THE ANALYSIS OF VAR, DELTAS AND STATE PRICES: A NEW APPROACH

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Abstract

We provide a monotonic transformation of an initial diffusion with a level-dependent diffusion parameter that yields a second, deterministic diffusion parameter process. Altering the diffusion parameter while maintaining the original Brownian motion at the expense of the drift can be viewed as a counterpart to Girsanov's Theorem. The transformed process provides a tractable basis for the analysis of the initial probability distribution, and hence provides insights into the value-at-risk (VAR), hedging and valuation of alternate investment strategies. Restrictions on the initial process imply theoretical bounds on VAR, position deltas and state prices, and an empirical bound on option deltas.

1. Introduction

Three questions that present a continuing challenge to investors are closely related. Investors want to know the distribution of possible future returns from a given investment strategy: What is the value-at-risk (VAR) in the event of a price collapse with, say, a 1-in-20 chance of occurrence. This question must often be addressed in the absence of sharp information about the drift and diffusion parameters of the process describing changes in the value of the portfolio of interest. The investor will typically have only coarse knowledge of the complex relation between the portfolio's volatility and its value. We will show how to use limited knowledge of the true process to place bounds on the VAR.

The second question of interest to investors concerns the 'correct' price to pay for an asset today. Answering this question also involves determining the probability of the asset's price finishing above any given level. Valuing an asset is equivalent to valuing the portfolio of state-contingent claims contained therein. It is well-known that the price of a state claim paying one dollar if and only if the asset's price finishes above a given level can be determined as the discounted probability of that event occurring under the *risk-neutral* version of the price process. Although the drift of the risk-neutral process is known from the risk-free rate, an investor's limited knowledge of the functional form of the diffusion parameter of the true price process implies only limited knowledge of that same parameter of the risk-neutral process. We will show how that limited knowledge can be used to place bounds on values.

Finally, investor's want to be able to hedge their portfolios. What is the share-equivalent risk or position delta of a complex portfolio of equities and derivatives thereon? It is not generally well-known that position deltas can be determined from knowledge of the probability of the underlying asset finishing above any given level under a particular stochastic process that we will term the *delta* process. Limited knowledge of the functional form of the diffusion parameter of the true price process will be seen to imply only limited knowledge of both the drift and diffusion parameters of the delta process. But that limited knowledge can still be used to provide a computational check on hedge ratios.

This paper contains a new and general technique useful in determining the probabilities of events in diffusion settings, and hence insights into VAR, valuation and hedging. Suppose an asset's price follows a one-dimensional diffusion with a diffusion parameter that depends on the level of the process. We show that the probability that the asset's price will finish above a given critical level is equivalent to the probability that a second process with a deterministic diffusion parameter and a known starting value will finish above a related critical level. (The drift of this second process is a function of the drift and diffusion parameters of the original process.) We also show that the probability of interest can instead be determined as the probability that, given its known starting value, a third process whose diffusion parameter is directly proportional to its level will finish above a different related critical level. (The drift of this third process is a function of the drift and diffusion parameters of the original process.) The second and third processes are each derived by applying a monotonic change of variables to the original process such that the new process has the desired simple diffusion parameter. By changing the diffusion parameter at the expense of the drift while maintaining the underlying Brownian motion, each change of variables provides a counterpart to Girsanov's Theorem. Girsanov's Theorem changes the drift at the expense of the probability measure, while maintaining the diffusion parameter. The analysis of the probability that the original process will finish above some level can be made more tractable if one examines instead the probability of the equivalent event under either the deterministic diffusion parameter or the deterministic volatility parameter transformations of the original process. Our main result, Theorem 2, uses the transformed process with a deterministic diffusion parameter to provide an expression for the probability of the original process reaching a critical level.

The properties of both deltas and the prices of state claims are the subject of much current research. On deltas, see Bergman (1983), Carr (1993), Bates (1995) and Bergman, Grundy and Wiener (1996). For theoretical and empirical work on state prices see Breeden and Litzenberger (1978), Bick and Reisman (1994), Derman and Kani (1994), Dupire (1994), Rubinstein (1994), Aït-Sahalia and Lo (1995), Jackwerth and Rubinstein (1995), Rady (1995), and Dumas, Fleming and Whaley (1996). By applying Theorem 2 to these closely related problems we are able to determine general properties of state prices and deltas. These properties are dependent on some knowledge of the functional form of the underlying asset's diffusion parameter. For example, given knowledge that the diffusion parameter is non-decreasing in the asset's value, we can determine bounds on deltas and state prices. These bounds only require knowledge of the asset's volatility at its current price level. Tighter bounds applicable when the volatility parameter is non-decreasing are also derived.

Section 2 contains the paper's theoretical contribution to the analysis of probabilities in diffusion settings. The remainder of the paper applies Theorem 2 to the three related questions of the hedging, pricing, and VAR of alternate investment strategies. Section 3 defines the delta process and shows that a call option's delta is equivalent to its probability of finishing in-the-money under the delta process. Section 4 re-expresses this probability as the probability of the equivalent event that a particular diffusion process with a known starting value and a deterministic diffusion parameter finishes above a level determined by the option's exercise price. Section 5 derives a bound on a call's delta that will be satisfied for the broad class of underlying stock price processes with a diffusion parameter that is non-decreasing in the level of the process. (This class includes the Black-Scholes and constant elasticity of variance diffusions.) Section 6 demonstrates that for this class of underlying price processes, the delta of an at-the-money option is always at least 1/2. We also demonstrate that for processes outside this class the delta of an at-the-money call need not be greater than 1/2. The second application of Theorem 2, the analysis of state prices, is contained in Section 7. Section 8 determines conditions under which the information about state prices implicit in even a limited set of option prices can be used to place an empirical bound on option deltas. Section 9 applies Theorem 2 to the analysis of VAR. Section 10 summarizes our results and suggests possible extensions of this research.

2. Properties of Probabilities in Diffusion Settings

Let $\xi_\tau^{y,t}$ denote the time τ value of a diffusion that at time $t < \tau$ starts at the level y and then obeys the stochastic differential equation (SDE)

$$\begin{aligned} d\xi_\tau &= \mu(\xi_\tau, \tau)d\tau + \sigma(\xi_\tau, \tau)\xi_\tau dB_\tau \\ &= \mu(\xi_\tau, \tau)d\tau + z(\xi_\tau, \tau)dB_\tau. \end{aligned} \tag{1}$$

We follow the finance literature and refer to the function $\sigma(\cdot)$ as the *volatility* parameter. Following Karlin and Taylor (1981, p. 159) we refer to the product $z(\cdot) := \sigma(\cdot)\xi$ as the *diffusion* parameter. We use numerical subscripts to denote partial derivatives. Thus, for example $z_{11}(\xi, t)$ is the second partial of the diffusion parameter with respect to its first argument, the level of the process. In addition to imposing Lipschitz and growth restrictions on the parameters μ and z that assure the existence of a unique Ito process satisfying (1) for each possible starting value in $\mathbb{R}^+$, whenever we apply Theorem 2 (developed below) we will also assume that μ and z satisfy:

Assumptions Set. (i) z is once differentiable in ξ and once differentiable in t. (ii) $\sigma(\xi, t) > 0$ for all $\xi > 0$ and all t. (iii) μ and z are such that for $y > 0$, $\xi_\tau^{y,t} > 0$ for all t and τ.[1]

Consider the probability of the event that at time T the process has reached a level at or above k; $\Pr\left(\xi_T^{y,t} > k\right)$. One thing that can make the analysis of this probability challenging is

[1] Assumption (iii) of the Assumption Set does more than simply rule out non-negative prices for limited-liability assets. It precludes a stock going bankrupt. We impose assumption (iii) only because of the resultant tractability of the analysis. Developing a variant of the result in Theorem 2 that is applicable when assumptions (ii) and (iii) are relaxed remains an interesting challenge. For some work in this direction see Section 1.3 of Nelson and Ramaswamy (1990).

the dependence of the diffusion parameter, $z(\xi, t)$, on the level of the process ξ. The analysis is potentially simpler if the diffusion parameter is either deterministic or directly proportional to the level of the process (in which case, the volatility parameter is deterministic). Thus our interest in transforming the original problem into one involving a process whose diffusion parameter takes a simple form. We first note that the probability of interest is equal to the probability of the following equivalent event.

Proposition 1. (Preservation of Probability) *Suppose ξ starts at y at time t and follows the diffusion in* (1). *If the function $F(\xi, t)$ is strictly increasing in ξ, then*

$$\Pr\left(\xi_T^{y,t} > k\right) = \Pr\left(F(\xi_T^{y,t}, T) > F(k, T)\right).$$

When F is twice differentiable in ξ and once differentiable in t, the random variable $F(\xi, t)$ will follow a diffusion. Consider one such specification of the F function:

$$F(\xi, t) := \int_{A(t)}^{\xi} \frac{a(t)}{z(x, t)} dx, \tag{2}$$

where $a(t)$ and $A(t)$ are smooth functions of time, with $a(t) > 0$ and $A(t) > 0$ for all t. We will show that for this specification of the F function, the diffusion parameter of the process describing changes in F is deterministic.[2] Applying this specification of the F function to the original process can be thought of as a generalization of the familiar technique of taking the natural log of a price process when the price follows geometric Brownian motion.[3]

Theorem 1. *Suppose ξ starts at y at time t and follows the diffusion in* (1) *with μ and z satisfying the restrictions of the Assumption Set. For the function $F(\xi, t)$ defined in (2) the dynamic of F is given by*

$$dF_\tau = \psi(F_\tau, \tau)d\tau + a(\tau)dB_\tau,$$

where the functional form of ψ, the drift parameter of the F process, is provided in the Appendix.

Theorem 1 is proved in Appendix A.

[2] Nelson and Ramaswamy (1990) have previously used such an F function to transform the SDE in (1) so as to obtain a second process that they use as the basis of their development of a computationally simple binomial approximation to the original process in (1).

