OPTIMAL CONSUMPTION CHOICES
FOR A "LARGE" INVESTOR

by

Domenico Cuoco
Jaksa Cvitanic

4-96

RODNEY L. WHITE CENTER FOR FINANCIAL RESEARCH
The Wharton School
University of Pennsylvania
3254 Steinberg Hall-Dietrich Hall
Philadelphia, PA 19104-6367
(215) 898-7616

Optimal Consumption Choices for a "Large" Investor*

Domenico Cuoco
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104
cuoco@wharton.upenn.edu

Jakša Cvitanić
Department of Statistics
Columbia University
New York, NY 10027
cj@stat.columbia.edu

Last revised: April 1996

Abstract

This paper examines the optimal consumption and investment problem for a "large" investor, whose portfolio choices affect the instantaneous expected returns on the traded assets. Alternatively, our analysis can be interpreted in terms of an optimal growth problem with nonlinear technologies. Existence of optimal policies is established using martingale and duality techniques under general assumptions on the securities' price process and the investor's preferences. As an illustration of our characterization result, explicit solutions are provided for specific examples involving an agent with logarithmic utilities, and a generalized two-factor version of the CCAPM is derived. The analogy of the consumption problem examined in this paper to the consumption problem with constraints on the portfolio choices is emphasized.

*We are grateful to Suleyman Basak and to participants in the finance workshop at the University of British Columbia, the economic theory workshop at Cornell University and the Micro Lunch Seminar at the Wharton School for their comments.

1. Introduction

This paper examines the optimal consumption and investment problem for a "large" investor, whose portfolio choices affect the drift of the securities' price process. The impact of the investor's position on prices is specified exogenously and may arise because of size only or because other agents in the market believe that the "large" trader has superior information. Existence and characterization results for the optimal choices are obtained using martingale and duality techniques similar to those employed by He and Pearson (1991), Karatzas, Lehoczky, Shreve and Xu (1991), Xu and Shreve (1992a), Cvitanić and Karatzas (1992), He and Pagès (1993), among others, to analyze the consumption/investment problem in markets with constraints. In fact, the present framework with policy-dependent prices includes (at least formally) the case of portfolio constraints as a special case.

The correspondence of the optimal consumption problem with portfolio constraints to one with policy-dependent price drifts was first pointed out by El Karoui, Peng and Quenez (1995). They used the theory of backward stochastic differential equations, independently developed by Pardoux and Peng (1990) and Duffie and Epstein (1992), to analyze the problem of a "large" investor trying to maximize a recursive utility or to hedge a contingent claim. In particular, they provided a martingale characterization of optimal consumption plans and of the minimal cost to hedge a contingent claim, but did not prove the existence of an optimal consumption policy. Cvitanić and Ma (1994) pointed out that if the payoff of the contingent claim is not exogenously fixed, but is allowed to depend on the future price of some traded asset, then the hedging problem translates into finding a solution to a forward-backward stochastic differential equation. For the Markovian case, the four-step scheme of Ma, Protter and Yong (1994) was used to obtain a partial differential equation identifying the minimal hedging cost. Cvitanić (1995) surveys this literature and briefly discusses the difficulty (nonconvexity) that arises when trying to apply the duality approach of Cvitanić and Karatzas (1992) to establish the existence of an optimal consumption policy for a "large" investor. Needles to say, the existence of optimal policies is an important concern if a general dependence of price drifts on the investor portfolio choices is to be allowed, as arbitrage opportunities, or "market manipulation trading strategies", might arise. Jarrow (1992, 1994) has examined this issue in a discrete-time infinite-horizon economy with policy-dependent prices.

In this paper, we use martingale and duality techniques to provide sufficient conditions for the existence of an optimal consumption plan for a "large" investor and to characterize the optimal plan. The martingale duality approach maps the primal consumption/investment problem into a dual minimization problem that solves for the individual's shadow state-prices (intertemporal marginal rates of substitution). To cope with nonconvexity, we depart from the approach in the above-mentioned papers dealing with portfolio constraints, and establish the existence of optimal consumption/investment policies by formulating the dual problem directly over the space of shadow state-prices, rather than over sample paths of w on $[0,T]$. We interpret the sigma-field $\mathcal{F}_t$ as representing the information of the individual at time t and the probability measure P as representing his beliefs. All the stochastic processes to appear in the sequel are progressively measurable with respect to $\mathbf{F}$ and all the equalities involving random variables are understood to hold P-a.s..[2]

Consumption space. There is a single perishable good (the numeraire). The consumption space $\mathcal{C}$ is given by the set of adapted consumption rate processes c with $\int_0^T |c(t)|\,dt < \infty$. The individual consumption set will be shortly specified as a subset of the non-negative orthant $\mathcal{C}_+$.

Securities market. The investment opportunities are represented by $n+1$ long-lived securities. The first security (the "bond") is a locally riskless savings account earning the instantaneous interest rate process r, so that its value process B evolves according to

$$B(t,\omega) = B(0) + \int_0^t r(\tau,\omega) B(\tau,\omega)\,d\tau. \tag{1}$$

The remaining n assets are risky. Letting $S = (S_1,\ldots,S_n)$ denote their price process and $D = (D_1,\ldots,D_n)$ their cumulative dividend process, we assume that $S+D$ is an Itô process:

$$S(t,\omega) + D(t,\omega) = S(0) + \int_0^t I_S(\tau,\omega)\mu(\tau,\omega)\,d\tau + \int_0^t I_S(\tau,\omega)\sigma(\tau,\omega)\,dw(\tau,\omega), \tag{2}$$

where $I_S(t)$ denotes the $n \times n$ diagonal matrix with elements $S(t)$.

We allow explicit dependence of the price process for the traded securities on the portfolio strategy chosen by the investor by assuming that

$$r(t,\omega) = \bar{r}(t,\omega) + \breve{r}(\alpha(t,\omega), \theta(t,\omega), t, \omega),$$

and

$$\mu(t,\omega) = \bar{\mu}(t,\omega) + \breve{\mu}(\alpha(t,\omega), \theta(t,\omega), t, \omega)$$

for some functions $\breve{r} : \mathbb{R} \times \mathbb{R}^n \times [0,T] \times \Omega \to \overline{\mathbb{R}}$ and $\breve{\mu} : \mathbb{R} \times \mathbb{R}^n \times [0,T] \times \Omega \to \overline{\mathbb{R}}^n$, where α and $\theta = (\theta_1,\ldots,\theta_n)$ denote, respectively, the dollar amounts invested in the bond and in the n risky asset. Without loss of generality, we assume that $\breve{r}(0,0,t,\omega) = 0$ and $\breve{\mu}(0,0,t,\omega) = 0$ for all $(t,\omega) \in [0,T] \times \Omega$. Clearly, the case $\breve{r} \equiv 0$ and $\breve{\mu} \equiv 0$ corresponds to the usual setting with exogenously fixed prices. In the sequel we will suppress the explicit dependence on the state ω whenever no possibility of confusion arises.

