A SPECIALIST'S QUOTED DEPTH AS A STRATEGIC CHOICE VARIABLE

by

Kenneth A. Kavajecz

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RODNEY L. WHITE CENTER FOR FINANCIAL RESEARCH
The Wharton School
University of Pennsylvania
3254 Steinberg Hall-Dietrich Hall
Philadelphia, PA 19104-6367
(215) 898-7616

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Kenneth A. Kavajecz
Department of Finance
The Wharton School of the University of Pennsylvania
2300 Steinberg Hall-Dietrich Hall
Philadelphia, PA 19104-6367

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Abstract

There are two distinct components to a specialist's price schedule, prices and depths. This paper presents a model of a specialist's problem of choosing prices and depths jointly in order to maximize profits. Closed form solutions are provided for both constrained and unconstrained versions of the model. The contribution of this work is twofold. First, the model demonstrates the strategic importance of depths for the specialist and highlights its effect on overall liquidity. Second, the joint responses of prices and depths to various concerns of the specialist may be useful in differentiating between competing microstructure effects. Comparative static results show how depths respond to changes in: (1) the amount of asymmetric information, (2) uncertainty about the terminal value, (3) the prior probability assessments of future prices and (4) the distribution of liquidity trades.

I. Introduction

At the most basic level, the study of how assets are traded in financial markets must include an analysis of the components of transactions, that is, price and quantity. The market microstructure field of finance has produced a considerable amount of research describing how specialists or market makers determine their prices; examples include: Kyle (1985) and Glosten and Milgrom (1985). In general, the research shows that specialists set their prices taking into consideration such items as inventory costs and adverse selection costs. Surprisingly, little work has been done on the quantity component of a specialist's price schedule. The quantity component is important because overall liquidity cannot be determined without assessing the depth of the market, and the choice of depth is potentially an important tool for managing inventory and mitigating adverse selection problems.

This paper presents closed form solutions for both unconstrained and constrained versions of a monopoly specialist's problem of choosing prices and depths jointly to maximize profits in a market with adverse selection. The unconstrained version has the specialist controlling both prices and depths whereas the constrained versions have the specialist controlling either prices or depths. The contribution of this work is twofold. First, the model improves upon earlier theoretical work by endogenizing the quantity component of the specialist's price schedule as well as maintaining the actual form of the specialist's price schedules posted on the New York Stock Exchange (NYSE). Consequently, the model demonstrates the strategic importance of depths for the specialist and highlights its effect on overall liquidity. Specifically, the results demonstrate that a specialist has a unique optimal depth and that the depth is a useful tool to address many of a specialist's concerns. Second, the distinct joint responses of prices and depths to various concerns of the specialist can be used to differentiate between competing microstructure effects. The comparative static results show how both prices and depths respond to changes in: (1) the amount of asymmetric information, (2) uncertainty about the terminal value, (3) the prior probabilities of future prices and (4) the distribution of liquidity trades.

Before proceeding it is important to be explicit about the definition of depth as it is used throughout the paper. Specialists on the NYSE post volume quotations (or depths) in addition to bid and ask prices. Strictly speaking, the specialist's depths specify that she is willing to purchase (sell) shares at the bid (ask) for any amount at or below the bid (ask) depth.[1] As a practical matter, the ability to set the depth parameter provides another tool for the specialist. For example, the specialist could manage her inventory by increasing the bid (ask) depth when her position is lower (higher) than the desired level . The depth quotes could also be used to mitigate the adverse selection problem. The approach of a significant information event might leave the specialist open to a barrage of informed traders. Under these circumstances, the specialist might collapse her quotes to one round lot or have the quotes merely reflect the interest in the limit order book as in Rock (1996). This way she is able to minimize the profit potential of the informed traders at her expense. Finally, the specialist may also 'experiment', in the Leach and Madhavan (1993) sense, with the depths in order to induce informed traders to reveal their information. The question is, given the ability to use prices and/or depths, how does the specialist chose to act?

The remainder of the paper is organized as follows. Section II reviews earlier theoretical work. Section III contains the unconstrained model and comparative statics. Section IV contains the two constrained models. Section V outlines possible extensions and Section VI concludes.

II. Earlier Theoretical Work

In order to place this paper within the context of earlier work, I review theoretical research relating to the study of depths. The first theory piece on the subject is Easley and O'Hara (1987) which demonstrates that large trades may execute at worse prices due to adverse selection costs. Their model is

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[1] It is widely known that the specialist is often willing to transact at better prices than what is quoted or willing to honor the quoted prices for more than the amount specified in the depth. Since these deviations from the strict price schedule are discretionary, they are removed from consideration to focus attention on the joint choice of prices and depths.

one of the first to allow variable quantities (small and large) to be traded. The model has two risk neutral market makers along with informed and uninformed traders. The competing market makers post 'no regret' prices for each quantity, while the informed traders maximize profits conditional on those price schedules. This model does not allow market makers to limit the quantity they are willing to honor (post depths), rather they must adjust their prices to induce the volume of trading they desire for small and large trades individually. Therefore, although multiple quantities are allowed, maximum quantities can not be controlled independently of price.

The second theory piece on the subject is Glosten (1989). The Glosten model generalizes the problem by solving for the complete optimal price schedule under the assumptions of negative exponential utility and normally distributed random variables. In the model, the market maker maximizes profits by quoting a price schedule that specifies the price per share as a function of the quantity traded, given the strategies of the market participants. His results demonstrate that a monopolist specialist is able to provide a more liquid market compared to a group of competitive market makers because the monopolist is able to average profits across trades. Again, there is no role for quoted depth as a choice variable, rather the depths are set implicitly by the prices.

More recently, researchers have focused on modeling the specialist's choice of depth. Charoenwong and Chung (1994) develop a model of a specialist's choice of depth in a continuous time setting; however, they assume that the specialist's quoted prices are exogenous. In particular, they assume that the ask (bid) price is simply the expected value plus (minus) some profit margin. Consequently, their focus is on the choice of depth abstracting from any joint interaction with prices.

Dupont (1995) improves upon Charoenwong and Chung (1994) by modeling the specialist's *joint* choice of prices and depths. Dupont's focus is on the adverse selection problem and how the relation between price and depth changes if the specialist is risk adverse rather than risk neutral. His simulation results show that depths, in absolute value, are decreasing in the information quality of the traders and, in addition, depths are more sensitive than the bid-ask spread to a change in the quality of the informed trader's information.

III. The Unconstrained Model

Consider a one-period model of the trading of a single risky asset. The payoff to the risky asset is the random variable θ. The terminal value of the security is θ_1 with probability μ and θ_2 with probability $1-\mu$ where $\theta_1 < \theta_2$. There is a continuum of traders who are differentiated by the amount of private information they possess; λ of the trading population are perfectly informed about the payoff of the risky asset (the informed) and $1-\lambda$ of the trading population possess no private information of the terminal payoff of the risky asset (the uninformed), where $0 < \lambda < 1$.

In addition to the traders there is a single risk-neutral monopolist specialist who makes a market in the risky asset. Her objective is to post a price schedule in order to maximize profits; however, the price schedule must have a special form. In particular, the specialist must post a bid (ask) price and a quantity up to which she will honor the posted bid (ask) price, above which there will be no trade.

The chronology of the trading process is as follows. The specialist posts the price schedule consisting of a bid and ask price (denoted b and a, respectively where $b<a$) along with the corresponding bid and ask depth quotes (denoted β and α, respectively). In doing so, the specialist takes into consideration the likelihood of trading with both types of traders and the amount that they are likely to trade. A trader is chosen at random from the population of market participants and decides to trade or not to trade. If the trader decides to trade, he chooses a quantity, where the quantity is constrained to be less than or equal to the relevant depth quote.

