THEORY OF RATIONAL OPTION PRICING: II

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Abstract

This paper investigates the properties of contingent claim prices in a one dimensional diffusion world and establishes that (i) the delta of any claim is bounded above (below) by the sup (inf) of its delta at maturity, and (ii), if its payoff is convex (concave), then its current value is convex (concave) in the current value of the underlying. These properties are used as the foundation for a detailed study of the properties of option prices. Interestingly, although an upward shift in the term structure of interest rates will always increase a call's value, a decline in the present value of the exercise price can be associated with a decline in the call price. We provide a new bound on the values of calls on dividend-paying assets. We establish that when the underlying's instantaneous volatility is bounded above (below), the call price is bounded above (below) by its Black-Scholes value evaluated at the bounding volatility level. This leads to a new bound on a call's delta. We also show that if changes in the value of the underlying follow a multidimensional diffusion (i.e., a stochastic volatility world), or are discontinuous or non-Markovian, then call option prices can exhibit properties very different from those of a Black-Scholes world: they can be decreasing, concave functions of the value of the underlying.

I. Introduction

The Merton (1973) *Theory of Rational Option Pricing* emphasizes the distinction between distribution-free bounds on option prices and properties of option prices conditional on distributional assumptions. Merton's meticulous analysis was one of the keystones of the subsequent explosion of interest in option pricing by both academics and financial markets. An explosion of such power that now, looking back, and reading in the *Theory* that "[b]ecause options are specialized and relatively unimportant securities, the amount of time and space devoted to the development of a pricing theory might be questioned" one can not help but smile at Merton's self-deprecating nascence. Now it seems intuitive that a call option is a wasting asset, that a call's value is an increasing convex function of the underlying stock price, and that a replicating strategy will involve increased borrowing to purchase additional stock as the stock price rises, while selling stock and repaying borrowing as the stock declines. As will be shown, much of this intuition is intimately related to the convexity of option prices. Yet Merton's proof of convexity is careful to highlight its underlying assumption, that the stock's distribution is such that the option price is homogeneous of degree one in the stock and exercise prices. In fact, Appendix 1 of the *Theory* provides an almost universally overlooked example in which the stock's distribution is such that the homogeneity property is not satisfied, and, over some range, the option price is concave in the value of the underlying.

Our research has two goals focused on the intimate relation between call price properties and the underlying asset's return distribution. Our first goal is to demonstrate that the bulk of the common intuition is, in fact, rigorously correct in the type of world typically assumed in the option pricing literature; i.e., when the underlying asset follows a diffusion whose volatility depends only on time and the *contemporaneous* level of the stock price. We refer to this as a one-dimensional diffusion world. In such a world, calls are always convex in the stock price, though they need not satisfy the homogeneity property. Still some caveats are in order: Although an upward shift in the term structure of interest rates will increase a call's value, a decline in the present value of the exercise price can be associated with a *decline* in the call price; and, a call's elasticity need not be everywhere increasing with the passage of time, or everywhere decreasing in the level of the stock price.

Our second goal is to sow seeds of doubt concerning a one-dimensional diffusion view of stock price dynamics. We suggest that is quite reasonable to model a stock's current volatility as reflecting the firm's past investment and financing decisions. Whenever those decisions were influenced by the health of the company, and were not subsequently continuously and "appropriately" adjusted, a stock's current volatility will reflect the firm's past health. The current volatility will then be linked to past stock prices. For such non-Markovian stock price processes, a call option can be a 'bloating' asset, which, over some range of stock prices, is a decreasing, concave function of the underlying's value, and is replicated by shorting the underlying, and selling increasing amounts of stock as the stock price rises. Further, an upward shift in the entire term structure can decrease a call's value. We also show that these 'unfamiliar' properties of option prices can arise if instead of relaxing the Markovian assumption we relax the continuity assumption of a diffusion world. We provide examples of decreasing, concave call prices when the underlying follows either a diffusion-jump process or a binomial process. In the one-dimensional diffusion case, the continuity and Markovian properties of a diffusion are together sufficient to guarantee that the common intuition of option pricing properties is well founded. But when changes in the underlying are driven by a multi-dimensional diffusion, as in a stochastic volatility world, we show that call prices can be, over some range, decreasing and concave in the value of the underlying.

