CAPITAL STRUCTURE, CALL POLICIES AND FLOTATION COSTS: A DOG CHASING ITS TAIL

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Capital Structure, Call Policies and Flotation Costs: A Dog Chasing Its Tail[1]

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ABSTRACT

This paper presents a characterization of callable bond pricing and call decision when there are transactions costs. To keep capital structure constant a firm that has outstanding callable bonds refinances them with similarly structured callable bonds. Since refinancing is costly, firms will delay the call decision. Given that the firm's cash flows differ from investors' cash flows by transaction costs, the valuation of the callable bond will be different for the firm and for investors. We find that investors' valuation function exhibits three important empirical regularities for low interest rates: Inverse convexity, negative duration and market (investors') prices higher than call prices. In addition, for low interest rates, the market valuation of the bonds has a hump.

We have assumed that the firm will replace the outstanding bond with an identically structured bond in order to simplify the problem of analyzing multiple refundings. The firm will be replacing a seasoned bond with a new one. It will therefore be pasting a pricing function with itself at two different times to expiration. We can say that the new issue is the head and the seasoned bond the tail because it is at the end of its life. Following this procedure we collapse into a single step the problem of figuring out when to replace a callable bond with another callable bond that needs to be priced before pricing the former. This exchange of bonds will occur at a lower rate than the normal call rate when cash in hand is used. Small transaction costs will justify waiting past the call price if the firm wants to keep a callable bond in its capital structure. Replacing a callable bond with another callable bond also allows the analysis of multiple future refundings.

We conclude that transaction costs <u>alone</u> may be enough to explain the overvaluation of callable bonds with respect to the call price. We use a general one-factor interest rate process in continuous time that nests most of the popular one-factor interest rate models used by researchers and practitioners.

By comparing the refunding characteristics for two different alternatives we shed light into the problem of optimal capital structure. When refunding is costly, the indifference among funding sources disappears. Once an alternative source has been chosen, the firm is in a sense locked to that source because it is costly to change. The firm will therefore choose a capital structure that will minimize refunding costs. Transactions cost make capital structure irreversible, implying that capital structure matters.

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I. Introduction

This paper is about capital structure and callable bonds. The literature on callable bonds has two paths: The first one tries to answer the question of why corporations attach call options to bonds, see for example Bodie and Taggart (1978). It could easily be said that the answer has not been found yet. The second path addresses a more practical question, once a corporation issues callable bonds, when should they be called? The answer depends on what the corporation plans to do in the future. The corporation needs to take into account the future refinancings as well[2]. The solution to the problem of pricing callable bonds and the call decision are inextricably tied. To price a callable bond it is necessary to determine when it will be called. This dynamic problem needs to be solved for the life of the corporation.

This paper prices callable bonds with stochastic interest rates and in the presence of flotation costs incurred when issuing debt to obtain the funds to call the outstanding bond. The paper also determines the optimal call decision under these conditions for a general interest rate process. This is the first paper to do both things for a general interest rate process. Corporations have to solve this problem because the majority of corporate bonds still outstanding are callable. In addition, most of the new treasury bonds issued are callable and the treasury has to decide when to call those bonds. The problem of why corporations attach call options to bonds will be left unanswered in the paper.

Brennan and Schwartz (1977) in a path breaking paper solved the pricing of callable bonds without transaction costs or call premium for a simple interest rate process given by: $dr = \sigma dW$.

[2]We are not aware of any recently published work that solves this problem.

Weingartner (1967) solved the problem of pricing callable bonds in a perfect foresight model. He was the first to realize that not only the current refinancing decision is important. It is necessary to look at all future refinancings as well. Recently, Mauer (1993) solved part of this problem for infinitely lived bonds and a particular interest rate process given by $dr = \alpha r^2 dt + \sigma r^{3/2} dW$ which allowed him to obtain closed-form solutions. He did not address the problem of future refinancings.

We assume a general interest rate process that subsumes many single-factor processes used in the literature. We also present the critical interest rate at which bonds should be called. The value of a callable bond with flotation costs is different for the borrower and the investor because some of the cashflows that the borrower has do not go to the investor.

We price callable bonds in two steps, first we price them from the point of view of the borrower who not only has to pay a call price to the investor but also has to pay underwriting costs. Once the pricing for the borrower is done, the critical interest rate is used to price the bond from the point of view of investors who only receive the call price.

These two valuations are very different and the market valuation will have four basic properties for low interest rates: (a) the market bond price will exceed the call price by an amount similar in magnitude to the flotation costs. Since these costs are non-trivial (3%-5%) the market price can be significantly larger than the call price. (b) the market price presents a non-monotonic region ("hump") before it is called; (c) the bond has negative duration; and (d) the bond exhibits inverse convexity.

Dunn and Spatt (1986) proposed a similar model in which not only the next refinancing had to be considered, but all future refinancings as well. They extended Timmis (1985) who only looked at the case of a single refinancing. The main contribution of our paper is the analysis of multiple refunding by replacing a callable bond with another callable bond. If the second bond is properly priced, then all future refundings will have been taken into account. The new bond has imbedded in it an option to refund in the future. In addition, we can do this without actually having to perform the pricing of multiple refundings as Weingartner (1967) did.

Weingartner (1967) solved the problem of refinancing callable bonds with flotation costs in a perfect foresight model. He was the first to realize that the firm needed to know all future refinancings when it decided to refinance. He analyzed the recursive problem of having to consider the optimal refinancing of the subsequent bond when solving for the value of the current bond. The problem becomes one of infinite recursion because he assumed an infinitely lived firm. For example, for a 17% coupon bond there would be a need to analyze all bonds with lower coupons. An infinite number of bonds will have to be priced. When studying the decision to call a bond, one would have to keep track of when all subsequent bonds will be called. This is possible in a perfect foresight model, but it would be rather complicated with stochastic interest rates.

In this paper we replace a callable bond with another callable bond. This procedure collapses into a single step the problem of looking at all future refinancings as in Weingartner (1967). This procedure correctly incorporates all future decisions to call because the bond used to provide the funds for the refinancing is also callable.

Empirical evidence shows that corporations delay calling callable bonds and that they trade at prices that are somewhat higher than call price. Unless one considers flotation costs, there is no reason to delay calling the bond. The only alternative explanation to the delay and overpricing might be a clientele model where corporations do not call because they are concerned about investors' future behavior regarding new callable issues by the firm. To properly test this hypothesis one would need to collect data about the original buyers of a corporation's bonds as well as similar information about all subsequent new issues[3]. Not only is it beyond the scope of this paper to test this hypothesis, we believe that given the data requirements this would be an impossible undertaking. The other empirical evidence regarding callable bonds is that the overpricing is not too large. Actually, Vu (1986) finds no evidence of large valuation over call price. In fact, he reports that only one of the bonds in his sample was called when the market price exceeded the call price by more than 2%. Given the results in Vu (1986) we propose that transaction costs alone could explain the instances in which market value exceeds call price.

The economic explanation of the result that transaction costs alone could be enough to explain overvaluation is fairly simple. Without refinancing costs[4] the firm would float new bonds

[3] Recent court evidence sugests that clientele arguments may not be valid. Borrowers are trying to call their bonds at any cost. Texas-New Mexico Power is trying to call "noncallable" bonds using an "Eminent Domain" covenant and a possible reposession by the government (Wall Street Journal October 1995). In droping interest rate environments boorowers will refinance rationally. The call protection period (which we call blackout period) and the call premium in callable bonds has increased significantly. Both facts are a clear indication that investors have to rely on these mechanisms to prevent borrowers from calling their bonds, not threat of future exclusion of the borrower from the marketplace.

[4] Refinancing costs are both the call premium and proper flotation costs. We are trying to explain why the use of non-callable debt to finance a call would by itself delay the call decision.

to replace bonds issued at par as soon as interest rates drop by a small amount. This is because the firm does not have to worry about giving up the option to refinance later at even lower rates as would be the case if the refinancing was done with non-callable debt[5]. Any single refunding with non-callable bonds will neglect all future refunding.

The presence of an initial blackout period (the call protection period)[6] for the newly issued bond will reduce the refinancing opportunities. Theories that explain the inclusion of call features in bonds have very little to say about the delay in the call decision. These theories could be categorized in the following groups: tax advantages, added management flexibility, and asymmetric information. They could also be included in two groups: zero sum (asymmetric information and growth opportunities), and non zero sum, (tax advantages). These are well-known models that explain the inclusion of call features in corporate bonds. There is evidence that none fully explains the inclusion of call options in bonds (see for example Crabbe and Helwege 1993), but this is of no relevance for the analysis of the decision of when to call. This is the first paper to fully solve the problem of multiple future refundings with stochastic rates[7].

Without call premium or refinancing transactions costs using non-callable bonds to call current callable bonds would delay the call decision because the firm would give up completely the option to refinance later at an even lower rate.

[5]Mauer (1993) proposed this framework to price the call option.

[6]In an equilibrium model the choice of call premium (the difference between the call price and face value) and call protection period (our blackout period) would be determined endogenously between borrowers and investors. The length of the blackout period increased significantly during the 1980's. The call premium also increased in this period. Actually, after 1990 the fraction of new issues with call peatures has decreased significantly.

[7]The recent decrease in the number of callable bonds seems to be clear evidence that these are not the reasons for the inclusion of callable provisions in corporate bonds. We conjecture that

The remainder of the paper is organized as follows: Section II presents the model and the modeling choices. We emphasize the economic significance of the assumptions made. Section III explains and solves the complex dynamic programing problem. Section IV provides numerical results based on a simple base case. We present results for many comparative dynamic exercises as well as a comparison between cash-calling the bond and our bond switching strategy. Finally, section V provides a summary and concluding remarks with ideas for further research.

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it is the development of swap markets what explains this change. Swap markets have allowed managers to separately sell bonds and buy interest rate options. Peter Ritchkin sugested this argument to us.

II. Economic environment and other modeling assumptions

Callable bonds are a part of the capital structure problem that still remains unresolved. There are two basic unanswered questions regarding callable bonds, the first one is why they are included in the capital structure and the second is when they should be optimally called. The second question is when corporations should call outstanding callable bonds. There seems to be no consensus as to what the answer to the first question may be. This paper resolves the second question and provides the first case of optimal call decision with multiple future refinancings. The problem is one of finding the critical rates at which the current bond should be replaced by another bond. If the bond being used to refinance is a callable bond, we will call it a bond switch. The problem of pricing a callable bond with future refunding is quite complicated. To solve this problem we need to make certain assumptions:

A0) There is an infinitely lived firm which has as its objective to minimize the value of outstanding coupon bonds given by $G^B(r,t;r_n)$. The superscript B stands for borrower; r_n is the nominal rate on the bond. This is a standard assumption in the capital structure literature, see for example Flannery (1986) and the references cited there. The bond has current maturity of t (it has t years until expiration), and it was originally a T-year bond. The coupon is paid continuously at a rate $pd\tau$ per unit of time. In general it might be the case that $p \neq r_n$. This will be important when we replace one bond with another.

The bond specifications are standard, and the continuous coupon rate was assumed for simplicity. The general results are not affected in the least by this assumption. The result of the minimization problem will be a critical interest rate for each point in time. This critical rate will tell the firm when to refinance.

A1) The interest rate process is a one factor Ito process given by the following stochastic differential equation at current time τ:

(1) $$dr = k(L - r)d\tau + \sigma r^{\gamma/2} dW.$$

This is a standard process that is more general than the one assumed by Cox, Ingersoll and Ross (1985) (CIR). In their model γ is equal to one (1) but we are able to change that parameter freely. For the process to be correctly specified it is necessary to impose restrictions on the parameters L, k and γ. For interest rates never to become negative, L, k and γ need to be non negative. We also assume $\gamma \leq 2$ to satisfy the growth condition so that bond prices do not explode. By changing the different parameters in this process, we can obtain many single-factor models. In this sense it has the same spirit as the process assumed by Chan, Karolyi, Longstaff and Sanders (1992)[8]. The parameter k represents the speed of reversion of this mean reverting process. The parameter L measures the long term value of the instantaneous interest rate. Finally, $\sigma r^{\gamma/2}$ is the volatility of the process and dW is the increment of a standard Wiener process.

[8]It is the same process. We have decided to present it in the standard form used by CIR with a long-term interest rate and a speed of reversion parameter.

A2) The bond being priced is callable at a call price $cp(G^B,t)^9$ which could in general be any function of time and/or bond price. The bond can be called after an initial blackout period $[0,t_B]$. The blackout period is often referred to as the call protection period[10].

A3) Default risk is zero. There are no additional stochastic variables that affect bond prices in this economy other than market risk. With respect to market risk see assumption (A7).

A4) The firm prefers to replace a callable bond with another callable bond, as it is often done. This assumption is consistent with most of the theories developed to explain the presence of callable bonds in capital structure[11].

If, for example, the firm chose callable bonds in its capital structure to be able to take advantage of growth opportunities, then it will choose callable bonds in its capital structure again if these growth opportunities have not been realized when the bond is called[12]. This is a very important

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[9]In practice corporations need to give their bond holders 30 to 60 days' notice. Sometimes the bonds can only be called on coupon dates. Adding these complexities would contribute very little to the understanding of the problem.

[10]Firms have been required to include two kinds of call restrictions. One is a standard refunding constraint that prevents the firm from issuing new bonds to pay the outstanding bonds (non-refundable bonds). This restriction allows firms to call the bonds as long as it is not done by the issuance of new bonds. The other one prevents the corporation from calling the bonds at all (non-redeemable). We are not making that distinction in this paper. Most work in this area has been done with this implicit assumption. Brennan and Schwartz (1977b) set the presedent. In our case, during the blackout period the bond is non redeemable.

[11]See for example Barnea, Haugen, and Senbet (1980) or Bodie and Taggard (1978) for theories regarding forgone future investment oportunities; Robins and Schatzberg (1986) for theories regarding mangerial signalling models.

[12]Similar arguments could be made for the other theories of capital structure.

assumption because it is the one that allows us to solve the problem of multiple future refinancings in one step.

A5) The only difference between the new bond and the outstanding bond is maturity. The maturity of the new bond is equal to the original maturity T. This assumption is used only to find the critical rate at which bonds should be switched and it is made for computational ease. The actual bond used to call may be structured with coupon rates that reflect current market conditions.

A6) The firm incurs flotation costs of $f(G^B(.))$ when calling a bond. $f(.)$ is a cost function and $G^B(.)$ is the market price of the bond being replaced. In financial markets it is costly to sell bonds. Underwriting costs are of the order of 3-5%.

A7) All financial instruments that depend on interest rates as given in (A1) will be priced using a market risk adjustment of λr, where the constant λ[13] is the price of interest rate risk.

A8) In addition to flotation costs, there are no corporate taxes or other cash flows related to these bonds. The results will not be significantly altered if marginal taxes are symmetric between borrowers and investors. Having taxes in this model will only make matters less clear.

A9) Investors have no taxes or transaction costs and possess the same information as the firm. In particular, investors know when the firm will refinance. This implies that they know the cost structure of the refunding decision $f(G^B(.))$ and the critical rate.

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[13]Having a stochastic market risk factor will significantly complicate matters without contributing much to the understanding of the problem at hand.

When investors solve their problem, they first solve the borrower's problem and therefore have the borrower's critical interest rate to use in their valuation problem.