Assumption 1. *The process $\bar{r}$ satisfies*

$$\int_0^T \bar{r}(t)^-\,dt < K_r \tag{3}$$

for some $K_r > 0$, where $x^- = \max(0,-x)$ denotes the negative part of the real number x.

[2] A process $X = \{X(t) : t \in [0,T]\}$ is said to be *progressively measurable* with respect to the filtration $\mathbf{F} = \{F_t\}$ if for every $t \in [0,T]$, the map $(s,\omega) \mapsto X(s,\omega)$ from $([0,t] \times \Omega, \mathcal{B}([0,t]) \otimes \mathcal{F}_t)$ into $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is measurable, where $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$ denotes the product σ-field of the Borel σ-field on $[0,t]$ and $\mathcal{F}_t$, and $\mathcal{B}(\mathbb{R}^n)$ denotes the Borel σ-field on $\mathbb{R}^n$. Recalling that a process X is *adapted* to $\mathbf{F}$ if $X(t)$ is an $\mathcal{F}_t$-measurable random variable for all $t \in [0,T]$, we have that every progressively measurable process is adapted. Conversely, any adapted process with right- or left-continuous paths is progressively measurable.

and there exist constants $\delta \in (0,1)$ *and* $\gamma \in (0,\infty)$ *such that*

$$\delta u_c(c,t) \geq u_c(\gamma c,t) \qquad \forall (c,t) \in (0,\infty) \times [0,T]. \tag{11}$$

Finally, $u(c,\cdot)$ *is continuous and decreasing on* $[0,T]$ *for all* $c > 0$.

Remark 1. Condition (10) is well understood and it implies in particular that the derivative function $u_c(\cdot,t)$ has a continuous and strictly decreasing inverse $f(\cdot,t)$ mapping $(0,\infty)$ onto itself. Condition (11) has the purpose of guaranteeing that certain functionals to be introduced in the sequel can be differentiated under the integral sign. It is easily verified that this condition holds for the utility functions $u(c,t) = \rho(t)\log c$ or $u(c,t) = \rho(t)\frac{c^{1-b}}{1-b}$, $b > 0$, $b \neq 1$. Also, taking $c = f(y,t)$ in (11), applying $f(\cdot,t)$ to both sides and iterating shows that the following property holds

$$\forall \delta \in (0,\infty),\ \exists \gamma \in (0,\infty) \text{ such that } f(\delta y,t) \leq \gamma f(y,t),\ \forall (y,t) \in (0,\infty) \times [0,T]. \tag{12}$$

The agent is endowed with some initial wealth $W_0 \geq 0$ and a *bounded* stochastic income stream $y \in C_+^*$.

3. Feasible consumption processes

A consumption process $c \in C_+^*$ is said to be *feasible* if there exists an admissible trading strategy $(\alpha,\theta) \in \Theta$ such that letting

$$W(t) = \alpha(t) + \sum_{k=1}^{n} \theta_k(t)$$

denote the value of the agent's portfolio at time t, we have

$$W(t) = W_0 + \int_0^t [\alpha(\tau)\bar{r}(\tau) + \theta(\tau)^\top \bar{\mu}(\tau) + g(\alpha(\tau),\theta(\tau),\tau)]\, d\tau \tag{13}$$
$$+ \int_0^t \theta(\tau)^\top \sigma(\tau)\, dw(\tau) - \int_0^t (c(\tau) - y(\tau))\, d\tau$$

$$W(t) \geq -K \exp\Big(\int_0^t \bar{r}(\tau)\, d\tau\Big) \tag{14}$$

$$W(T) \geq 0 \tag{15}$$

for all $t \in [0,T]$ and some $K \in \mathbb{R}_+$, where

$$g(\alpha,\theta,t,\omega) = \alpha \breve{r}(\alpha,\theta,t,\omega) + \theta^\top \breve{\mu}(\alpha,\theta,t,\omega). \tag{16}$$

If the above conditions are satisfied, the trading strategy (α,θ) is said to *finance* c.

Equation (13) is the usual dynamic budget constraint: it states that the wealth at any time $t \in [0,T]$ equals the initial wealth, plus the trading gains, minus the cumulative net consumption. The only difference from the standard setting is that the stochastic integral equation for the wealth process is non-linear in the trading strategy (α,θ): this non-linearity arises from the effect of trading strategies on prices and is captured by the function g.

Remark 2. It follows from Theorem 5 in Rockafellar (1974) that

$$g(\alpha, \theta, t, \omega) = \inf_{\nu \in \mathcal{N}_t(\omega)} [\tilde{g}(\nu, t, \omega) + \alpha \nu_0 + \theta^\top \nu_-].$$

By the compactness of $\mathcal{N}_t(\omega)$ and the lower semicontinuity of $\tilde{g}$, this implies that for any $(\alpha, \theta) \in \Theta$ there exists a $\nu \in \mathcal{N}$ such that

$$g(\alpha(t), \theta(t), t) = \tilde{g}(\nu(t), t) + \alpha(t)\nu_0(t) + \theta(t)^\top \nu_-(t). \tag{18}$$

4. Examples

The following examples motivate the analysis in the present paper and illustrate the connection with the optimal consumption problem with constraints on the investment policies. For purposes of comparison, we start from the standard complete-market setting in which prices are exogenously fixed.

Standard setting: In this case $\breve{r} \equiv 0$ and $\breve{\mu} \equiv 0$, which implies $g \equiv 0$ and

$$\tilde{g}(\nu, t, \omega) = \begin{cases} 0 & \text{if } \nu = 0; \\ \infty & \text{otherwise.} \end{cases}$$

Therefore

$$\mathcal{N}_t(\omega) = \{0\} \quad \text{for all } (t, \omega) \in [0, T] \times \Omega.$$

This is the setting examined by Karatzas, Lehoczky and Shreve (1987) and Cox and Huang (1989, 1991).