Since the informed trader has perfect information about the terminal value of the risky asset, he will have an unbounded demand if the specialist has 'mispriced' the asset; however, the specialist's price schedule precludes any trading of quantities greater than the posted bid and ask depths. In addition, the fact that there is a continuum of informed traders means that each trader acts as if he will have at most one chance to trade. Given these constraints, informed trader j will trade according to:

$$q_j^I = \begin{cases} -\beta & \text{if } b > \theta^* \\ |\alpha| & \text{if } a < \theta^* \\ 0 & \text{otherwise} \end{cases}$$

where θ^* represents the true value of the risky asset.[2]

The uninformed possess no private information; therefore, their motivation for trading cannot be information driven, rather they can be thought of as a collection of traders all with various (exogenously determined) motivations and propensities for trading. Uninformed trader j can be described by a pair of numbers (e_j, r_j) which represent the quantity which he wishes to trade (his endowment) and his reservation value, respectively. Positive (negative) values of e_j imply that the trader wants to sell (buy) some amount of the risky asset. In addition, high (low) values of r_j imply a high (low) valuation of the risky asset. Given the above assumptions, each individual uninformed trader places his order according to the following strategy:

$$q_j^u = \begin{cases} -\min\,[\,\beta\,,e_j\,] & \text{if } e_j > 0 \text{ and } b > r_j \\ +\min\,[\,|\alpha|,|e_j|\,] & \text{if } e_j < 0 \text{ and } a < r_j \\ 0 & \text{otherwise} \end{cases}$$

The strategy suggests that an uninformed trader will buy (sell) if his reservation price is higher (lower) than the ask (bid) and the amount that is traded is constrained to be no more than the ask (bid) volume quote.[3] The trade quantities of the uninformed traders have cumulative distribution function F(x) with

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[2]The depths are quoted from the specialist's perspective. Since the specialist increases (decreases) her inventory when she transacts at the bid (ask), $\beta > 0$ and $\alpha < 0$.

[3] Notice that this formulation does not suffer from the problem addressed by Paul (1994). The uninformed traders in this model are different from the exogenous liquidity traders in other models because they are unwilling to take an infinite expected loss to conduct their transactions.

support $[t_1, t_2]$ where $t_1 < 0 < t_2$. Similarly, the reservation values have a cumulative distribution function G(x) with support $[\theta_1, \theta_2]$. Moreover, I assume that the trade quantities and reservation values are independent. Conditional on a price schedule (b,β,a,α), the trades of the uninformed population come from the following distribution:

$$q^u(b,\beta,a,\alpha) = \begin{cases} -\beta & wp \quad [1-F(\beta)]\cdot G(b) \\ \in(-\beta,0) & wp \quad [F(\beta)-F(0)]\cdot G(b) \\ |\alpha| & wp \quad F(\alpha)\cdot[1-G(a)] \\ \in(0,|\alpha|) & wp \quad [F(0)-F(\alpha)]\cdot[1-G(a)] \\ 0 & wp \quad [1-[1-F(0)]\cdot G(b)-F(0)\cdot(1-G(a))]] \end{cases}$$

Taking the strategies of the informed and uninformed as given, the specialist maximizes her expected profit using the following objective function:

$$\begin{gathered} E[\pi(b,\beta,a,\alpha)] = \\ \mu\lambda[\beta(\theta_1-b)] + (1-\mu)\lambda[\alpha(\theta_2-a)] + \\ \mu(1-\lambda)\ [\ [1-F(\beta)]G(b)\beta(\theta_1-b) + \\ [F(\beta)-F(0)]G(b)E[q|q\in(0,\beta)](\theta_1-b) + \\ F(\alpha)[1-G(a)]\alpha(\theta_1-a) + \\ [F(0)-F(\alpha)][1-G(a)]E[q|q\in(\alpha,0)](\theta_1-a)\ ] + \\ (1-\mu)(1-\lambda)[\ [1-F(\beta)]G(b)\beta(\theta_2-b) + \\ [F(\beta)-F(0)]G(b)E[q|q\in(0,\beta)](\theta_2-b) + \\ F(\alpha)[1-G(a)]\alpha(\theta_2-a) + \\ [F(0)-F(\alpha)][1-G(a)]E[q|q\in(\alpha,0)](\theta_2-a)\ ] \end{gathered}$$

The first two terms represent the expected loss from transacting with the informed traders and the remaining terms represent the expected profit or loss due to transacting with the uninformed traders. The objective function makes implicit assumptions about the relation between the specialist's choice variables. These assumptions can be summarized by the following two restrictions and will be verified later in equilibrium: (1) $t_1 \leq \alpha < 0 < \beta \leq t_2$, and (2) $\theta_1 < b < a < \theta_2$. In order to proceed with solving the model, more structure must be placed on the problem. I assume that e_j is distributed uniformly from t_1 to

t_2 and r_j is distributed uniformly from θ_1 to θ_2, i.e., $e_j \sim U[t_1, t_2]$, and $r_j \sim U[\theta_1, \theta_2]$.[4] Using the uniform distribution assumptions, the specialist's maximization problem can restated as follows:

$$
\begin{aligned}
&E[\pi(b,\beta,a,\alpha)] = \\
&\mu\lambda\beta(\theta_1 - b) + (1-\mu)\lambda\alpha(\theta_2 - a) \\
&+ (1-\lambda)\left(\frac{t_2-\beta}{t_2-t_1}\right)\left(\frac{b-\theta_1}{\theta_2-\theta_1}\right)\beta\,\{\mu(\theta_1-b)+(1-\mu)(\theta_2-b)\} \\
&+ (1-\lambda)\left(\frac{\beta}{t_2-t_1}\right)\left(\frac{b-\theta_1}{\theta_2-\theta_1}\right)\left(\frac{1}{2}\beta\right)\{\mu(\theta_1-b)+(1-\mu)(\theta_2-b)\} \\
&+ (1-\lambda)\left(\frac{\alpha-t_1}{t_2-t_1}\right)\left(\frac{\theta_2-a}{\theta_2-\theta_1}\right)\alpha\,\{\mu(\theta_1-a)+(1-\mu)(\theta_2-a)\} \\
&+ (1-\lambda)\left(\frac{-\alpha}{t_2-t_1}\right)\left(\frac{\theta_2-a}{\theta_2-\theta_1}\right)\left(\frac{1}{2}\alpha\right)\{\mu(\theta_1-a)+(1-\mu)(\theta_2-a)\}
\end{aligned}
$$

Remember that the depths take on the sign of the specialist's implied inventory change; therefore, the bid depth is positive and the ask depth is negative, i.e., $\beta > 0$ and $\alpha < 0$. Moreover, the objective function takes into account the fact that setting depths greater than the support of the uninformed trader's endowment is a strictly dominated strategy.

An equilibrium in this model is a set of four functions that describe the price schedule of the specialist given the strategies of the traders and the implicit assumptions embedded in the objective function itself.

[4]A number of distributional assumptions were attempted; the only one that proved tractable was the uniform assumption.

Proposition 1: If μ satisfies the condition:

$$\left(\frac{\lambda - \lambda\left(\frac{t_2}{t_1}\right)}{1 - \lambda\left(\frac{t_2}{t_1}\right)}\right) < \mu < \left(\frac{1 - \lambda}{1 - \lambda\left(\frac{t_1}{t_2}\right)}\right)$$

then the following is the unique equilibrium for the unconstrained one-period game:

$$b_u^* = \frac{1}{4}(3E_\mu[\theta] + \theta_1) - \frac{1}{4}(1-\mu)(\theta_2 - \theta_1)\sqrt{1 + \left(\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)(t_2 - t_1)}$$

$$\beta_u^* = \frac{3}{2}t_2 - \frac{1}{2}t_2\sqrt{1 + \left(\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)(t_2 - t_1)}$$

$$a_u^* = \frac{1}{4}(3E_\mu[\theta] + \theta_2) + \frac{1}{4}\mu(\theta_2 - \theta_1)\sqrt{1 - \left(\frac{8}{t_1}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)(t_2 - t_1)}$$

$$\alpha_u^* = \frac{3}{2}t_1 - \frac{1}{2}t_1\sqrt{1 - \left(\frac{8}{t_1}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)(t_2 - t_1)}$$

Proof: See Appendix 1.