One assumption is non-conventional and therefore needs further explanation: The replacement of a callable bond with an identically structured callable bond. This assumption was made to avoid the infinite regress problem of Weingartner (1967) in which one needs to solve for all future refinancings before solving for the current one. By replacing a callable bond with another callable bond we solve, in one step, all future refinancings. To replace a bond with another we need the price of the new bond. Unless this is known, there is little gained by this procedure. For example, to price a 17% callable bond we will have to determine the price of all callable bonds with coupons below 17%. In addition, to price each of those bonds, the price of lower coupon callable bonds will have to be determined. We propose to use the price of the current bond to find the critical rate in order to eliminate these steps. The price of the new bond has to be enough to finance the purchase of the old bond and the flotation costs. This is a "value matching" condition as explained in Delgado (1991) and the references there. Since we are using the price of the same bond, we are pasting a bond with a new maturity, with the same bond at a different maturity. We call this the pasting of the dog with its tail. The head is the new bond with original maturity, and the tail is the outstanding old bond with shorter maturity. We are basically pasting a bond with itself.

The only purpose of this assumption is to obtain the critical interest rate at which bonds should be exchanged. This assumption should not be taken literally because the structure of the new bond will be such that it matches market conditions at the time of issuance. In particular, the coupon it pays may be different from the previous one. If market conditions change in the interim, bonds will sell at prices other than par even if the corporation arranges with its underwriters the characteristics of the bond so that it sells at par.

Refunding with a callable bond should also be contrasted with a parallel one made by Mauer (1993) where a callable bond was replaced by a non-callable bond. The difference in his strategy is that it does not incorporate future refinancings. To highlight this difference further, assume a perfect world where there are no taxes, or costs. Further, assume that there is no call premium. In this world, if a callable bond is going to be replaced with another callable bond it will be done as soon as its price increases by any small amount above par. If, in contrast, the callable bond is going to be replaced by a non-callable bond, then rates will be allowed to drop further before the bond is called because the firm will be giving up the option to refinance at lower rates. Assumption (A4) guarantees that the firm maintains this option to refinance later at lower rates. We allow the analysis of future refinancings that gives the corporation the option to refinance again in the future. In a market with transactions costs it matters what instrument is used to refinance an outstanding callable bond. The switching strategy that we propose in this paper is the only alternative that we know of to solve the problem of multiple refinancings and infinite regress in a world with stochastic interest rates and transaction costs.

This analysis also helps to understand why results regarding overvaluation are so significant. If the firm were to maintain its policy of refinancing as soon as rates dropped by a small amount, there would be a significant accumulation of flotation costs with repeated refundings. The refunding costs would grow very quickly. To avoid most of those costs, the firm waits until the benefits obtained from one refinancing compensate for the current and future flotation costs properly discounted. Our switching strategy allows the firm to significantly reduce these refunding costs.

Given the assumption for this economy, the evolution of ***any*** coupon bond will be given by the following partial differential equation (PDE)[14]:

$$(2) \qquad 0.5\,\sigma^2 r^{\gamma} G_{rr}(\cdot) + (kL - (L+\lambda)r)\,G_r(\cdot) - G_t(\cdot) + p = r\,G(\cdot).$$

This relationship is obtained by equating the risk adjusted return on a bond with the risk free rate. The expected return on bonds is equal to the risk free rate once the risk adjustment factor has been included (λr). Including the risk adjustment factor allows our specification to be general as well as to incorporate a pure expectations hypothesis as in Brennan and Schwartz (1977) if we make λ=0. Equation (2) must be satisfied by every coupon bond (remember that p is the continuously paid coupon rate) in this partial equilibrium one-factor model of the economy.

In this economy, the only way to differentiate one bond from another is to look at the bond covenants and imbedded options[15]. These represent boundary behavior that the bond must satisfy. These boundary conditions make callable bonds different from any other bond. Since we are interested in the study of callable bonds, we will state the boundary conditions that the bond must satisfy.

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[14]This is a standard equation and a similar equation can be found in Brennan and Schwartz (1977).

[15]By making $p = 0$ a pure discount bond could be studied. See Delgado and Dumas (1992) for a similar treatment in a exchange rate band framework.

Before doing that we would like to draw attention to the difference between the valuation of the bond by the issuing corporation (borrower) and the valuation of the same bond by the market (investors). The main difference between these valuations is the fact that flotation costs are paid by the corporation and go to third parties as opposed to going to investors. The crucial difference between these two valuations has been generally absent in the literature until now, with the exception of Stanton (1993) who has a similar setup[16].

Investors will be assumed to know the costs incurred by the firm when it refinances (A9). They would solve the problem from the point of view of the firm and use the critical interest rate that triggers refinancing as a boundary condition to solve their own problem; r^* will be that critical rate. With that process in mind the boundary conditions for the firm will be stated first:

(3) a). $$\lim_{r \to \infty} G^B(r,t;r_n) = 0,$$

the economic value of a bond when rates increase without bounds is zero.

b). $$G^B(r,0;r_n) = 1,$$

at maturity, the borrower returns the face value of the bond, if the bond has not already been called. We normalize the face value to one. Finally, a differential boundary condition is obtained at the reflecting barrier when $r = 0$:

c). $$kL\,G_r(\cdot) - G_t(\cdot) + p = 0 \quad \text{if} \quad r = 0.$$

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[16]Timmis (1985) is the earliest reference that we have found to this difference although he and Stanton (1993) analyze the behavior of individual mortgages as opposed to callable bonds. In addition, they only consider single refinancings.

The natural boundary of the interest rate process assumed in (1) provides this additional boundary condition for the bond.

d). $G^B(r, \tau \leq t_B; r_n)$ unrestricted during the blackout period,

the blackout period condition prevents the corporation from calling the bond. At the end of the initial blackout period the bond is called if $r = r^*$. We approximate the value $G^B(r=0, \tau \leq t_B; r_n)$ during the blackout period as the present value of $G^B(r=0, \tau = t_B; r_n)$ plus the coupon stream setting to zero the probability of the short-term rate entering the continuation region[17] at $\tau = t_B$, conditional on $r_t = 0, \tau < t_B$. This approximation works quite well but causes small numerical distortions for $r < 0.004$ (forty basis points). Figure 4 shows the effects of this approximation in graphical form.

So far, there are four conditions and, in principle, for this bond, that should be enough to solve the PDE given in (2). Indeed, they are the boundary conditions for a straight, non-callable, non-convertible bond. In addition, there is the call boundary condition if the firm decides to replace a callable bond with another callable bond. In this case the following condition must be satisfied:

(4) $$(1 + f(G^B(\,)\,) * G^B(r^*, t; r_n) = G^B(r^*, T; r_n),$$

where $G^B(r^*, T; r_n)$ includes $f(G^B(\cdot))$, the transaction costs function, which are the flotation costs incurred when calling the bond. This equation states: The funds to compensate the

[17]The continuation region for this problem is the region in the interest rate domain where the bond is not called, that is, the process is allowed to "continue". In terms of interest rates, the continuation region is the region where interest rates are higher than the critical rate and therefore the corporation does not have to call the bond.

underwriters and to purchase the outstanding bond have to come from the sale of the new bond. Our choice of refunding instrument allows the critical rate to incorporate not only the current refinancing but all future refundings as well.

Another way of stating the same relationship is to say that what the firm owes after refinancing should be equal to what it used to owe. The price of the new bond has to exceed the old bond by an amount equal to refunding costs represented by $f(G^B(\cdot))$. Thus, at the point of switching, the marginal benefit from refinancing should be the same as the marginal cost. In other words, under the switching strategy there should be no net cashflow. Since equation (2) was obtained using the expected capital appreciation of the bond, the firm is actually taking into account not only the present refinancing decision, but all future ones as well. Only Weingartner (1967) has done a refunding problem will infinite refundings.

The implicit assumption is that the objective of the firm is to minimize the value of the bonds to favor shareholders (A0). It is beyond the scope of the current paper to study the optimality of the capital structure or the optimality of the bond minimization strategy. Brennan and Schwartz (1977b) and most researchers since then have used identical specifications. Our specification is consistent with most capital structure models that explain the presence of callable bonds in the capital structure.

What is new in this paper is the comparison of the usual call policy with assumption (A5) that not only will a callable bond be replaced by another callable bond but that the stated coupon rate is the same. Although the usual call policy might be preferred in the absence of transactions costs, switching directly from a callable bond into another one will save one transaction if the firm wants a callable bond in its capital structure. The implication of our assumption is that the firm will owe no more than what it owes before it refinances. This allows for a solution of the infinite regress problem of Weingartner (1967)[18]. In principle, it means that the firm will be replacing the outstanding bonds with identical bonds, except that they will have the initial maturity and will be selling at a premium because of the lower interest rate in the current market conditions. But our replacement strategy need not be taken literally, because all we are after is the critical interest rate at which the firm will call the bond. Indeed, the new bond may be issued at rates which are consistent with current market condition when the refinancing decision is made. In terms of the numerical implementations it means that condition (4) will be used to locate the free boundary where it is optimal to call the bond. Unlike Dunn and Spatt (1986) we have to find only one continuation region and one critical interest rate. This is a significant improvement in simplicity and efficiency in the calculation.

For clarity and simplicity a flat call premium schedule will be assumed after the initial blackout period. Adding a deterministic general call schedule would be simple to implement but would add nothing to the understanding of the problem. Also, only proportional transaction costs will be used[19].

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[18]Dunn and Spatt (1986) also propose a solution technique similar to Weingartner (1967).

[19]For this particular problem it makes little difference whether the costs are fixed or proportional. In a more general setting, the nature of transactions costs may affect the result of the solution. The behavior for fixed costs will be different than under proporional costs. With fixed costs the switch may have to be done sooner than with proportional costs.

These two assumptions imply that

(5) $f(G^B(\cdot)) = G^B(\cdot)\, f(t)$

where $f(t)$ represents the transaction (flotation) costs which are assumed proportional to the value of the bond being issued. We chose the transaction costs of the refinancing decision to depend on time because at maturity the bond should be refinanced in any event. Therefore, the incremental transaction costs due to the call, which affects the switching decision, depends on the time left to maturity for the bond being replaced. For simplicity we assume that $f(t)$ is proportional to the time until expiration.

The Investor's Problem

So far we have discussed the firm's problem. From the investors' perspective, the only difference is in the boundary condition (4) which they will replace with

6. a) $G^M(r^*,t;\, r_n) = (1 + \pi)$ [the superscript M indicates market values],

investors are only going to receive the call price. The call premium, π, is assumed constant (remember that the face was normalized to 1). Investors do not receive the refunding cost for the issuance of the new bonds. They go to a third party, the underwriters. In this respect our model falls into the zero sum class. Borrowers will try to obtain as much as they can from investors. Investors would have demanded the inclusion of the call premium and the blackout period. This is why it is often called the call protection period. Investors know what the corporation will try to do, and they demand the inclusion of call protection clauses in callable bonds. An important difference is presented in our model, there is a third party that has no say in the process of calling. When callable bonds are designed, underwriters may influence the behavior of the issuer to maximize their fees. Since we are only analyzing the call decision, their role is passive. Actually, the role assumed for the investors is also passive. In an equilibrium model everything would be determined endogenously, including the call premium and blackout period. We have analyzed the callable bond problem from the point of view of the borrower and the investor. We have found that the key to the solution of the problem is the careful statement of the boundary conditions both for the borrower and the investor.

II. b. The Frictionless Case:

In an economy without transactions costs (or call premium), taxes, information asymmetries or other market imperfections, it would not make any difference whether the firm maintains its capital structure constant or not. The solution to the model will therefore be obtained by making $f=\pi=0$ (no flotation costs or call premium) and using equation (6b) as the firm's boundary condition

6. b) $G^M(r^*, t) = G^B(r^*, t) = 1$,

this implies that the firm is using cash to call the bonds and that investors only receive the face amount. How the firm obtains the funds would be irrelevant in this frictionless economy. The Modigliani Miller conditions would be satisfied and capital structure would be irrelevant. This is the simplest refinancing of callable bonds as solved by Brennan and Schwartz (1977). In the current paper it is assumed that the firm refinances the callable bond with another callable bond of similar specification, also keeping constant the amount of money owed[20].

The space and time we have spent analyzing and discussing the boundary conditions is essential because, as was said earlier, in this economy all coupon bonds satisfy the PDE given in (2). The only difference between one bond and another is the set of boundary conditions that

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[20]This is very important because the argument used to support the claim that it does not matter how a corporation finances a call says that if a holder of a callable bond needs to be paid \$1.09, then it does not matter how this money is obtained. There will be a non-callable bond that sells at par (\$1.00), but there would also be a callable bond that sells at \$0.95 (the difference is the price the firm pays for the call option). By selling 1.1474 callable bonds instead of 1.09 non-callable bonds the firm can call the current bond using callable bonds. However this financing strategy minimizes expected costs over the interval $(0,t)$. Our switching strategy minimizes costs over the interval $(0,T_C+T)$. Were T_C is the point in time at which the switching takes place.

must be satisfied. Imposing the wrong boundary condition would give the wrong answer. It would not be correct to impose a boundary condition that says that the bond should be called when the *market* price (as opposed to the issuer's valuation) is equal to one plus the premium plus transactions costs. This would result in a model where at the critical interest rate the value of the bond also includes the transactions costs. Such a model would not satisfy the empirical regularities that the current model captures, but more importantly it would not correct.

The latter condition does not incorporate the fact that investors do not receive the transaction costs. The market valuation should be made with the right amount of funds that investors receive, and that is only face plus call premium. These funds are received when the firm finds it optimal to refinance, subject to the investment bank flotation costs.

The correct boundary condition for the borrower is given by equation (4) and for the investors by equation (6). They allow the model to explain market valuations larger than call premium (by an amount similar in size to transactions costs).

The typical argument used to support the claim that the observed market value of callable bonds could not be explained by transactions costs alone could be summarize as follows. The company's valuation G^B is larger than the market valuation, G^M, but it cannot be too much bigger than the call price because costs have to be discounted. The difference between the two is the transaction cost f that the borrower pays a third party. This implies that G^B is an upper bound for G^M. Since G^B will never be higher than the call premium by more than $f\,G^M$, it will be barely larger than call price. Empirical evidence by Vu (1986) supports the case that overvaluations are not too large.

Under the bond switching policy the proper boundary condition for the borrower's valuation is not $1 + \pi + f$[21], but $G^B(r^*, T; r_n)$ to reflect the fact that the company is switching to this new bond and that it is the only source of funds. Since the nominal rate r_n is higher than the market rate r^* at which the bond is being refinanced, the bond is selling at a premium and $G^B(r^*, T; r_n)$ could be larger than the call boundary condition. This allows more room for G^M to be larger than the call price. In general, G^M will be larger than the call price by an amount very similar to the transactions costs involved when refinancing. For the numerical exercises performed for this paper[22], $G^B(r^*, T; r_n)$ is very similar in size to the call boundary condition, but the difference between $G^B(r^*, T; r_n)$ and $G^M(r^*, T; r_n)$ at the switching point is only somewhat smaller than f.

The PDE in equation (2) was solved by finite differences after making a change of variable from $r \in [0,\infty)$ to $S \in [0,1]$. The number of time points=200, number of interest rate points=200 [for a transformation $r = ((1-S)/XS)$]. X was chosen equal to 10 so that most of the interest rate points fell bellow 15%. The solution of the problem consists in numerically finding the value of the callable bond for the borrower and the investor. Appendix 1 presents all the steps for the general numerical procedure. We have a Gauss procedure to solve both problems. We first solve the borrower's problem and then the investor's problem once we have the critical rate for the borrower. To solve the pasting problem of a bond with itself we first solve the problem assuming that we refund with cash. After that we have a value for the bond when it is first issued, that is,

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[21]This is the standard call boundary condition.