[3] Section *4.1* demonstrates that when the original process is geometric Brownian motion, the F function given in (2) does in fact simplify to the natural log function.

Note that the functions $a(t)$ and $A(t)$ in (2) are choices to be made in defining F. The choice of the function $a(t)$ is the choice of the deterministic diffusion parameter in the SDE describing the dynamic of F. Given the value of ξ at time t, the choice of the function $A(t)$ can be thought of as the choice of the time t starting value of the F process.

Consider a second specification of the F function denoted by $\mathcal{F}$:

$$\mathcal{F}(\xi, t) := \exp\left(\int_{A(t)}^{\xi} \frac{a(t)}{z(x,t)} dx \right), \tag{3}$$

where again $a(t)$ and $A(t)$ are smooth functions of time with $a(t) > 0$ and $A(t) > 0$ for all t. For this $\mathcal{F}$ function, the diffusion parameter of the diffusion describing the dynamic of $\mathcal{F}$ is directly proportional to $\mathcal{F}$; i.e., the volatility function of the diffusion describing changes in $\mathcal{F}_t$ is deterministic.

Theorem 1′. *Suppose ξ starts at y at time t and follows the diffusion in (1) with μ and z satisfying the restrictions of the Assumption Set. For the function $\mathcal{F}(\xi, t)$ defined in (3), the dynamic of $\mathcal{F}$ is given by*

$$d\mathcal{F}_\tau = \theta(\mathcal{F}_\tau, \tau)d\tau + a(\tau)\mathcal{F}_\tau dB_\tau,$$

where the functional form of θ, the drift parameter of the $\mathcal{F}$ process, is provided in the Appendix.

Theorem 1′ is proved in Appendix A.

Consider now the probability that a process with a level dependent diffusion parameter will exceed k at date T. We can compute one of two equivalent probabilities: The probability that a second process with a known starting value and a *deterministic diffusion parameter* will exceed the amount $F(k, T)$ at date T; or, the probability that a third process with a different known starting value and a *deterministic volatility parameter* will exceed the amount $\mathcal{F}(k, T)$ at date T. One can think of both the F function in (2) and the $\mathcal{F}$ function in (3) as counterparts to Girsanov's Theorem, an important theorem in the theory of contingent claims pricing (see Duffie (1992, p. 237), and the analysis in Harrison and Kreps (1979) and Harrison and Pliska (1981)). Rather than transforming the drift while maintaining the diffusion parameter at the expense of a change in the probability measure (as in Girsanov's Theorem), one can transform the diffusion parameter while maintaining the Brownian motion at the expense of a change in the drift.

The F function in (2) can be used along with Proposition 1 to develop a new expression for $\Pr\left(\xi_T^{y,t} > k\right)$ that we state as Theorem 2. This new expression is an expression for the probability of the equivalent event that $F(\xi_T^{y,t}, T) > F(k, T)$.

Theorem 2. *Suppose ξ starts at y at time t and follows the diffusion in (1) with μ and z satisfying the restrictions of the Assumption Set. Let $a(t)$ and $A(t)$ be smooth functions of t with $a(t) > 0$ for all t. Define the function F as in (2). The probability that $\xi_T^{y,t}$ exceeds k can be expressed as*

$$\Pr\left(\xi_T^{y,t} > k\right) = \Pr\left(\frac{F(y,t) - F(k,T) + \int_t^T \left(a(\tau)\left(\frac{\mu(\xi_\tau,\tau)}{z(\xi_\tau,\tau)} - \frac{1}{2}z_1(\xi_\tau,\tau)\right) + F_2(\xi_\tau,\tau)\right)d\tau}{\sqrt{\int_t^T [a(\tau)]^2 d\tau}} > \mathcal{X}\right)$$

where the random variable $\mathcal{X} := -\frac{\int_t^T a(\tau)dB_\tau}{\sqrt{\int_t^T [a(\tau)]^2 d\tau}}$ is distributed $\mathcal{N}(0,1)$.

Theorem 2 is proved in Appendix A.

2.1. Two examples of the choice of $a(t)$ and $A(t)$

Recall that both the deterministic diffusion parameter of the dynamic of F, namely the function $a(t)$, and the lower bound of integration in (2), $A(t)$, are choices that can be made when applying Theorem 2. The appropriate choice in any setting will be that which yields the most tractable analysis. As shown in Appendix A the dynamic of F is given by

$$dF(\xi_\tau,\tau) = \left(a(\tau)\left(\frac{\mu(\xi_\tau,\tau)}{z(\xi_\tau,\tau)} - \frac{1}{2}z_1(\xi_\tau,\tau)\right) + F_2(\xi_\tau,t)\right)d\tau + a(\tau)dB_\tau. \tag{4}$$

Although the diffusion parameter of the F process has a simple form, the drift parameter may be even more complex than that of the original process. Thus we can be interested in choosing $a(t)$ and $A(t)$ with a view to simplifying the drift parameter of (4). In the following two examples we consider choices of $a(t)$ and $A(t)$ such that the F_2 component of the drift in (4) is either identically zero or of known sign.

As our first example, consider the case where the time dependence of the volatility function in (1) takes the form $\sigma(\xi,t) := m(t)n(\xi)$. This simple form includes two interesting special cases:

volatility does not depend on time when $m(t) \equiv 1$; and volatility is a deterministic function of time when $n(\xi) \equiv 1$. When $\sigma(\xi, t) := m(t)n(\xi)$ we can choose $a(t)$ and $A(t)$ such that $F_2(\xi, t) = 0$ for all ξ and t. To see this set $a(t) = m(t)$ and $A(t) = k$. Then the function F takes the form

$$F(\xi, t) = \int_k^\xi \frac{m(t)}{m(t)n(x)x} dx = \int_k^\xi \frac{1}{n(x)x} dx,$$

with the immediate property that for all t,

$$F_2(\xi, t) = 0.$$

As a second example suppose that the volatility parameter satisfies the more general restriction that there exists a uniform upper bound $U(t)$ with the property that for any time t and for all ξ,

$$\frac{\sigma_2(\xi, t)}{\sigma(\xi, t)} \leq U(t).$$

The existence of such a bound imposes only very weak restrictions on the functional form of the volatility parameter. Given such a bound it is always possible to choose $a(t)$ and $A(t)$ such that $F_2(\xi, t) \geq 0$ for all ξ and t. To see this set $A(t) = k$, and this time choose $a(t)$ as the solution to the ordinary differential equation

$$\frac{a_1(t)}{a(t)} = U(t);$$

i.e., as

$$a(t) = a(0) exp\left(\int_0^t U(\tau) d\tau\right).$$

Given these choices for $a(t)$ and $A(t)$ we obtain

$$\begin{aligned} F_2(\xi, t) &= \int_{A(t)}^\xi \frac{a_1(t)z(x,t) - a(t)z_2(x,t)}{[z(x,t)]^2} dx - \frac{a(t)}{z(A(t), t)} A_1(t) \\ &= \int_k^\xi \frac{a(t)(U(t)z(x,t) - z_2(x,t))}{[z(x,t)]^2} dx \\ &= \int_k^\xi \left(\frac{a(t)}{z(x,t)} \left(U(t) - \frac{z_2(x,t)}{z(x,t)}\right)\right) dx \\ &= \int_k^\xi \left(\frac{a(t)}{z(x,t)} \left(U(t) - \frac{\sigma_2(x,t)}{\sigma(x,t)}\right)\right) dx, \end{aligned}$$

which is non-negative since $U(t) - \frac{\sigma_2(x,t)}{\sigma(x,t)} \geq 0$ for all x and t.[4] Similarly, when there exists a lower bounding function $L(t)$ such that for all ξ and any time t,

$$\frac{\sigma_2(\xi,t)}{\sigma(\xi,t)} \geq L(t),$$

setting $A(t) = k$ and $a(t)$ as the solution to $a_1(t) = a(t)L(t)$ will ensure that $F_2(\xi,t) \leq 0$ for all ξ and t.

We now apply Theorem 2 to the analysis of the related problems of hedging, pricing, and determining the likelihood of possible payoffs from, an investment strategy

3. Some Properties of Position Deltas

In order to apply Theorem 2 to the analysis of deltas, we must first determine conditions under which the delta of a position can be shown to be determined by the probability distribution of the realization of a diffusion process at the position's terminal date. Let s_t denote the time t price of the asset underlying the position. The risk-neutral process for this underlying asset is given by

$$\begin{aligned} ds_\tau &= r(\tau)s_\tau d\tau + \sigma(s_\tau,\tau)s_\tau dB_\tau \\ &= r(\tau)s_\tau d\tau + z(s_\tau,\tau)dB_\tau, \end{aligned} \tag{5}$$

where $r(t)$ is the instantaneous risk-free rate at time t. Let $v(s,t)$ denote the nominal price of a European contingent claim on the underlying asset. When, in particular, a call option is considered, c instead of v will be used to denote its nominal price. The contractual payoff function is $g(\cdot)$, meaning that if the underlying price is s at the expiration date T, then the contingent claim will pay off $g(s)$. Therefore, to prevent arbitrage, $v(s,T) = g(s)$. Under mild regularity conditions the claim's price solves the p.d.e.:

$$r(t)v_1(s,t)s - r(t)v(s,t) + v_2(s,t) + \frac{1}{2}[z(s,t)]^2 v_{11}(s,t) = 0 \tag{6}$$

[4] Our first example, $\sigma(\xi,t) = m(t)n(\xi)$, can also be described in this way simply by choosing $U(t) = \frac{m_1(t)}{m(t)}$, and hence $a(t) = m(t)$.

subject to the terminal condition $v(s,T) = g(s)$.