The restriction on μ is equivalent to requiring the specialist to keep both sides of the market open. The left hand inequality covers the ask side of the market, $a_u^* < \theta_2$ and $\alpha_u^* < 0$, and the right hand side covers the bid side of the market, $b_u^* > \theta_1$, $\beta_u^* > 0$. There are two ways of interpreting the restriction on the primatives (μ, λ, t_1, t_2). The first interpretation is that the restriction requires sufficient uncertainty about the final payoff of the risky asset, i.e., μ not to close too zero or one, so that the specialist cannot afford to forgo the expected profit on one side of the market by shutting that side down. The second interpretation argues that the restriction requires that there not be so many informed traders in the trading population as to make the specialist's position unprofitable, i.e., λ close to zero. In order to better understand the optimal price and depth functions, it is useful to highlight the special case where there are no informed traders. If $\lambda = 0$, the radicals disappear and the price schedule reduces to $b_u^* = 1/2(\theta_1 + E_\mu[\theta])$, $\beta_u^* = \bar{t}_2$, $a_u^* = 1/2(E_\mu[\theta] + \theta_2)$, $\alpha_u^* = t_1$ where $0 < \mu < 1$. Consequently, the prices are just the midpoints between the expected value and the respective terminal values and the depths allow all traders to trade their desired quantities.

There is a rich set of comparative static results that can be obtained from the posited equilibrium. A standing assumption for all the forthcoming Corollaries is that the parameter restrictions placed on μ in Proposition 1 still hold.

Corollary 1.1: As the percentage of informed traders (λ) increases, the bid-ask spread increases and both the bid and ask depths decrease. Formally stated:

$$\frac{\partial b_u^*}{\partial \lambda} < 0, \quad \frac{\partial a_u^*}{\partial \lambda} > 0 \quad and \quad \frac{\partial \beta_u^*}{\partial \lambda} < 0, \quad \frac{\partial \alpha_u^*}{\partial \lambda} > 0$$

Proof: See Appendix 1.

As the adverse selection problem becomes more acute, the standard result of a widening bid-ask spread obtains and in addition, both depths narrow. In other words, as the adverse selection problem worsens the specialist is unwilling to supply as much liquidity to the market, either in terms of prices or in terms of quantities. This result is consistent with the theory work done by Dupont (1995). It is also consistent with the empirical work done by Lee, Mucklow and Ready (1993). In general, their work shows that both spreads and depths are needed to infer changes in liquidity. Specifically, they demonstrate that quoted spreads widen and quoted depths drop in advance of quarterly earnings announcements. Moreover, they report that the quoted spread widens an average of 1.4% the day preceding the announcement and 8.2% on the announcement while the depths fall an average of 5.3% and 4.4% for the same time periods. They conclude that the specialist and other liquidity providers actively manage information asymmetry risk by adjusting both spreads and depths.

<u>Corollary 1.2:</u> Given an increase in the mean preserving spread of the possible terminal values of the risky asset, $(\theta_2 - \theta_1)$, the bid-ask spread increases and the depths remain unchanged.

$$\frac{\partial b_u^*}{\partial(\theta_2 - \theta_1)} < 0, \quad \frac{\partial a_u^*}{\partial(\theta_2 - \theta_1)} > 0 \quad \textit{and} \quad \frac{\partial \beta_u^*}{\partial(\theta_2 - \theta_1)} = 0, \quad \frac{\partial \alpha_u^*}{\partial(\theta_2 - \theta_1)} = 0$$

Proof: See Appendix 1.

An increase in the spread of possible terminal values can be thought of as an increase in the amount of uncertainty that the specialist faces. That is, the specialist is, in some sense, less sure of the possible terminal value; put differently, the cost of being wrong about the terminal value increases. Notice that changes in $(\theta_2 - \theta_1)$ widen the bid-ask spread but, unlike Corollary 1.1, have no effect on the depth parameters. In this model, depths are used to mitigate the adverse selection problem but are not used to address uncertainty about the terminal value.

<u>Corollary 1.3:</u> As the probability that $\theta = \theta_1$, (μ), increases, the bid and ask prices decrease and the bid depth decreases while the ask depth increases.

$$\frac{\partial b_u^*}{\partial \mu} < 0, \quad \frac{\partial a_u^*}{\partial \mu} < 0 \quad \textit{and} \quad \frac{\partial \beta_u^*}{\partial \mu} < 0, \quad \frac{\partial \alpha_u^*}{\partial \mu} < 0$$

Proof: See Appendix 1.

Changes in the prior probability measure merely shift prices toward the more likely terminal value and redistribute depth toward the side of the market with the less likely terminal value. For instance, an increase in μ causes both the bid and ask prices to fall in order to readjust to the new expected value of the risky asset, the bid depth falls in order to curtail probable losses, and the ask depth increases to allow larger trades with a higher probability of profit.

Corollary 1.4: Changes in the distribution of desired liquidity trades by increasing either t_1 or t_2, where $t_1 < 0 < t_2$, raise prices and increase the bid depth while decreasing the ask depth:

$$\frac{\partial b_u^*}{\partial t_1} > 0, \quad \frac{\partial a_u^*}{\partial t_1} > 0 \quad \textit{and} \quad \frac{\partial \beta_u^*}{\partial t_1} > 0, \quad \frac{\partial \alpha_u^*}{\partial t_1} > 0$$

$$\frac{\partial b_u^*}{\partial t_2} > 0, \quad \frac{\partial a_u^*}{\partial t_2} > 0 \quad \textit{and} \quad \frac{\partial \beta_u^*}{\partial t_2} > 0, \quad \frac{\partial \alpha_u^*}{\partial t_2} > 0$$

Proof: See Appendix 1.

Although not shown, it is easy to verify that price reactions wash out if the change in the distribution of endowments results in $t_1 = -t_2$; therefore, changes in the distribution of desired liquidity trades induce a change in prices as long as the buy and sell sides are not perfectly symmetric.

A comparison of the comparative static results shows that the specialist has a unique price/depth response for various changes in the trading environment. The consideration of either prices or depths alone is insufficient to discriminate between these effects. For instance, notice that considering only prices, it would be impossible to distinguish between a change in λ and a change in $(\theta_2 - \theta_1)$ (Corollaries 1.1 and 1.2) or a change in μ and a change in (t_1, t_2) (Corollaries 1.3 and 1.4). However, three distinct reactions can be established from the model by appraising *both* prices and depths. The first is a widening (narrowing) of the bid-ask spread and a decrease (increase) in the depths. This reaction is attributable to changes in the information asymmetry. The second is a change in the bid-ask spread with no change in the depths. Changes in price uncertainty drive this reaction. Lastly, the bid-ask spread may shift downward (upward) accompanied by a reallocation of depth from the bid (ask) to the ask (bid). There are two factors that could produce this reaction that are indistinguishable from one another: changes in the probability assessments for the risky asset and a redistribution of liquidity trades. Although inventory effects are excluded from this model, it is likely that inventory imbalances could cause a similar reaction. In addition, the comparative static results suggest that the specialist may have 'preferred' applications for prices and depths. For example, a change in the distribution of prices is handled by adjusting the bid and ask (Corollary 1.2), whereas a symmetric change in the distribution of liquidity trades is handled by adjusting the depths (Corollary 1.4).

