -dimensional counterparts. Associated with each agent is an L -dimensional innovation process $W_z^n \equiv (W_1^n, \dots, W_L^n)^\top$, $n = 1, 2$, related by

$$dW_z^2(t) = dW_z^1(t) + \bar{\mu}(t) dt, \quad \bar{\mu}(t) \equiv \sigma_z(t)^{-1}(\mu_z^1(t) - \mu_z^2(t)),$$

where μ_z^n is the vector of mean growths of the extraneous processes, as perceived by agent n . The vector process $\bar{\mu}$ captures the disagreement between the two agents about the mean growths of the L extraneous processes. Given that there are $L + 1$ dimensions of uncertainty that may affect equilibrium, in addition to the bond we assume there to be $L + 1$ zero net supply risky securities and hence an L -dimensional perceived market price of nonfundamental risk, $\theta_z^n \equiv (\theta_1^n, \dots, \theta_L^n)^\top$, for each agent. In Propositions 10 and 11 we characterize the equilibrium dynamics.

Proposition 10. *In an equilibrium where L -dimensional extraneous uncertainty matters, the market prices of fundamental, nonfundamental and total risk perceived by the agents are given by*

$$\theta_\epsilon^n(t) = A(t) \sigma_\epsilon(t); \quad \theta_\ell^1(t) = \frac{A(t)}{a_2(t)} \bar{\mu}_\ell(t); \quad \theta_\ell^2(t) = -\frac{A(t)}{a_1(t)} \bar{\mu}_\ell(t), \quad \ell = 1, \dots, L,$$

$$\|\theta^n(t)\| = A(t) \sqrt{\sigma_\epsilon(t)^2 + \sum_{\ell=1}^L (\bar{\mu}_\ell(t)/a_m(t))^2}, \quad (n, m) = (1, 2), (2, 1),$$

and the agents' consumption volatilities by

$$\sigma_{c_n^\epsilon}(t) = \frac{A(t)}{a_n(t)} \sigma_\epsilon(t); \quad \sigma_{c_n^\ell}(t) = \frac{A(t)}{a_n(t) a_m(t)} \bar{\mu}_\ell(t), \quad \ell = 1, \dots, L,$$

$$\|\theta_{c_n^*}(t)\| = \frac{A(t)}{a_n(t)} \sqrt{\sigma_\epsilon(t)^2 + \sum_{\ell=1}^L (\bar{\mu}_\ell(t)/a_m(t))^2}, \quad (n, m) = (1, 2), (2, 1).$$

These expressions have the same form as those for 1-dimensional extraneous uncertainty, the main difference being that the extraneous disagreement risk vector $\bar{\mu}$ is now multidimensional and driven by disagreement about multiple extraneous processes. The implication is that multiple additional terms appear in the total volatilities of the state prices, $\|\theta^n\|$, and individual agents' consumption streams, $\|\sigma_{c_n^*}\|$. Hence, for given $a_1(t), a_2(t)$, these volatilities increase further with every additional dimension of extraneous uncertainty that matters in equilibrium.

Proposition 11. *In an equilibrium where L -dimensional extraneous uncertainty matters, the interest rate is given by*

$$r(t) = A(t) \mu_\epsilon(t) - \frac{1}{2} A(t) B(t) \sigma_\epsilon(t)^2 + \frac{A(t)^2}{a_1(t) a_2(t)} \sum_{\ell=1}^L \bar{\mu}_\ell(t)^2 - \frac{1}{2} \frac{A(t)^3}{a_1(t)^2 a_2(t)^2} [b_1(t) + b_2(t)] \sum_{\ell=1}^L \bar{\mu}_\ell(t)^2,$$

and the agents' perceived consumption growths are given by

$$\mu_{c_n^*}^n(t) = \frac{A(t)}{a_n(t)} \mu_\epsilon(t) + \frac{1}{2} \frac{A(t)^2}{a_n(t)} \left[\frac{b_n(t)}{a_n(t)} - \frac{B(t)}{A(t)} \right] \sigma_\epsilon(t)^2 + \left(\frac{A(t)}{a_n(t)} \right)^2 \frac{1}{a_m(t)} \sum_{\ell=1}^L \bar{\mu}_\ell(t)^2$$

$$+ \frac{1}{2} \left(\frac{A(t)}{a_n(t) a_m(t)} \right)^3 [a_n(t) b_n(t) - a_m(t) b_m(t)] \sum_{\ell=1}^L \bar{\mu}_\ell(t)^2, \quad (n, m) = (1, 2), (2, 1).$$

There now appear $2L$ additional terms in the interest rate and the consumption growth formulae as compared with the benchmark case, two for each dimension of extraneous uncertainty. Hence, for given $a_n(t), b_n(t)$, the effect of extraneous uncertainty is increasing in the number of dimensions of that uncertainty believed to affect the real quantities of the economy.