The plan of the paper is as follows. Section II demonstrates that under the standard one-dimensional diffusion assumption, and for certain restricted forms of a multi-dimensional diffusion, a call option's price is an increasing, convex function of the stock price. Section III examines further properties of a call option's price and its replicating portfolio in a one-dimensional diffusion world; in particular, the relation between a call's delta and its sensitivity to changes in the exercise price. Still within a one-dimensional diffusion world Section IV explores the comparative statics of the effect on call prices of changes in the term structure of interest rates, in dividend policy, and in the functional form of the relation between the volatility and the contemporaneous stock price and time. With respect to dividend policy, we are able to bound the relative values of call's on otherwise equivalent dividend- and non-dividend-paying assets. With respect to volatility we present two fascinating results. First, we demonstrate an equivalence between (i) a comparison of different functional forms for the relation between instantaneous volatility and the contemporaneous stock price and time and (ii) increasing risk in the Rothschild-Stiglitz sense. Second, we show that when the instantaneous volatility is bounded above (below), the call price is bounded above (below) by the Black Scholes (1973) model value evaluated at the bounding volatility level. An immediate implication is that we can place upper and lower bounds on the stock positions necessary to hedge a given option position irrespective of the functional form of the bounded link between instantaneous volatility and the contemporaneous stock price and time. Section V examines the relation between the value of a put option and the value of the underlying asset, and provides some sufficient conditions for the put to become worthless as the underlying becomes infinitely valuable.

Section VI challenges one's intuition about the properties of option prices by examining option prices in a stochastic volatility world, a discontinuous Markovian world, and a continuous Non-Markovian world. Non-Markovian worlds are shown to arise naturally when the underlying firm will make investment and/or financing decisions prior to the maturity of options written on the firm's stock. Our examples demonstrate that call prices can be decreasing and concave in the value of the underlying, and that a put need not become valueless as the underlying becomes infinitely valuable. Section VII contains our closing remarks.

II. A Diffusion Process for the Underlying Asset

Consider a contingent claim maturing at time T. Let s_t denote the time t value of the underlying asset.[1] The stochastic process describing changes in s_t is either a one-dimensional or a multi-dimensional diffusion.

Definition 1. *The value of the underlying asset will be said to follow a one-dimensional diffusion when*

$$ds_t = \alpha(\cdot)dt + \sigma(s_t, t)s_t dB_t. \tag{1}$$

The instantaneous volatility, $\sigma(\cdot)$, is a function of s_t and t only, while the drift parameter, $\alpha(\cdot)$, is not necessarily so restricted. B_t denotes a standard Brownian motion.

We follow the finance literature and refer to $\sigma(\cdot)$, rather than $\sigma(\cdot)s$, as the volatility. Following Karlin and Taylor (1981, p. 159) we refer to the product $\sigma(\cdot)s$ as the *diffusion parameter*. The functions $\alpha(\cdot)$ and $\sigma(\cdot)$ are assumed to satisfy whatever regularity conditions are necessary for (1) to be a well-defined stochastic differential equation.[2] We refer to the special case when volatility is deterministic, $\sigma(t)$, as a Black-Scholes world.

The term 'stochastic volatility' has come to be associated in the finance literature with a setting where changes in $\sigma(\cdot)$ are driven by a random variable, y_t, other than the contemporaneous level of s, though clearly the volatility in the one-dimensional case need

[1] We use τ to denote time when t is already used to denote an earlier date. We use w to denote time when both t and τ have been used to denote earlier dates.

[2] See Chapter 6 of Arnold (1992).

not be deterministic. The dimension of y may be greater than or equal to unity. For ease of exposition only, we consider the case where y is a one-dimensional diffusion, and hence changes in s_t are driven by a two-dimensional diffusion. Our results for the case where changes in s_t are driven by a two-dimensional diffusion can be easily extended to the case where y is multi-dimensional.

Definition 2. *The value of the underlying asset will be said to follow a two-dimensional diffusion when*

$$ds_t = \alpha(\cdot)dt + \sigma(s_t, y_t, t)s_t dB_t^1, \tag{2a}$$

$$dy_t = \beta(s_t, y_t, t)dt + \theta(s_t, y_t, t)dB_t^2, \tag{2b}$$

Superscripts on dB_t are indices not powers.

$$dB_t^1 dB_t^2 = \rho(s, y, t)dt. \tag{2c}$$

We use the shorthand 'one-dimensional case' and 'two-dimensional case' to distinguish definitions 1 and 2. We consider only deterministic interest rates, $r(t)$.[3] Unless otherwise noted, we assume that the underlying asset pays no dividends over the life of the option. We consider only European-style contingent claims.