[22]See Figure 6 which shows the difference between the market and the borrowers valuations.

we have a value of $G^B(r^*, T; r_n)$. Once this value is found, we proceed with an iterative procedure until the initial value of the bond is found and it converges to the value shown in the graphs. This is a standard convergence procedure used in numerical analysis. We stop the iterations when $G^B(r^*, T; r_n)$ does not change in value for an additional iteration. We have actually solved for $\Delta G^B(r^*, T; r_n) = \lim_{N \to \infty} \left\{ G_N^B(r, T; r_n) - G_{(N-1)}^B(r, T; r_n) \right\} = 0$. This is a fixed point in the space of

functions $G^B(r^*, T; r_n)$. We stop when the difference between one iteration and the next is only 0.0001 at any of the discrete values of S. In general, this only takes a few iterations. The number of iterations depends on the parameter values, but it is usually less than 20.

III Results

After having discussed the model and the solution technique we can present and analyze some of the numerical results. As a reference point we choose the benchmark parameter values given in the following table. The parameters are estimates from Chan, Karolyi, Longstaff, and Sanders (1992) (CKLS) and we use 25 years as bond maturity.

Volatility Coefficient	γ	1.5
Long-term value of the ***instantaneous*** rate	L	0.02
Speed of reversion parameter	k	0.2
Instantaneous rate volatility	σ	0.045
Interest risk premium	λ	0.02
Coupon rate	p	0.08
Initial maturity of the bond	T	25
Flotation or Refunding Cost	f	0.03
Blackout Period	t_B	3 years[23]
Call premium	π	0.06[24]
Amount that needs to be raised	$1+f+\pi$	1.09

[23]The blackout period has increased since the mid 1980's. It used to be the case that bonds had a small "call protection" period. It has significantly increased in the 1990's. Corporate callable bonds have different blackout periods than Treasury bonds. We chose 3 years because this is a value more representative of currenlty outstanding bonds, not necesarily newly issued bonds.

[24]The call premium is ussually non-constant for corporate bonds. As the expiartion approaches there is less need for the "call-protection" achieved by the call premium. A similar historical evolution has been observed for the premium as there was for the blackout period. In the early 1980's bonds were called at par. Currently bonds are sold with call premiums of up to 10%. We chose 6% to represent the pool of outstanding bonds.

Our goal was to study the behavior of the refunding of callable bonds and why the market price of a callable bond can be larger than the call price. We need to perform some comparative static exercises to better understand our model. In addition to the standard comparative exercises we are going to compare the value of bonds under two different refunding strategies. The strategies we choose to compare are our own, which we will call the switching strategy, and a standard call strategy where the bond is refunded with cash[25]. For the latter strategy we need to change a boundary condition. The cash-call boundary condition is $G^B(r^*, t; r_n) = 1 + \pi$. This equation replaces equation (4) for the case of bond switching. When the corporation calls with cash it will delay calling even further because it gives up its option to refinance in the future. In an economy with transaction costs or other market imperfections capital structure is relevant. If we look at the problem that the corporation faces, it matters what it is used to refund a callable bond. Refinancing with one instrument will give the corporation a set of future choices that a different alternative may not provide. If it is costly to change capital structure then the choice may be irreversible to a degree. If we look at the mathematical problem that the corporation faces we conclude that the choice of refunding instrument will be reflected in the choice of boundary

[25]In the remainder of the paper we will call this the cash-call strategy.

condition to replace (4). Different instruments will give the corporation different boundary conditions[26].

We present our results in graphical and tabular form. Figure 1 shows the value of a callable bond as a function of time to maturity and instantaneous interest rates. The call occurs when the value of a new bond provides funds to pay the old bond as well as to compensate underwriters the following amount: $G^B(r^*, T; r_n)f$[27]. The introduction of a three-year initial blackout period, in Figure 2, allows for the bond price during the blackout period to be initially higher than the refinancing costs at very low interest rates. Note that the partial derivative of the bond value with respect to time is discontinuous at ($t = t_B$[28], $r = 0$), where it jumps from $-p$ to zero.

To start the description of the results we will first look at the vector of critical rates (r^*) as a function of time to expiration (t). If the firm wishes to maintain a callable bond in its capital structure it may decide to forgo the opportunity of calling the bond until it is optimal to exchange the existing callable bond for a new one. To solve the multiple refunding problem we assume that the new bond has all the original characteristics of the old bond (assumption (A5)). In a world without transactions costs such a strategy is clearly dominated by an outright cash-call; however,

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[26]Our choice of refunding instrument will give us results consistent with that choice. For example the borrower is not only refunding, but it is also buying a longer refunding option. This is what corporations have to analyze when refunding is decided.

[27]Note that the interest rate scale is ordinal. The graph with this scale is clearer than the one with the cardinal numbers.

[28]t_B is the end of the blackout period.

we will show that it reduces transactions costs. This is a better strategy even if transaction costs are small because not going through an intermediate bond between the two callable ones saves one transaction.

Figure 3 contrasts the critical call rate with the critical switching rate. It is apparent that bond switching becomes optimal at rates lower than the cash-call rates. In the interval between the two critical rates, holding the original callable bond is suboptimal if transaction costs are not taken into account. To measure the cost of waiting until the lower critical rate is reached, Table I reports the values of the callable bond under the two call policies for a variety of initial interest rates and blackout periods. Panel A presents the results for the switching strategy, panel B for the call strategy and panel C the difference. By construction, the difference is zero for r=0. After that, the difference decreases monotonically with interest rates.

The cost of using the critical rate of the switching strategy rather than the lower cash-call policy is very small unless the short term rate is approaching the call region. The difference in call value under the two policies is highlighted in Figure 4, which shows the difference between the bond values computed under the two policies. In the call region the difference between the two bonds is close to transaction costs[29]. As rates increase, the price difference drops rather quickly to zero. Also, after the blackout period, as maturity nears, the difference decreases because the advantage of the switching strategy decreases as maturity nears. As we said before, at maturity the bond has to be refinanced in any event. Figure 5 also shows a series of cross sections at nine

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[29]For extremely low values of the interest rate (r < 0.004) and at the time of first issue, there are numerical inaccuracies that cause an area of relative high differences. This only happens during the blackout period because of the approximate boundary condition that we use there.

and a half years after the end of the initial blackout. This figure shows the pattern previously described when interest rates increase. Similarly, Table II presents the results for two maturities, 12.5 and 6.25 years to maturity. The difference is constant until rates reach 7.15% and then decreases monotonically to zero for higher rates.

For moderate flotation costs following the switching policy dominates the usual cash-call policy. Changing the blackout period has almost no effect on the results. Finally, Figure 6 shows two sets of cross sections: one for $G^M(r)$ and the other for $G^B(r)$ at a given point in time evidencing the hump in $G^M(r)$ at low rates. $G^B(r)$ is indeed an upper bound for $G^M(r)$ but the latter is very close to the former. This is one of the most significant results of this paper because it shows that for interest rates in the neighborhood of the critical refinancing rate the ***market*** price of a callable non-convertible bond presents the empirical regularities that we anticipated in $G^M(r)$: (a) a region of non-monotonic value (the "hump"), (b) inverse convexity, and (c) negative duration for low rates[30]. The market allows prices to increase beyond call price because it knows that the firm will delay refinancing further due to transactions costs. As rates continue to decrease the market price starts to decrease because eventually investors will only receive the call price.

Figures 7 and 8 show critical rates for 10 different values of the call or refinancing costs (the sum of the call premium and the flotation costs to switch). From the point of view of the corporation (the borrower), it does not make any difference whether the call costs include

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[30]Convexity is the standard term used to refer to the non-linearity in bond prices. Regular convexity is therefore the curvature that straight bonds have with respect to yields. A bond with inverse-convexity is one that exhibits the opposite behavior.

flotation costs as well as a call premium. The only difference between the two is the fact that the call premium is part of the debt contract and cannot be changed or altered, while transaction costs could be negotiated with the underwriters.

For the firm, call premium and transaction costs just make it costly to refinance. To understand better the effects of the call premium and transactions costs as deterrents of refinancing we have changed the call or refinancing costs ($f + \pi$) from 0.001 (10 basis points (bps)) to 0.1210 (1210 bps).

There are two striking and significant effects of this experiment. First of all, the amount that interest rates need to fall for the bonds to be called the day the bonds are first callable after the blackout period (when time to maturity is 22 years) is directly affected by the costs (this is the intersection of the critical rate curve with the vertical axis). This value changes from 7.99% needed at 10 (bps) to 7.01% at 1210 bps. The other result is the time until expiration for which NO decrease in interest rates will trigger refinancing (the intercept with the horizontal axis). For refinancing costs equal to 10 bps the firm will refinance even one week before expiration, while for costs equal to 1210 bps, once bonds have eighteen months until expiration, there is no positive rate low enough that will trigger refinancing because the firm will not recover the costs of refinancing.

This behavior explains clearly why we observe decreasing call premium schedules or even an initial blackout period when refinancing is not possible (this could be interpreted as infinite refinancing costs). As expiration nears, the importance of refinancing costs increases dramatically.

Table III presents the most salient values of the same experiment performed with a blackout period of three years as in our benchmark case. In addition to the blackout period there is an "optimal" no-switch region where replacing the outstanding bond will not pay. The initial blackout period causes a small increase in the delay.

We can now look again at Figures 7 and 8 and observe an important result regarding the costs of refinancing and the relationship for small values of refinancing costs (again, this is both call premium and transaction costs per se). As typical with transaction costs models in continuous time, the effect of costs per unit of costs decreases as refinancing costs increase. This means that the strongest effect is felt for the first epsilon of transaction costs (that is, in real markets, the first basis point is the most important). As we increase refinancing costs, the delay before refinancing increases at a decreasing rate. Given results in other papers (see for example Delgado and Dumas (1994)) it is tempting to conjecture that there will be initially a cubic relationship between the delay of refinancing and refinancing costs. Due to the existence of the contractual call premium, refinancing costs are larger than 500 basis points and such a relationship would be of limited practical use. This analysis, however, highlights the effect of pure transaction costs (as opposed to refinancing costs that include call premium) on the refinancing decision. Since most of the effect of refinancing costs is due to the built-in call premium, transaction costs are not as important as they could be without the call premium. This should not be taken to mean that they are irrelevant. Because transaction costs go to underwriters, as opposed to investors, they create the "hump" in the market price of callable bonds, and delay further the call decision (see Figure 6 and our previous discussion of that figure).

Our results regarding the behavior of the critical rate as a function of the time to expiration should be contrasted with results obtained when the interest rate process allows for negative interest rates as it would be the case with a simple Vasicek (1977) model. Figure 9 shows the critical rate for different values of volatility for the following particular interest rate process $dr = \sigma dW$, which is the one used by Brennan and Schwartz (1977). Two very important features of this process should be highlighted. The most striking one is the fact that the relationship between the critical rate and time to expiration is actually inverted; that is, the critical rate increases as we get closer to expiration (time to expiration goes to zero). With $k=L=\gamma=0$, the critical rate increases as the bond gets closer to expiration because the process $dr = \sigma dW$ allows for negative interest rates. Actually, rates can be infinitely negative (to perfectly replicate Brennan and Schwartz (1977) we also assumed zero refinancing costs, that is zero call premium and zero flotation costs, but we have kept our three year blackout period). This means that the critical rate will intercept the two axises just as before, but the one with the horizontal axis will give us the smallest maturity for which it is optimal to refinance.

As figure 9 shows, all curves reach the coupon rate of 8% at maturity to indicate that rates do not have to be smaller than 8% to optimally refinance. The critical rate decreases from that point on as maturity increases. For $\sigma=0.049$ we see that the intercept at $t=22$ (when the bond is callable for the first time) is 1.54%. The economic rationale of this behavior is that interest rates are allowed to become negative.

As maturity increases the critical rate decreases. If we increase volatility, there are values of volatility for which it does not matter how low interest rates became the day the bond is callable for the first time. It will never be optimal to refinance for positive interest rates. For these curves, there is an intercept with the horizontal axis which represents the minimum expiration for which it is optimal to refinance at positive interest rates. This is due to the fact that the longer the maturity the larger the probability that between now and expiration interest rates will become negative. As is the case in all option valuation models increases in time to expiration and volatility produce results in the same direction: the value of the option increases.

Table IV presents these results in a compact manner. The "optimal" blackout region is represented by the zeroes that start for some values of σ after eighteen months to indicate that for most of its life the bond will not be called. This analysis implies that for these models the choice of interest rate process is crucial. In particular, whether the process allows for negative interest rates because under those conditions the results are rather counterintuitive. The critical refinancing rate increases as maturity approaches. This result is not only present for the Brennan and Schwartz's model, Vasicek's (1977) model also has the same properties because it also allows negative rates.

The second type of experiment that is important to perform is the change in volatility for our basic case. We can see the effects that volatility has on the critical interest rate. Figures 10 and 11 present the results of these experiments. As expected, increases in volatility produce a decrease in the critical rate (or equivalently they delay the call decision). These curves were obtained by changing volatility (σ) from 0.001 to 0.151. An important feature of this behavior should be pointed out: The effect of increases in volatility decrease with time to maturity. They decrease so much that the differences in the critical rates for significantly different volatilities vanish at one point. In the case of Figures 10 and 11 this happens at about ten months to expiration. This is the point after which it does not matter how low interest rates get. It will not be optimal to refinance. Basically, what increases in volatility do is to rotate clockwise the critical rate schedule using the intercept with the horizontal axis as a fulcrum. As volatility is increased, the interest rate needed to trigger refinancing at the point at which the bond is first callable (22 years to maturity) increases almost proportionately to volatility while after ten months to maturity there will be no effect. Table V presents results numerically for the most relevant values. What is important to note is that for small times to maturity (two years or less) σ has very little effect on the critical rates (the values in each row are the same).

An additional experiment that sheds light on the understanding of refinancing with transaction costs is one where the mean of the interest rate process is changed. Note that the interest rate process assumed in equation (1) is in general mean reverting with long term instantaneous mean L and speed of reversion k. As we increase the long term mean of the interest rate process, the critical interest rate at which it is optimal to refinance also increases because the probability of lower rates decreases. Figure 12 and Table VI give us these results. The most relevant fact is the existence of a non-monotonic critical rate (it is more clear in this figure but it can also be seen in figure 10). As maturity approaches the advantages of switching decrease because at expiration the bond will have to be refinanced. This is contrasted with the fact that as maturity approaches transaction costs have more impact on the switching decision because there is less time to amortize the costs. The combination of these two forces causes the non-monotonic behavior. It is important to note that changes in the market price of risk (λ) have very similar results to changes in L. A look at equation (2) will confirm this statement.

The most relevant exercise to be performed is, of course, the market price of callable bonds at different maturities and the determination of whether they are significantly larger than call price. As Figure 6 shows, the market price of callable bonds not only can be greater than the call price, but the amount of the overpricing can be as large as the size of the flotation costs. For the parameter values we have chosen, the overpricing is of the order of 3%. Two of the most important results of this paper are: the confirmation that transaction costs alone can explain callable bond overpricing given that flotation costs are non-trivial and that 3% to 5% is a rather conservative range for flotation costs. Also, as discussed before, we obtain negative duration and inverse convexity for the market price of the bond.