7. Conclusion

We develop a continuous-time, pure-exchange, general equilibrium model where nonfundamental risk matters and gets priced, and investigate the implications of such nonfundamental risk. In equilibrium, the nonfundamental risk matters as a result of agents' heterogeneous beliefs about extraneous processes. Equilibrium is determined in terms of a representative agent's utility function with stochastic weights driven by the agents' initial wealths, disagreements about extraneous processes and nonfundamental uncertainty. We provide a full characterization of agents' perceived state price densities and consumption allocations by deriving explicit representations for market prices of fundamental and nonfundamental risk, interest rate, agents' consumption volatilities and growths. A conclusion is that, for given agents' risk tolerances, the agents' perceived state price densities and consumption streams are more volatile than they would be if extraneous risk did not matter. Further comparisons of equilibria with and without nonfundamental risk are carried out.

Appendix: Proofs

Proof of Proposition 1. This is a variation on Karatzas, Lehoczky, and Shreve (1990), to incorporate heterogeneous state price densities across agents. The first equality in (10), together with (8)–(9), imply (11)–(12), by substituting $\xi^2(t) = \xi^1(t)/\eta(t)$ and making use of the fact that the inverse of $U'(c; \Lambda)$ is $J(h; \Lambda) = I_1(h/\lambda_1) + I_2(h/\lambda_2)$. The two agents' budget constraints are equivalent, and using the property of a representative agent's utility function that $U'(c; \nu\Lambda) = \nu U'(c; \Lambda)$, $\nu > 0$, and taking $\nu = y_1$, they both only determine the ratio y_1/y_2 . (Agents' weights $1/y_n$ are only determined up to a multiplicative constant.) Substitution of (11) into (8) yields (14). To prove the last statement, \hat{c}_1 , and \hat{c}_2 given by (8), together with (11), imply the first equality in (10). The proof that the first equality implies the remaining two is a similarly modified version of that in Karatzas, Lehoczky, and Shreve (1990) (to account for η). Q.E.D.

Proof of Proposition 2. Common beliefs imply $\bar{\mu}(t) = 0$ from (4) and hence $\eta(t) = \eta(0) = 1$. Then from (11)–(12), the processes ξ^n depend only on ϵ , and from (14), so do c_n^* , $n = 1, 2$. Applying Itô's Lemma to (11)–(12) would imply $\theta_z^n(t) = 0$, $n = 1, 2$. Q.E.D.

Proof of Proposition 3. Applying Itô's Lemma to (8) and equating uncertainty terms yields

$$\frac{1}{a_n(t)} \theta_\epsilon^n(t) = \sigma_{c_n^* \epsilon}(t), \quad \frac{1}{a_n(t)} \theta_z^n(t) = \sigma_{c_n^* z}(t), \quad n = 1, 2. \quad (\text{A1})$$

Summing over agents and recalling (6)–(7) yields

$$\left[\frac{1}{a_1(t)} + \frac{1}{a_2(t)} \right] \theta_\epsilon^1(t) = \sigma_{c_1^* \epsilon}(t) + \sigma_{c_2^* \epsilon}(t) = \sigma_\epsilon(t), \quad (\text{A2})$$

$$\left[\frac{1}{a_1(t)} + \frac{1}{a_2(t)} \right] \theta_z^1(t) - \frac{1}{a_2(t)} \bar{\mu}(t) = \sigma_{c_1^* z}(t) + \sigma_{c_2^* z}(t) = 0, \quad (\text{A3})$$

where the last equality in (A2) and (A3) follows by applying Itô's Lemma to clearing in the consumption good market and equating uncertainty terms.

Differentiating $c_1^*(t) = I_1(y_1 U'(\epsilon(t); 1/y_1, \eta(t)/y_2))$ with respect to $\epsilon(t)$ and manipulating yields $\partial c_1^*(t)/\partial \epsilon(t) = A(t)/a_1(t)$; similarly, for agent 2, $\partial c_2^*(t)/\partial \epsilon(t) = A(t)/a_2(t)$. Then, using $\partial c_1^*(t)/\partial \epsilon(t) + \partial c_2^*(t)/\partial \epsilon(t) = 1$ (implied by $c_1^*(t) + c_2^*(t) = \epsilon(t)$), we deduce $1/A(t) = 1/a_1(t) + 1/a_2(t)$. Substituting into (A2)–(A3) and rearranging yields the desired expressions for θ_ϵ^1 and θ_z^1 . Then, using (6)–(7), we derive the corresponding expressions for agent 2. Finally, we derive $\|\theta^n(t)\|$ from its definition $\|\theta^n(t)\| \equiv \sqrt{(\theta_\epsilon^n(t))^2 + (\theta_z^n(t))^2}$. Q.E.D.