Let $c(s,t)$ denote the time t value of a contingent claim in the one-dimensional case. Although we will often take $c(\cdot)$ to be the value of a call option, the notation c is intentionally mnemonic for 'contingent claim'. Subindices denote partial derivatives: $c_1(s,t)$ is the first partial of the claim's price with respect to the stock price; $c_{11}(s,t)$ is the second partial with respect to the stock price; $c_2(s,t)$ is the first partial with respect to time, etc. We consider only limited liability underlying assets. Hence for a call option, $c(0,t) = 0$. The contingent claim's time T payoff is given by $c(s,T) = g(s)$. We assume explicitly that the value of the claim can be expressed, using the Feynman-Kac Theorem,

[3] Some, but not all, of our results generalize to the case where interest rates are stochastic.

as the discounted expectation of its payoff under a risk-neutral probability measure.[4] In the one-dimensional case, $c(s,t)$ is then given by

$$c(s,t) = E\left\{e^{-\int_t^T r(\tau)d\tau} g\left(\xi_T^{s,t}\right)\right\}, \tag{3}$$

where $\xi^{s,t}$ solves the SDE

$$d\xi_\tau = r(\tau)\xi_\tau d\tau + \sigma(\xi_\tau, \tau)\xi_\tau dB_\tau \tag{4}$$

with initial condition s at time t.

In the two-dimensional case, we assume that the price of volatility risk takes the form $\lambda(s,y,t)$, and that the claim's time T payoff is given by $c(s,y,T) = g(s)$. Given the restriction on the function λ, the time t value of a contingent claim will have the form $c(s,y,t)$.

A. The Intuitive Link Between A Diffusion Process and Properties of Option Prices.

The intuition underlying the properties of contingent claim prices developed in this section is most clear in the one-dimensional case. For the one-dimensional case Figure 1 depicts potential paths for the ξ_t process *for a given realization of the B_t process.* The crucial observation is that, *for a given realization of the B_t process*, if the ξ_t process starts from $\xi_t = s'$, the subsequent level of the process at any time is at least as great as the level the process would have attained at that time had it started from any value $s'' < s'$. To understand why this is so, consider the following thought experiment. Suppose the path with the lower starting value were to attain a level above the path with the higher starting value. Since sample paths of ξ_t are continuous[5], the two paths must have met

[4] Rather than assuming some particular set of restrictions on the diffusion parameters in (1) and (2) known to be sufficient for the applicability of the Feynman-Kac Theorem, we prefer to implicitly consider the full set of diffusion parameters consistent with the Theorem.

[5] With probability one.

at some earlier time, t'. But, given the process in (4), the two sample paths will have become identical from time t' on. Therefore the path starting from s'' cannot exceed the path starting from s'. An immediate implication of this 'no-crossing' property is that the distribution of ξ_T conditional on $\xi_t = s'$ first-order stochastically dominates the distribution of ξ_T conditional on $\xi_t = s''$. Thus if a contingent claim's payoff is non-decreasing (non-increasing) in the maturity date value of the underlying asset, the claim's current price must be non-decreasing (non-increasing) in the current value of the underlying.

Our demonstration of the no-crossing property required that the risk-neutralized process for s_t be both continuous and Markovian. A diffusion is, by definition, both a continuous and a Markovian process: see Karlin and Taylor (1981, p. 157). If one considers a stochastic process that is either discontinuous or non-Markovian, then the process need not possess the no-crossing property. Section VI will show that the value of a call can then be decreasing in the value of the underlying.

Conditions sufficient to guarantee that, in the two-dimensional case, the value of a call is an increasing function of the value of the underlying will be shown to be conditions under which, despite its dependence on the contemporaneous y, an analogous no-crossing property continues to hold for the risk-neutralized stock price process.

B. Sufficient Conditions for a Call's Replicating Portfolio to be Long in the Underlying, But by Less than One Share.

Theorem 1 establishes bounds on the first partial of the value of any contingent claim with respect to the value of the underlying asset that are applicable in the one-dimensional case, and in certain restricted versions of the two-dimensional case. The Theorem as stated is applicable whenever the function $g(\cdot)$ is everywhere differentiable. At any points of non-differentiability, one should first replace the original g function by a new smooth function of s arbitrarily close to the original function.