Theorem 3 establishes that a contingent claim's delta can be expressed as the expectation of its delta at maturity under a particular diffusion process for the underlying asset.[5]

Theorem 3. (Bergman (1983)) *Suppose the risk-neutral process for the underlying asset is given by (5). Consider a European contingent claim on this asset whose time T contractual payoff, g, is differentiable on its domain. The delta of this claim is given by*

$$v_1(s,t) = E\{g_1(\xi_T^{s,t})\},$$

where the dynamic of ξ_τ is described by

$$d\xi_\tau = (r(\tau)\xi_\tau + z_1(\xi_\tau,\tau)z(\xi_\tau,\tau))\,d\tau + z(\xi_\tau,\tau)dB_\tau.$$

Theorem 3 is proved in Appendix A.

The stochastic process that determines a claim's delta, *the delta process*, is neither the true process nor the risk-neutral process.

Definition. *When the risk-neutral process is given by (5), the following process will be said to be the corresponding delta process:*

$$d\xi_\tau = (r(\tau)\xi_\tau + z_1(\xi_\tau,\tau)z(\xi_\tau,\tau))\,d\tau + z(\xi_\tau,\tau)dB_\tau. \tag{7}$$

The diffusion parameter of the delta process is common to the true process, the risk-neutral process and the delta process. The drift of the delta process is the sum of the risk-neutral drift plus a term that depends on the common diffusion parameter. Recalling from the Assumption Set that $\sigma(\cdot)$, and hence $z(\cdot)$, are non-negative, the drift of the delta process will exceed (be less than) the risk-neutral drift when $z_1(s,t)$ is positive (negative). In the case of a call option, Theorem 3 specializes to an equality between a call's delta and its probability of finishing in-the-money under the delta process.

[5] An extension of Theorem 3 to the case of a multi-dimensional diffusion setting (i.e., a setting with stochastic volatility) can be found in the appendix to Bergman, Grundy and Wiener (1996). An independent derivation of Theorem 3 in a deterministic volatility setting can be found in Carr (1993).

Theorem 4. *The delta of a call equals its probability of finishing in-the-money under the delta process. Suppose the risk-neutral process for the underlying asset is given by (5). The delta of a call option is then given by*

$$c_1(s,t) = \Pr\left(\xi_T^{s,t} > k\right),$$

where changes in ξ_τ *are described by*

$$d\xi_\tau = \left(r(\tau)\xi_\tau + z_1(\xi_\tau,\tau)z(\xi_\tau,\tau)\right)d\tau + z(\xi_\tau,\tau)dB_\tau.$$

Proof: For a call option, $c(s,T) = g(s) = \max[0, s-k]$ and

$$g_1(s) = \begin{cases} 1, & \text{if } s > k; \\ 0, & \text{if } s < k. \end{cases}$$

Hence, from Theorem 3,

$$c_1(s,t) = E\{g_1(\xi_T^{s,t})\} = \Pr\left(\xi_T^{s,t} > k\right). \quad \blacksquare$$

4. Theorem 2 and Option Deltas

4.1. Theorem 2 and option deltas in a Black-Scholes setting

As an illustration of both Theorems 2 and 4, we derive the delta of a call option in the familiar Black-Scholes setting. In a Black-Scholes setting the asset's volatility is a deterministic function, $\sigma(t)$. Hence $z_1(s,t) = \sigma(t)$, and the delta process in (7) takes the form

$$d\xi_\tau = \left(r(\tau) + [\sigma(\tau)]^2\right)\xi_\tau d\tau + \sigma(\tau)\xi_\tau dB_\tau. \tag{8}$$

The instantaneous drift rate in (8) exceeds the contemporaneous risk-free rate by the deterministic amount $[\sigma(t)]^2$. The delta of the call in a Black-Scholes setting is simply the probability that $\xi_T^{s,t} > k$ under the delta process in (8). We use Theorem 2 to find an expression for this probability by determining the probability of the equivalent event $F(\xi_T^{s,t}, T) > F(k,T)$.

In applying Theorem 2, set $a(t) = \sigma(t)$ and $A(t) = k$. For this choice of $a(t)$ and $A(t)$ the function F in Theorem 2 takes the simple form:

$$F(\xi,t) = \int_k^\xi \frac{\sigma(t)}{\sigma(t)x}dx = \int_k^\xi \frac{1}{x}dx = \ln\left(\frac{\xi}{k}\right).$$

Thus $F(k,t)=0$ for all t, and $F(s,t)=\ln(s/k)$ for all t. Recall that the function μ in Theorem 2 corresponds to the drift of the original process whose probability of finishing above k we wish to determine; in this case, the original process of interest is the delta process. Since for all ξ and t, $F_2(\xi,t)=0$, the drift of the F process in (4) is then

$$a(\tau)\left(\frac{\mu(\xi_\tau,\tau)}{z(\xi_\tau,\tau)}-\frac{1}{2}z_1(\xi_\tau,\tau)\right)=\sigma(\tau)\left(\frac{\left(r(\tau)+[\sigma(\tau)]^2\right)\xi_\tau}{\sigma(\tau)\xi_\tau}-\frac{1}{2}\sigma(\tau)\right)=r(\tau)+\frac{1}{2}[\sigma(\tau)]^2.$$

Substitution in Theorem 2 gives

$$\Pr\left(\xi_T^{s,t}>k\right)=\Pr\left(\frac{\ln(s/k)+\int_t^T\left(r(\tau)+\frac{1}{2}[\sigma(\tau)]^2\right)d\tau}{\sqrt{\int_t^T[\sigma(\tau)]^2d\tau}}>\mathcal{X}\right).$$

Since the left-hand-side of the preceding inequality is not a random variable, we have

$$\Pr\left(\xi_T^{s,t}>k\right)=N\left(\frac{\ln(s/k)+\int_t^T\left(r(\tau)+\frac{1}{2}[\sigma(\tau)]^2\right)d\tau}{\sqrt{\int_t^T[\sigma(\tau)]^2d\tau}}\right),$$

which is the familiar expression for the hedge ratio in a Black-Scholes setting.

4.2. Theorem 2 and option deltas in a general setting

We now wish to apply Theorem 2 to the general determination of option hedge ratios given the risk-neutral process in (5).

Lemma 1. (Deltas) *Suppose that the risk-neutral process for the underlying asset starts at s at time t and follows the diffusion given in (5). For F as defined in Theorem 2, the call's delta can be expressed as*

$$c_1(s,t)=\Pr\left(\frac{F(s,t)-F(k,T)+\int_t^T\left(a(\tau)\left(\frac{r(\tau)}{\sigma(\xi_\tau,\tau)}+\frac{1}{2}z_1(\xi_\tau,\tau)\right)+F_2(\xi_\tau,\tau)\right)d\tau}{\sqrt{\int_t^T[a(\tau)]^2d\tau}}>\mathcal{X}\right)\qquad(9)$$

where the random variable $\mathcal{X}:=-\frac{\int_t^T a(\tau)dB_\tau}{\sqrt{\int_t^T[a(\tau)]^2d\tau}}$ *is distributed* $\mathcal{N}(0,1)$.

Lemma 1 is proved in Appendix A.

To calculate deltas using (9) one must know the diffusion parameter $z(s,t)$. But if one knew $z(s,t)$, we could then solve (6) to price options and calculate deltas numerically. What we wish to show is that expression (9) can provide useful information about an option's delta when one has only limited knowledge concerning the functional form of $z(s,t)$.

5. Deltas when the Diffusion Parameter is Non-Decreasing in the Underlying's Value

Lemma 1 allows an exploration of the properties of deltas for broad classes of diffusion parameters. Recall that in a Black-Scholes setting, $z_1(s,t) = \sigma(t) > 0$; i.e., the diffusion parameter is increasing in s for all s and t. We wish to examine the deltas of call options in more general settings with the property that $z_1(s,t) \geq 0$ for all s and t. We first consider the meaning of such a non-negative partial derivative.

$$\begin{aligned} z_1(s,t) &= \frac{\partial \sigma(s,t)s}{\partial s} = \sigma(s,t)\left(1 + \frac{\sigma_1(s,t)s}{\sigma(s,t)}\right) \\ &= \sigma(s,t)\left(1 + \omega(s,t)\right), \end{aligned}$$

where

$$\omega(s,t) := \frac{\sigma_1(s,t)s}{\sigma(s,t)}.$$

Thus a sufficient condition for $z_1(s,t) \geq 0$ for all s and t is that $\sigma_1(s,t) \geq 0$ for all s and t. The necessary condition for $z_1(s,t) \geq 0$ for all s and t is that $\omega(s,t)$, the elasticity of volatility with respect to the price of the underlying, be not less than negative unity: σ_1 can be negative, but not 'too' negative.[6,7]

When one views a call's delta as its probability of finishing-in-the-money under the delta process, the import of a restriction on the sign of $z_1(s,t)$ is not immediately obvious. But suppose one considers the probability of the equivalent event under the F transformation of the delta process. In the dynamic of F given in (4), μ and z are the drift and diffusion parameters of the process to which the F transformation is applied. The drift of the delta process is

$$\mu(\xi,t) = r(t)\xi + z_1(\xi,t)z(\xi,t).$$

[6] Consider the constant elasticity of variance (CEV) setting studied in Cox (1975) in which the diffusion parameter takes the form $z(s,t) := \hat{\sigma}(t)s^{\rho}$, where $\hat{\sigma}(t)$ is the time t volatility given $s_t = 1$ and $0 \leq \rho \leq 1$. The elasticity of volatility is $\omega(s,t) = \rho - 1$. Since $\rho \geq 0$, $\omega(s,t) \geq -1$.

[7] Note that for a risky, limited-liability asset, $z(0,t) = 0$, $\sigma(s,t) \geq 0$ for all s and t, and $\sigma(s,t) > 0$ for some s and t. For such an asset, it cannot be that $z_1(s,t) \leq 0$ for all s and t.