Price pressure: Let $\breve{r} \equiv 0$ and

$$\breve{\mu}(\alpha, \theta, t, \omega) = \begin{cases} -\dfrac{a(t, \omega)\theta}{|\theta|} & \text{if } \theta \neq 0; \\ 0 & \text{otherwise} \end{cases}$$

for some nonnegative bounded process a, so that buying a risky asset depresses its expected return, while shorting it increases the expected return. In this case it is easily verified that

$$g(\alpha, \theta, t, \omega) = -a(t, \omega)|\theta|$$

(the expected return on wealth decreases in a concave fashion with the absolute amounts invested in risky assets) and

$$\tilde{g}(\nu, t, \omega) = \begin{cases} 0 & \text{if } \nu_0 = 0 \text{ and } |\nu| \leq a(t, \omega); \\ \infty & \text{otherwise.} \end{cases}$$

Therefore,

$$\mathcal{N}_t(\omega) = \{\nu \in \mathbb{R}^{n+1} : \nu_0 = 0 \text{ and } |\nu| \leq a(t, \omega)\}.$$

More generally, if

$$\breve{\mu}(\alpha, \theta, t, \omega) = \begin{cases} -\dfrac{A(t, \omega)\theta}{|\theta|} & \text{if } \theta \neq 0; \\ 0 & \text{otherwise} \end{cases}$$

is the *barrier cone* of $-K$. This problem with convex constraints on the investment strategies was dealt with by Cuoco (1995), building on previous work by He and Pearson (1991), Karatzas, Lehoczky, Shreve and Xu (1991), Xu and Shreve (1992a), and Cvitanić and Karatzas (1992, 1993). We remark that this setting is not included in our analysis, since the sets $\mathcal{N}_t(\omega)$ are convex cones, and hence not bounded. Nevertheless, the feasibility (hedging) result of Theorem 1 below, and the subsequent remark, are still valid, as shown in the above mentioned literature, provided that free disposal of wealth is allowed.

5. No-Arbitrage State-Price Densities

Since security prices and the individual income stream are allowed to be possibly non-Markovian processes, dynamic programming techniques cannot be applied to analyze the individual consumption problem. Therefore we will derive a martingale characterization of the optimal policies using the duality techniques developed by He and Pearson (1991), Karatzas, Lehoczky, Shreve and Xu (1991), He and Pagès (1991) and Cvitanić and Karatzas (1992) to analyze the optimal consumption problem with constrained investment policies. In order to formulate the proper dual shadow state-price problem, we start by identifying the set of state-price densities for the economy consistent with the absence of arbitrage opportunities.[5]

For an arbitrary process $\nu \in \mathcal{N}$, define the exponential martingale

$$\xi_\nu(t) = \exp\left(\int_0^t \kappa_\nu(\tau)^\top dw(\tau) - \frac{1}{2}\int_0^t |\kappa_\nu(\tau)|^2\, d\tau\right), \tag{19}$$

and the discount factor

$$\beta_\nu(t) = \exp\left(-\int_0^t (\bar{r}(\tau) + \nu_0(\tau))\, d\tau\right),$$

where

$$\kappa_\nu(t) = -\sigma(t)^{-1}\Big(\bar{\mu}(t) + \nu_-(t) - (\bar{r}(t) + \nu_0(t))\bar{1}\Big),$$

and let

$$\pi_\nu(t) \doteq \beta_\nu(t)\xi_\nu(t). \tag{20}$$

Clearly, each π_ν with $\nu \in \mathcal{N}$ would represent the unique state-price density in a shadow economy in which the portfolio policy (α, θ) of the "large" investor was known to be such that $\check{r}(\alpha(t), \theta(t), t) = \nu_0(t)$ and $\check{\mu}(\alpha(t), \theta(t), t) = \nu_-(t)$ for all $t \in [0, T]$. More generally, the following lemma shows that any process π_ν with $\nu \in \mathcal{N}$ can be interpreted as a shadow state-price density for the economy.

Lemma 1. *If $c \in C_+^*$ is a feasible consumption process, then*

$$\mathrm{E}\left[\int_0^T \pi_\nu(t)(c(t) - y(t))\, dt\right] \le W_0 + \mathrm{E}\left[\int_0^T \pi_\nu(t)\tilde{g}(\nu(t), t)\, dt\right] \tag{21}$$

holds for all $\nu \in \mathcal{N}$.

[5] In our setting, an *arbitrage opportunity* is a nonzero consumption process $c \in C_+^*$ that is feasible with zero initial wealth and zero income.

By the above lemma, the fact that $0 \in \mathcal{N}$ is sufficient to rule out the existence of arbitrage opportunities. We will refer to (21) as a *static budget constraint.* Karatzas, Lehoczky and Shreve (1987) and Cox and Huang (1991) have shown that, in the standard setting where $\breve{r} \equiv 0$, $\breve{\mu} \equiv 0$ and $\mathcal{N} = \{0\}$, a consumption process is feasible if *and only if* it satisfies a static budget constraint with respect to the unique state-price density π_0. The following theorem gives a general version of this result by showing that the satisfaction of a budget constraint with respect to each π_ν with $\nu \in \mathcal{N}$ is sufficient to guarantee feasibility in our setting.

Theorem 1. *Let $c \in \mathcal{C}_+^*$ be a consumption process and suppose that there exists a process $\nu^* \in \mathcal{N}$ such that for all $\nu \in \mathcal{N}$:*

$$\mathrm{E}\left[\int_0^T \pi_\nu(t)(c(t) - y(t) - \tilde{g}(\nu(t),t))\,dt\right] \le \mathrm{E}\left[\int_0^T \pi_{\nu^*}(t)(c(t) - y(t) - \tilde{g}(\nu^*(t),t))\,dt\right] = W_0. \tag{24}$$

Then c is feasible and the optimal wealth process is given by

$$W_{\nu^*}(t) = \pi_{\nu^*}(t)^{-1}\,\mathrm{E}\left[\int_t^T \pi_{\nu^*}(\tau)(c(\tau) - y(\tau) - \tilde{g}(\nu^*(\tau),\tau))\,dt \;\Big|\; \mathcal{F}_t\right]. \tag{25}$$

PROOF. See Appendix A. □

Remark 3. In fact, using the methods of El Karoui, Peng and Quenez (1995) or Cvitanić and Karatzas (1993), it is possible to show a stronger result: the minimal initial wealth W_0 required for financing a given consumption process c and a given terminal wealth W, assuming the endowment stream y, is given by

$$W_0 = \sup_{\nu \in \mathcal{N}} \mathrm{E}\left[\int_0^T \pi_\nu(t)(c(t) - y(t) - \tilde{g}(\nu(t),t))\,dt + \pi_\nu(T)W\right].$$

6. Optimal Consumption Policies

Letting c^* denote the optimal consumption policy, Theorem 1 suggests that there should exist a Lagrangian multiplier $\psi^* > 0$ such that (c^*, ψ^*, ν^*) is a saddle point of the map

$$\mathcal{L}(c,\psi,\nu) = U(c) - \psi\left(\mathrm{E}\left[\int_0^T \pi_\nu(t)(c(t) - y(t) - \tilde{g}(\nu(t),t))\,dt\right] - W_0\right), \tag{26}$$

where we maximize with respect to c and minimize with respect to (ψ, ν).

Let

$$\tilde{u}(y,t) \equiv \max_{c \ge 0}[u(c,t) - yc] = u(f(y,t),t) - yf(y,t) \tag{27}$$

denote the convex conjugate of $-u(-c,t)$. The following lemma collects some properties of the function $\tilde{u}$ that will be used repeatedly in the sequel.