IV. Constrained Models

This sections presents results on how the model performs under two separate constraints. In the first model, the price is constrained to be some proportional spread around the expected value and the second model constrains the depths to be some fraction of the maximum liquidity trade. The reasons for investigating these constraints are to assess whether prices and depths are used as complements or substitutes and to observe how the specialist adapts to extreme trading conditions. Specifically, constraining the prices addresses how the specialist's depth quotes are affected by minimum tick size restrictions. Moreover, constraining the depths addresses how specialist's prices are affected by size guarantee requirements. In each model, the resulting choice of depths and prices are presented. The results presented here match closely with empirical work done by other researchers.

Constrained Prices

Consider the one period model described earlier, where the specialist must post a price schedule for the risky asset that maximizes her profit given that λ of the trading population are perfectly informed and $1-\lambda$ of the trading population are uninformed. Instead of allowing the specialist the ability to change prices, suppose that prices are exogenously set according to the following equations:

$$a = (1+\gamma)(\mu\theta_1 + (1-\mu)\theta_2)$$

$$b = (1-\gamma)(\mu\theta_1 + (1-\mu)\theta_2)$$

The parameter γ is assumed to be an exogenous constant which forces a proportional spread of $2\gamma(\mu\theta_1 + (1-\mu)\theta_2)$ centered around the expected value of the risky asset. Substituting the above constraints into the unconstrained model yields an objective function with only two choice variables, the bid and ask depths. As in the unconstrained model the bid and ask depths are independent of each other so the maximization problem reduces to two separate single-variable maximization problems.

Proposition 2: If γ satisfies the condition:

$$0 < \gamma < \min\left[\frac{(1-\mu)(\theta_2-\theta_1)}{\mu\theta_1 + (1-\mu)\theta_2}, \frac{\mu(\theta_2-\theta_1)}{\mu\theta_1 + (1-\mu)\theta_2}\right]$$

then the following is the unique equilibrium for the *price*-constrained one-period game:

$$\beta_c^* = t_2 - \left(\frac{\lambda}{1-\lambda}\right)\left(\frac{(t_2-t_1)(\theta_2-\theta_1)\mu}{\gamma(\mu\theta_1 + (1-\mu)\theta_2)}\right)$$

$$\alpha_c^* = t_1 + \left(\frac{\lambda}{1-\lambda}\right)\left(\frac{(t_2-t_1)(\theta_2-\theta_1)(1-\mu)}{\gamma(\mu\theta_1 + (1-\mu)\theta_2)}\right)$$

Proof: See Appendix 2.

The condition in Proposition 2 ensures that both sides of the market remain open. A comparison of these results with those of the unconstrained model are telling. Like the unconstrained model, if there are no informed traders, ($\lambda=0$) the specialist allows all quantities to be traded. Unlike the unconstrained model, a change in *any* of the primatives will change the depth quotes. This suggests that the depth quotes can be used as a substitute for the bid and ask prices. Furthermore, for a given change in one of the primatives, there is a larger change in the constrained depths than in the unconstrained depths. Notice that this constrained model is analogous to the model posed by Charoenwong and Chung (1994), in that the only two choice variables are the bid and ask depths.

Corollary 2.1: As the spread narrows, that is, γ decreases, both the bid and ask depths decrease.

$$\frac{\partial \beta_c^*}{\partial \gamma} > 0 \quad and \quad \frac{\partial \alpha_c^*}{\partial \gamma} < 0$$

Proof: See Appendix 2.

This insight has been revealed empirically in the Harris (1994) paper. He explores the effect of having a minimum tick size on the trading environment. In particular, he explores the relation between discrete spreads, depths and volume. His regression results show that a smaller minimum price variation, which implies a smaller minimum spread, would decrease displayed quotation size or depth. This finding is consistent with Corollary 2.1 where we see that as the constrained spread decreases around the expected value, the specialist compensates by reducing the number of shares she is willing to trade.

Constrained Depths

Again consider the one period model described earlier; however, now suppose that the specialist has full control over her prices but the depth figures are fixed according to the following equations:

$$\alpha = \rho t_1$$

$$\beta = \rho t_2$$

The parameter ρ is assumed to be an exogenous constant which forces the depths to be some fraction of the maximum trade size desired by the uninformed traders. Like before, substituting the above constraints into the unconstrained model yields an objective function with two independent choice variables, the bid and ask prices.

Proposition 3: If $0 < \rho$, then the following is the unique equilibrium for the *depth*-constrained one-period game:

$$b_c^* = \frac{1}{2}(\mu\theta_1 + (1-\mu)\theta_2 + \theta_1) - \frac{1}{2}\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{(t_2 - t_1)(\theta_2 - \theta_1)\mu}{(1 - \frac{1}{2}\rho)t_2}\right)$$

$$a_c^* = \frac{1}{2}(\mu\theta_1 + (1-\mu)\theta_2 + \theta_2) + \frac{1}{2}\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{(t_2 - t_1)(\theta_2 - \theta_1)(1-\mu)}{(1 - \frac{1}{2}\rho)t_2}\right)$$

Proof: See Appendix 3.

As was demonstrated in the price constrained case, the depth-constrained model yields the same result as the unconstrained model when $\lambda = 0$. Likewise, the bid and ask prices act as substitutes for the depths in that they are affected by changes in any of the primatives, and their reactions are larger for a given change in the primatives.

Corollary 3.1: As the depths increase, that is, ρ approaches 1, the spread widens.

$$\frac{\partial b_c^*}{\partial \rho} < 0 \quad and \quad \frac{\partial a_c^*}{\partial \rho} > 0$$

Proof: See Appendix 3.

The results shown here are consistent with those of Reiss and Werner (1994). Their research compares the transaction costs in two dealer markets, the NASDAQ and the London Stock Exchange (SEAQ). They find that the SEAQ's spreads are much wider than the spreads on the NASDAQ. Moreover, SEAQ market makers rarely change their prices, opting instead to transact within the spread much of the time. One explanation for this difference is that NASDAQ market makers have much smaller size guarantee restrictions whereas SEAQ market makers must stand ready to accept trades amounting to two or more percent of average daily trading volume. The evidence on the SEAQ is consistent with the model's prediction that constraining the depth quotes to be unusually large causes the spread to widen.

V. Extensions

There are many extensions that can be done with this work. First, the model ignores inventory costs. These were eliminated from the model for tractability in obtaining the closed form solutions. This does not rule out the use of simulation to solve the more complicated model with inventory costs. Second, the model ignores trading at prices different from those posted, as well as trading quantities larger than those posted. The strict adherence to the price schedule could be relaxed to allow for greater flexibility in the schedule. This line of work would be helpful in better determining the relation between the quoted schedule and the effective schedule. Third, the model could be extended to include a limit order book, allowing for an even greater strategic role for the specialist's price schedule. The specialist's ability to manipulate both prices and depths together with the limit order precedence rule permit the specialist to pass off unwanted trades onto the limit order book, e.g., Kavajecz (1995). Lastly, the model suggests that there may be 'preferred' applications for the use of prices and depths. Knowledge of these responses may help to identify empirically the specific concerns, like inventory management costs or adverse selection costs, to which the specialist is reacting.