Substitution for μ in (4) gives the drift of the F process as

$$a(\tau)\left(\frac{r(\tau)}{\sigma(\xi_\tau,\tau)}+\frac{1}{2}z_1(\xi_\tau,\tau)\right)+F_2(\xi_\tau,\tau).$$

As was shown in subsection *2.1.*, the existence of a uniform upper bound $U(t)$ such that for all t and s,

$$\frac{\sigma_2(s,t)}{\sigma(s,t)}\leq U(t),$$

implies that it is possible to set the non-negative choice parameters of the F function, $a(t)$ and $A(t)$, such that $F_2(\xi,t)\geq 0$ for all ξ and t. Given such a choice of $a(t)$ and $A(t)$, the drift of the F process will then be at least

$$a(\tau)\left(\frac{r(\tau)}{\sigma(\xi_\tau,\tau)}+\frac{1}{2}z_1(\xi_\tau,\tau)\right).$$

When $z_1(\xi,t)\geq 0$ for all ξ and t, the drift of the F process is at least $a(\tau)r(\tau)/\sigma(\xi_\tau,\tau)$. Given the existence of such a bounding U function, it follows that when $r(t)\geq 0$ for all t, the restriction that $z_1(s,t)\geq 0$ for all s and t implies that certain properties of a call's delta can be determined from the properties of the probability distribution of a random variable that follows a diffusion process that has both a deterministic diffusion parameter and a non-negative drift. One implication of such a non-negative drift is established in Lemma 2.

Lemma 2. (Deltas) *Suppose that the risk-neutral process for the underlying asset starts at s at time t and follows the diffusion given in (5). Suppose that the volatility parameter is such that there exists a uniform upper bound, $U(t)$, with the property that for all s and t,*

$$U(t)\geq\frac{\sigma_2(s,t)}{\sigma(s,t)}.$$

If $z_1(s,t)\geq 0$ for all s and t, then, provided $r(t)\geq 0$ for all t, the delta of a call option will be bounded below as:

$$c_1(s,t)\geq N\left(\frac{s-k}{s\,\sigma(s,t)\sqrt{\int_t^T[exp\left(\int_t^\tau U(w)dw\right)]^2d\tau}}\right). \tag{10}$$

If the condition $z_1(s,t)\geq 0$ for all s and t, is replaced by the stronger condition that $\sigma_1(s,t)\geq 0$ for all s and t, then, provided $r(t)\geq -\frac{1}{2}[\sigma(s,t)]^2$ for all s and t, the delta of a call will be bounded below as:

$$c_1(s,t) \geq N\left(\frac{\ln(s/k)}{\sigma(s,t)\sqrt{\int_t^T [exp\left(\int_t^\tau U(w)dw\right)]^2 d\tau}}\right). \tag{11}$$

When $\sigma(s,t)$ is time independent, the term $\sqrt{\int_t^T [exp\left(\int_t^\tau U(w)dw\right)]^2 d\tau}$ appearing in (10) and (11) simplifies to $\sqrt{T-t}$.

Lemma 2 is proved in Appendix A.

The restriction that $r(t) \geq 0$ for all t is made explicit in Lemma 2 since the risk-neutral process in (5) may describe a real price process. Real risk-free rates can be negative. To implement the bound in Lemma 2 we need only know $\sigma(s,t)$, the underlying asset's volatility at its current price level. Under the conditions of Lemma 2 (Deltas), we immediately obtain a simple bound on the delta of an at-the-money option.

6. The Delta of an At-the-Money Option

6.1. At-the-money deltas when $z_1(s,t) \geq 0$ for all s and t

In the familiar Black-Scholes setting with non-negative interest rates, the delta of an at-the-money option is at least 1/2. Under the weak restriction that z be such that a bounding U function exists, this '1/2 at-the-money' property is true of deltas in almost all one-dimensional diffusion settings with $z_1(s,t) \geq 0$ for all s and t. By definition, an at-the-money option has $s = k$ and we have as an immediate corollary to Lemma 2:

Corollary. (Deltas) *Under the conditions in Lemma 2 (Deltas), the delta of an at-the-money call option is always at least 1/2.*

The probability of the delta process finishing above its starting value is the same as the probability that the F process finishes above its starting value. Under the conditions of Lemma 2 it is possible to choose $a(t)$ and $A(t)$ when specifying F such that the drift of F is always non-negative. The probability that F then finishes above its starting value is at least 1/2.

6.2. At-the-money deltas when $z_1(s,t)$ can be negative

It is important to note that the delta of an at-the-money option is *not* always at least 1/2. The restriction that $z_1(s,t) \geq 0$ for all s and t is an important precondition of the above Corollary. When this restriction is not satisfied, then, unlike the result in a Black-Scholes setting, the delta of an at-the-money option can be less than 1/2. We demonstrate this result in the following example. For simplicity we assume in this example that, for all t, $r(t) = 0$. Consider an asset price process that at all times in the interval $[T - 0.1, T]$ follows the diffusion

$$s_\tau = \sigma(s_\tau, \tau) s_\tau dB_\tau,$$

with (12)

$$[\sigma(s,t)]^2 = \frac{e^{2(1-s)}}{1 + s^2(T-t)e^{2(1-s)} + 2s\ln(s) - s(T-t)e^{2(1-s)} + s^2[\ln(s)]^2 - \frac{1}{4}s^2(T-t)^2e^{4(1-s)}}.$$

We show in Appendix B that this process is well defined for $\tau \in [T - 0.1, T]$. Figure 1a depicts this volatility as a function of s for a fixed t. For sufficiently high values of s, the volatility declines faster than $1/s$, and hence the diffusion parameter depicted in Figure 1b is decreasing in s.

Consider a call option on this asset with an exercise price $k = 1$ and a date T maturity. The price of this option is given by the solution of the p.d.e.

$$c_2(s,t) + \frac{1}{2}[\sigma(s,t)]^2 s^2 c_{11}(s,t) = 0$$

subject to the terminal condition $c(s,T) = \max[0, s - 1]$. An analytical solution for the value of this call exists and is given by

$$c(s,t) = sN(\mathrm{d}_1) - N(\mathrm{d}_2),$$

where

$$\mathrm{d}_1 := \frac{\ln(s) + \frac{1}{2}(T-t)e^{2(1-s)}}{\sqrt{T-t}\, e^{1-s}}$$

and

$$\mathrm{d}_2 := \mathrm{d}_1 - \sqrt{T-t}\, e^{1-s}.$$

Thus we can determine analytically that when this call option is at-the-money, its delta is given by

$$c_1(s,t)\Big|_{s=k=1} = N\left(\frac{\sqrt{T-t}}{2}\right) - \frac{\sqrt{T-t}}{\sqrt{2\pi}}e^{-(T-t)/8}. \tag{13}$$

Figure 2 graphs the right-hand-side of (13) as a function of $T-t$. It is clear from (13) that for *all* $t \in [T-0.1, T)$, the delta of this at-the-money call option is strictly *less* than 1/2.

7. Theorem 2 and State Prices

In this section we examine properties of state prices in a one-dimensional diffusion world. Let $\pi(s,t,k,T)$ denote the time t price of a state claim written on a non-dividend-paying asset worth s at time t. The state claim pays one dollar at time T if and only if at that date the asset's value exceeds k. The price of a state claim is simply the discounted expectation that the underlying asset's price will finish above k under the risk-neutral process; i.e.,

$$\pi(s,t,k,T) = e^{-\int_t^T r(\tau)d\tau}\Pr\left(\xi_T^{s,t} > k\right),$$

where ξ follows the diffusion

$$d\xi_\tau = r(\tau)\xi_\tau d\tau + z(\xi_\tau,\tau)dB_\tau.$$

Recognizing that the price of a state-contingent claim is simply a discounted probability calculated under the risk-neutral measure, we can apply Theorem 2 to the problem of pricing state-contingent claims. The proof of the following Lemma mirrors that of Lemma 1 (Deltas).

Lemma 1. (State Prices) *Suppose that the risk-neutral process for the underlying asset starts at s at time t and follows the diffusion given in (5). For F as defined in Theorem 2, the value of a state claim paying one dollar if and only if the underlying's price exceeds k at date T, $\pi(s,t,k,T)$, is given by*

$$e^{-\int_t^T r(\tau)d\tau}\Pr\left(\frac{F(s,t) - F(k,T) + \int_t^T \left(a(\tau)\left(\frac{r(\tau)}{\sigma(\xi_\tau,\tau)} - \frac{1}{2}z_1(\xi_\tau,\tau)\right) + F_2(\xi_\tau,\tau)\right)d\tau}{\sqrt{\int_t^T [a(\tau)]^2 d\tau}} > \mathcal{X}\right) \tag{14}$$

where the random variable $\mathcal{X} := -\frac{\int_t^T a(\tau)dB_\tau}{\sqrt{\int_t^T [a(\tau)]^2 d\tau}}$ is distributed $\mathcal{N}(0,1)$.

Comparing the expressions for the value of a state-claim in (14) and an option's delta in (9), we see two differences for a given k. First, valuing the state claim involves multiplying the relevant probability by $e^{-\int_k^s r(\tau)d\tau}$ to recognize the time-value of money. Second, and of importance in this study, the term $+\frac{1}{2}z_1(\xi_\tau, \tau)$ in (9) becomes $-\frac{1}{2}z_1(\xi_\tau, \tau)$ in (14); i.e, the term is higher by $z_1(\xi, t)$ when working with deltas. This is a reflection of that fact that while state prices are determined by the risk-neutral process, deltas are determined by the delta process whose drift exceeds the risk-free drift by $z_1(\xi, t)z(\xi, t)$.

Now consider the implications for state prices of the Section 5 restriction that $z_1(s, t) \geq 0$ for all s and t. Since z_1 enters positively into expression (9) for deltas, but enters negatively into expression (12) for state prices, a given restriction on z_1 that imposes a lower bound on deltas can be used to impose an upper bound on state prices. The proof of the upper bound contained in Lemma 2 (State Prices) mirrors that of the proof of the lower bound in Lemma 2 (Deltas).