Theorem 1. *If changes in s_t are described either by* (i) *a one-dimensional diffusion, or* (ii) *a two-dimensional diffusion with the property that the drift and diffusion parameters of the risk-neutralized process for y do not depend on s, then for all s, y and t,*

$$\inf_q g_1(q) \le c_1(s,y,t) \le \sup_q g_1(q).$$

Proof: Given condition (i), the value of the call can be expressed as in (3) and (4). Consider the path followed by ξ_τ *for a given realization of the process* B_τ. The level of ξ_τ attained at time T with initial condition s' at time t, $\xi_T^{s',t}$, is at least as great as the level attained when the process starts with initial condition $s'' < s'$ at time t, $\xi_T^{s'',t}$. Let $\mathcal{X}$ denote the difference in the levels attained. $\mathcal{X}$ is a non-negative random variable with the property that

$$E\{\mathcal{X}\} = E\{\xi_T^{s',t}\} - E\{\xi_T^{s'',t}\} = e^{\int_t^T r(\tau)d\tau}(s' - s'').$$

$$\begin{aligned}
c(s',t) &= E\{e^{-\int_t^T r(\tau)d\tau} g(\xi_T^{s'',t} + \mathcal{X})\} \\
&\ge E\{e^{-\int_t^T r(\tau)d\tau} g(\xi_T^{s'',t})\} + E\{e^{-\int_t^T r(\tau)d\tau} \inf_q g_1(q)\mathcal{X}\} \\
&= c(s'',t) + \inf_q g_1(q)(s' - s''); \\
\text{i.e.,}\quad \frac{c(s',t) - c(s'',t)}{s' - s''} &\ge \inf_q g_1(q).
\end{aligned}$$

Similarly, one can demonstrate that

$$\frac{c(s',t) - c(s'',t)}{s' - s''} \le \sup_q g_1(q).$$

Now consider condition (ii). For the two-dimensional case, we again assume explicitly that the value of the call can be expressed, using the Feynmac-Kac Theorem, as its discounted expected payoff under a risk-neutral probability measure. (See Appendix A.)

$$c(s,y,t) = E\{e^{-\int_t^T r(\tau)d\tau} g(\xi_T^{1\,s,y,t})\}, \tag{5}$$

where the superscripts '1' and '2' on ξ denote indices not powers, and $\xi^{1^{s,y,t}}$ and $\xi^{2^{s,y,t}}$ solve the system of SDE's

$$d\xi_\tau^1 = r(\tau)\xi_\tau^1 d\tau + \sigma(\xi_\tau^1, \xi_\tau^2, \tau)\xi_\tau^1 dB_\tau^1, \tag{6a}$$

$$d\xi_\tau^2 = \big(\beta(\xi_\tau^1, \xi_\tau^2, \tau) - \lambda(\xi_\tau^1, \xi_\tau^2, \tau)\theta(\xi_\tau^1, \xi_\tau^2, \tau)\big)d\tau + \theta(\xi_\tau^1, \xi_\tau^2, \tau)dB_\tau^2, \tag{6b}$$

with initial conditions s and y respectively at time t. Further, the correlation between innovations in the two Brownian motions, B^1 and B^2, is such that $dB_\tau^1 dB_\tau^2 = \rho(\xi_\tau^1, \xi_\tau^2, \tau)d\tau$. The SDE's in (6) describe the 'risk-neutralized' processes for s and y.

Condition (ii) implies that there exists functions G^1 and G^2, where the superscripts denote indices not powers, such that

$$G^1(\xi^2, \tau) = \beta(\xi^1, \xi^2, \tau) - \lambda(\xi^1, \xi^2, \tau)\theta(\xi^1, \xi^2, \tau),$$

$$\text{and } G^2(\xi^2, \tau) = [\theta(\xi^1, \xi^2, \tau)]^2.$$

In this case, expression (6) simplifies to

$$d\xi_\tau^1 = r(\tau)\xi_\tau^1 d\tau + \sigma(\xi_\tau^1, \xi_\tau^2, \tau)\xi_\tau^1 dB_\tau^1, \tag{6a'}$$

$$d\xi_\tau^2 = G^1(\xi_\tau^2, \tau)d\tau + \sqrt{G^2(\xi_\tau^2, \tau)}dB_\tau^2. \tag{6b'}$$