VI. Conclusion

The role of quoted depth in a specialist's price schedule has received little research attention. Yet it is an important aspect since approximately half of the updated price schedules posted by specialists on the NYSE result in only changes to the quoted depth figures. The model presented here demonstrates that quoted depth is used strategically by the specialist. In particular, the model provides a unique analytic solution to a specialist's choice of prices and depths. For example, the unconstrained model predicts that (1) spreads will widen and depths will fall in response to an increase in the amount of adverse selection, (2) spreads will widen and depths will remain unchanged in the wake of an increase in price uncertainty, and (3) prices will shift upward and depth will be shifted from the ask side to the bid side of the market in the event that either expectations about future prices becomes more optimistic or there is an increase in the percentage of desired sales by liquidity traders. The constrained models show that constraining one of these choice variables causes the specialist to compensate with the other, implying that prices and depths are used as substitutes. In particular, a narrow bid-ask spread induces small depth quotes whereas large depth quotes induce a wide bid-ask spread.

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Appendix 1: Proof of Proposition 1 and Comparative Static Results

The specialist's unconstrained one period maximization problem is the following:

$$
\begin{aligned}
&E[\pi(b,\beta,a,\alpha)] = \\
&\mu\lambda\beta(\theta_1 - b) + (1-\mu)\lambda\alpha(\theta_2 - a) \\
&+ (1-\lambda)\left(\frac{t_2-\beta}{t_2-t_1}\right)\left(\frac{b-\theta_1}{\theta_2-\theta_1}\right)\beta\,(\mu(\theta_1-b)+(1-\mu)(\theta_2-b)) \\
&+ (1-\lambda)\left(\frac{\beta-0}{t_2-t_1}\right)\left(\frac{b-\theta_1}{\theta_2-\theta_1}\right)\left(\frac{1}{2}\beta\right)(\mu(\theta_1-b)+(1-\mu)(\theta_2-b)) \\
&+ (1-\lambda)\left(\frac{\alpha-t_1}{t_2-t_1}\right)\left(\frac{\theta_2-a}{\theta_2-\theta_1}\right)\alpha\,(\mu(\theta_1-a)+(1-\mu)(\theta_2-a)) \\
&+ (1-\lambda)\left(\frac{0-\alpha}{t_2-t_1}\right)\left(\frac{\theta_2-a}{\theta_2-\theta_1}\right)\left(\frac{1}{2}\alpha\right)(\mu(\theta_1-a)+(1-\mu)(\theta_2-a))
\end{aligned}
$$

The objective function can be simplified by collecting terms and defining some simplifying variables. Notice that the bid and ask sides of the market are completely separate and can be solved independently.

$$
\begin{aligned}
&E[\pi(b,\beta,a,\alpha)] = \\
&\mu\lambda\beta(\theta_1 - b) + M_1(b-\theta_1)(E_\mu[\theta]-b)(t_2\beta - \frac{1}{2}\beta^2) \\
&(1-\mu)\lambda\alpha(\theta_2 - a) + M_1(\theta_2-a)(E_\mu[\theta]-a)(\frac{1}{2}\alpha^2 - t_1\alpha)
\end{aligned}
$$

$$
\textit{where}\quad
\begin{cases}
E_\mu[\theta] = \mu\theta_1 + (1-\mu)\theta_2 \\
M_1 = \left(\dfrac{1-\lambda}{(t_2-t_1)(\theta_2-\theta_1)}\right)
\end{cases}
$$

<u>First and Second Order Conditions</u>

The following are the first order conditions associated with an interior maximum:

$$
\frac{\partial E[\pi]}{\partial b} = -\mu\lambda\beta + M_1(t_2\beta - \frac{1}{2}\beta^2)[(E_\mu[\theta]-b)+(b-\theta_1)(-1)] = 0
$$

$$
\frac{\partial E[\pi]}{\partial \beta} = \mu\lambda(\theta_1 - b) + M_1(b-\theta_1)(E_\mu[\theta]-b)(t_2-\beta) = 0
$$

$$
\frac{\partial E[\pi]}{\partial a} = -(1-\mu)\lambda\alpha + M_1(\frac{1}{2}\alpha^2 - t_1\alpha)[(-1)(E_\mu[\theta]-a)+(\theta_2-a)(-1)] = 0
$$

$$
\frac{\partial E[\pi]}{\partial \alpha} = (1-\mu)\lambda(\theta_2 - a) + M_1(\theta_2-a)(E_\mu[\theta]-a)(\alpha - t_1) = 0
$$

These simplified expressions can be derived from the first order conditions assuming the restrictions listed below hold. I proceed assuming they do hold and verify later that in equilibrium the restrictions are satisfied.

$$b = \frac{1}{2}(E_\mu[\theta] + \theta_1) - \frac{\mu\lambda}{2M_1(t_2 - \frac{1}{2}\beta)} \qquad (1)$$

$$\beta = t_2 - \frac{\mu\lambda}{M_1(E_\mu[\theta] - b)} \qquad (2)$$

$$a = \frac{1}{2}(E_\mu[\theta] + \theta_2) + \frac{(1-\mu)\lambda}{2M_1(\frac{1}{2}\alpha - t_1)} \qquad (3)$$

$$\alpha = t_1 - \frac{(1-\mu)\lambda}{M_1(E_\mu[\theta] - a)} \qquad (4)$$

$$\textit{given the following restrictions}: \begin{cases} b - \theta_1 \neq 0 & (R1) \\ \beta \neq 0 & (R2) \\ \theta_2 - a \neq 0 & (R3) \\ \alpha \neq 0 & (R4) \end{cases}$$

The second order sufficient conditions to ensure a unique maximum are that the hessian of second derivatives be negative definite. In order for the hessian to be negative definite, the naturally ordered principle minors of the hessian must alternate in sign. Because of the separability between the bid and ask sides of the specialist's problem, it suffices to check each side of the market separately.

$$\frac{\partial^2 E[\pi]}{\partial b^2} = M_1(t_2\beta - \frac{1}{2}\beta^2)(-2) < 0$$

$$\frac{\partial^2 E[\pi]}{\partial b\,\partial\beta} = -\mu\lambda + M_1(t_2 - \beta)\{(E_\mu[\theta] - b) + (b - \theta_1)(-1)\}$$

$$\frac{\partial^2 E[\pi]}{\partial\beta^2} = M_1(b - \theta_1)(E_\mu[\theta] - b)(-1) < 0$$

$$\frac{\partial^2 E[\pi]}{\partial\beta\,\partial b} = -\mu\lambda + M_1(t_2 - \beta)\{(E_\mu[\theta] - b) + (b - \theta_1)(-1)\}$$

$$\frac{\partial^2 E[\pi]}{\partial a^2} = M_1(\frac{1}{2}\alpha^2 - t_1\alpha)(2) < 0$$

$$\frac{\partial^2 E[\pi]}{\partial a\,\partial\alpha} = -(1-\mu)\lambda + M_1(\alpha - t_1)\{(-1)(E_\mu[\theta] - a) + (\theta_2 - a)(-1)\}$$

$$\frac{\partial^2 E[\pi]}{\partial\alpha^2} = M_1(\theta_2 - a)(E_\mu[\theta] - a)(1) < 0$$

$$\frac{\partial^2 E[\pi]}{\partial\alpha\,\partial a} = -(1-\mu)\lambda + M_1(\alpha - t_1)\{(-1)(E_\mu[\theta] - a) + (\theta_2 - a)(-1)\}$$

It is easy to verify that the second derivatives with respect to b and β are negative. The final condition to check is:

$$\det\begin{bmatrix} \dfrac{\partial^2 E[\pi]}{\partial b^2} & \dfrac{\partial^2 E[\pi]}{\partial b\,\partial\beta} \\[2ex] \dfrac{\partial^2 E[\pi]}{\partial\beta\,\partial b} & \dfrac{\partial^2 E[\pi]}{\partial\beta^2} \end{bmatrix} > 0$$

$$\leftrightarrow\ 2M_1^2(b-\theta_1)(E_\mu[\theta]-b)(t_2\beta - \frac{1}{2}\beta^2) - [-\mu\lambda + M_1(t_2-\beta)(E_\mu[\theta]+\theta_1-2b)]^2 > 0$$

$$\leftrightarrow\ (1-\mu)(\theta_2-\theta_1)(2t_2-\beta)\beta - (b-\theta_1)t_2^2 > 0$$

Substitution of the optimal b and β reduce the above equation to one of the conditions given in Proposition 1, namely that $\mu < (1-\lambda)/(1-\lambda\,(t_1/t_2))$. Verification of the second order conditions for the ask side of the market is done in the analogous way.