Lemma 2. (State Prices) *Suppose that the risk-neutral process for the underlying asset starts at s at time t and follows the diffusion given in (5). Suppose that the volatility parameter is such that there exists a uniform lower bound, $L(t)$, with the property that for all s and t,*

$$L(t) \leq \frac{\sigma_2(s,t)}{\sigma(s,t)}.$$

If $z_1(s,t) \geq 0$ for all s and t, then, provided $r(t) \leq 0$ for all t, the price of a state claim, $\pi(s,t,k,T)$, will be bounded above as:

$$\pi(s,t,k,T) \leq e^{-\int_t^T r(\tau)d\tau} N\left(\frac{s-k}{s\,\sigma(s,t)\sqrt{\int_t^T [exp\left(\int_t^\tau L(w)dw\right)]^2 d\tau}}\right). \tag{15}$$

If the condition $z_1(s,t) \geq 0$ for all s and t, is replaced by the stronger condition that $\sigma_1(s,t) \geq 0$ for all s and t, then, provided $r(t) \leq \frac{1}{2}[\sigma(s,t)]^2$ for all s and t, the price of a state claim will be bounded above as

$$\pi(s,t,k,T) \leq e^{-\int_t^T r(\tau)d\tau} N\left(\frac{\ln(s/k)}{\sigma(s,t)\sqrt{\int_t^T [exp\left(\int_t^\tau L(w)dw\right)]^2 d\tau}}\right). \tag{16}$$

When $\sigma(s,t)$ is time independent, the term $\sqrt{\int_t^T [exp\left(\int_t^\tau L(w)dw\right)]^2 d\tau}$ appearing in (15) and (16) simplifies to $\sqrt{T-t}$.

Under the conditions of Lemma 2 (State Prices) we immediately obtain a simple bound on the price of a state claim that pays one dollar if and only if the underlying asset finishes above its starting value.

Corollary. (State Prices) *Consider a state claim that pays off one dollar if and only if the underlying asset finishes above its starting value. Under the conditions of Lemma 2 (State Prices), the price of such a claim is never greater than the discounted value of fifty cents.*

We have seen how the term $z_1(\xi_\tau, \tau)$ enters positively in expression (9) for deltas and negatively in expression (12) for state prices. This suggests the possibility of bounding option deltas in terms of state prices whenever $z_1(s,t) \geq 0$ for all s and t.

8. A Relation Between Deltas and State Prices

We introduce the notation $c(s,t,k,T)$ to make explicit the dependence of the option's value on its exercise price.

Theorem 5. *Suppose that the risk-neutral process for the underlying asset starts at s at time t and follows the diffusion in (5). If $z_1(s,t) \geq 0$ for all s and t, then a call's delta exceeds $e^{\int_t^T r(\tau)d\tau}$ times the corresponding state price; i.e.*

$$c_1(s,t,k,T) \geq e^{\int_t^T r(\tau)d\tau}\pi(s,t,k,T). \tag{17}$$

Theorem 5 is proved in Appendix A.

Theorem 5 can be applied whenever observed option prices provide a lower bound on $\pi(s,t,k,T)$. Given a rich set of observed option prices, one could use the approach in Bick and Reisman (1994), Derman and Kani (1994), Dupire (1994), and Rubinstein (1994) to numerically determine an option's delta in the one-dimensional diffusion setting of interest here. But what if one can only observe the prices of two call options that differ in their exercise prices, but are otherwise equivalent? When can the information in these two observed option prices be used to bound the delta of the option with the lower exercise price?[8]

[8] One bound is provided by Proposition 2 of Bergman, Grundy and Wiener (1996). Proposition 2 establishes that when the underlying asset follows a one-dimensional diffusion (as in the setting considered in this paper), $c_1(s,t) \geq c(s,t)/s$ for all s and t.

Breeden and Litzenberger (1978) show that

$$\pi(s,t,k,T) = -c_3(s,t,k,T). \tag{18}$$

An option's price must be a convex function of its exercise price, since otherwise the prices offer a simple arbitrage opportunity. Hence the observed prices of two options with exercise prices k' and $k'' > k'$ yield

$$-c_3(s,t,k',T) \geq \frac{c(s,t,k',T) - c(s,t,k'',T)}{k'' - k'}. \tag{19}$$

This empirical bound on the state price is depicted in Figure 3. Combining (17), (18) and (19) we can translate the bound on the relevant state price into an empirically determined bound on the option's delta whenever $z_1(s,t) \geq 0$ for all s and t:

$$c_1(s,t,k',T) \geq e^{\int_t^T r(\tau)d\tau} \frac{c(s,t,k',T) - c(s,t,k'',T)}{k'' - k'}.$$

9. Theorem 2 and the Determination of Value-at-Risk

A natural application of Theorem 2 is to the analysis of the likelihood that a portfolio's value will exceed some critical level on a future date T. To illustrate this application suppose that the true price process for the underlying porfolio is a time-homogeneous diffusion of the form:

$$ds_\tau = \alpha(s_\tau)s_\tau d\tau + \sigma(s_\tau)s_\tau dB_\tau.$$

Suppose an investor does not know the exact functional forms of the drift and diffusion parameters, $\alpha(\cdot)$ and $\sigma(\cdot)$, but is confident that $\underline{\alpha} \leq \alpha(s) \leq \overline{\alpha}$ and $\underline{\sigma} \leq \sigma(s) \leq \overline{\sigma}$ for all s. The investor plans to hold the portfolio from t to T. Suppose she is interested in determining the size of the possible losses that have, say, at least a 1-in-20 chance of occurrence. This question can be answered if one can determine a bound on the probability that, for any given level k of interest, the time T value of her portfolio will exceed the value of k. What additional information about the volatility parameter will allow her to determine such a bound? Lemma 3 provides one answer.

Lemma 3. *Suppose that the true price process for the underlying asset is given by*

$$d\xi_\tau = \alpha(\xi_\tau)\xi_\tau d\tau + \sigma(\xi_\tau)\xi_\tau dB_\tau,$$

with $0 \le \underline{\alpha} \le \alpha(\xi) \le \overline{\alpha}$ *and* $\underline{\sigma} \le \sigma(\xi) \le \overline{\sigma}$ *for all* ξ.

If $\sigma_1(\xi) \ge 0$ *for all* ξ, *then*

$$\Pr\left(\xi_T^{s,t} > k\right) \le \begin{cases} N\left(\frac{\ln(s/k)}{\underline{\sigma}\sqrt{T-t}} + \left(\overline{\alpha}/\underline{\sigma} - \frac{1}{2}\underline{\sigma}\right)\sqrt{T-t}\right), & \text{if } s > k; \\ N\left(\frac{\ln(s/k)}{\overline{\sigma}\sqrt{T-t}} + \left(\overline{\alpha}/\underline{\sigma} - \frac{1}{2}\underline{\sigma}\right)\sqrt{T-t}\right), & \text{if } s < k. \end{cases}$$

If $\sigma_1(\xi) \le 0$ *for all* ξ, *then*

$$\Pr\left(\xi_T^{s,t} > k\right) \ge \begin{cases} N\left(\frac{\ln(s/k)}{\overline{\sigma}\sqrt{T-t}} + \left(\underline{\alpha}/\overline{\sigma} - \frac{1}{2}\overline{\sigma}\right)\sqrt{T-t}\right), & \text{if } s > k; \\ N\left(\frac{\ln(s/k)}{\underline{\sigma}\sqrt{T-t}} + \left(\underline{\alpha}/\overline{\sigma} - \frac{1}{2}\overline{\sigma}\right)\sqrt{T-t}\right), & \text{if } s < k. \end{cases}$$

The proof of the Lemma follows by a straightforward application of Theorem 2 to the true price process.[9] Note that if the underlying asset's price follows a CEV diffusion, $\sigma_1(s) \le 0$ for all s. Note also that when $\underline{\sigma} = \overline{\sigma} = \sigma$, Lemma 3 simplifies to the natural bound:

$$N\left(\frac{\ln(s/k) + \left(\underline{\alpha} - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \le \Pr\left(\xi_T^{s,t} > k\right) \le N\left(\frac{\ln(s/k) + \left(\overline{\alpha} - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

10. Summary and Possible Extensions

This paper develops a new technique for the study of probability distributions in diffusion settings. We provide two different monotonic transformations of an initial diffusion process with a level-dependent diffusion parameter. Under the first transformation the new diffusion process has a deterministic diffusion parameter. Under the second, the new diffusion process has a deterministic volatility parameter. Since both transformations are monotonic, the probability of the original process finishing above some critical level is identical to the probability of the equivalent events that the transformed processes finish above their corresponding transformed critical levels. Our

[9] When neither a restriction that $\underline{\alpha} \ge 0$ or a restriction that $\overline{\alpha} \ge 0$ is imposed, a more general variant of the Lemma applies with $\max\left[\overline{\alpha}/\underline{\sigma}, \overline{\alpha}/\overline{\sigma}\right]$ replacing $\overline{\alpha}/\underline{\sigma}$ and $\min\left[\underline{\alpha}/\underline{\sigma}, \underline{\alpha}/\overline{\sigma}\right]$ replacing $\underline{\alpha}/\overline{\sigma}$.

transformations can be viewed as counterparts to Girsanov's Theorem. Rather than maintaining the diffusion parameter while altering the drift at the expense of the probability measure, our transformations maintain the original Brownian motion while altering the diffusion parameter at the expense of the drift.

Theorem 2 develops an expression for the likelihood of an initial diffusion process finishing above some critical level in terms of the likelihood that a second diffusion process with a deterministic diffusion parameter will finish above a related critical level. We believe Theorem 2 can be useful in the analysis of many interesting problems. We illustrate three such applications. The first application considered is to the study of position deltas. A call's delta is shown to be its probability of finishing in-the-money under a process related to, but different from, the underlying asset's risk-neutral process. We term this process the delta process. We then transform the delta process to obtain a new process with a deterministic diffusion parameter and apply Theorem 2. Theorem 2 allows us to develop a lower bound on option deltas applicable when the diffusion parameter of the underlying asset's price dynamic is non-decreasing in the asset price. We obtain a tighter bound when the underlying's volatility parameter is non-decreasing in the asset price. Direct application of the first bound allows us to generalize the familiar Black-Scholes result that the delta of an at-the-money option is always at least 1/2 to all settings where the diffusion parameter is non-decreasing in the underlying's value. Note that an at-the-money option's delta does *not* always exceed 1/2. We provide an analytic solution to a particular call option pricing problem in which the underlying asset's diffusion parameter is first increasing and then decreasing in the price of the underlying asset. For this particular price process, the delta of an at-the-money call option with an exercise price of unity is, in direct contrast to the result in a Black-Scholes setting, always *less* than 1/2.