Proof of Proposition 4. (A1) and (15)–(16) yield (19)–(20). (21) is immediate. Q.E.D.

Proof of Proposition 5. Applying Itô's Lemma to agents' first order conditions (8) and equating drift terms yields

$$\frac{r(t)}{a_n(t)} = \mu_{c_n^*}^n(t) - \frac{1}{2} b_n(t) \sigma_{c_n^* \epsilon}(t)^2 - \frac{1}{2} b_n(t) \sigma_{c_n^* z}(t)^2, \quad n = 1, 2. \quad (\text{A4})$$

Applying Itô's Lemma to clearing in the consumption good market and equating drift terms (under agent 1's probability space) yields

$$\mu_{c_1^*}^1(t) + \mu_{c_2^*}^2(t) = \mu_\epsilon(t) - \bar{\mu}(t) \sigma_{c_2^* z}(t) = \mu_\epsilon(t) + \frac{A(t)}{a_1(t) a_2(t)} \bar{\mu}(t)^2, \quad (\text{A5})$$

where the second equality makes use of (20). Summing (A4) over agents and using (A5) and the fact that $1/A(t) = 1/a_1(t) + 1/a_2(t)$ yields

$$\begin{aligned} r(t) = A(t) \mu_\epsilon(t) - \frac{1}{2} A(t)^3 \left(\frac{b_1(t)}{a_1(t)^2} + \frac{b_2(t)}{a_2(t)^2} \right) \sigma_\epsilon(t)^2 \\ + \frac{A(t)^2}{a_1(t) a_2(t)} \bar{\mu}(t)^2 + \frac{1}{2} \frac{A(t)^3}{a_1(t)^2 a_2(t)^2} (b_1(t) + b_2(t)) \bar{\mu}(t)^2. \end{aligned} \quad (\text{A6})$$

Differentiating $c_1^*(t) = I_1(y_1 U'(\epsilon(t); 1/y_1, \eta(t)/y_2))$ twice with respect to $\epsilon(t)$, using $\partial c_1^*(t)/\partial \epsilon(t) = A(t)/a_1(t)$ yields $\partial^2 c_1^*(t)/\partial \epsilon^2(t) = (A(t)/a_1(t))^2 b_1(t) - (A(t)/a_1(t)) B(t)$; similarly for agent 2 we obtain $\partial^2 c_2^*(t)/\partial \epsilon^2(t) = (A(t)/a_2(t))^2 b_2(t) - (A(t)/a_2(t)) B(t)$. Then using $\partial^2 c_1^*(t)/\partial \epsilon^2(t) + \partial^2 c_2^*(t)/\partial \epsilon^2(t) = 0$ (implied by $c_1^*(t) + c_2^*(t) = \epsilon(t)$), we deduce $B(t)/A(t)^2 = b_1(t)/a_1(t)^2 + b_2(t)/a_2(t)^2$, which when substituted into (A6), yields (23). Q.E.D.

Proof of Proposition 6. Substitution of (19), (20) and (23) into (A4) yields for $n = 1, 2$

$$\begin{aligned} \mu_{c_n^*}^n(t) = \frac{A(t)}{a_n(t)} \mu_\epsilon(t) + \frac{1}{2} \frac{A(t)^3}{a_n(t)} \left(\frac{b_1(t)}{a_1(t)^2} + \frac{b_2(t)}{a_2(t)^2} \right) \sigma_\epsilon(t)^2 + \frac{A(t)^2}{a_1(t) a_2(t)} \bar{\mu}(t)^2 \\ + \frac{1}{2} \frac{A(t)^3}{a_1(t)^2 a_2(t)^2} (b_1(t) + b_2(t)) \bar{\mu}(t)^2 - \frac{1}{2} b_n(t) \left(\frac{A(t)}{a_n(t)} \right)^2 \sigma_\epsilon(t)^2 - \frac{1}{2} b_n(t) \left(\frac{A(t)}{a_1(t) a_2(t)} \right)^2 \bar{\mu}(t)^2, \end{aligned}$$

which after some manipulation, yields (24). Q.E.D.

Proof of Proposition 7. Substitution of the power or log utility into (11)–(14), (15)–(17), and (23)–(24) yields the desired results. Q.E.D.

Proof of Propositions 8–11. The proofs are N -agent or L -dimensional versions of Propositions 3–6. Q.E.D.

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