Lemma 2. *The function $\tilde{u}(\cdot,t) : (0,\infty) \to \mathbb{R}$ is strictly decreasing and strictly convex for all $t \in [0,T]$, with $\frac{\partial}{\partial y}\tilde{u}(y,t) = -f(y,t)$. Moreover*

$$\tilde{u}(0+,t) = u(\infty,t), \qquad \tilde{u}(\infty,t) = u(0+,t).$$

PROOF. See, e.g., Karatzas, Lehoczky, Shreve and Xu (1991), p. 707. □

Theorem 3. *Assume that*

(a) $u(\infty, t) = \infty$ *for all* $t \in [0, T]$ *and* $u(c,t)^- \leq k(1 + c^{1-b})$ *on* $(0, \infty) \times [0, T]$ *for some* $k \geq 0$, $b \geq 1$*;*

(b) *either* $W_0 > 0$ *or* $y/B > \varepsilon$ $(\lambda \times P) -$ *a.e. for some* $\varepsilon > 0$*;*

(c) *for all* $\psi \in (0, \infty)$, *there exists a* $\nu \in \mathcal{N}$ *such that* $J(\psi, \nu) < \infty$.

Then the minimum in (P^*) *is attained and hence a minimax state-price density exists. If in addition*

(d) $cu_c(c,t) \leq a + (1-b)u(c,t)$ *on* $(0,\infty) \times [0,T]$ *for some* $a \geq 0$, $b > 0$,

then condition (30) of Theorem 2 is also satisfied, and hence there exists a constrained optimal consumption/investment policy.

PROOF. See Appendix B. □

Remark 4. Since the dual functional J is not convex in ν, existence of a solution to the dual problem is proved in the Appendix by reformulating (P^*) as a minimization problem directly over the set of state-price densities π_ν rather than over the set of "Lagrangian multipliers" ν representing them.

Remark 5. Conditions (a) and (c) of Theorem 3 are in particular satisfied if either $u(c,t)$ is bounded below on $(0, \infty) \times [0, T]$ and $u(\infty, t) = \infty$ for all $t \in [0, T]$, or $u(c,t) = \rho(t) \log c$ for some bounded measurable function $\rho : [0, T] \mapsto (0, \bar{\rho}]$. Also, since $0 \in \mathcal{N}$, a sufficient condition for assumption (c) is that $J(\psi, 0) < \infty$ for all $\psi \in (0, \infty)$. In particular, if $u(c,t)$ is nonnegative and satisfies the growth condition $u(c,t) \leq k(1 + c^{1-b})$ for some $k > 0$, $b \in (0,1)$, then (c) will also hold (cf. Karatzas, Lehoczky, Shreve and Xu (1991), Remark 11.9).

Remark 6. Proceeding as in Cuoco (1995), it would have also been possible to establish the existence of an optimal consumption policy without resorting to duality. However, we will see in the next sections that the dual problem offers computational advantages.

7. Explicit Solution for a Logarithmic Investor

Suppose that $u(c,t) = e^{-\rho t} \log(c)$ and that $y \equiv 0$, so that the investor is only endowed with some positive amount of wealth W_0. Also, assume that $\tilde{g} \equiv 0$ on its effective domain. Then we have

$$\tilde{u}(y,t) = \max_{c \geq 0} [e^{-\rho t} \log(c) - yc] = -e^{-\rho t}(1 + \rho t + \log(y)),$$

and the dual problem becomes

$$\begin{aligned} \min_{(\psi,\nu) \in (0,\infty) \times \mathcal{N}} & \mathrm{E}\Big[-\int_0^T e^{-\rho t} \Big(1 + \rho t + \log(\psi \pi_\nu(t))\Big)\, dt + \psi W_0 \Big] \\ = \quad & T e^{-\rho T} - 2\,\frac{1 - e^{-\rho T}}{\rho} + \mathrm{E}\Big[\int_0^T e^{-\rho t} \Big(\int_0^t \bar{r}(\tau)\, d\tau \Big)\, dt \Big] \end{aligned}$$

In particular, with a single risky asset ($n = 1$), the above implies

$$\nu_-^*(t) = \begin{cases} -(\bar{\mu}(t) - \bar{r}(t)) & \text{if } |A(t)^{-1}(\bar{\mu}(t) - \bar{r}(t))| \leq 1; \\ -\operatorname{sign}(\bar{\mu}(t) - \bar{r}(t))A(t) & \text{otherwise} \end{cases}$$

and

$$\frac{\theta(t)}{W(t)} = \begin{cases} \sigma(t)^{-2}(\bar{\mu}(t) - \bar{r}(t) + A(t)) & \text{if } \bar{\mu}(t) - \bar{r}(t) \leq -A(t); \\ 0 & \text{if } -A(t) < \bar{\mu}(t) - \bar{r}(t) < A(t); \\ \sigma(t)^{-2}(\bar{\mu}(t) - \bar{r}(t) - A(t)) & \text{if } \bar{\mu}(t) - \bar{r}(t) \geq A(t). \end{cases}$$

This shows that the fraction of wealth invested in the risky asset is always lower than what it would be absent the negative price pressure effect ($A(t) = 0$). However, the marginal propensity to consume is unchanged.

Different borrowing and lending rates: As an additional example, consider the case of different borrowing and lending rates, which corresponds to

$$\mathcal{N}_t(\omega) = \{\nu \in \mathbb{R}^{n+1} : 0 \leq \nu_0 \leq \bar{R}(t,\omega) - \bar{r}(t,\omega) \text{ and } \nu_- = 0\}.$$

In this case, it is easily verified that the solution to (33) is given by $\nu^* = (\nu_0^*, 0)$, where

$$\nu_0^*(t) = \begin{cases} \bar{R}(t) - \bar{r}(t) & \text{if } (\bar{R}(t) - \bar{r}(t))|\sigma(t)^{-1}\bar{1}|^2 \leq -\kappa_0(t)^\top\sigma(t)^{-1}\bar{1} - 1; \\ -\dfrac{\kappa_0(t)^\top\sigma(t)^{-1}\bar{1} + 1}{|\sigma(t)^{-1}\bar{1}|^2} & \text{if } 0 \leq -\kappa_0(t)^\top\sigma(t)^{-1}\bar{1} - 1 \leq (\bar{R}(t) - \bar{r}(t))|\sigma(t)^{-1}\bar{1}|^2; \\ 0 & \text{otherwise.} \end{cases}$$

This result first appeared in an Appendix of Cvitanić and Karatzas (1992).