Our second application of Theorem 2 is to the pricing of state-contingent claims. State prices are simply discounted probabilities of finishing above some level under the risk-neutral process. We use Theorem 2 to determine an upper bound on state prices when the diffusion parameter is non-decreasing in the underlying's price, and a tighter bound applicable when the volatility parameter is non-decreasing in the underlying's price. We also show how the bounds on state prices implied by even a coarse grid of observed option prices can be used to bound option deltas whenever the diffusion parameter is non-decreasing. Our final application of Theorem 2 is to the analysis of the likelihood of incurring a gain or loss of a given size from some investment strategy. We provide sufficient conditions under which coarse knowledge of the drift and diffusion parameters of the process describing changes in value of the portfolio of interest can be used to place bounds on the value-at-risk in the event of a price collapse with, say, at least a 1-in-20 chance of occurrence.

The bounds derived from Theorem 2 reflect restrictions on the functional form of either the diffusion parameter or the volatility parameter. Determining the economics of the interesting set of such restrictions is part of our ongoing research in this area. Recognizing that

$$c(s,t,k,T) = \int_0^s c_1(x,t,k,T)dx$$

and

$$c(s,t,k,t) = \int_k^\infty (x-k)\pi(s,t,x,T)dx,$$

we are also working to translate our bounds on deltas and our bounds on state prices into bounds on option prices. A natural empirical extension of this line of research is to the analysis of the link between the functional form of the underlying asset's diffusion parameter and the existence and shape of implied volatility smiles.

Appendix A: Proofs of Theorems and Lemmas

Proof of Theorem 1

Given assumption (i) of the Assumption Set, the partial derivatives of F are given by

$$F_1(\xi, t) = \frac{a(t)}{z(\xi, t)},$$
$$F_{11}(\xi, t) = -\frac{a(t) z_1(\xi, t)}{[z(\xi, t)]^2},$$

and

$$F_2(\xi, t) = \int_{A(t)}^{\xi} \frac{a_1(t) z(x, t) - a(t) z_2(x, t)}{[z(x, t)]^2} dx - \frac{a(t)}{z(A(t), t)} A_1(t).$$

Applying Ito's Lemma to F we have

$$\begin{aligned} dF(\xi_\tau, \tau) &= F_1(\xi_\tau, \tau) d\xi_\tau + \frac{1}{2} F_{11}(\xi_\tau, \tau) [d\xi_\tau]^2 + F_2(\xi_\tau, \tau) d\tau \\ &= \left(a(\tau) \left(\frac{\mu(\xi_\tau, \tau)}{z(\xi_\tau, \tau)} - \frac{1}{2} z_1(\xi_\tau, \tau) \right) + F_2(\xi_\tau, t) \right) d\tau + a(\tau) dB_\tau. \qquad (A1) \end{aligned}$$

Since $F(\xi, t)$ is strictly increasing in ξ for all t, there exists an inverse function γ such that $\xi = \gamma(F(\xi, t), t)$ for all ξ and t. Thus the drift parameter in $(A1)$ can be expressed as $\psi(F_\tau, \tau)$. The diffusion parameter of the F_τ process is the deterministic value $a(\tau)$. ∎

Proof of Theorem 1′

The partial derivatives of $\mathcal{F}$ are given by

$$\mathcal{F}_1(\xi, t) = \mathcal{F}(\xi, t) \frac{a(t)}{z(\xi, t)},$$
$$\mathcal{F}_{11}(\xi, t) = \mathcal{F}(\xi, t) \left(\frac{a(t)\left(a(t) - z_1(\xi, t)\right)}{[z(\xi, t)]^2} \right),$$

and

$$\mathcal{F}_2(\xi, t) = \mathcal{F}(\xi, t) \left(\int_{A(t)}^{\xi} \frac{a_1(t) z(x, t) - a(t) z_2(x, t)}{[z(x, t)]^2} dx - \frac{a(t)}{z(A(t), t)} A_1(t) \right).$$

Applying Ito's Lemma to $\mathcal{F}$ we have

$$d\mathcal{F}(\xi_\tau, \tau) = \left(a(\tau) \mathcal{F}(\xi_\tau, \tau) \left(\frac{\mu(\xi_\tau, \tau)}{z(\xi_\tau, \tau)} + \frac{1}{2} \left(a(\tau) - z_1(\xi_\tau, \tau)\right) \right) + \mathcal{F}_2(\xi_\tau, t) \right) d\tau + a(\tau) \mathcal{F}(\xi_\tau, \tau) dB_\tau. \qquad (A2)$$

Since $\mathcal{F}(\xi, t)$ is strictly increasing in ξ for all t, there exists an inverse function φ such that $\xi = \varphi(\mathcal{F}(\xi, t), t)$ for all ξ and t. Thus the drift parameter in $(A2)$ can be expressed as $\theta(\mathcal{F}_\tau, \tau)$. The volatility parameter of the $\mathcal{F}_\tau$ process is the deterministic value $a(\tau)$. ∎

Proof of Theorem 2

$$F\left(\xi_T^{y,t}, T\right) = F(y,t) + \int_t^T \left(a(\tau)\left(\frac{\mu(\xi_\tau,\tau)}{z(\xi_\tau,\tau)} - \frac{1}{2}z_1(\xi_\tau,\tau)\right) + F_2(\xi_\tau,\tau)\right)d\tau + \int_t^T a(\tau)dB_\tau.$$

Thus

$$\Pr\left(\xi_T^{y,t} > k\right) = \Pr\left(F\left(\xi_T^{y,t}, T\right) > F(k,T)\right) = \Pr\left(F(y,t) + \int_t^T dF_\tau > F(k,T)\right)$$

$$= \Pr\left(F(y,t) + \int_t^T \left(a(\tau)\left(\frac{\mu(\xi_\tau,\tau)}{z(\xi_\tau,\tau)} - \frac{1}{2}z_1(\xi_\tau,\tau)\right) + F_2(\xi_\tau,\tau)\right)d\tau + \int_t^T a(\tau)dB_\tau > F(k,T)\right)$$

$$= \Pr\left(\frac{F(y,t) - F(k,T) + \int_t^T \left(a(\tau)\left(\frac{\mu(\xi_\tau,\tau)}{z(\xi_\tau,\tau)} - \frac{1}{2}z_1(\xi_\tau,\tau)\right) + F_2(\xi_\tau,\tau)\right)d\tau}{\sqrt{\int_t^T [a(\tau)]^2 d\tau}} > \mathcal{X}\right),$$

where the random variable $\mathcal{X} := -\frac{\int_t^T a(\tau)dB_\tau}{\sqrt{\int_t^T [a(\tau)]^2 d\tau}}$ is distributed $\mathcal{N}(0,1)$. ∎

Note that the random variables compared in the inequality are not independent.

Proof of Theorem 3

Taking the partial of (6) with respect to s gives

$$r(t)v_{11}(s,t)s + v_{12}(s,t) + z_1(s,t)z(s,t)v_{11}(s,t) + \frac{1}{2}[z(s,t)]^2 v_{111}(s,t) = 0. \qquad (A3)$$

Let f be the first partial of the contingent claim's value with respect to the value of the underlying; $f(s,t) := v_1(s,t)$. The p.d.e. in $(A3)$ can then be rewritten as

$$(r(t)s + z_1(s,t)z(s,t))\, f_1(s,t) + f_2(s,t) + \frac{1}{2}[z(s,t)]^2 f_{11}(s,t) = 0.$$

Assuming that $r(t)s + z_1(s,t)z(s,t)$ and $z(s,t)$ each satisfy Lipschitz and growth conditions, the Feynman-Kac Theorem can be used to express $v_1(s,t)$ as

$$v_1(s,t) = E\{g_1(\xi_T^{s,t})\},$$

where the dynamic of ξ_τ is described by

$$d\xi_\tau = \left(r(\tau)\xi_\tau + z_1(\xi_\tau,\tau)z(\xi_\tau,\tau)\right)d\tau + z(\xi_\tau,\tau)dB_\tau. \quad \blacksquare$$

Proof of Lemma 1 (Deltas)

In the application of Theorem 2 to the analysis of the probability of the option finishing in-the-money under the delta process in (7), the μ function of Theorem 2 is simply the drift of the delta process; i.e.,

$$\mu(\xi,t) = r(t)\xi + z_1(\xi,t)z(\xi,t).$$

Thus

$$\begin{aligned} a(\tau)\left(\frac{\mu(\xi_\tau,\tau)}{z(\xi_\tau,\tau)} - \frac{1}{2}z_1(\xi_\tau,\tau)\right) &= a(\tau)\left(\frac{r(\tau)\xi_\tau + z_1(\xi_\tau,\tau)z(\xi_\tau,\tau)}{z(\xi_\tau,\tau)} - \frac{1}{2}z_1(\xi_\tau,\tau)\right) \\ &= a(\tau)\left(\frac{r(\tau)\xi_\tau}{\sigma(\xi_\tau,\tau)\xi_\tau} + \frac{1}{2}z_1(\xi_\tau,\tau)\right). \end{aligned}$$

Substituting in Theorem 2 immediately establishes expression (9) of Lemma 2. $\blacksquare$

Proof of Lemma 2 (Deltas)

Following the steps in subsection *2.1.*, setting $A(t) = k$ and $a(t) = a(0)exp\left(\int_0^t U(\tau)d\tau\right)$ in the specification of the F function gives $F_2(\xi,t) = 0$ for all ξ and t and $F(k,T) = 0$ for all t. Expression (9) of Lemma 1 then simplifies to

$$c_1(s,t) = \Pr\left(\frac{F(s,t) + \int_t^T \left(a(\tau)\left(\frac{r(\tau)}{\sigma(\xi_\tau,\tau)} + \frac{1}{2}z_1(\xi_\tau,\tau)\right)\right)d\tau}{\sqrt{\int_t^T [a(\tau)]^2 d\tau}} > \mathcal{X}\right) \tag{A4}$$

where the random variable $\mathcal{X} := -\frac{\int_t^T a(\tau)dB_\tau}{\sqrt{\int_t^T [a(\tau)]^2 d\tau}}$ *is distributed* $\mathcal{N}(0,1)$.