IV Conclusions

This paper has shown that with transactions costs alone and very reasonable parameter values (estimated by Chan, Karoly, Longstaff and Sanders (1992)) it is possible to obtain market prices of callable bonds that exceed call price by an amount similar in magnitude to the costs of raising funds to call the outstanding bonds. It has also shown that in an economy with transaction costs capital structure may matter because firms cannot costlessly change the instruments they are currently holding. In a sense, capital structure is irreversible. In this sense, the capital structure problem therefore resembles the irreversible investment problem. Market imperfections, transaction costs in this case, make corporations change their behavior significantly compared to the frictionless conditions. In a frictionless market corporations are indifferent about capital structure, in a market with imperfections capital structure matters.

The choice of interest rate process drastically affects the results obtained. In particular, if the assumed process allows for negative rates one can obtain counterintuitive results. With the possibility of negative rates the borrower has an incentive to delay refinancing in the expectation that future rates might be significantly lower (negative). As maturity approaches, the probability of negative rates decreases and the incentive to wait for lower rates also decreases. In the presence of transactions costs this tendency is tempered by the time needed to take advantage of the reduced interest rate payments.

Since a model that implies negative rates is not realistic, our conclusions regarding the effect of flotation costs do not take them into consideration. We find that flotation costs cause a "hump" in the market price of callable bonds. In addition, market prices can be larger than call prices by an amount similar in magnitude to the flotation costs. Finally, transaction costs reduce the value of the critical rate at which it is optimal to refinance bonds. This extends the period of time during which callable bonds will not be refinanced regardless of how low interest rates became.

We have also provided an example of when capital structure matters. We have compared the refunding behavior for two instruments, callable bonds and cash. Given an economy with transactions costs, we have shown that it makes a difference what instrument is used to provide the funds for refunding. If capital structure cannot be changed costlessly, the firm is in a sense locked to a given capital structure. This irreversibility gives the firms different incentives to refund. In mathematical terms it means that the boundary conditions are very different among different instruments. We have not solved a problem of optimal capital structure; however, we have provided strong evidence that capital structure matters.

Unanswered questions left for future empirical research is the measure of the overpricing of callable bonds over call price and the determination of whether this overpricing can be significantly larger than reasonable flotation costs. In terms of the capital structure questions that we have raised, it would be interesting to analyze the optimal capital structure of a firm that is given a limited choice of instruments to refinance with. Given costly changes in capital structure we conjecture that there will be an optimal capital structure that will be achieved by minimizing the refunding costs for the life of the firm. It is beyond the scope of this paper to perform these empirical tests or to extend the model to one of optimal capital structure.

Table I: A Comparison of switching against cash-calling. Other than those changed, parameter values are those of the benchmark case. This table presents the values of bonds when first issued as a function of market interest rates and size of Blackout period. The first column shows the market short term rate in percentages and the first row the size of the initial blackout period in years. This table presents the value of a bond as it is first issued for the two alternatives we study. The difference is always positive indicating that the value of the bond is higher with the switching strategy than with the cash-call. Increases in the blackout period produce increases in the value of the bond when interest are zero. This is because the blackout period prevents the borrower from calling the bonds. The difference between the values of the bonds diminishes significantly as interest rates increase. For interest rates of 20% there is almost no difference in the value of the bonds.

A: BOND SWITCHING

$r \setminus T_B$	*5.25*	*4.875*	*4.5*	*4.125*	*3.75*	*3.375*	*3*	*2.625*	*2.25*	*1.875*
0.00 %	1.51865	1.488782	1.458908	1.429028	1.399143	1.369252	1.339355	1.309453	1.279545	1.249632
1.03	1.424492	1.402781	1.380864	1.358733	1.336379	1.313796	1.290973	1.267902	1.244572	1.220972
3.02	1.348162	1.329997	1.311804	1.293591	1.275368	1.257147	1.238939	1.220759	1.202622	1.184544
5.03	1.275377	1.260447	1.245659	1.231041	1.216621	1.202434	1.188516	1.174908	1.161656	1.148814
7.01	1.207995	1.195925	1.184156	1.172735	1.161710	1.151138	1.141084	1.131619	1.122825	1.114795
9.10	1.140779	1.131426	1.122533	1.114165	1.106394	1.099302	1.092982	1.087533	1.083059	1.079656
10.07	1.111154	1.102953	1.095281	1.088211	1.081824	1.076210	1.071461	1.067665	1.064882	1.063104
12.11	1.051356	1.045384	1.040069	1.035489	1.031715	1.028800	1.026749	1.025491	1.024860	1.024625
14.04	0.998017	0.993885	0.990472	0.987818	0.985920	0.984712	0.984058	0.983775	0.983684	0.983665
16.01	0.946508	0.943918	0.942002	0.940714	0.939955	0.939578	0.939429	0.939384	0.939375	0.939374
18.36	0.888451	0.887163	0.886357	0.885920	0.885722	0.885651	0.885631	0.885627	0.885627	0.885627
20.19	0.845323	0.844651	0.844290	0.844126	0.844065	0.844048	0.844045	0.844044	0.844044	0.844044

B: CASH-CALLING THE BOND

$r \setminus T_B$	*5.25*	*4.875*	*4.5*	*4.125*	*3.75*	*3.375*	*3*	*2.625*	*2.25*	*1.875*
0.00 %	1.518650	1.488782	1.458908	1.429028	1.399143	1.369252	1.339355	1.309453	1.279545	1.249632
1.03	1.402639	1.380376	1.357903	1.335211	1.312295	1.289145	1.265753	1.242111	1.218208	1.194033
3.02	1.327638	1.308904	1.290131	1.271327	1.252502	1.233666	1.214830	1.196007	1.177212	1.158461
5.03	1.256115	1.240603	1.225217	1.209983	1.194927	1.180081	1.165481	1.151165	1.137178	1.123568
7.01	1.189897	1.177237	1.164856	1.152797	1.141107	1.129840	1.119058	1.108828	1.099229	1.090348
9.10	1.123838	1.113889	1.104373	1.095350	1.086890	1.079072	1.071986	1.065731	1.060418	1.056160
10.07	1.094722	1.085924	1.077625	1.069895	1.062813	1.056468	1.050957	1.046381	1.042831	1.040361
12.11	1.035951	1.029385	1.023447	1.018219	1.013782	1.010212	1.007553	1.005787	1.004801	1.004377
14.04	0.983540	0.978838	0.974846	0.971625	0.969203	0.967556	0.966585	0.966117	0.965946	0.965903
16.01	0.932973	0.929882	0.927496	0.925803	0.924734	0.924155	0.923901	0.923816	0.923796	0.923793
18.36	0.876089	0.874439	0.873345	0.872709	0.872395	0.872269	0.872231	0.872223	0.872221	0.872221
20.19	0.833925	0.833012	0.832486	0.832228	0.832124	0.832091	0.832083	0.832082	0.832082	0.832082

C: DIFFERENCE BETWEEN SWITCHING AND CASH-CALLING

$r \setminus T_B$	*5.25*	*4.875*	*4.5*	*4.125*	*3.75*	*3.375*	*3*	*2.625*	*2.25*	*1.875*
0.00 %	0	0	0	0	0	0	0	0	0	0
1.03	0.021853	0.022405	0.022961	0.023522	0.024084	0.024651	0.025220	0.025791	0.026364	0.026939
3.02	0.020524	0.021093	0.021673	0.022264	0.022866	0.023481	0.024109	0.024752	0.025410	0.026083
5.03	0.019262	0.019844	0.020442	0.021058	0.021694	0.022353	0.023035	0.023743	0.024478	0.025246
7.01	0.018098	0.018688	0.019300	0.019938	0.020603	0.021298	0.022026	0.022791	0.023596	0.024447
9.10	0.016941	0.017537	0.018160	0.018815	0.019504	0.020230	0.020996	0.021802	0.022641	0.023496
10.07	0.016432	0.017029	0.017656	0.018316	0.019011	0.019742	0.020504	0.021284	0.022051	0.022743
12.11	0.015405	0.015999	0.016622	0.017270	0.017933	0.018588	0.019196	0.019704	0.020059	0.020248
14.04	0.014477	0.015047	0.015626	0.016193	0.016717	0.017156	0.017473	0.017658	0.017738	0.017762
16.01	0.013535	0.014036	0.014506	0.014911	0.015221	0.015423	0.015528	0.015568	0.015579	0.015581
18.36	0.012362	0.012724	0.013012	0.013211	0.013327	0.013382	0.013400	0.013404	0.013406	0.013406
20.19	0.011398	0.011639	0.011804	0.011898	0.011941	0.011957	0.011962	0.011962	0.011962	0.011962

Table II. Price Differences Between a cash-called bond and a switched bond. Other than those changed, parameter values are those of the benchmark case. The first column shows the interest rate in percentages and the top row the time until expiration of the bonds. Until rates are about 7.15% the difference is almost constant. After that, the difference quickly drops to zero as rates increase. The difference decreases at all levels of interest rates as expiration approaches.

$r \setminus t$	*12.5*	*6.25*
0.000000	0.026231	0.026231
0.071518	0.026231	0.026231
0.079094	0.025000	0.025047
0.080699	0.024770	0.024810
0.089225	0.023600	0.023529
0.091037	0.023362	0.023237
0.098685	0.022396	0.021808
0.100703	0.022150	0.021358
0.109243	0.021151	0.019015
0.111503	0.020897	0.018272
0.118614	0.020122	0.015658
0.121101	0.019860	0.014672
0.128944	0.019060	0.011505

Table III: Critical Switching Rates as Refinancing Costs are Changed, and as Functions of Maturity. Other than those changed, parameter values are those of the benchmark case. The first column represents the time to maturity of the bond in years and the first row the total refinancing costs $\pi + f$. The zeroes after 22 years represent the initial blackout period. The zeroes close to expiration are "optimal" non-switching because costs will not be recovered. The critical rate decreases to justify spending more to refund as refinancing costs increase. The critical rate decreases because there is less time to recoup the expense of refunding as maturity nears. The exercise of the call option needs considers the present value of the coupons for the current bond.

$t \setminus 1+\pi+f$	*0.001*	*0.008*	*0.016*	*0.023*	*0.031*	*0.038*	*0.046*	*0.053*	*0.061*	*0.068*
0.125	0	0	0	0	0	0	0	0	0	0
0.25	0.076	0.016	0	0	0	0	0	0	0	0
0.375	0.079	0.048	0.018	0	0	0	0	0	0	0
0.5	0.079	0.06	0.039	0.018	0	0	0	0	0	0
0.625	0.079	0.066	0.05	0.034	0.019	0	0	0	0	0
0.75	0.079	0.07	0.057	0.044	0.032	0.019	8E-04	0	0	0
0.875	0.079	0.073	0.062	0.051	0.04	0.029	0.019	0.008	0	0
1	0.079	0.074	0.066	0.056	0.047	0.038	0.028	0.019	0.01	0
2	0.079	0.079	0.078	0.074	0.07	0.066	0.062	0.057	0.051	0.047
3	0.079	0.079	0.078	0.078	0.076	0.074	0.073	0.07	0.067	0.065
4	0.079	0.079	0.078	0.078	0.078	0.076	0.076	0.074	0.074	0.073
22	0	0	0	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0

Table IV. Critical Switching Rates when volatility is changed for the case when $L = k = \gamma = 0$. Also, there are no transaction costs or Call Premium, but the initial Blackout Period was set to three years. The first column shows time to maturity in years while the first row the values of sigma. The "optimal" policy is not to call when time to maturity is large because the probability of negative rates in this case increases with time to maturity. The critical rate increases monotonically as maturity approaches because the probability of negative rates decreases. All critical rates converge at expiration to 8%. These results are amplified as volatility increases. Of course, larger volatility implies that the probability of negative rates is larger.

t / σ	*0.04*	*0.049*	*0.058*	*0.067*	*0.076*	*0.085*	*0.094*	*0.103*	*0.112*	*0.121*
0.125	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800
0.25	0.0791	0.0791	0.0791	0.0791	0.0791	0.0791	0.0791	0.0791	0.0775	0.0775
0.5	0.0715	0.0715	0.0701	0.0687	0.0673	0.0659	0.0633	0.0620	0.0595	0.0571
0.75	0.0659	0.0673	0.0646	0.0620	0.0595	0.0559	0.0525	0.0493	0.0462	0.0413
1	0.0620	0.0633	0.0608	0.0571	0.0525	0.0493	0.0452	0.0404	0.0351	0.0287
1.5	0.0548	0.0583	0.0525	0.0482	0.0432	0.0377	0.0310	0.0229	0.0142	0.0026
1.75	0.0525	0.0559	0.0503	0.0442	0.0385	0.0318	0.0243	0.0148	0.0026	0
2.125	0.0493	0.0525	0.0462	0.0394	0.0326	0.0250	0.0148	0.0017	0	0
2.625	0.0452	0.0482	0.0413	0.0342	0.0257	0.0159	0.0013	0	0	0
3.375	0.0394	0.0442	0.0359	0.0272	0.0165	0.0008	0	0	0	0
4.5	0.0334	0.0385	0.0287	0.0178	0.0017	0	0	0	0	0
6.75	0.0243	0.0302	0.0184	0.0008	0	0	0	0	0	0
13.625	0.0103	0.0190	0.0004	0	0	0	0	0	0	0
22	0.0053	0.0154	0	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0

Table V. Critical Rates as we change sigma (σ) for different times to maturity. Other than those changed, parameter values are those of the benchmark case. The first column contains the times to maturity in years while the first row shows the values of sigma. Note that for times to maturity less than two years the critical rate does not depend on the level of volatility. Basically, the time left is so small that the probability of drastically lower rates is not affected by the level of sigma. Unlike table IV, critical interest rates decease as maturity nears because there is less time to recoup the refunding cots.

t \ σ	*0.04*	*0.049*	*0.058*	*0.067*	*0.076*	*0.085*	*0.094*	*0.103*	*0.112*	*0.121*
0.125	0	0	0	0	0	0	0	0	0	0
0.75	0	0	0	0	0	0	0	0	0	0
1	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093
1.5	0.0368	0.0368	0.0368	0.0368	0.0368	0.0368	0.0368	0.0368	0.0368	0.0359
2	0.0503	0.0503	0.0503	0.0503	0.0503	0.0503	0.0503	0.0503	0.0503	0.0493
3	0.0659	0.0673	0.0659	0.0659	0.0646	0.0646	0.0633	0.0633	0.0620	0.0608
4	0.0730	0.0730	0.0715	0.0715	0.0701	0.0687	0.0687	0.0673	0.0659	0.0659
5	0.0730	0.0730	0.0730	0.0730	0.0715	0.0715	0.0701	0.0687	0.0673	0.0673
6	0.0730	0.0730	0.0730	0.0730	0.0715	0.0715	0.0701	0.0701	0.0687	0.0673
22	0	0	0	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0

Table VI: Critical switching rates as the long term value of the instantaneous rates (L) is changed. Other than those changed, parameter values are those of the benchmark case. As L is decreased from 721 basis points to 1 basis point the critical rate also decreases because the interest rate will tend towards that value. Note the non-monotonic relationship for the critical rate. For L equal to one basis point the highest critical rate is 744 basis points at six years to maturity. For L equal to 321 the highest value is 730 at ten years.