In particular, with a single risky asset the above implies

$$\nu_0^*(t) = \begin{cases} \bar{R}(t) - \bar{r}(t) & \text{if } \sigma(t)^{-2}(\mu(t) - \bar{R}(t)) \geq 1; \\ \mu(t) - \bar{r}(t) - \sigma(t)^2 & \text{if } \sigma(t)^{-2}(\mu(t) - \bar{R}(t)) \leq 1 \leq \sigma(t)^{-2}(\mu(t) - \bar{r}(t)); \\ 0 & \text{otherwise} \end{cases}$$

and

$$\frac{\theta(t)}{W(t)} = \begin{cases} \sigma(t)^{-2}(\mu(t) - \bar{R}(t)) & \text{if } \sigma(t)^{-2}(\mu(t) - \bar{R}(t)) \geq 1; \\ 1 & \text{if } \sigma(t)^{-2}(\mu(t) - \bar{R}(t)) \leq 1 \leq \sigma(t)^{-2}(\mu(t) - \bar{r}(t)); \\ \sigma(t)^{-2}(\mu(t) - \bar{r}(t)) & \text{otherwise.} \end{cases}$$

As expected, the fraction of wealth invested in the risky asset is always no greater than what it would be without a spread between borrowing and lending rates ($\bar{R} = \bar{r}$). As long as $\mu(t) \leq \bar{r}(t) + \sigma(t)^2$, the agent behaves exactly as in the standard setting, investing the fraction $\sigma(t)^{-2}(\mu(t) - \bar{r}(t)) \leq 1$ of his wealth in the stock and lending. However, when $\mu(t) > \bar{r}(t) + \sigma(t)^2$ the agent deviates from his optimal policy in the standard setting. In particular, for $\bar{r}(t) + \sigma(t)^2 < \mu(t) \leq \bar{R}(t) + \sigma(t)^2$, the agent keeps all of his wealth invested in the risky asset and neither borrows nor lends. Borrowing only occurs when $\mu(t) > \bar{R}(t) + \sigma(t)^2$. As in the previous example, the marginal propensity to consume is unchanged.

PROOF. Let $u_{ic}(t) = u_{ic}(c_i(t), t)$. Under the stated assumptions, c_i is an Itô process for $i = 1, 2$, so that Itô's lemma gives

$$u_{ic}(t) = u_{ic}(0) + \int_0^t u_{icc}(c_i(\tau), \tau)\, dc_i(\tau) + \int_0^t u_{ict}(c_i(\tau), \tau)\, d\tau + \frac{1}{2}\int_0^t u_{iccc}(c_i(\tau), \tau)\, d[c_i, c_i](\tau).$$

Since the quadratic covariation between a continuous semimartingale and a process of finite variation is zero (Jacod and Shiryaev (1987), Proposition I.4.49), we have

$$[u_{ic}, S](t) \;=\; \Big[\int u_{icc}\, dc_i, S\Big](t) \;=\; \int_0^t u_{icc}(c_i(\tau), \tau)\, d[c_i, S](\tau),$$

where the last equality follows from Theorems I.4.40 and I.4.52 in Jacod and Shiryaev (1987). On the other hand, since $u_{ic} = \psi_i \pi_i$, we also have

$$\begin{aligned}
[u_{1c}, S](t) &= \psi_1 [\pi_1, S](t) \\
&= -\psi_1 \int_0^t \pi_1(\tau) I_S(\tau)\Big(\bar{\mu}(\tau) + \nu_-(\tau) - (\bar{r}(\tau) + \nu_0(\tau))\bar{1}\Big)\, d\tau \\
&= -\int_0^t u_{1c}(\tau) I_S(\tau)\Big(\mu(\tau) - \breve{\mu}(\tau) + \nu_-(\tau) - (r(\tau) - \breve{r}(\tau) + \nu_0(\tau))\bar{1}\Big)\, d\tau
\end{aligned}$$

and

$$\begin{aligned}
[u_{2c}, S](t) &= \psi_2 [\pi_2, S](t) \\
&= -\psi_2 \int_0^t \pi_2(\tau) I_S(\tau)\Big(\mu(\tau) - r(\tau)\bar{1}\Big)\, d\tau \\
&= -\int_0^t u_{2c}(\tau) I_S(\tau)\Big(\mu(\tau) - r(\tau)\bar{1}\Big)\, d\tau
\end{aligned}$$

The last three equations imply

$$\alpha_1(t)\Big(\mu(t) - r(t)\bar{1}\Big) \;=\; I_S(t)^{-1}\,\frac{d}{dt}[c_1, S](t) + \alpha_1(t)\Big(\breve{\mu}(t) - \nu_-(t) - (\breve{r}(t) - \nu_0(t))\bar{1}\Big)$$

and

$$\alpha_2(t)\Big(\mu(t) - r(t)\bar{1}\Big) \;=\; I_S(t)^{-1}\,\frac{d}{dt}[c_2, S](t),$$

so that

$$\begin{aligned}
\Big(\alpha_1(t) + \alpha_2(t)\Big)\Big(\mu(t) - r(t)\bar{1}\Big) &= I_S(t)^{-1}\,\frac{d}{dt}[S, C](t) + \alpha_1(t)\Big(\breve{\mu}(t) - \nu_-(t) - (\breve{r}(t) - \nu_0(t))\bar{1}\Big) \\
&= I_S(t)^{-1}\,\frac{d}{dt}[S, C](t) + \alpha_1(t) I_S(t)^{-1}\,\frac{d}{dt}[S, Y](t),
\end{aligned}$$

where the last equality follows from the fact that

$$Y(t) = \log\Big(\frac{u_{1c}(t)}{u_{2c}(t)}\Big) = \log\Big(\frac{\psi_1\pi_1(t)}{\psi_2\pi_2(t)}\Big). \qquad \square$$

$t \in [0, T]$. This extends to the intertemporal CCAPM a result first obtained by Brennan (1971) and Black (1972) for the static CAPM: with different borrowing and lending rates, the CCAPM still holds, but the expected instantaneous return $\hat{r}$ on a zero-beta portfolio replaces the instantaneous interest rate.

Finally, we remark that in the case of portfolio constraints ($(\alpha, \theta) \in K$) we have

$$\begin{aligned} \pi_1(t) &= \exp\Big(- \int_0^t \Big[\sigma(\tau)^{-1}\Big(\mu(\tau) + \nu_-(\tau) - (\bar{r}(\tau) + \nu_0(\tau))\bar{1}\Big)\Big]^\top dw(\tau) \\ &\quad - \int_0^t \Big(\bar{r}(\tau) + \nu_0(\tau) + \frac{1}{2}\Big|\sigma(\tau)^{-1}\Big(\mu(\tau) + \nu_-(\tau) - (\bar{r}(\tau) + \nu_0(\tau))\bar{1}\Big)\Big|^2\Big) d\tau\Big) \end{aligned}$$

for some process ν taking values in the barrier cone of $-K$, and

$$I_S(t)^{-1} \frac{d}{dt}[S, Y](t) = -(\nu_-(t) - \nu_0(t))\bar{1},$$

so that (34) can be rewritten as

$$\mu(t) - r(t)\bar{1} = \frac{I_S(t)^{-1}\operatorname{cov}(dS(t), dC(t))}{\alpha_1(t) + \alpha_2(t)} - \frac{\alpha_1(t)(\nu_-(t) - \nu_0(t))\bar{1})}{\alpha_1(t) + \alpha_2(t)},$$

which recovers the constrained CCAPM of Cuoco (1995).