The integrand of the integral in the numerator of the expression on the left-hand-side of the inequality in $(A4)$ is non-negative given the conditions that $r(t) \geq 0$ for all t, and $z_1(s,t) \geq 0$ for all s and t. Thus we have

$$c_1(s,t) \geq \Pr\left(\frac{\int_k^s \frac{a(t)}{z(x,t)}dx}{\sqrt{\int_t^T [a(\tau)]^2 d\tau}} > \mathcal{X}\right). \tag{A5}$$

The condition that $z_1(s,t) \geq 0$ for all s and t implies $z(x,t) > \sigma(s,t)s$ for $x >$s, and $z(x,t) < \sigma(s,t)s$ for $x < s$. Therefore

$$\int_k^s \frac{a(t)}{z(x,t)}\, dx \geq \int_k^s \frac{a(t)}{\sigma(s,t)s}\, dx = \frac{a(t)(s-k)}{\sigma(s,t)s},$$

and hence

$$c_1(s,t) \geq \Pr\left(\frac{a(t)(s-k)}{\sigma(s,t)s\sqrt{\int_t^T [a(\tau)]^2 d\tau}} > \mathcal{X}\right).$$

Substituting for $a(\tau)$ and noting that the left-hand-side of the inequality is not a random variable gives the desired result.

The integrand of the integral in the numerator of the expression on the left-hand-side of the inequality in $(A4)$ can be written as

$$a(\tau)\left(\frac{r(\tau) + \frac{1}{2}[\sigma(\xi_\tau,\tau)]^2}{\sigma(\xi_\tau,\tau)} + \frac{1}{2}\sigma_1(\xi_\tau,\tau)\xi_\tau\right).$$

If $\sigma_1(s,t) \geq 0$ for all s and t, the integrand will be non-negative provided $r(t) + \frac{1}{2}[\sigma(s,t)]^2 \geq 0$ for all s and t. The relation in $(A5)$ will again be satisfied. When $\sigma_1(s,t) \geq 0$ for all s and t, the integral term in $(A5)$ can be bounded as

$$\int_k^s \frac{a(t)}{\sigma(x,t)x}\, dx \geq \int_k^s \frac{a(t)}{\sigma(s,t)x}\, dx = \frac{a(t)\ln(s/k)}{\sigma(s,t)},$$

and hence

$$c_1(s,t) \geq \Pr\left(\frac{a(t)\ln(s/k)}{\sigma(s,t)\sqrt{\int_t^T [a(\tau)]^2 d\tau}} > \mathcal{X}\right).$$

Substituting for $a(\tau)$ and noting that the left-hand-side of the inequality is not a random variable gives the desired result.

Finally note that when $\sigma(s,t)$ is time independent, the uniform upper bound restriction is satisfied by $U(t) = 0$ for all t. Hence

$$\sqrt{\int_t^T [exp\left(\int_t^T U(w)dw\right)]^2 d\tau} = \sqrt{\int_t^T [exp(0)]^2 d\tau} = \sqrt{\int_t^T 1 d\tau} = \sqrt{T-t}. \quad \blacksquare$$

Proof of Theorem 5

Compare the drift of the risk-neutral process in (5) and the drift of the delta process in (7). Given the restriction on the sign of z_1, the drift in (7) exceeds that in (5). The result then follows from an application of Proposition 2.18 of Chapter 5 of Karatzas and Shreve (1991). ∎

Appendix B: Demonstration that the SDE in (12) satisfies both a Lipschitz and a Growth Condition

Note first that $[\sigma(s,t)]^2$ as defined in (12) is positive and finite for all $s \geq 0$ and all $t \in [T-0.1, T]$. This follows from the fact that the denominator in (12) is always positive. In fact, the denominator is always greater than $0.02\overline{6}$. To demonstrate this lower bound on the denominator we introduce a new variable $\mathcal{T}$.

$$\mathcal{T} = \mathcal{T}(s,t) := (T-t)e^{2(1-s)}.$$

We denote the denominator by $\mathcal{Q}$ and rewrite it as

$$\mathcal{Q}(s,\mathcal{T}) = 1 + s^2\mathcal{T} + 2s\ln(s) - s\mathcal{T} + s^2[\ln(s)]^2 - \frac{1}{4}s^2\mathcal{T}^2.$$

We can bound the sum of the three terms involving $\mathcal{T}$ as

$$\begin{aligned} s^2\mathcal{T} - s\mathcal{T} - \frac{1}{4}s^2\mathcal{T}^2 &= s\mathcal{T}\left(s - 1 - \frac{1}{4}s\mathcal{T}\right) \\ = s\mathcal{T}\left(s\left(1-\frac{\mathcal{T}}{4}\right)-1\right) &= s\mathcal{T}\frac{4-\mathcal{T}}{4}\left(s - \frac{4}{4-\mathcal{T}}\right) \\ \geq \frac{2\mathcal{T}}{4-\mathcal{T}}\frac{4-\mathcal{T}}{4}\frac{-2}{4-\mathcal{T}} &= -\frac{\mathcal{T}}{4-\mathcal{T}} > -\frac{1}{3}, \end{aligned}$$

since $0 \leq \mathcal{T} \leq 0.1e^2 < 1$ and the function $\frac{-\mathcal{T}}{4-\mathcal{T}}$ is monotonically decreasing in this region. Thus

$$\mathcal{Q}(s,\mathcal{T}) > \frac{2}{3} + 2s\ln(s) + s^2[\ln(s)]^2.$$

Using the fact that $s\ln(s) > -0.4$ for all values of s we have $\mathcal{Q}(s,\mathcal{T}) > 0.02\overline{6}$.

Now note that the diffusion parameter, $z(s,t) = \sigma(s,t)s$, must satisfy Lipschitz and growth conditions. Consider the respective Lipschitz and growth conditions given in conditions (E.2) and (E.3) of Duffie (1992, p. 240). That a growth condition of the form (E.3) is satisfied for all $t \in [T-0.1, T]$ follows immediately from the twin observations that the numerator of the expression for $[\sigma(s,t)]^2$ in (12) is a decreasing function of s, while the denominator is bounded from below. To demonstrate that a Lipschitz condition of the form in (E.2) is satisfied for all $t \in [T-0.1, T]$, we demonstrate that there exists a constant $\mathcal{K}$ such that

$$z_1(s,t) \leq \mathcal{K}.$$

First we write

$$\sigma(s,t)s = \frac{P(s)}{\sqrt{\mathcal{Q}(s, \mathcal{T}(s,t))}}$$

where $P(s) := se^{1-s}$. Then

$$\frac{\partial[\sigma(s,t)s]}{\partial s} = \frac{P_1 \mathcal{Q}^{1/2} - \frac{1}{2} P \mathcal{Q}^{-1/2} \frac{\partial \mathcal{Q}(s,\mathcal{T}(s,t))}{\partial s}}{\mathcal{Q}}. \qquad (B1)$$

Recalling that $\mathcal{Q}(s, \mathcal{T})$ is positive and bounded from below, we have that the first term in the expression in $(B1)$,

$$\frac{P_1(s)}{\sqrt{\mathcal{Q}(s,\mathcal{T})}} = \frac{e^{1-s}(1-s)}{\sqrt{\mathcal{Q}(s,\mathcal{T})}},$$

is bounded. The second term in $(B1)$,

$$\frac{P(s)}{2[\mathcal{Q}(s,\mathcal{T})]^{3/2}} \frac{\partial \mathcal{Q}(s, \mathcal{T}(s,t))}{\partial s}$$

can be rewritten as

$$\frac{se^{1-s}}{2[\mathcal{Q}(s,\mathcal{T})]^{3/2}}(4s\mathcal{T} + 2\ln(s) + 2 - \mathcal{T} + 2s[\ln(s)]^2 + 2s\ln(s) - \frac{1}{2}s\mathcal{T}^2 - 2s^2\mathcal{T} + s^2\mathcal{T}^2),$$

This expression is a continuous function that tends to zero when $s \to +\infty$ and is bounded when $s \to 0$. Thus it is bounded from above and below. The difference between the two terms comprising $(B1)$ is then a continuous function that is bounded from above and below, and the Lipschitz condition is satisfied. Thus we have established that the SDE in (12) is well-defined over the relevant time horizon. ∎