t \ *L*	*0.0001*	*0.0081*	*0.0161*	*0.0241*	*0.0321*	*0.0401*	*0.0481*	*0.0561*	*0.0641*	*0.0721*
0	0	0	0	0	0	0	0	0	0	0
0.875	0.010878	0.010878	0.010346	0.009820	0.008787	0.008278	0.007278	0.006786	0.001708	0.001708
1	0.020287	0.020287	0.019644	0.018382	0.017763	0.017151	0.015949	0.015359	0.014775	0.013629
1.25	0.034231	0.034231	0.032589	0.031785	0.030992	0.030210	0.028678	0.027927	0.026455	0.025734
1.5	0.044163	0.044163	0.042234	0.041292	0.040363	0.038546	0.037658	0.036783	0.035069	0.034231
1.75	0.051407	0.051407	0.050322	0.048204	0.047170	0.045150	0.044163	0.043191	0.041292	0.039448
2	0.057107	0.057107	0.055929	0.054770	0.052510	0.051407	0.049255	0.048204	0.046152	0.044163
2.25	0.062026	0.062026	0.060764	0.059524	0.057107	0.055929	0.053631	0.051407	0.050322	0.048204
2.5	0.065948	0.065948	0.064617	0.063310	0.060764	0.059524	0.057107	0.054770	0.052510	0.051407
2.75	0.070087	0.070087	0.068682	0.065948	0.064617	0.062026	0.059524	0.058305	0.055929	0.053631
3	0.071518	0.071518	0.070087	0.068682	0.067302	0.064617	0.062026	0.060764	0.058305	0.055929
3.5	0.074462	0.074462	0.072976	0.071518	0.070087	0.068682	0.065948	0.064617	0.062026	0.058305
4	0.074462	0.074462	0.072976	0.072976	0.072976	0.071518	0.068682	0.067302	0.064617	0.060764
4.5	0.074462	0.074462	0.074462	0.072976	0.072976	0.071518	0.070087	0.068682	0.065948	0.063310
5	0.074462	0.074462	0.072976	0.072976	0.072976	0.072976	0.071518	0.070087	0.067302	0.064617
6	0.074462	0.074462	0.072976	0.072976	0.072976	0.072976	0.072976	0.071518	0.070087	0.067302
10	0.072976	0.072976	0.072976	0.072976	0.072976	0.072976	0.071518	0.071518	0.071518	0.070087
20	0.072976	0.072976	0.072976	0.071518	0.071518	0.071518	0.071518	0.070087	0.070087	0.068682
22	0	0	0	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0

Captions for figures

Figure 1: This figure presents the value of a callable bond from the point of view of the Borrower. The vertical axis has the value of the bond as a fraction of face value. The other two axises are the time to expiration and the interest rate. In the interest rate axis we have decided to do a transformation to clearly present the results. This transformation is just the ordinal values of the interest rate vector. The plateau in the figure is the area where the bond should be called. Since this happens for low interest rates, below 7% in our case, a cardinal scale would have given us a small region with interest rates below 7%. In the figure the value of 200 is for interest rates equal to zero. The value of one (1) in the interest rate axis is for interest rate equal to 250%. The relevant part of the graph is for small interest rates. Our transformation allows us to see more area in the "low" interest rate region.

Figure 2: This figure presents the borrower's value of a callable bond with a blackout period. The vertical axis has the value of the bond as a fraction of face value. The other two axises are the time to expiration and the interest rate. In the interest rate axis we have decided to do a transformation to clearly present the results. This transformation is just the ordinal values of the interest rate vector.

Figure 3: This figure presents the critical rates for the simple cash-calling strategy and our proposed switching strategy. The vertical axis presents the critical interest rates in decimals. This is a regular scale as opposed to the ordinal scale used in Figures 1 and 2. The Horizontal axis presents the time to maturity. The continuous line represents the switching strategy. With the cash-call strategy the call decision is delayed further because the borrower gives up the option to refinance later.

Figure 4: This figure presents the difference in value between the bond using the switching strategy and the cash-call strategy. The vertical axis has the difference between the values of the bond. We are plotting the value of the bond with the switching strategy minus the value of the bond under the cash calling strategy. This difference is a fraction of face value. The other two axises are the time to expiration and the interest rate. In the interest rate axis we have decided to do a transformation to clearly present the results. This transformation is just the ordinal values of the interest rate vector. For small interest rates, $r < 0.004$, and within the blackout period our procedure presents small inaccuracies. The slope after the blackout period is due to our declining refunding costs.

Figure 5: This figure presents a cross section of the difference in value surface between the bond using the switching strategy and the cash-call strategy. The vertical axis has the difference between the values of the bond. We are plotting the value of the bond with the switching strategy minus the value of the bond under the cash-call strategy. This difference is a fraction of face value. The interest rate axis is standard and the scale is decimal. The cross sections start at 9.5 years to maturity.

Figure 6: This figure presents a cross section of the value of the bond from the point of view of the borrower and the investor. The vertical axis has the value of the bond as a fraction of face value. The interest rate axis is standard and the scale is decimal. Given that the borrower has to pay not only the investor but the flotation costs as well the borrower's valuation is always higher than the investor. Market valuation can exceed the call price of 1.06 in our case. As the figure shows, the overpricing is of the same order of magnitude as refunding costs.

Figure 7: This figure presents the critical rates for our switching strategy for different values of refunding costs. The vertical axis presents the critical interest rates in decimals. This is a regular scale as opposed to the ordinal scale used in Figures 1 and 2. The Horizontal axis presents the time to maturity. As refunding costs increase the call decision is delayed further. The lower curves represent higher costs.

Figure 8: This figure presents, in three dimensions, the critical rates for our switching strategy for different values of refunding costs. The vertical axis presents the critical interest rates in decimals. The other two axises represent the time to expiration in years and the refinancing costs. The scale is actually one plus the refunding costs. As refunding costs increase the call decision is delayed further. The lower part of the surface represents higher costs.

Figure 9: This figure presents the critical rates for our switching strategy for different values of volatility, sigma (σ). This experiment was performed for a special case where the mean of the interest rate process equals zero. This case replicates the situation analyzed by Brennan and Schwartz (1977b). The vertical axis presents the critical interest rates in decimals. The Horizontal axis presents the time to maturity. As volatility, sigma (σ), increases the call decision is delayed further. Note however that for all levels of volatility the critical rate is decreasing with maturity. This is a result of having zero mean which implies possible negative rates. The further away from maturity the more likely there will be negative rates. The higher the possibility of negative rates the lower the critical rate.

Figure 10: This figure presents the critical rates for our switching strategy for different values of volatility, sigma (σ). This case is for the values of the parameters as given by our table. The vertical axis presents the critical interest rates in decimals. The Horizontal axis presents the time to maturity. As volatility, sigma (σ), increases the call decision is delayed further. Note however that the effect of volatility is, as expected, stronger as we move further away from expiration.

Figure 11: This figure presents, in three dimensions, the critical rates for our switching strategy for different values of volatility, sigma (σ). The vertical axis presents the critical interest rates in decimals. The other two axises represent the time to expiration in years and volatility, sigma (σ). As volatility, sigma (σ), increases the call decision is delayed further. The lower part of the surface represents higher volatility, sigma (σ).

Figure 12: This figure presents the critical rates for our switching strategy for different values of the long term value of the instantaneous interest rate (L). This is for the values of the parameters as given by our table. The vertical axis presents the critical interest rates in decimals. The Horizontal axis presents the time to maturity. As the long term value of the instantaneous interest rate (L) decreases the call decision is delayed further. The effects of changes in the long term value of instantaneous volatility are more important the further away we are from expiration.

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Figure 1: Borrower's Bond Value
No Balckout Period

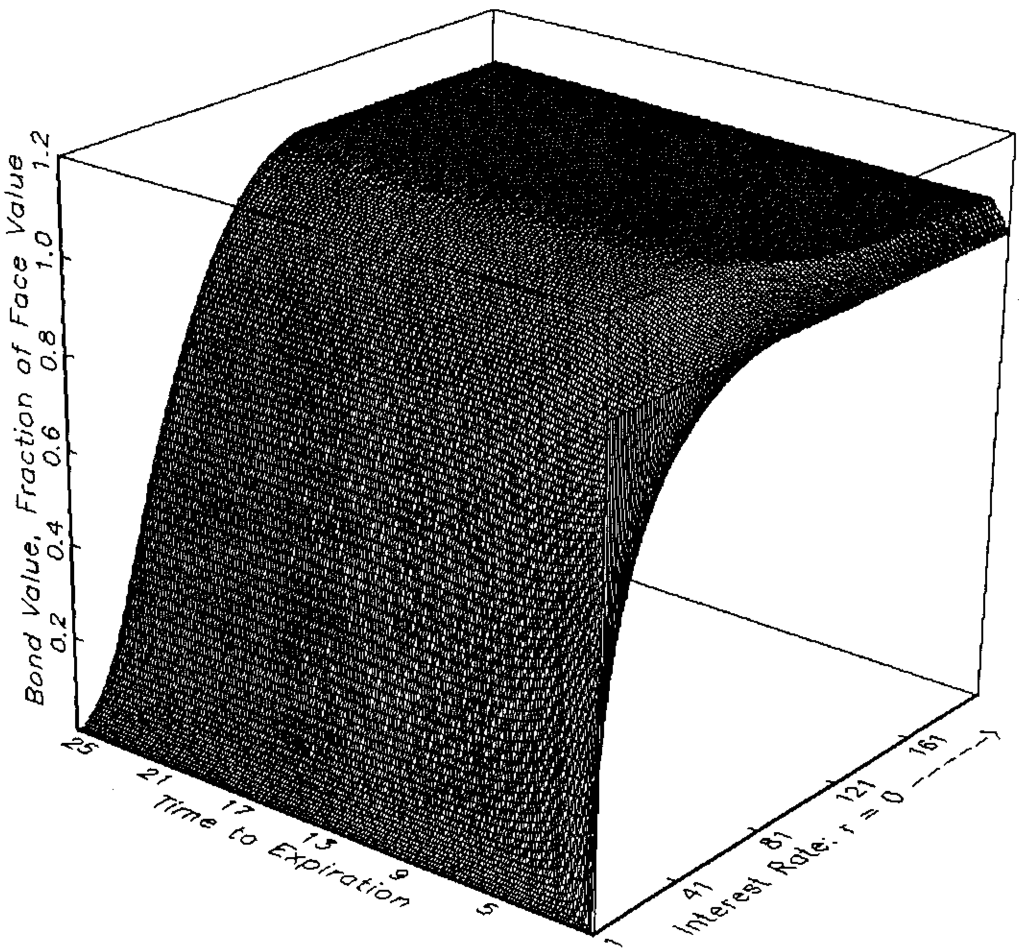

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Figure 2: Borrower's Bond Value
Three Year Balckout Period

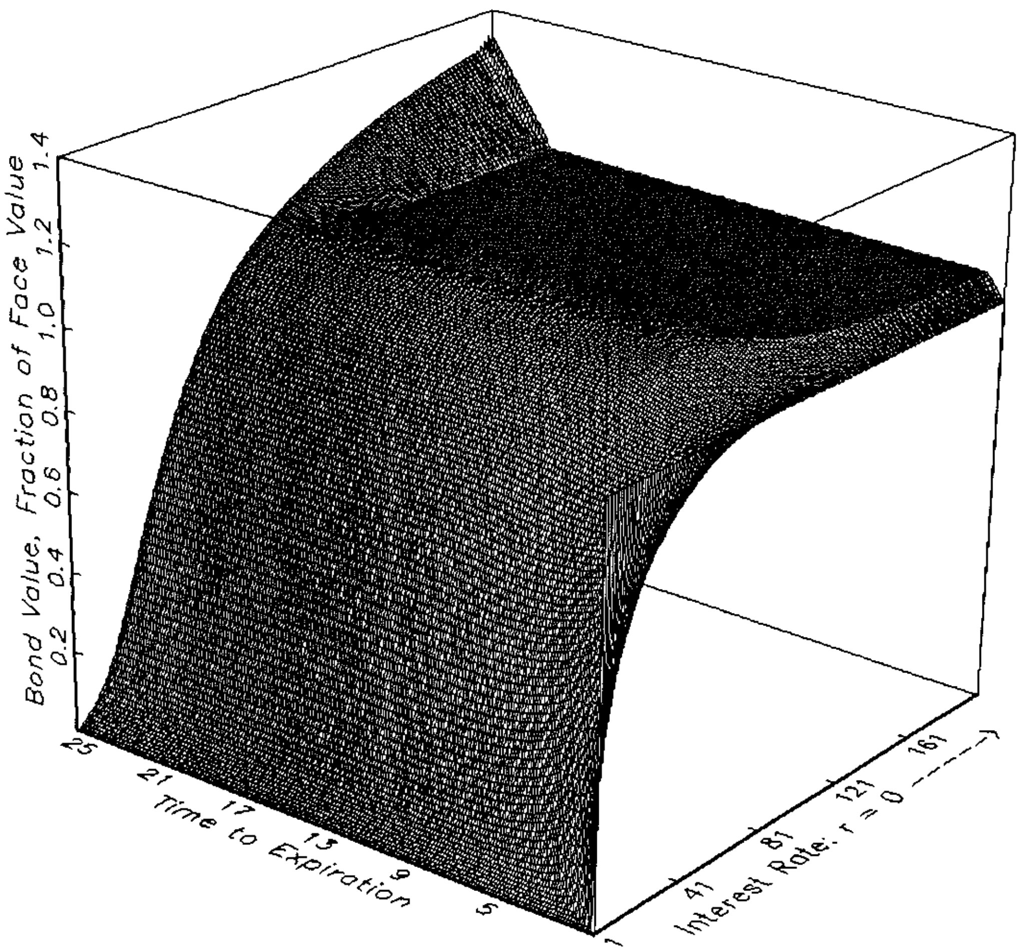

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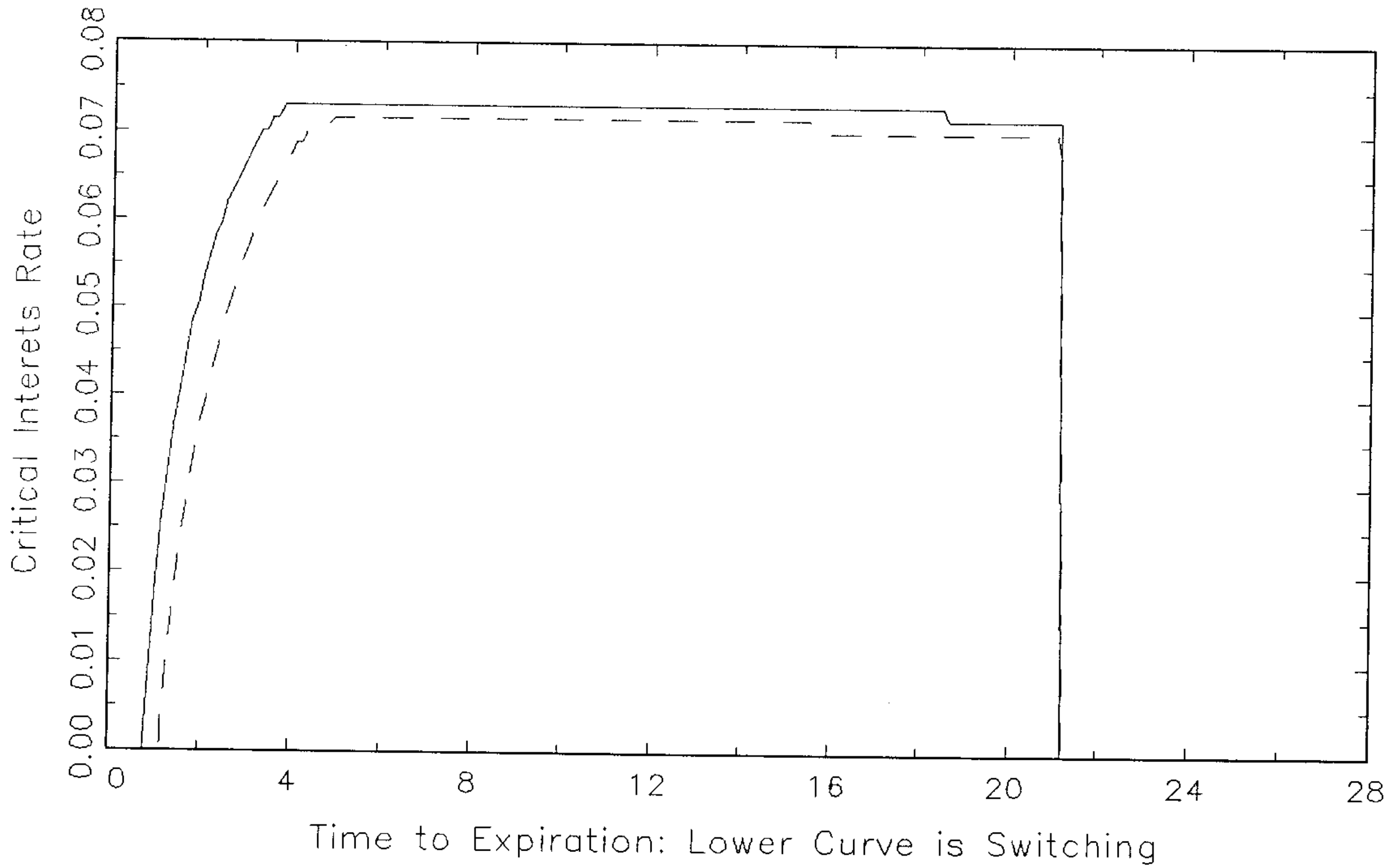
Figure 3: Critical Rates Compared
Call and Switching Strategies