Bid Side Solution

I proceed by solving for the bid price and the bid depth. The ask side is not shown due to the symmetry of the problem, however it is available from the author upon request.

Substituting equation (2) into equation (1) yields:

$$-\mu\lambda + M_1\left(t_2 - \frac{1}{2}\left(t_2 - \frac{\mu\lambda}{2M_1(E_\mu[\theta]-b)}\right)\right)(E_\mu[\theta]+\theta_1-2b) = 0$$

After a few lines of algebra the above reduces to:

$$2b^2 - (E_\mu[\theta]+\theta_1)b + \left((E_\mu[\theta]^2 + E_\mu[\theta]\theta_1) - \left(\frac{\mu\lambda}{t_2M_1}\right)(E_\mu[\theta]-\theta_1)\right) = 0$$

Applying the quadratic equation and substituting for intermediate variables gives:

$$b = \frac{1}{4}(3E_\mu[\theta]+\theta_1) \pm \frac{1}{4}(1-\mu)(\theta_2-\theta_1)\sqrt{1+\left(\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)(t_2-t_1)}$$

The root that I am interested in is the negative root. To see why the positive root is not optimal, consider the following case. Suppose $\lambda = 0$, then using the positive root, $b = E_\mu[\theta]$. Substituting b into the specialist's profit function shows that the specialist yields zero profits; however, if $b < E_\mu[\theta]$, the specialist would make positive profits. Therefore, the negative root is optimal.

Switching over to the bid depth, I substitute equation (1) into equation (2) which yields:

$$-\mu\lambda + M_1\left(E_\mu[\theta] - \left(\frac{1}{2}(E_\mu[\theta] + \theta_1) - \frac{\mu\lambda}{2M_1(t_2 - \frac{1}{2}\beta)}\right)\right)(t_2 - \beta) = 0$$

A bit of algebra can verify:

$$\frac{1}{2}\beta^2 - \frac{3}{2}t_2\beta + \left(t_2^2 - \frac{\mu\lambda t_2}{M_1(E_\mu[\theta] - \theta_1)}\right) = 0$$

Again applying the quadratic formula and substituting for intermediate variables gives:

$$\beta = \frac{3}{2}t_2 \pm \frac{1}{2}t_2\sqrt{1 + \left(\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)(t_2 - t_1)}$$

Here, the positive root can be eliminated as a solution candidate. Since the first term $(3/2)t_2$ is greater than t_2, it is outside the feasible range for β. The only way for $\beta^* < t_2$, is for the optimal solution to be the negative root.

It is easy to verify that the condition on μ in Proposition 1 ensure the restrictions R1 through R4 by substituting $(1 - \lambda)/(1 - \lambda(t_1/t_2))$ for μ into b^* and β^*.

QED.

Proof of Corallary 1.1:

$$\frac{\partial b_u^*}{\partial \lambda} = -\frac{1}{4}(1-\mu)(\theta_2-\theta_1)\left(\frac{\left(\frac{8}{t_2}\right)\left(\frac{\mu}{1-\mu}\right)(t_2-t_1)\left\{\frac{1}{1-\lambda}+\frac{\lambda}{(1-\lambda)^2}\right\}}{2\sqrt{1+\left(\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)(t_2-t_1)}}\right) < 0$$

$$\frac{\partial a_u^*}{\partial \lambda} = +\frac{1}{4}\mu(\theta_2-\theta_1)\left(\frac{\left(\frac{-8}{t_1}\right)\left(\frac{1-\mu}{\mu}\right)(t_2-t_1)\left(\frac{1}{(1-\lambda)^2}\right)}{2\sqrt{1-\left(\frac{8}{t_1}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)(t_2-t_1)}}\right) > 0$$

$$\frac{\partial \beta_u^*}{\partial \lambda} = -\frac{1}{2}t_2\left(\frac{\left(\frac{8}{t_2}\right)\left(\frac{\mu}{1-\mu}\right)(t_2-t_1)\left(\frac{1}{(1-\lambda)^2}\right)}{2\sqrt{1+\left(\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)(t_2-t_1)}}\right) < 0$$

$$\frac{\partial \alpha_u^*}{\partial \lambda} = -\frac{1}{2}t_1\left(\frac{\left(\frac{-8}{t_1}\right)\left(\frac{1-\mu}{\mu}\right)(t_2-t_1)\left(\frac{1}{(1-\lambda)^2}\right)}{2\sqrt{1-\left(\frac{8}{t_1}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)(t_2-t_1)}}\right) > 0$$

Proof of Corallary 1.2:

$$\frac{\partial b_u^*}{\partial(\theta_2-\theta_1)} = \frac{\partial}{\partial(\theta_2-\theta_1)}\left(\frac{3}{4}(1-\mu)(\theta_2-\theta_1)+\theta_1-\frac{1}{4}(1-\mu)(\theta_2-\theta_1)\sqrt{1+\left(\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)(t_2-t_1)}\right)$$

$$= \frac{3}{4}(1-\mu)+\frac{\partial\theta_1}{\partial(\theta_2-\theta_1)}-\frac{1}{4}(1-\mu)\sqrt{1+\left(\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)(t_2-t_1)}$$

Aside :

Assume a mean preserving spread for θ_1, θ_2

Let $E_\mu[\theta] = M$ *and Spread* $= S$

$$M = \mu\theta_1 + (1-\mu)\theta_2$$

$$S = \theta_2 - \theta_1$$

Solving for θ_1 *yields* $\theta_1 = M - (1-\mu)S$

$$\therefore \frac{\partial\theta_1}{\partial(\theta_2-\theta_1)} = -(1-\mu)$$

$$\frac{\partial b_u^*}{\partial(\theta_2-\theta_1)} = \frac{-1}{4}(1-\mu)-\frac{1}{4}(1-\mu)\sqrt{1+\left(\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)(t_2-t_1)} < 0$$

$$\frac{\partial a_u^*}{\partial(\theta_2-\theta_1)} = \frac{\partial}{\partial(\theta_2-\theta_1)}\left(\frac{-3}{4}\mu(\theta_2-\theta_1)+\theta_2+\frac{1}{4}\mu(\theta_2-\theta_1)\sqrt{1-\left(\frac{8}{t_1}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)(t_2-t_1)}\right)$$

$$= \frac{-3}{4}\mu+\frac{\partial\theta_2}{\partial(\theta_2-\theta_1)}+\frac{1}{4}\mu\sqrt{1-\left(\frac{8}{t_1}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)(t_2-t_1)}$$

$$= \frac{-3}{4}\mu+\mu+\frac{1}{4}\mu\sqrt{1-\left(\frac{8}{t_1}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)(t_2-t_1)} > 0$$

$$\frac{\partial\beta_u^*}{\partial(\theta_2-\theta_1)} = 0$$

$$\frac{\partial\alpha_u^*}{\partial(\theta_2-\theta_1)} = 0$$

Proof of Corallary 1.3:

$$\frac{\partial b_u^*}{\partial \mu} = \frac{1}{4}(\theta_2-\theta_1)\left(-3+\sqrt{1+\left(\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)(t_2-t_1)}-\frac{\left(\frac{4}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1}{1-\mu}\right)(t_2-t_1)}{\sqrt{1+\left(\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)(t_2-t_1)}}\right)$$

$$= \frac{1}{4}(\theta_2-\theta_1)\left(-3+\frac{1+\left(\frac{4}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{2\mu-1}{1-\mu}\right)(t_2-t_1)}{\sqrt{1+\left(\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)(t_2-t_1)}}\right) < 0$$