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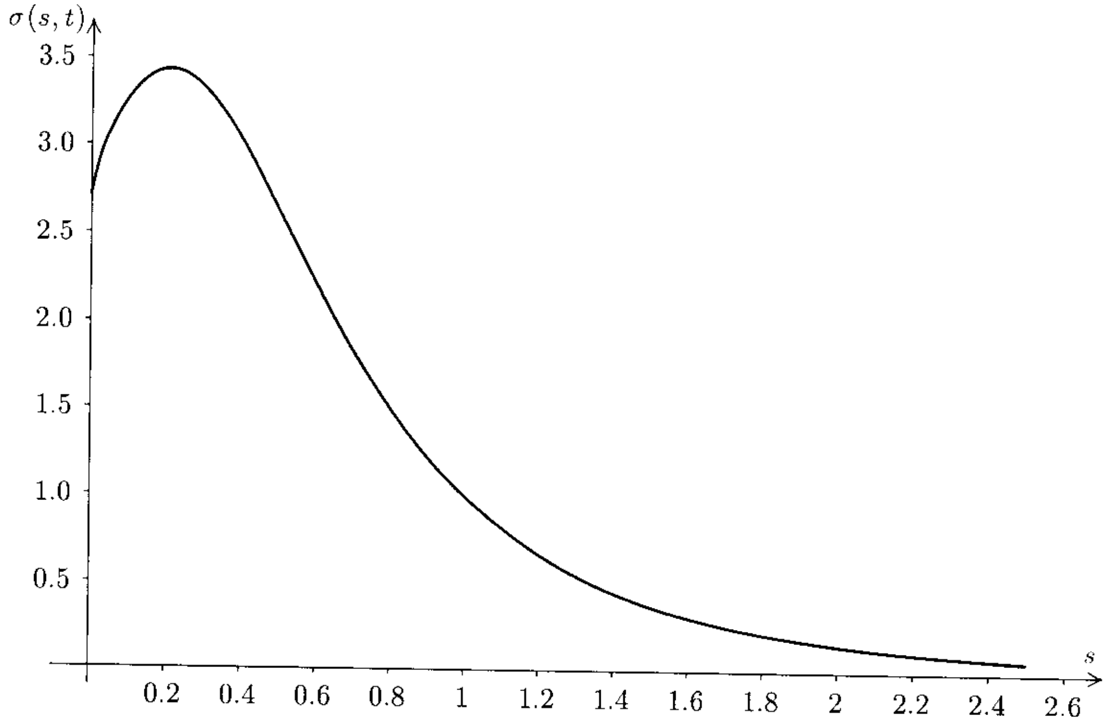

Figure 1a. A volatility function, $\sigma(s,t)$, such that the diffusion parameter, $z(s,t) = \sigma(s,t)\,s$, is not everywhere non-decreasing in s. Volatility depicted at time $t = T - 0.05$, when $\sigma(s,t)$ is defined by

$$[\sigma(s,t)]^2 = \frac{e^{2(1-s)}}{1 + s^2(T-t)e^{2(1-s)} + 2s\ln(s) - s(T-t)e^{2(1-s)} + s^2[\ln(s)]^2 - \frac{1}{4}s^2(T-t)^2e^{4(1-s)}}.$$

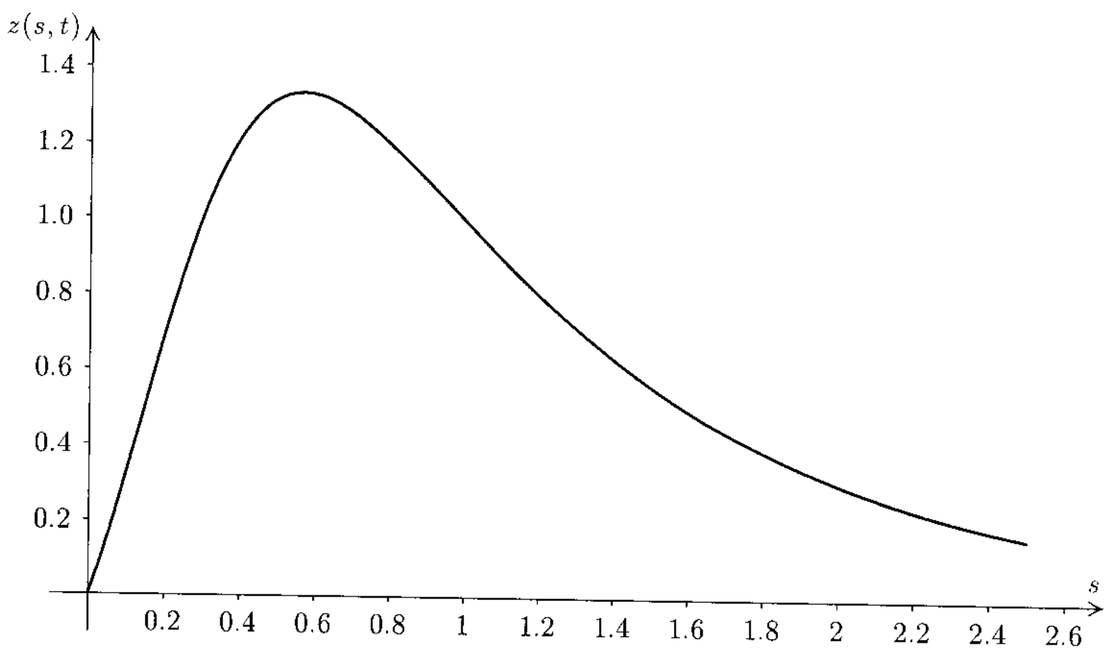

Figure 1b. Diffusion parameter, $z(s,t)$, that is not everywhere non-decreasing in s. $z(s,t) = \sigma(s,t)\,s$, where $\sigma(s,t)$ is as depicted in Figure 1a.

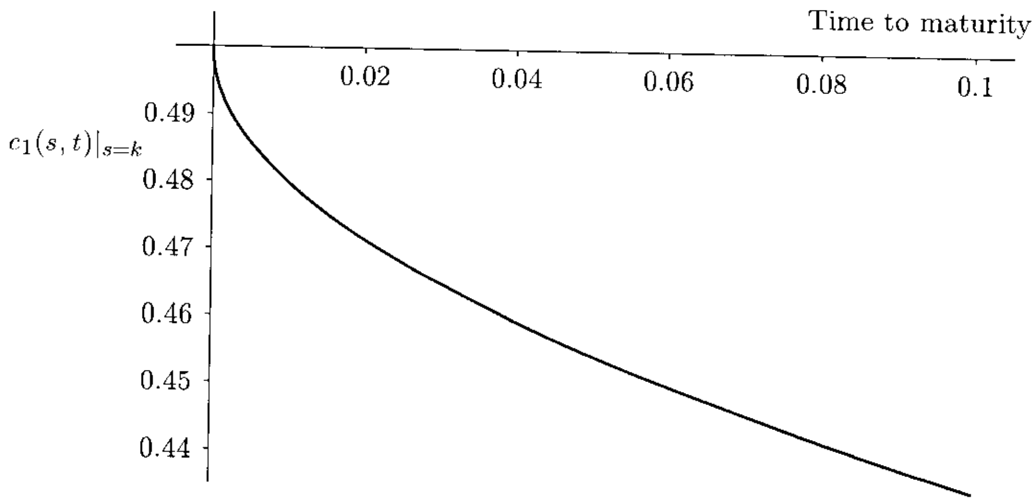

Figure 2. Delta of an at-the-money call with $s = k = 1$. The risk-free rate is zero. The call with maturity date T is written on an asset whose price s_t at all times $t \in [T - 0.1, T]$ follows the risk-neutral diffusion:

$$s_\tau = \sigma(s_\tau, \tau) s_\tau dB_\tau,$$

whose squared volatility parameter is given by

$$[\sigma(s,t)]^2 = \frac{e^{2(1-s)}}{1 + s^2(T-t)e^{2(1-s)} + 2s\ln(s) - s(T-t)e^{2(1-s)} + s^2[\ln(s)]^2 - \frac{1}{4}s^2(T-t)^2e^{4(1-s)}}.$$

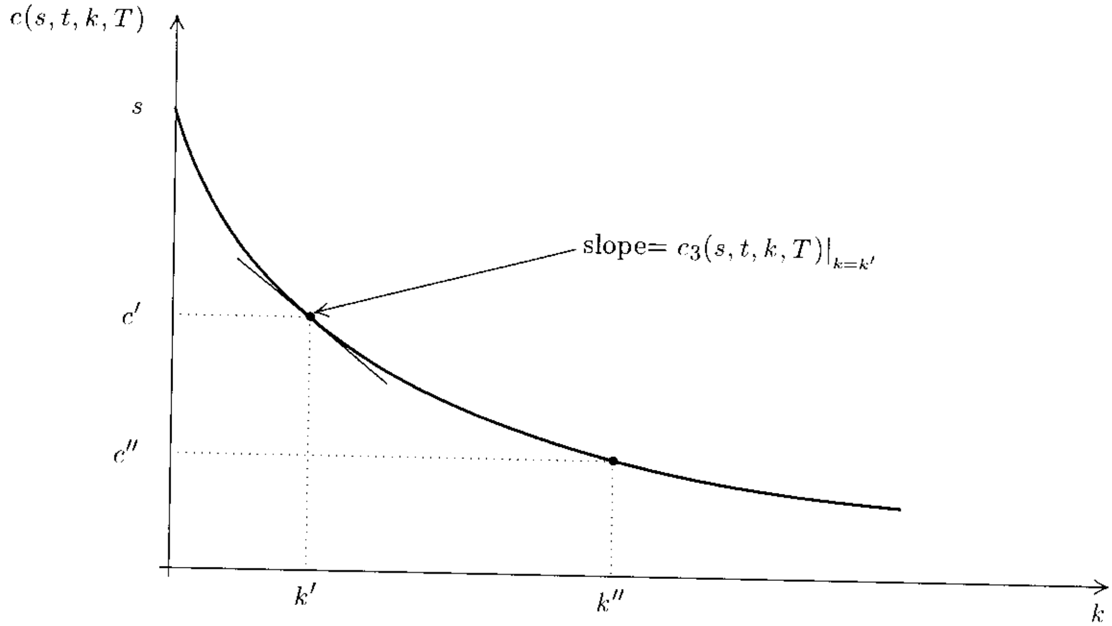

Figure 3. Illustration of the bound on $c_3(s,t,k,T)|_{k=k'}$ implied by (i) the no-arbitrage relation that an option's price must be a convex function of its exercise price and (ii) the observed prices of options with exercise prices of k' and $k'' > k'$. $c(s,t,k,T)$ is the time t price of a call option with a date T maturity and exercise price of k written on an asset worth s.

$$-c_3(s,t,k,T)\big|_{k=k'} \geq \frac{c(s,t,k',T) - c(s,t,k'',T)}{k'' - k'}.$$