GAUSS Fri Mar 3 16:11:21 1995

Figure 4: Bond Price Differences: Call vs. Switching

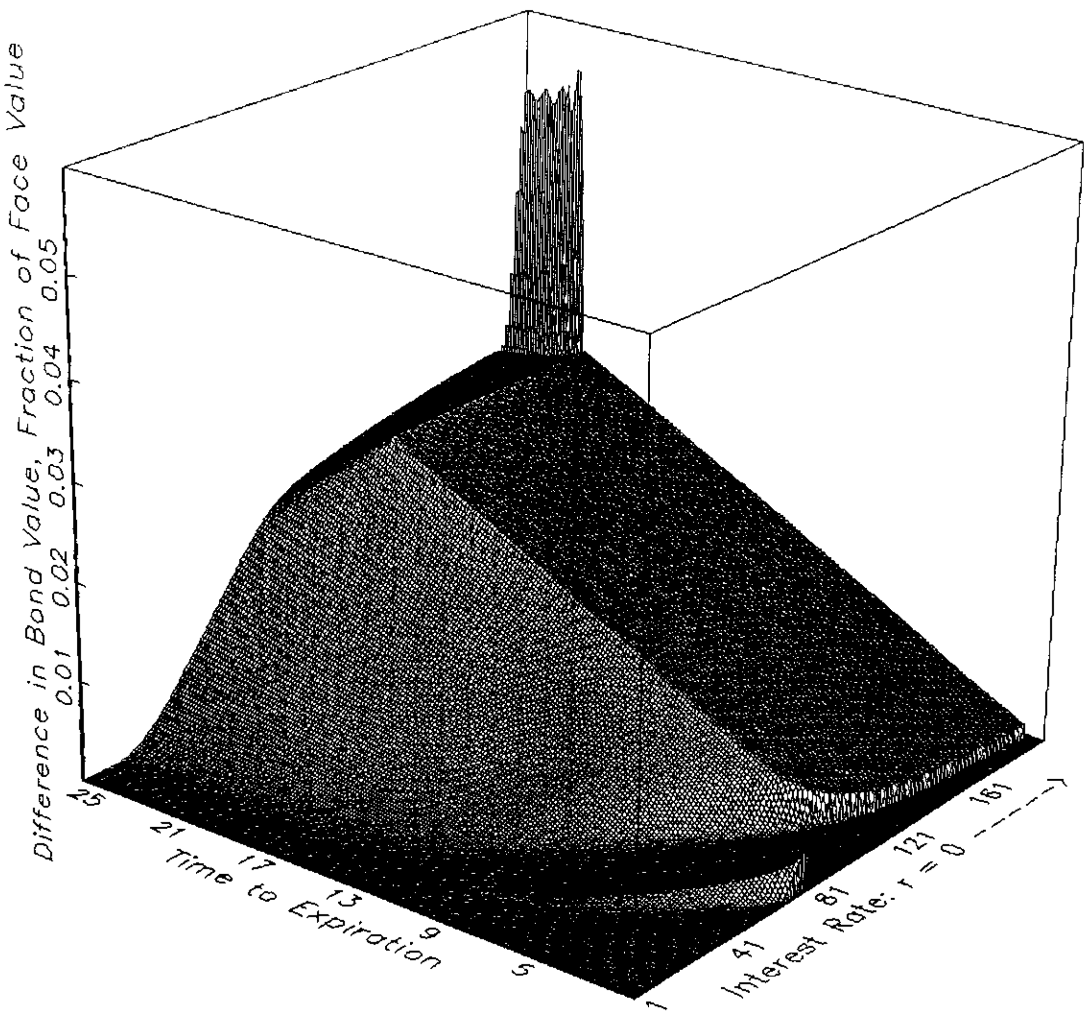


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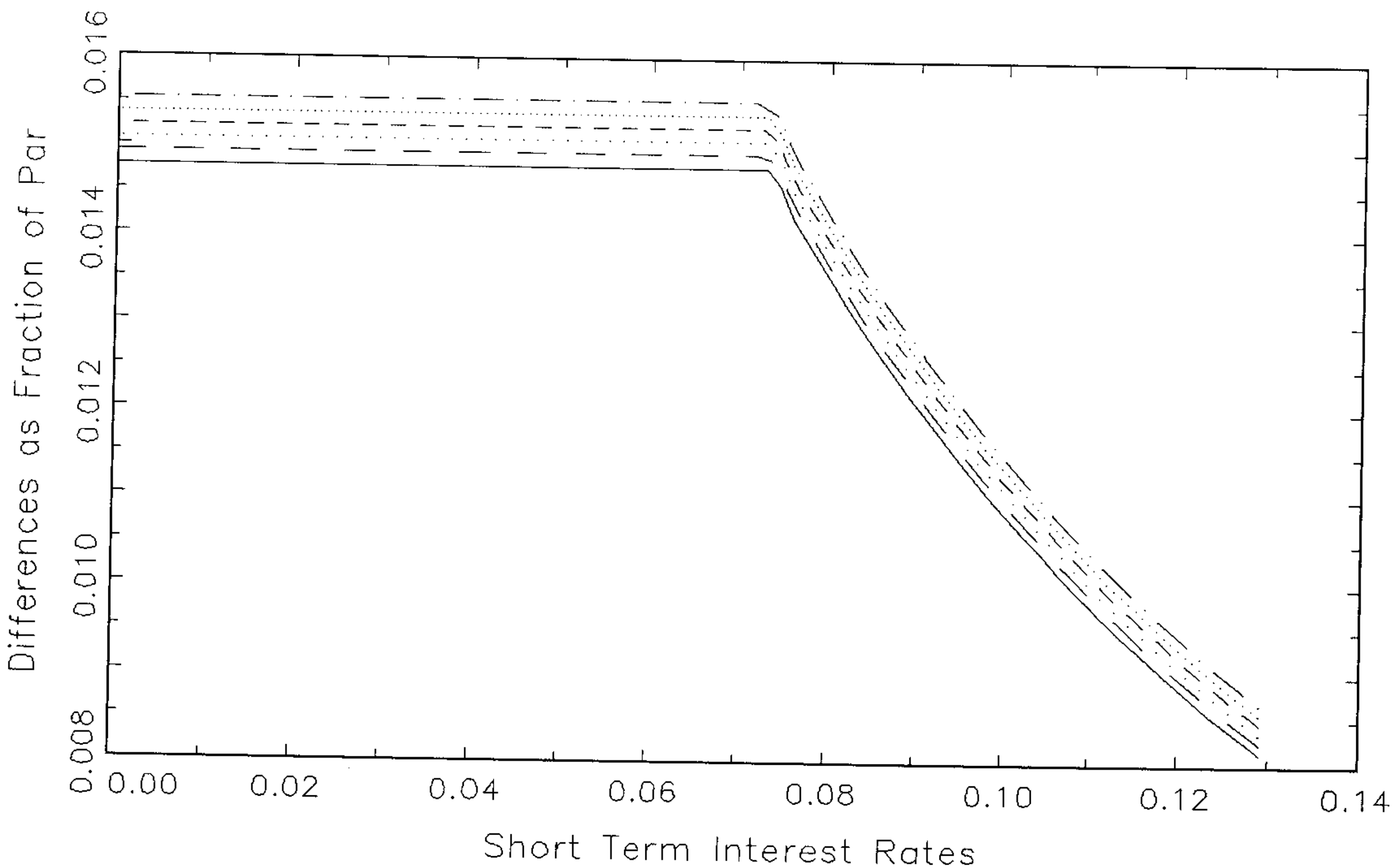
Figure 5: Crossections (6) of Price Differences: Call vs. Switching
Time to Matirity Equal to 12.5 years



GAUSS Wed Mar 1 14:15:56 1995

Figure 6: Crossections (10 each) of Bond Prices
Top Curve is value for Borrower, bottom one for Investor

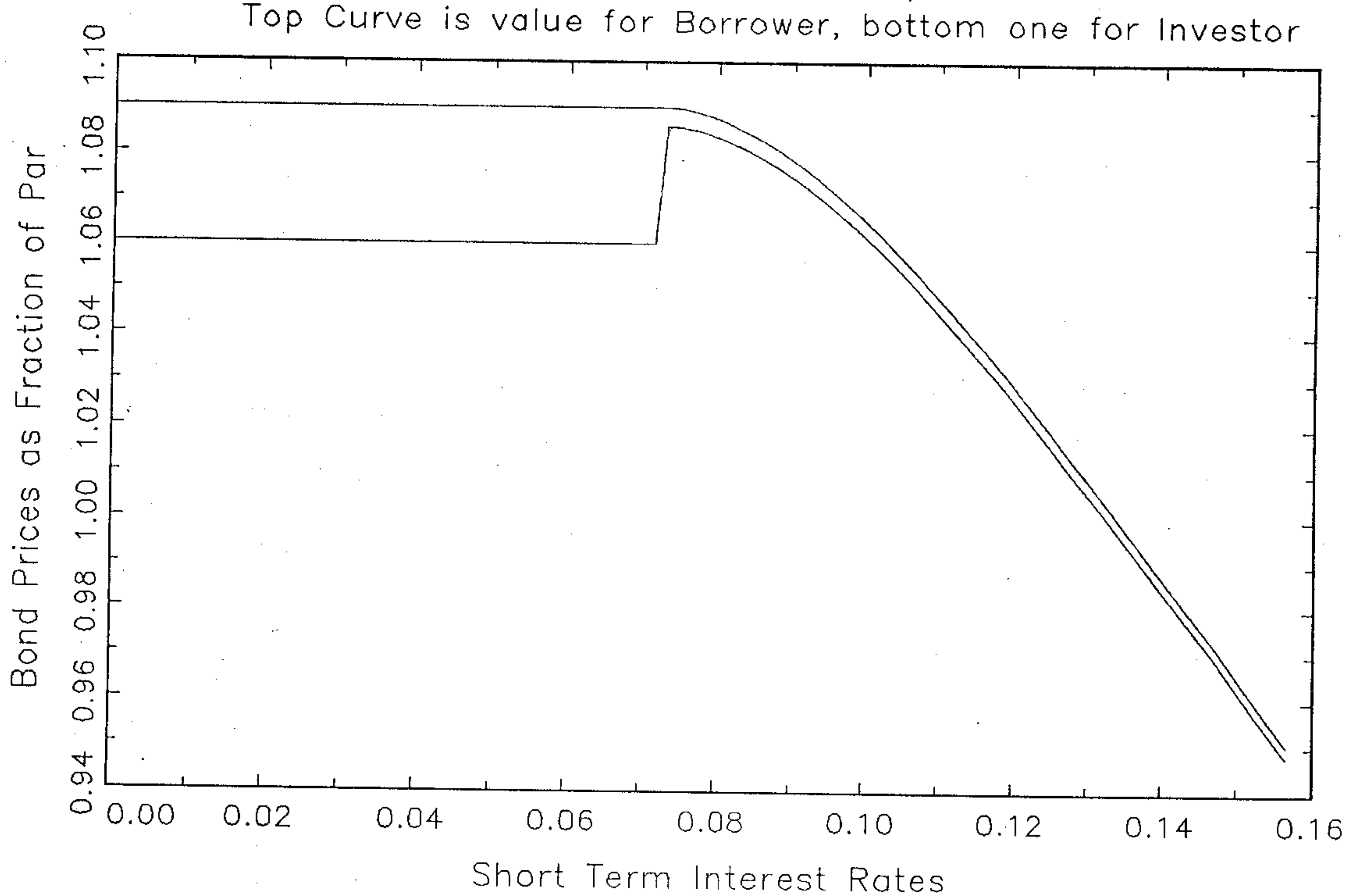


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Figure 7: Critical Rates Compared
Changes in Refinancing Cost