9. Concluding Remarks

This paper has examined the individual's optimal consumption and investment problem for a "large" investor, whose portfolio choices affect the assets' expected returns. The main result is related to the existence of optimal policies under fairly general assumptions on the security price coefficients and on the income process. As already pointed out, even if we have assumed no bequest function for final wealth, the introduction of such a function can be easily accommodated, and in fact would simplify the statement of some results. Of course, the case in which the agent is maximizing the expected utility from final wealth only could be treated similarly.

for $\varepsilon \in (0,1)$, we have $\nu_{\varepsilon,n} \in \mathcal{N}$ (because of the convexity of the sets $\mathcal{N}_t$).

Letting

$$w_n(\varepsilon) = \mathrm{E}\Big[\int_0^T \pi_{\nu_{\varepsilon,n}}(t)(c(t) - y(t) - \tilde{g}(\nu_{\varepsilon,n}(t), t)\, dt\Big]$$

it can be shown that

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \frac{w_n(0) - w_n(\varepsilon)}{\varepsilon} \qquad\qquad (38)\\
&= \lim_{\varepsilon\downarrow 0} \mathrm{E}\Big[\int_0^T \pi_{\nu^*}(t)\frac{c(t) - y(t) - \tilde{g}(\nu^*(t), t)}{\varepsilon}\Big(1 - \frac{\pi_{\nu_{\varepsilon,n}}(t)}{\pi_{\nu^*}(t)}\Big)\, dt \\
&\qquad + \int_0^T \pi_{\nu_{\varepsilon,n}}(t)\frac{\tilde{g}(\nu_{\varepsilon,n}(t), t) - \tilde{g}(\nu^*(t), t)}{\varepsilon}\, dt\Big] \\
&\le \lim_{\varepsilon\downarrow 0} \mathrm{E}\Big[\int_0^T \pi_{\nu^*}(t)\frac{c(t) - y(t) - \tilde{g}(\nu^*(t), t)}{\varepsilon}\Big(1 - \frac{\pi_{\nu_{\varepsilon,n}}(t)}{\pi_{\nu^*}(t)}\Big)\, dt \\
&\qquad + \int_0^{\tau_n} \pi_{\nu_{\varepsilon,n}}(t)(\tilde{g}(\nu(t), t) - \tilde{g}(\nu^*(t), t))\, dt\Big] \\
&= \mathrm{E}\Big[\int_0^T \zeta(t\wedge\tau_n)\pi_{\nu^*}(t)(c(t) - y(t) - \tilde{g}(\nu^*(t), t))\, dt \\
&\qquad + \int_0^{\tau_n} \pi_{\nu^*}(t)(\tilde{g}(\nu(t), t) - \tilde{g}(\nu^*(t), t))\, dt\Big] \\
&= \mathrm{E}\Big[\int_0^{\tau_n} \zeta(t)\pi_{\nu^*}(t)(c(t) - y(t) - \tilde{g}(\nu^*(t), t))\, dt + \zeta(\tau_n)\pi_{\nu^*}(\tau_n)W_{\nu^*}(\tau_n) \\
&\qquad + \int_0^{\tau_n} \pi_{\nu^*}(t)(\tilde{g}(\nu(t), t) - \tilde{g}(\nu^*(t), t))\, dt\Big],
\end{aligned}$$

where the first inequality follows from the convexity of $\tilde{g}$, the second equality follows from the dominated convergence theorem, and the third equality follows from the definition of W_{ν^*}. On the other hand, by Itô's lemma:

$$\begin{aligned}
\zeta(\tau_n)\pi_{\nu^*}(\tau_n)W_{\nu^*}(\tau_n) &= \int_0^{\tau_n} \zeta(t)\pi_{\nu^*}(t)\Big(\sigma(t)^\top\theta(t) + W_{\nu^*}(t)\kappa_{\nu^*}(t)\Big)^\top dw(t) \\
&- \int_0^{\tau_n} \zeta(t)\pi_{\nu^*}(t)(c(t) - y(t) - \tilde{g}(\nu^*(t), t))\, dt + \int_0^{\tau_n} \pi_{\nu^*}(t)W_{\nu^*}(t)(\nu_0(t) - \nu_0^*(t))\, dt \\
&+ \int_0^{\tau_n} \pi_{\nu^*}(t)W_{\nu^*}(t)\Big(\sigma(t)^{-1}(\nu_-(t) - \nu_-^*(t) - (\nu_0(t) - \nu_0^*(t))\bar{1})\Big)^\top\Big(dw(\tau) - \kappa_{\nu^*}(\tau)\, d\tau\Big) \\
&+ \int_0^{\tau_n} \pi_{\nu^*}(t)\Big(\sigma(t)^{-1}(\nu_-(t) - \nu_-^*(t) - (\nu_0(t) - \nu_0^*(t))\bar{1})\Big)^\top\Big(\sigma(t)^\top\theta(t) + W_{\nu^*}(t)\kappa_{\nu^*}(t)\Big)\, dt.
\end{aligned}$$

Taking expectations and using the fact that the stochastic integrals in the above expression are martingales shows that

$$\begin{aligned}
&\mathrm{E}\Big[\int_0^{\tau_n} \zeta(t)\pi_{\nu^*}(t)(c(t) - y(t) - \tilde{g}(\nu^*(t), t))\, dt + \zeta(\tau_n)\pi_{\nu^*}(\tau_n)W_{\nu^*}(\tau_n)\Big] \\
&\quad = \mathrm{E}\Big[\int_0^{\tau_n} \pi_{\nu^*}(t)\Big(\alpha(t)(\nu_0(t) - \nu_0^*(t)) + \theta(t)^\top(\nu_-(t) - \nu_-^*(t))\Big)\, dt\Big]
\end{aligned}$$

where the second equality follows from Lebesgue's dominated convergence theorem and (39), using the fact that

$$\left|\frac{\tilde{u}\Big((\psi^*+\varepsilon)\pi_{\nu^*}(t),t\Big)-\tilde{u}\Big(\psi^*\pi_{\nu^*}(t),t\Big)}{\varepsilon}\right| \leq \frac{\tilde{u}\Big((\psi^*-|\varepsilon|)\pi_{\nu^*}(t),t\Big)-\tilde{u}\Big(\psi^*\pi_{\nu^*}(t),t\Big)}{|\varepsilon|}$$
$$\leq \pi_{\nu^*}(t)f\Big((\psi^*-|\varepsilon|)\pi_{\nu^*}(t),t\Big) \leq \pi_{\nu^*}(t)f\Big((\psi^*/2)\pi_{\nu^*}(t),t\Big)$$

for $|\varepsilon| < \psi^*/2$, because $\tilde{u}(\cdot,t)$ is decreasing and convex, $\frac{\partial}{\partial y}\tilde{u}(y,t) = -f(y,t)$, and $f(\cdot,t)$ is decreasing. Therefore

$$\mathrm{E}\Big[\int_0^T \pi_{\nu^*}(t)(c^*(t)-y(t)-\tilde{g}(\nu^*(t),t))\,dt\Big] = W_0. \tag{40}$$

Next, let $c \in \mathcal{C}_+^*$ be any feasible consumption process. Since by concavity

$$u(f(y,t),t)-u(c,t) \geq y[f(y,t)-c] \qquad \forall c>0, y>0, \tag{41}$$

we have from (21)

$$U(c^*)-U(c) = \mathrm{E}\Big[\int_0^T (u(c^*(t),t)-u(c(t),t))\,dt\Big] \geq \psi^*\,\mathrm{E}\Big[\int_0^T \pi_{\nu^*}(t)(c^*(t)-c(t))\,dt\Big] \geq 0.$$

Hence, c^* must be optimal provided it is feasible.