For any given realization of the B_τ^2 process, condition (ii) guarantees that the path followed by ξ_τ^2 is independent of the initial condition for ξ_τ^1. Now consider two paths for ξ_τ^1 differing only in their initial conditions. Given the realization of the B_τ^2 process, we construct each path for ξ_τ^1 from innovations dB_τ^1 constructed as $dB_\tau^1 = \rho(\xi_\tau^1, \xi_\tau^2, \tau)dB_\tau^2 + (1 - [\rho(\xi_\tau^1, \xi_\tau^2, \tau)]^2)^{1/2} dB_\tau^3$, where B_τ^3 is a standard Brownian motion independent of B_τ^2. Now suppose these two paths for ξ_τ^1 did come together at, say, time t'. Could they cross? Consider the SDE in $(6a')$. By construction, both paths for ξ_τ^1 always share a common realization of the ξ_τ^2 process and hence, must become identical subsequent to t'. Suppose $s'' < s'$. Starting from $\{\xi_t^1 = s', \xi_t^2 = y\}$, the subsequent level of ξ_τ^1 attained at time T is at least as great as the level attained when the processes start from $\{s'', y\}$. Again let $\mathcal{X}$ denote the difference in the levels of ξ^1_T attained.

$$E\{\mathcal{X}\} = E\{\xi^{1\,s',y,t}_T\} - E\{\xi^{1\,s'',y,t}_T\} = e^{\int_t^T r(\tau)d\tau}(s' - s'').$$

$$\begin{aligned} c(s', y, t) &= E\{e^{-\int_t^T r(\tau)d\tau} g(\xi^{1\,s'',y,t}_T + \mathcal{X})\} \\ &\geq E\{e^{-\int_t^T r(\tau)d\tau} g(\xi^{1\,s'',y,t}_T)\} + E\{e^{-\int_t^T r(\tau)d\tau} \inf_q g_1(q)\mathcal{X}\}, \\ &= c(s'', y, t) + \inf_q g_1(q)(s' - s''). \end{aligned}$$

The remaining steps parallel those of condition (i). **QED**

Theorem 1 is an extension of Proposition 5(a) of Grundy (1991) to the case of general contingent claims and two-dimensional diffusions. Note that the condition (ii) restriction on the drift of the risk-neutralized process for y will be satisfied whenever y is the price of a traded asset, that drift being equal to $r(t)y$.

Theorem 1 extends easily to the case where for all $\tau \in [t, T]$, the underlying pays a continuous proportional dividend at the rate $\delta^{\mathcal{U}}(\tau)$ and the contingent claim pays a continuous proportional dividend at the rate $\delta^{\mathcal{O}}(\tau)$. In this case, under the conditions of Theorem 1 we have that for all s, y and t, the contingent claim's delta satisfies

$$e^{\int_t^T \left(\delta^{\mathcal{O}}(\tau) - \delta^{\mathcal{U}}(\tau)\right)d\tau} \inf_q g_1(q) \leq c_1(s, y, t) \leq e^{\int_t^T \left(\delta^{\mathcal{O}}(\tau) - \delta^{\mathcal{U}}(\tau)\right)d\tau} \sup_q g_1(q).$$

Proposition 1 (Bounds on the first partial of a call).
Under the conditions of Theorem 1, the partial of a call option with respect to the value of the underlying satisfies $0 \leq c_1(s, y, t) \leq 1$.

Proposition 1 is an immediate implication of Theorem 1, and itself immediately implies that a call option's replicating portfolio always involves a long position in the underlying stock, but never by more than one share.

For a call option, satisfaction of the inequality, $0 \leq c_1(s, y, t) \leq 1$ for all s, y and t, is a precondition for $c(s, y, t)$ to be convex in s for all s, y, and t.[6] Suppose that for some s' and y', $c_1(s', y', t) < 0$. Since $c(0, y', t) = 0$ and, for all s, y, and t, $c(s, y, t) \geq 0$, $c_1(s', y', t) < 0$ implies that there exists an $s'' \in (0, s')$ such that $c(0, y', t) < c(s'', y', t) > c(s', y', t)$, and hence $c(s, y', t)$ will be strictly concave in s over some region. See Figure 2a. Now suppose that for some s' and some y', $c_1(s', y', t) > 1$. If for all s, y and t, $c_{11}(s, y, t) \geq 0$, then, as illustrated by the dotted line in Figure 2b, for high enough s, $c(s, y', t)$ will violate the upper bound on its value: namely, that for all s, y and t, $c(s, y, t) \leq s$. Thus, as illustrated by the solid line, in order to preclude the violation of this upper bound there must then exist an $s'' > s'$ such that $c_1(s'', y', t) \leq 1$, and the call price will again be strictly concave over some region.