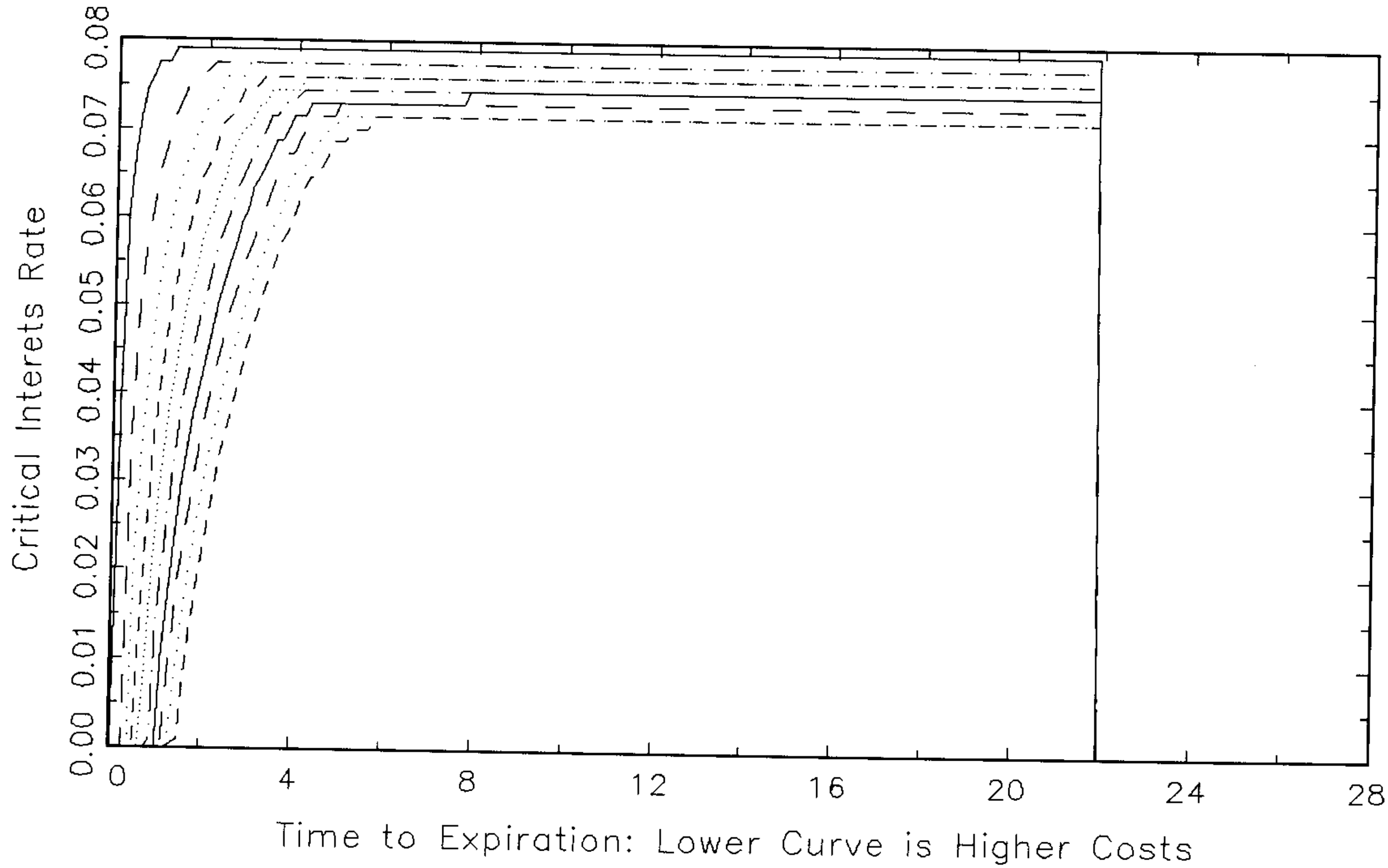

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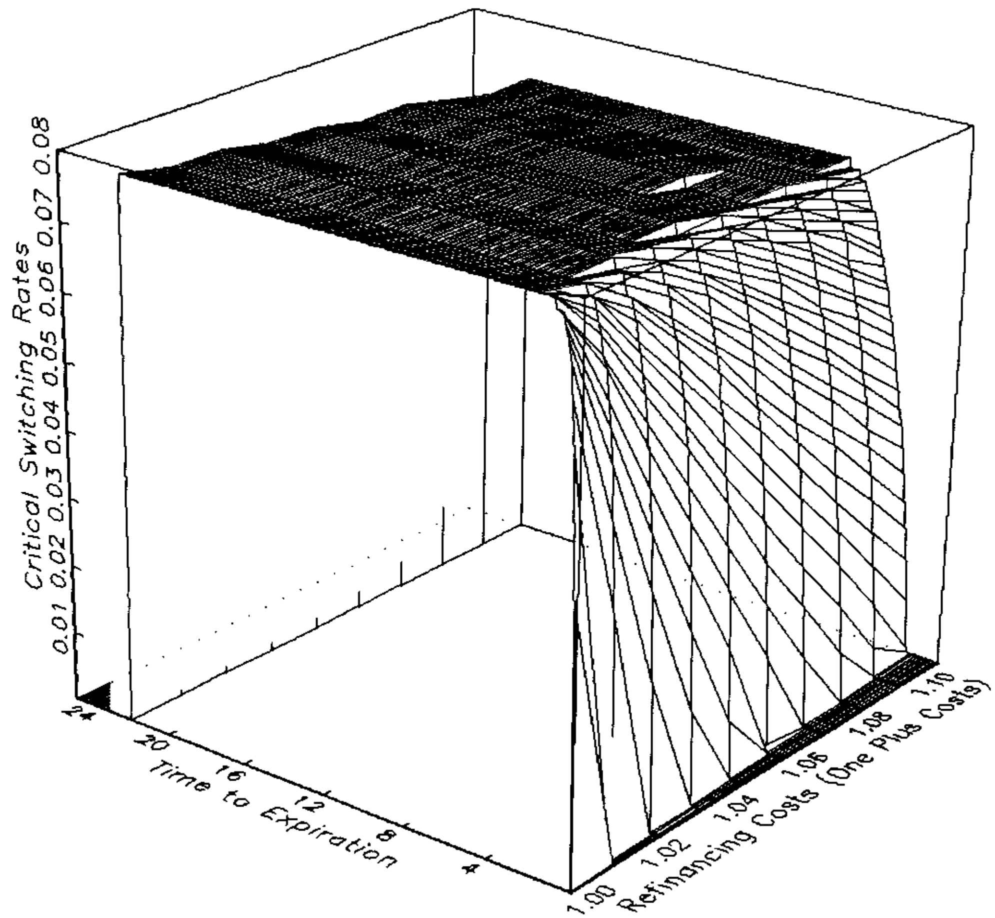
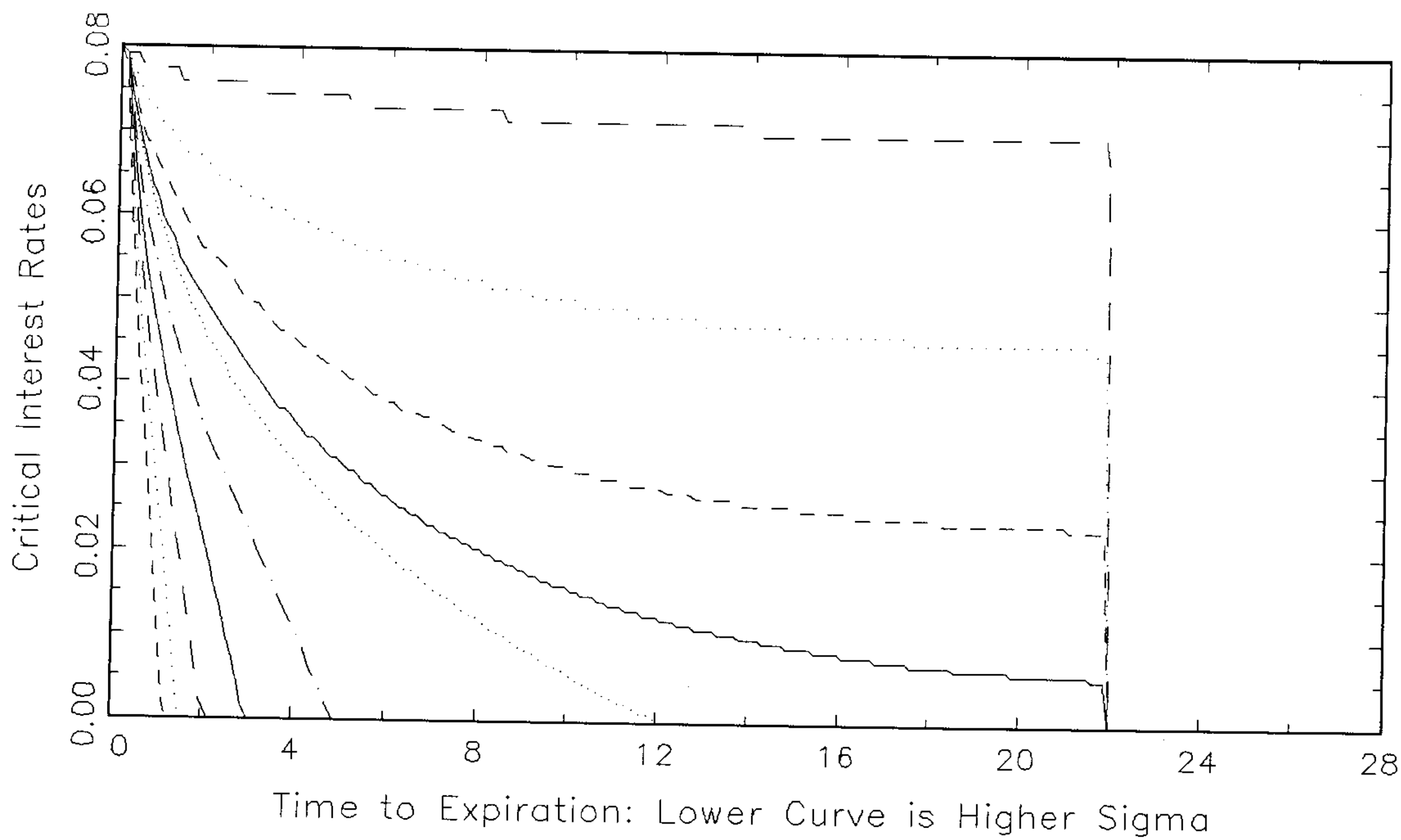
Figure 8: Critical Switching Rates
Varying Refinancing Costs:
Call Premium Plus Flotation Costs



GAUSS Fri Mar 3 15:50:09 1995

Figure 9: Critical Rates Compared For Changes in Volatility (Sigma)
k=L=gamma=0



GAUSS Tue May 30 00:09:06 1995

Figure 10: Critical Rates Compared For Changes in Volatility (Sigma)

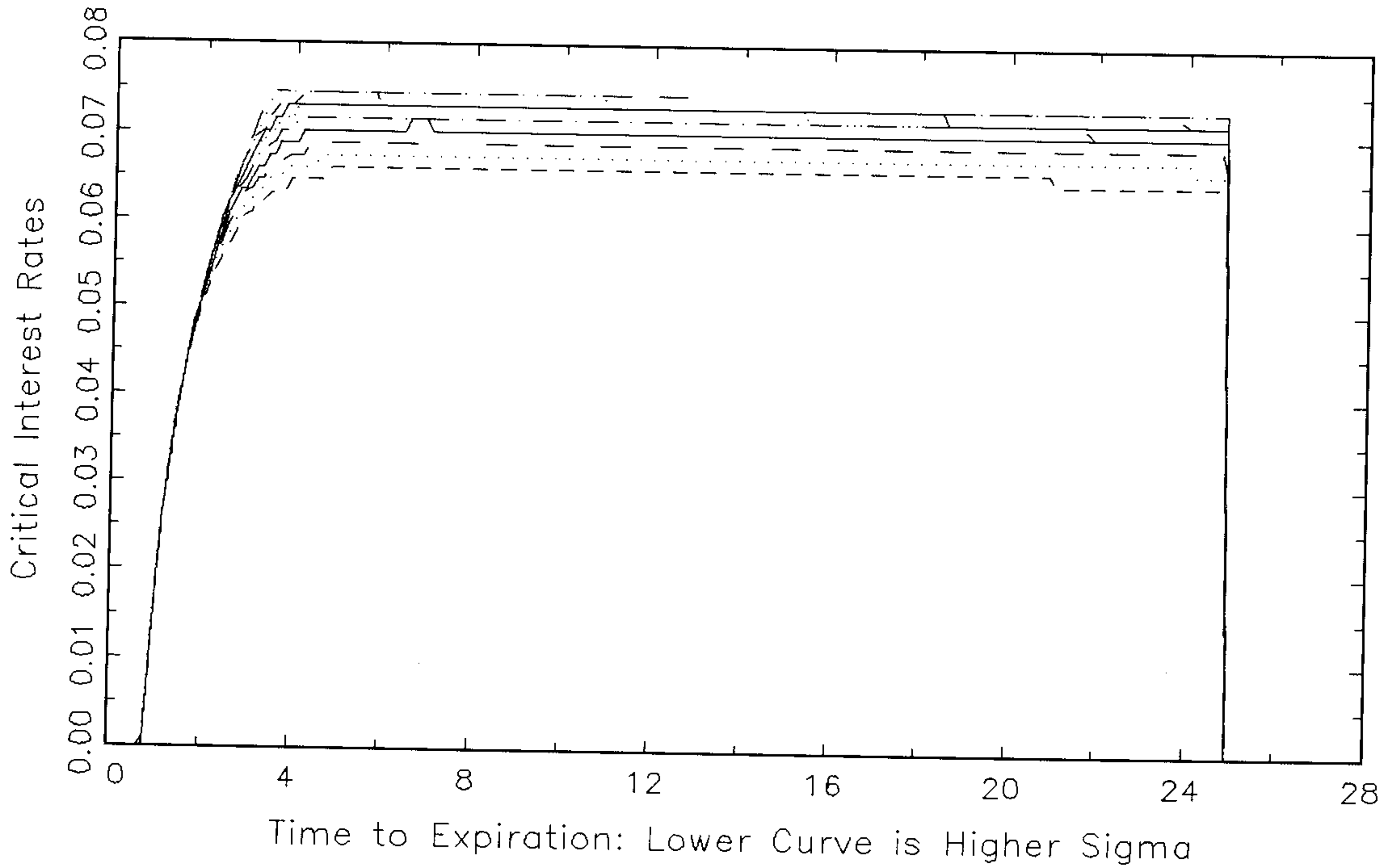


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Figure 11: Critical Switching Rates for Varying Sigmas