Step 3: By the martingale representation theorem, there exists an adapted process φ with $\int_0^T |\varphi(t)|^2\,dt < \infty$ a.s. such that

$$\pi_{\nu^*}(t)W^*(t) + \int_0^t \pi_{\nu^*}(\tau)(c^*(\tau)-y(\tau)-\tilde{g}(\nu^*(\tau),\tau))\,d\tau = W_0 + \int_0^t \varphi(\tau)^\top dw(\tau).$$

Defining the portfolio strategy θ by (36), it can be verified as in the proof of Theorem 1 that (14) and (15) are satisfied, and hence in order to prove that c^* is feasible we are only left to show that (13) holds. As in the proof of Theorem 1, this is equivalent to showing that (37) is satisfied.

Let $\nu \in \mathcal{N}$ be the process of (18) and define the process ζ, as well as the stopping times τ_n, as in the proof of Theorem 1. For $\varepsilon \in (0,1)$, let

$$\nu_{\varepsilon,n}(t) = \nu^*(t) + \varepsilon[\nu(t)-\nu^*(t)]1_{\{t\leq\tau_n\}}.$$

Using Lebesgue's dominated convergence theorem and Itô's lemma as in the proof of Theorem 1, it can be verified that

$$\lim_{\varepsilon\downarrow 0}\frac{J(\psi^*,\nu_{\varepsilon,n})-J(\psi^*,\nu^*)}{\varepsilon}$$
$$= \lim_{\varepsilon\downarrow 0}\mathrm{E}\Big[\int_0^T \frac{\tilde{u}(\psi^*\nu_{\varepsilon,n}(t),t)-\tilde{u}(\psi^*\pi_{\nu^*}(t),t)}{\varepsilon}\,dt$$
$$-\psi^*\int_0^T \pi_{\nu^*}(t)\frac{y(t)+\tilde{g}(\nu^*(t),t)}{\varepsilon}\Big(1-\frac{\pi_{\nu_{\varepsilon,n}}(t)}{\pi_{\nu^*}(t)}\Big)\,dt$$

Appendix B

This Appendix is devoted to the proof of Theorem 3.

Assume for simplicity that $\bar{r}(t) = 0$ and that $\sigma(t)$ is the identity matrix, for all $t \in [0, T]$. Also, assume that $\bar{\mu} \equiv 0$ (alternatively, one can work under the risk-neutral measure Q_0 instead of the original measure P). Since the dual functional $J(\psi, \nu)$ is not convex in ν, we start by reformulating the dual problem (P^*) as a minimization problem over a closed and convex subset of a L^2 space.

Let $\mathcal{M}$ denote the progressive σ-field (i.e., the smallest σ-field with respect to which all progressive processes are measurable), and let $L_n^2 = L_n^2([0,T] \times \Omega, \mathcal{M}, \lambda \times P)$ denote the space of all progressively measurable n-dimensional processes X with

$$\mathrm{E}\left[\int_0^T |X(t)|^2 \, dt\right] < \infty.$$

By the boundedness of $\mathcal{N}$, it is easily seen that the set $\Pi = \{\pi_\nu : \nu \in \mathcal{N}\}$ is uniformly bounded in L_1^2. Because of this square-integrability, the positivity of each $\pi_\nu \in \Pi$, and the uniqueness of the semimartingale representation

$$\pi_\nu(t) = 1 + \int_0^t \pi_\nu(\tau)\nu_0(\tau)\, d\tau + \int_0^t \pi_\nu(\tau)\nu_-(\tau)^\top \, dw(\tau), \tag{42}$$

ν is uniquely determined by π_ν (up to $(\lambda \times P)$-equivalence). The dual problem can therefore be regarded as a problem in (ψ, π_ν), and rewritten as

$$\inf_{(\psi,\pi_\nu)\in(0,\infty)\times\Pi} \tilde{J}(\psi, \pi_\nu) \tag{P^{**}}$$

where

$$\tilde{J}(\psi, \pi_\nu) = \mathrm{E}\left[\int_0^T \tilde{u}(\psi\pi_\nu(t), t)\, dt + \psi \int_0^T \pi_\nu(t)(y(t) + \tilde{g}(\nu(t), t))dt + \psi W_0\right].$$

It is easy to see that, under the assumptions of Theorem 3, a sufficient condition for the minimum in (P^{**}) to be attained is that for all $\psi \in (0, \infty)$ there exists a solution to the problem

$$\inf_{\pi\in\Pi} \tilde{J}(\psi, \pi). \tag{43}$$

In fact, letting $V(\psi)$ denote the value function in (43), it can be verified that V is strictly convex and continuous on $(0, \infty)$, and that it satisfies the coercitivity conditions $V(0+) = V(\infty) = \infty$. Therefore, V must attain a (unique) minimum on $(0, \infty)$, and hence (P^{**}) has a solution.

By Proposition 2.1.2 in Ekeland and Temam (1976), in order to prove that the infimum in (43) is attained it is sufficient to show that (i) Π is convex and closed in L_1^2, and (ii) $\tilde{J}(\psi, \cdot)$ is convex and lower semicontinuous on Π. Indeed, since any minimizing sequence $\{\pi_n\}$ is bounded in L_1^2, one can extract a (minimizing) subsequence, converging weakly to some $\pi^* \in \Pi$ (since closedness is preserved under weak convergence because of convexity). By

In the following we denote by $\mathrm{co}(A)$ the convex hull of a set A and by $\overline{\mathrm{co}}(A)$ its closed convex hull (see, e.g., Dunford and Schwartz (1988), Definition V.2.2). The next lemma implies in particular that Π is closed.