Aside

$$\left(-3+\frac{1+\left(\frac{4}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{2\mu-1}{1-\mu}\right)(t_2-t_1)}{\sqrt{1+\left(\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)(t_2-t_1)}}\right) < 0$$

Note it is easy to show that the above expression is less than zero for $\mu = 0$

Need to check the sign for the largest possible μ; $\mu = \left(\frac{1-\lambda}{1-\lambda\left(\frac{t_1}{t_2}\right)}\right)$

Substitution of μ *into the above expression yields*

$$-3+\frac{1}{3}\left(1+\left(\frac{4}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-2\lambda+\lambda\left(\frac{t_1}{t_2}\right)}{\lambda\left(1-\frac{t_1}{t_2}\right)}\right)(t_2-t_1)\right) < 0$$

$$1+4\left(\frac{1}{1-\lambda}\right)\left(1-2\lambda+\lambda\left(\frac{t_1}{t_2}\right)\right) < 9$$

$$\lambda\left(\frac{t_1}{t_2}\right) < 1$$

QED.

$$\frac{\partial \beta_u^*}{\partial \mu} = -\frac{1}{2}t_2\left(\frac{\left(\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1}{(1-\mu)^2}\right)(t_2-t_1)}{2\sqrt{1+\left(\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)(t_2-t_1)}}\right) < 0$$

<u>Proof of Corallary 1.3 (continued):</u>

$$\frac{\partial a_u^*}{\partial \mu} = \frac{1}{4}(\theta_2 - \theta_1)\left(-3 + \sqrt{1 - \left(\frac{8}{t_1}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)(t_2 - t_1)} + \frac{\left(-\frac{8}{t_1}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{-1}{\mu}\right)(t_2 - t_1)}{2\sqrt{1 - \left(\frac{8}{t_1}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)(t_2 - t_1)}} \right)$$

$$= \frac{1}{4}(\theta_2 - \theta_1)\left(-3 + \frac{1 + \left(\frac{4}{t_1}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{2\mu - 1}{\mu}\right)(t_2 - t_1)}{\sqrt{1 - \left(\frac{8}{t_1}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)(t_2 - t_1)}} \right) < 0$$

Aside

Note that $\partial \alpha_u^* / \partial \mu < 0$ *for* $\mu = 1$.

It remains to check the sign for the smallest feasible μ, $\mu = \left(\frac{\lambda - \lambda \frac{t_1}{t_2}}{1 - \lambda\left(\frac{t_2}{t_1}\right)} \right)$

Substitution of μ *into the above expression yields*

$$-3 + \frac{1}{3}\left(1 - 4\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{-1 + 2\lambda - \lambda\left(\frac{t_2}{t_1}\right)}{\lambda\left(1 - \frac{t_2}{t_1}\right)} \right)\left(1 - \frac{t_2}{t_1}\right) \right) < 0$$

$$\lambda\left(\frac{t_2}{t_1}\right) < 1 \qquad \textit{QED}.$$

$$\frac{\partial \alpha_u^*}{\partial \mu} = -\frac{1}{2} t_1 \left(\frac{\left(-\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1}{\mu^2}\right)(t_2 - t_1)}{2\sqrt{1 - \left(\frac{8}{t_1}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)(t_2 - t_1)}} \right) < 0$$

<u>Proof of Corallary 1.4:</u>

$$\frac{\partial b_u^*}{\partial t_1} = -\frac{1}{4}(1-\mu)(\theta_2-\theta_1)\left(\frac{\left(-\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)}{2\sqrt{1+\left(\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)(t_2-t_1)}}\right) > 0$$

$$\frac{\partial \beta_u^*}{\partial t_1} = -\frac{1}{2}t_2\left(\frac{\left(-\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)}{2\sqrt{1+\left(\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)(t_2-t_1)}}\right) > 0$$

$$\frac{\partial a_u^*}{\partial t_1} = \frac{1}{4}\mu(\theta_2-\theta_1)\left(\frac{8\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)\left(\frac{t_2}{t_1^2}\right)}{2\sqrt{1-\left(\frac{8}{t_1}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)(t_2-t_1)}}\right) > 0$$

$$\frac{\partial \alpha_u^*}{\partial t_1} = \frac{1}{2}\left(3-\frac{1+8\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)-4\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)\left(\frac{t_2}{t_1}\right)}{\sqrt{1-\left(\frac{8}{t_1}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)(t_2-t_1)}}\right) > 0$$

[*Where the bracketed expression in* $\frac{\partial \alpha_u^*}{\partial t_1}$ *can be signed for all* μ *satisfying*

$$\left(\frac{\lambda-\lambda\left(\frac{t_2}{t_1}\right)}{1-\lambda\left(\frac{t_2}{t_1}\right)}\right) < \mu$$

]

Proof of Corallary 1.4 (continued):

$$\frac{\partial b_u^*}{\partial t_2} = -\frac{1}{4}(1-\mu)(\theta_2-\theta_1)\left(\frac{8\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)\left(\frac{t_1}{t_2^2}\right)}{2\sqrt{1+\left(\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)(t_2-t_1)}}\right) > 0$$

$$\frac{\partial \beta_u^*}{\partial t_2} = \frac{1}{2}\left(3 - \frac{1+8\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right) - 4\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)\left(\frac{t_1}{t_2}\right)}{\sqrt{1+\left(\frac{8}{t_2}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{\mu}{1-\mu}\right)(t_2-t_1)}}\right) > 0$$

[*Where the bracketed expression in* $\frac{\partial \beta_u^*}{\partial t_2}$ *can be signed for all* μ *satisfying*

$$\mu < \left(\frac{1-\lambda}{1-\lambda\left(\frac{t_2}{t_1}\right)}\right)$$
]

$$\frac{\partial a_u^*}{\partial t_2} = \frac{1}{4}\mu(\theta_2-\theta_1)\left(\frac{\left(-\frac{8}{t_1}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)}{\sqrt{1-\left(\frac{8}{t_1}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)(t_2-t_1)}}\right) > 0$$

$$\frac{\partial \alpha_u^*}{\partial t_2} = -\frac{1}{2}t_1\left(\frac{\left(-\frac{8}{t_1}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)}{\sqrt{1-\left(\frac{8}{t_1}\right)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1-\mu}{\mu}\right)(t_2-t_1)}}\right) > 0$$

Appendix 2: Proof of Proposition 2 and Comparative Static Results

The specialist's unconstrained one period maximization problem is the following:

$$E[\pi(b,\beta,a,\alpha)] =$$

$$\mu\lambda\beta(\theta_1 - b) + M_1(b - \theta_1)(E_\mu[\theta] - b)(t_2\beta - \frac{1}{2}\beta^2)$$

$$(1-\mu)\lambda\alpha(\theta_2 - a) + M_1(\theta_2 - a)(E_\mu[\theta] - a)(\frac{1}{2}\alpha^2 - t_1\alpha)$$

$$\textit{where} \quad \begin{cases} E_\mu[\theta] = \mu\theta_1 + (1-\mu)\theta_2 \\ M_1 = \left(\dfrac{1-\lambda}{(t_2 - t_1)(\theta_2 - \theta_1)}\right) \end{cases}$$

Imposing the constraints on the prices shown below and substituting them into the objective function yields an optimization problem with the two choice variables β and α.