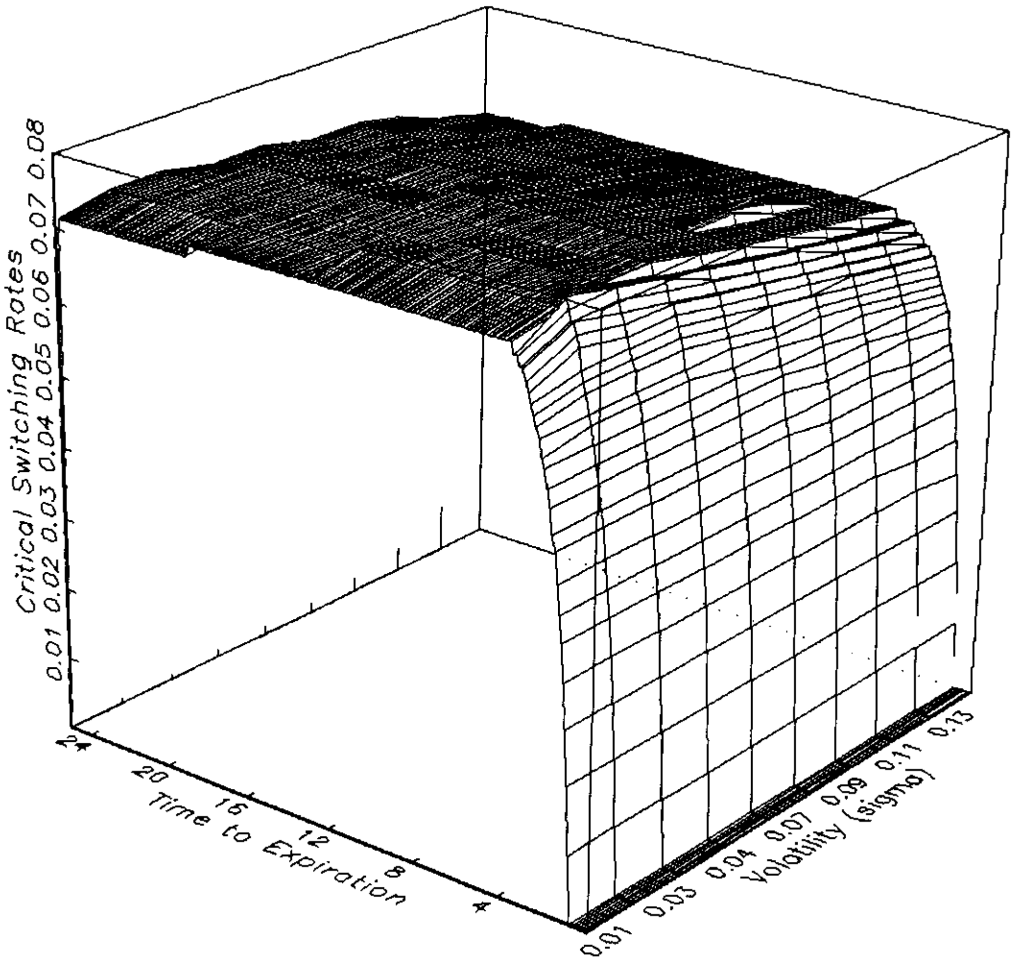

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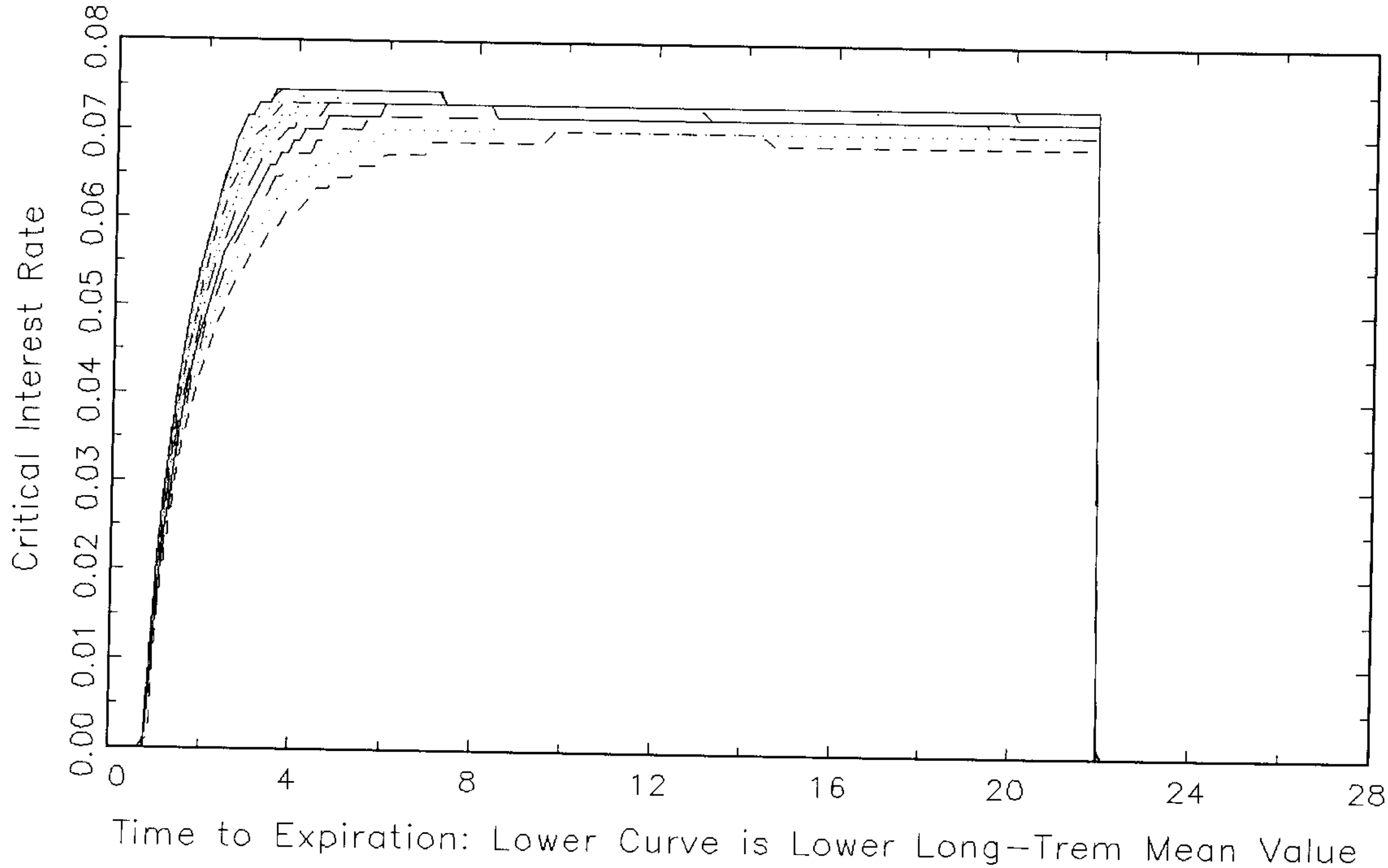
Figure 12: Critical Rates Compared For Changes in Long Term Mean (L)



Appendix 1: Change of variable discretization of the state and solution of the PDE. This appendix is somewhat linger than it needs to be. It will be shortened after the refereeing process.

This appendix solves for a simple transformation that allows to use a faster and more accurate Crank-Nicholson algorithm. The differential equation that needs to be solved is:

$$\frac{1}{2}\sigma^2 r G_{rr} + (K(L-r) - \lambda r)G_r - G_t - rG + P = 0$$

$G(r,t;r_n)$ is the security valuation.

r is the interest rate
t is the time to maturity
r_n is the contractural rate for the security,
additionally f(r,t) is the cost of refinancing.

The boundary conditions that must be satisfied by this PDE are:

$G(r,0;r_n) = 1$ At maturity the value of the bond is face (normalized to one).

$$\lim_{r\to\infty} G(r,t;r_n) = 0$$

$KL\,G_r - G_t + P = 0$ If not called at all

$G(r^*,t;r_n) = G(r^*,T;r_n)$ Call condition

Where we have assumed that the refinancing is made through a security of similar structure. T is the original time to maturity.

Since G(·) includes both the straight bond valuation and the call option, we do not have to go through the cumbersome procedure of Weingartner (1967) in order to consider multiple refundings.

Following Brennan & Schwartz (1977) we make the transformation $S = (1+r)^{-X}$ (see also Timmis (1985)). $X > 0$ so that $r\epsilon(0,\infty)$ is mapped into $S\epsilon(0,1)$ and the implementation of a Crank-Nicholson finite difference method can be done more accurately.

$Y(S,t;S_n) = G(r,t;r_n)$ also $S_n = (1+r_n)^{-X}$

$$\frac{dS}{dr} = -X(1+r)^{-X-1} \; ; \text{ also } S^{\frac{X+1}{X}} = (1+r)^{-X-1}$$

$$\frac{d^2S}{dr^2} = X(X+1)(1+r)^{-X-2} \text{ also } S^{\frac{X+2}{X}} = (1+r)^{-X-2}$$

Before we transform the PDE we need:

$$G_r = Y_S \frac{dS}{dr} = -XY_S S^{\frac{X+1}{X}}$$

$$G_{rr} = Y_{SS}\left(-XS^{\frac{X+1}{X}}\right)\frac{dS}{dr} - (X+1)Y_S S^{\frac{1}{X}}\frac{d^2S}{dr^2}$$

$$G_{rr} = X^2 Y_{SS} S^{\frac{2X+2}{X}} + Y_S X(X+1) S^{\frac{X+2}{X}}$$

$$G_t = Y_t$$

The transformed PDE is therefore

$$\frac{1}{2}\sigma^2\left[S^{-\frac{1}{x}} - 1\right]\left\{X^2 Y_{SS} S^{\frac{2X+2}{X}} + X(X+1)Y_S S^{\frac{X+2}{X}}\right\}$$

$$+\left(KL - (K+\lambda)\left(S^{-\frac{1}{X}} - 1\right)\right)\left\{-XY_S S^{\frac{X+1}{X}}\right\} - Y_t$$

$$+\left(-S^{-\frac{1}{X}} + 1\right)Y + P = 0$$

Alternatively we can write

$$A(S)Y_{SS} + B(S)Y_S + C(S)Y + P = Y_t$$

Where:

$$A(S) = \frac{1}{2}\sigma^2\left(S^{-\frac{1}{X}} - 1\right)X^2 S^{\frac{2X+2}{X}} = \frac{X^2}{2}\sigma^2\left(1 - S^{\frac{1}{X}}\right)S^{2+\frac{1}{X}}$$

$$B(S) = X(X+1)\frac{1}{2}\sigma^2\left[S^{\frac{X+1}{X}} - S^{\frac{X+2}{X}}\right] - XS^{\frac{X+1}{X}}\left(Km - (K+\lambda)\left(S^{\frac{-1}{X}} - 1\right)\right)$$

$$C(S) = \left(1 - S^{\frac{-1}{X}}\right)$$

Not functions of time.

The corresponding boundary conditions are:

$$Y(S, 0; S_n) = 0$$

$$\lim_{S \to 0} Y(S, t; S_n) = 0 \quad \text{or} \quad Y(0, t; S_n) = 0$$

$$Y(S,^* t; S_n) \leq Y(S,^* T; S_n) + C(S^*)$$

or

$$-KLXS^{\frac{X+1}{X}} Y_S - Y_t + P = 0 \qquad (at \quad S = 1) \quad i = M$$

Partition time from 0 to N and call the partition size d = T/N. Partition S from 0 to 1 and call the partition size h = 1/M. The index for time will be j and the index for space will be i. Or is compact from

i:	h = 1/M,	M number of state space points
j:	d = T/N,	N number of time points
S = 0, h, 2h, 3h 1:		i → S = ih
t = 0, d, 2d, 3d .....T:		j → t = jd

If we call Y_{ij} the value of the security at the point (ih, jd) we can write the following finite difference approximations to the derivatives.

$$(Y_{SS})_{ij} = \frac{Y_{i+1,j} - 2Y_{ij} + Y_{i-1,j}}{h^2}$$

$$(Y_S)_{ij} = \frac{Y_{i+1,j} - Y_{i-1,j}}{2d} \qquad i = 1, 2, \ldots\ M-1$$

$$(Y_t)_{ij} = \frac{Y_{i,j+1} - Y_{i,j-1}}{2d} \qquad j = 1, \ldots\ N$$

$$(Y_t)_{ij} = \frac{Y_{i,j} - Y_{i,j-1}}{d} \qquad j = 0$$

Calling

$A_i = A(S)$ $\qquad S = hi$
$B_i = B(S)$
$C_i = C(S)$

$$A_i(Y_{SS})_{ij} + B_i(Y_S)_{ij} + C_i(Y)_{ij} + P = (Y_t)_{ij}$$

$$\frac{A_i}{h^2}\left(Y_{i+1,j} - 2Y_{ij} + Y_{i-1,j}\right) + \frac{B_i}{2h}\left(Y_{i+1,j} - Y_{i-1,j}\right) + C_i Y_{ij} + P = \frac{(Y_{i,j} - Y_{i,j-1})}{d}$$

$$\left(\frac{A_i}{h^2} + \frac{B_i}{2h}\right) Y_{i+1,j} + \left(C_i - \frac{1}{d} - \frac{2A_i}{h^2}\right) Y_{i,j} + \left(\frac{A_i}{h^2} - \frac{B_i}{2h}\right) Y_{i-1,j} = -P - \frac{Y_{i,j-1}}{d}$$

Valid for $\qquad i = 1, 2, 3 \ldots M - 1$
$\qquad j = 1, 2, 3 \ldots N$

Boundary Conditions:

$Y_{i,0} = 0$
$Y_{0,j} = 0$

$$Y_{ij} \le Y^*_{ij} + C \qquad \left.\begin{aligned} C &= cF(t) \\ Y^*_{ij} &= lF(t) \end{aligned}\right\} (l+c)F(t)$$

If $\quad Y_{ij} < Y^*_{ij} + C$ then $-KLXY_S - Y_t + P = 0$ $\quad$ at $S = 1$

Define $\qquad c_i = \left(\frac{A_i}{h^2} + \frac{B_i}{2h}\right)$

$$b_i = \left(c_i - \frac{1}{d} - \frac{2A_i}{h^2} \right)$$

$$a_i = \left(\frac{A_i}{h^2} - \frac{B_i}{2h} \right)$$

$$d_{i,j} = - P - \frac{Y_{i,j-1}}{d}$$

We can rewrite the full system as

$$c_i \, Y_{i+1,j} + b_i \, Y_{ij} + a_i \, Y_{i-1,j} = d_{i,j}$$

Valid for $i = 1, 2, 3 \ldots N - 1$

$j = 1, 2, 3 \ldots M$

Note that $d_{i,j}$ is known, it only depends on values of $Y_{i,j-1}$, that is i-1.

For the boundary we have

$$- KLX \frac{Y_{N,j} - Y_{N-1,j}}{h} + P - \frac{Y_{N,j} - Y_{N,j-1}}{d} = 0$$

$$\begin{array}{lcl}
Y_{0,j} & = & d_{0,j} = 0 \\
a_1 Y_{0,j} + b_1 Y_{1,j} + c_1 Y_{2,j} & = & d_{1,j} \\
a_2 Y_{1,j} + b_2 Y_{2,j} + c_2 Y_{3,j} & = & d_{3,j} \\
\vdots & \vdots & \\
a_i Y_{i-1,j} + b_1 Y_{i,j} + c_i Y_{i+1,j} & = & d_{1,j} \\
\vdots & \vdots & \\
a_{N-1} Y_{N-2,j} + b_{N-1} Y_{N-1,j} + c_{N-1} Y_{N,j} & = & d_{N-1,j} \\
a_N Y_{N-1,j} + b_Y Y_{N,j} & = & d_{N,j} = -P
\end{array}$$

This system can be converted into a bidiagonal one:

$$Y_{i,j} \quad i \neq N \qquad Y_{i,i} = \psi(Y_{N,j})$$

Replacing equation zero into 1 we have:

$$b_1^* Y_{1j} + c_1 Y_{2,j} = d_{1,j}^*$$

where $b_1^* = b_1$ *and* $d_{1,j}^* = d_{1,j}$

multiply this equation by a_2 / b_1^*

$$a_2 Y_{1,j} + \frac{a_2 c_1}{b_1^*} Y_{2,j} = \frac{a_2 d_1^{*j}}{b_1^*} \tag{i}$$

rewrite equation 2

$$a_2 Y_{1,j} + b_2 Y_{2,j} + c_2 Y_{3,j} = d_{2,j} \tag{ii}$$

Substract (ii) from (i)

$$\left[b_2 - \frac{a_2 c_1}{b_1^*}\right] Y_{2,j} + c_2 Y_{3,j} = d_{2,j} - \frac{a_2 d_{1j}^*}{b_1^*} \quad \text{or recognizing } b_2^*$$

$$b_2^* Y_{2,j} + c_2 Y_{3,j} = d_{2,j}^*$$

$$b_2^* = b_2 - \frac{a_2 c_1}{b_1^*}; \quad \text{in general} \quad b_i^* = b_i - \frac{a_i c_{i-1}}{b_{i-1}^*}$$

$$d_{2,j}^* = d_{2,j} - \frac{a_2 d_1^*}{b_1^*}; \quad \text{in general} \quad d_{i,j}^* = d_{i,j} - \frac{a_i d_{i-1j}^*}{b_{i-1}^*}$$

We can therefore write FOR j = 1, ... M

$$Y_{0,j} = 0 \quad (1)$$

$$b_1^* Y_{1,j} + C_2 Y_{2,j} = d_{1,j}^* \quad (2)$$

$$b_2^* Y_{2,j} + C_2 Y_{3,j} = d_{2,j}^* \quad (3)$$

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$$b_i^* Y_{ij} + C_i Y_{i+1,j} = d_{i,j}^* \quad (i)$$

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$$b_{N-1}^* Y_{N-1,j} + C_{N-1} Y_{N,j} = d_{N-1,j}^* \quad (N-1)$$

$$b_N^* Y_{N,j} = d_{N,j}^* \quad (N)$$

Given $d_{N,j}^*$, we can get $Y_{N,j}$ from equation N. From N-1 we can get Y_{N-1} and so on.

Solving for $Y_{N-1,j}$ $$Y_{N-1,j} = \frac{d_{N-1,j}^* - C_{N-1} Y_{N,j}}{b_{N-1}^*}$$

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