Lemma B2. *Suppose that $\{\pi_{\nu_n}\} \subset \Pi$ converges to π in L_1^2. Then $\pi = \pi_\nu$ for some $\nu \in \mathcal{N}$ and*

$$(\pi_\nu, \pi_\nu \nu) \in \overline{\mathrm{co}}(\{(\pi_{\nu_n}, \pi_{\nu_n}\nu_n)\}).$$

PROOF. Suppose that $\{\pi_{\nu_n}\} \subset \Pi$ converges to π in L_1^2. The same argument used at the beginning of the proof of Lemma B1 then implies that $\pi > 0$. Since the set $\{(\pi_\nu, \pi_\nu \nu) : \nu \in \mathcal{N}\}$ is uniformly bounded in L_{n+2}^2, it is weakly sequentially compact (Dunford and Schwartz (1988), Theorem II.3.28), and we can then assume (by possibly passing to a subsequence) that there exists a process ν such that $\{(\pi_{\nu_n}, \pi_{\nu_n}\nu_n)\}$ converges to $\{(\pi, \pi\nu)\}$ weakly in L_{n+2}^2. By Mazur's Lemma (Dunford and Schwartz (1988), Corollary V.3.14), $\{(\pi, \pi\nu)\} \in \overline{\mathrm{co}}(\{(\pi_{\nu_n}, \pi_{\nu_n}\nu_n)\})$, and hence $(\pi, \pi\nu)$ belongs to the set $\hat{\Pi}$ of (44) by Lemma B1. Therefore, $\pi = \pi_\nu$.

□

Lemma B3. *The functional $\tilde{J}(\psi, \cdot)$ is convex and lower semicontinuous on Π.*

PROOF. It follows immediately from the convexity of the set $\hat{\Pi}$ in (44), that

$$\alpha \pi_{\nu_1} + (1-\alpha)\pi_{\nu_2} = \pi_\nu,$$

where

$$\nu = \frac{\alpha \pi_{\nu_1}}{\pi_\nu}\nu_1 + \frac{(1-\alpha)\pi_{\nu_2}}{\pi_\nu}\nu_2.$$

By the convexity of $\tilde{g}$, this implies

$$\pi_\nu(t)\tilde{g}(\nu(t), t) \le \alpha \pi_{\nu_1}(t)\tilde{g}(\nu_1(t), t) + (1-\alpha)\pi_{\nu_2}(t)\tilde{g}(\nu_2(t), t).$$

Together with convexity of $\tilde{u}$, this implies the convexity of $\tilde{J}(\psi, \cdot)$.

Next, we claim that $\tilde{J}(\psi, \cdot)$ is lower semicontinuous on Π. In fact, suppose that this is not the case. Then there is a $\alpha > 0$, a $\pi = \pi_\nu \in \Pi$ and a sequence $\{\pi_{\nu_n}\}$ converging to π in L_1^2 such that

$$J(\psi, \pi_{\nu_n}) \le \alpha < J(\psi, \pi) \qquad \text{for all } n.$$

By Lemma B2, we can then find a sequence $\{\pi_{\hat{\nu}_n}\} \subset \mathrm{co}(\{\pi_{\nu_n}\})$ such that $(\pi_{\hat{\nu}_n}, \hat{\nu}_n)$ converges to (π, ν) a.e., and it follows from the convexity of $\tilde{J}(\psi, \cdot)$ that $\tilde{J}(\psi, \pi_{\hat{\nu}_n}) \le \alpha$ for all n. On the other hand, since $\tilde{u}(y, t)$ is convex in its first argument and decreasing in its second argument, there exist constants a, b such that $\tilde{u}(y, t) \ge \tilde{u}(y, T) \ge -(a + by)$ for $y > 0$. We then have from Fatou's lemma and the uniform integrability of $\{\pi_{\hat{\nu}_n}\}$ that

$$\alpha < \tilde{J}(\psi, \pi) \le \liminf_{n \uparrow \infty} \tilde{J}(\psi, \pi_{\hat{\nu}_n}) \le \alpha,$$

where we have used the continuity of $\tilde{u}$ and the lower semicontinuity of $\tilde{g}$. The contradiction establishes the lower semicontinuity of $\tilde{J}(\psi, \cdot)$.

□

References

BASAK, S. (1995), "Dynamic Consumption and Portfolio Choice and Asset Pricing with Non-Price-Taking Agents", mimeo, The Wharton School, University of Pennsylvania.

BENEŠ, V.E. (1971), "Existence of Optimal Stochastic Control Laws", *SIAM Journal of Control* **8**, 446–475.

BLACK, F. (1972), "Capital Market Equilibrium with Restricted Borrowing", *Journal of Business*, **45**, 444-455.

BRENNAN, M.J. (1971), "Capital Market Equilibrium with Divergent Borrowing and Lending Rates", *Journal of Financial and Quantitative Analysis*, **6**, 1197–1205.

CHOW, Y.S. AND H. TEICHER (1988), *Probability Theory*, Springer-Verlag, New York.

COX, J.C., AND C.-F. HUANG (1989), "Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process", *Journal of Economic Theory* **49**, 33–83.

COX, J.C., AND C.-F. HUANG (1991), "A Variational Problem Arising in Financial Economics", *Journal of Mathematical Economics* **20**, 465–487.

CUOCO, D. (1995), "Optimal Consumption and Equilibrium Prices with Portfolio Constraints and Stochastic Income", forthcoming in *Journal of Economic Theory*.

CVITANIĆ, J. (1995), "Nonlinear Financial Markets: Hedging and Portfolio Optimization", mimeo, Department of Statistics, Columbia University.

CVITANIĆ, J. AND I. KARATZAS (1992), "Convex Duality in Constrained Portfolio Optimization", *Annals of Applied Probability* **2**, 767–818.

CVITANIĆ, J. AND I. KARATZAS (1993), "Hedging Contingent Claims with Constrained Portfolios ", *Annals of Applied Probability* **3**, 652-681.

CVITANIĆ, J. AND J. MA (1994), "Hedging Options for a Large Investor and Forward-Backward SDE's", mimeo, Department of Statistics, Columbia University.

DUFFIE, D. AND L.G. EPSTEIN (1992), "Stochastic Differential Utility", *Econometrica* **60**, 353–394.

DYBVIG, P.H. AND C.-F. HUANG (1989), "Nonnegative Wealth, Absence of Arbitrage, and Feasible Consumption Plans", *Review of Financial Studies* **1**, 377–401.

EKELAND, I., AND R. TEMAM (1976), *Convex Analysis and Variational Problems*, North Holland, Amsterdam.

EL KAROUI, N., S. PENG, AND M.C. QUENEZ (1995), "Backwards Stochastic Differential Equations in Finance and Optimization", forthcoming in *Mathematical Finance*.

HE, H. AND H.F. PAGÈS (1993), "Labor Income, Borrowing Constraints, and Equilibrium Asset Prices: A Duality Approach", *Economic Theory* **3**, 663-696.

HE, H. AND N.D. PEARSON (1991), "Consumption and Portfolio Policies with Incomplete Markets and Short-Sale Constraints: The Infinite Dimensional Case", *Journal of Economic Theory* **54**, 259–304.