$$a = (1+\gamma)(\mu\theta_1 + (1-\mu)\theta_2)$$

$$b = (1-\gamma)(\mu\theta_1 + (1-\mu)\theta_2)$$

$$\textit{where} \quad 0 < \gamma < \min\left[\frac{(1-\mu)(\theta_2 - \theta_1)}{\mu\theta_1 + (1-\mu)\theta_2}, \frac{\mu(\theta_2 - \theta_1)}{\mu\theta_1 + (1-\mu)\theta_2}\right]$$

$$\textit{implies} \quad \theta_1 < (1-\gamma)E_\mu[\theta] < (1+\gamma)E_\mu[\theta] < \theta_2$$

$$E[\pi(\beta,\alpha;\gamma)] =$$

$$\mu\lambda\beta(\theta_1 - (1-\gamma)E_\mu[\theta]) + M_1((1-\gamma)E_\mu[\theta] - \theta_1)(E_\mu[\theta] - (1-\gamma)E_\mu[\theta])(t_2\beta - \frac{1}{2}\beta^2)$$

$$+ (1-\mu)\lambda\alpha(\theta_2 - (1+\gamma)E_\mu[\theta]) + M_1(\theta_2 - (1+\gamma)E_\mu[\theta])(E_\mu[\theta] - (1+\gamma)E_\mu[\theta])(\frac{1}{2}\alpha^2 - t_1\alpha)$$

First and Second Order Conditions

The following are the first order conditions associated with an interior maximum:

$$\frac{\partial E[\pi]}{\partial \beta} = \mu \lambda (\theta_1 - (1-\gamma)E_\mu[\theta]) + M_1((1-\gamma)E_\mu[\theta] - \theta_1)(\gamma E_\mu[\theta])(t_2 - \beta) = 0$$

$$\frac{\partial^2 E[\pi]}{\partial \beta^2} = -M_1((1-\gamma)E_\mu[\theta] - \theta_1)(\gamma E_\mu[\theta]) < 0$$

$$\frac{\partial E[\pi]}{\partial \alpha} = (1-\mu)\lambda (\theta_2 - (1+\gamma)E_\mu[\theta]) + M_1(\theta_2 - (1+\gamma)E_\mu[\theta])(-\gamma E_\mu[\theta])(\alpha - t_1) = 0$$

$$\frac{\partial^2 E[\pi]}{\partial \alpha^2} = M_1(\theta_2 - (1+\gamma)E_\mu[\theta])(-\gamma E_\mu[\theta]) < 0$$

Constrained Depth Solutions

The first order conditions given above reduce to the following functions:

$$\beta_c^* = t_2 - \left(\frac{\lambda}{1-\lambda}\right)\left(\frac{(t_2 - t_1)(\theta_2 - \theta_1)\mu}{\gamma(\mu\theta_1 + (1-\mu)\theta_2)}\right)$$

$$\alpha_c^* = t_1 + \left(\frac{\lambda}{1-\lambda}\right)\left(\frac{(t_2 - t_1)(\theta_2 - \theta_1)(1-\mu)}{\gamma(\mu\theta_1 + (1-\mu)\theta_2)}\right)$$

QED.

Proof of Corallary 2.1

Comparative statics with respect to γ:

$$\frac{\partial \beta_c^*}{\partial \gamma} = \left(\frac{\lambda}{1-\lambda}\right)\left(\frac{(t_2 - t_1)(\theta_2 - \theta_1)\mu}{\mu\theta_1 + (1-\mu)\theta_2)\gamma^2}\right) > 0$$

$$\frac{\partial \alpha_c^*}{\partial \gamma} = -\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{(t_2 - t_1)(\theta_2 - \theta_1)(1-\mu)}{\mu\theta_1 + (1-\mu)\theta_2)\gamma^2}\right) < 0$$

Therefore a smaller spread (decrease in γ) reduces the depths. QED.

Appendix 3: Proof of Proposition 3 and Comparative Static Results

The specialist's unconstrained one period maximization problem is the following:

$$E[\pi(b,\beta,a,\alpha)] =$$

$$\mu\lambda\beta(\theta_1 - b) + M_1(b - \theta_1)(E_\mu[\theta] - b)(t_2\beta - \frac{1}{2}\beta^2)$$

$$(1-\mu)\lambda\alpha(\theta_2 - a) + M_1(\theta_2 - a)(E_\mu[\theta] - a)(\frac{1}{2}\alpha^2 - t_1\alpha)$$

$$\textit{where} \quad \begin{cases} E_\mu[\theta] = \mu\theta_1 + (1-\mu)\theta_2 \\ M_1 = \left(\dfrac{1-\lambda}{(t_2 - t_1)(\theta_2 - \theta_1)}\right) \end{cases}$$

Imposing the constraints on the depths shown below and substituting them into the objective function yields an optimization problem with the two choice variables b and a.

$$\alpha = \rho t_1$$

$$\beta = \rho t_2$$

$$\textit{where} \quad 0 < \rho \leq 1$$

$$E[\pi(b,a;\rho)] =$$

$$\mu\lambda\rho t_2(\theta_1 - b) + M_1(b - \theta_1)(E_\mu[\theta] - b)(t_2^2\rho - \frac{1}{2}\rho^2 t_2^2)$$

$$+ (1-\mu)\lambda\rho t_1(\theta_2 - a) + M_1(\theta_2 - a)(E_\mu[\theta] - a)(\frac{1}{2}\rho^2 t_1^2 - t_1^2\rho)$$

First and Second Order Conditions

The following are the first order conditions associated with an interior maximum:

$$\frac{\partial E[\pi]}{\partial b} = -\mu \lambda \rho t_2 + M_1(t_2^2 \rho - \frac{1}{2}\rho^2 t_2^2)(E_\mu[\theta] - 2b + \theta_1) = 0$$

$$\frac{\partial^2 E[\pi]}{\partial b^2} = M_1(t_2^2 \rho - \frac{1}{2}\rho^2 t_2^2)(-2)) < 0$$

$$\frac{\partial E[\pi]}{\partial a} = -(1-\mu)\lambda \rho t_1 + M_1(-E_\mu[\theta] + 2b - \theta_2)(\frac{1}{2}\rho^2 t_1^2 - t_1^2 \rho) = 0$$

$$\frac{\partial^2 E[\pi]}{\partial a^2} = M_1(2)(\frac{1}{2}\rho^2 t_1^2 - t_1^2 \rho) < 0$$

Constrained Depth Solutions

The first order conditions given above reduce to the following functions:

$$b_c^* = \frac{1}{2}(\mu\theta_1 + (1-\mu)\theta_2 + \theta_1) - \frac{1}{2}\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{(t_2 - t_1)(\theta_2 - \theta_1)\mu}{(1 - \frac{1}{2}\rho)t_2}\right)$$

$$a_c^* = \frac{1}{2}(\mu\theta_1 + (1-\mu)\theta_2 + \theta_2) + \frac{1}{2}\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{(t_2 - t_1)(\theta_2 - \theta_1)(1-\mu)}{(1 - \frac{1}{2}\rho)t_2}\right)$$

QED.

Proof of Corallary 3.1

Comparative statics with respect to ρ:

$$\frac{\partial b_c^*}{\partial \rho} = -\frac{1}{4}\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{(t_2 - t_1)(\theta_2 - \theta_1)\mu}{t_2(1 - \frac{1}{2}\rho)^2}\right) < 0$$

$$\frac{\partial a_c^*}{\partial \rho} = -\frac{1}{4}\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{(t_2 - t_1)(\theta_2 - \theta_1)(1-\mu)}{t_1(1 - \frac{1}{2}\rho)^2}\right) > 0$$

Therefore a larger depths (increase in ρ) increases the spread. QED.