C. Sufficient Conditions for Call Price Convexity.

The work of Merton (1973), Cox and Ross (1976), and Jagannathan (1984) establishes that, when the risk neutral process for the underlying is a proportional stochastic process, then any contingent claim whose payoff is convex (concave) in the maturity date value of the underlying will have a value that is convex (concave) in the current value of the underlying. In the one-dimensional case, a risk neutral proportional stochastic process implies a deterministic volatility function; i.e., a Black-Scholes world.[7] While a proportional risk neutral process is sufficient for the convexity of call prices, Theorem 2 shows that it is not necessary in either the one- or two-dimensional cases. In fact, Theorem 2 establishes that call price convexity is always true in the one-dimensional case.

[6] Footnote 16 of Chapter 8 of Jarrow and Rudd (1983) contains the results that for a call option, if c is everywhere convex in s, then $c_1(s, t) \leq 1$ and $c_1(s, t)s/c(s, t) \geq 1$.

[7] The statement that the underlying follows a risk neutral proportional process is equivalent to the statement that the value of a call option written on the underlying is homogeneous of degree one in s and K. To see intuitively why homogeneity implies convexity, imagine a doubling of s and K. The option would then double in value. But if only s were doubled and the exercise price were held constant, c would more than double.

Theorem 2. *Suppose changes in s_t are described by either* (i) *a one-dimensional diffusion, or* (ii) *a two-dimensional diffusion with the twin properties that* (a) *the drift and diffusion parameters of the risk-neutralized process for y do not depend on s and* (b) *the covariance between instantaneous changes in s and y is linear in s. Then, if a contingent claim's payoff is everywhere convex (concave) in the maturity date value of the underlying, the claim's current price is everywhere convex (concave) in the current value of the underlying.*

Proof: See Appendix A.

The method of proof contained in Appendix A is analogous to the above method of proof of Theorem 1. The result is obtained by combining the Feynman-Kac Theorem and a suitable no-crossing property. Appendix B contains an alternate geometric proof of call price convexity that is based on the stochastic maximum principle. This alternate proof is not superfluous. It proceeds by developing tools useful in the analysis of the properties of any region in which c is concave in s.[8]

Condition (ii) of Theorem 2 is more restrictive than condition (ii) of Theorem 1. To guarantee convexity we require the additional restriction, condition (ii)(b), that the instantaneous covariance between changes in s and y be linear in s. The instantaneous covariance between s and y is given by $\sigma(s,y,t)s\theta(s,y,t)\rho(s,y,t)$. That the function $\theta(\cdot)$ not depend on s is required by condition (ii)(a). Hence linearity of the covariance in s requires that the product $\sigma(s,y,t)\rho(s,y,t)$ not depend on s. This could occur in three ways. First, and pathologically, both $\sigma(\cdot)$ and $\rho(\cdot)$ may depend on s, but in such inverse ways that their product does not.[9] Second, $\rho(\cdot)$ may be zero for all s, y and t. Coupled with condition (ii)(a), this implies that instantaneous changes in the risk neutral process for y are independent of the contemporaneous s. Finally, both $\sigma(\cdot)$ and $\rho(\cdot)$ may not be dependent

[8] As will be seen in section VI, long-dated call options written on a corporation's stock can quite naturally possess such regions of concavity.

[9] We use 'pathological' in its strict mathematical sense, namely, that many readers are likely to consider this first possibility to be uninteresting.

on s. In this third case, there exists functions κ^1, κ^2, κ^3 and κ^4, where superscripts denote indices not powers, such that the risk neutral processes for s and y take the form:

$$d\xi_t^1 = r(t)\xi_t^1 dt + \kappa^1(\xi_t^2, t)\xi_t^1 dB_t^1,$$

$$d\xi_t^2 = \kappa^2(\xi_t^2, t)dt + \kappa^3(\xi_t^2, t)dB_t^2,$$

with $dB_t^1 dB_t^2 = \kappa^4(\xi_t^2, t)dt$; i.e., the risk-neutralized process for s is a proportional stochastic process.[10]

Theorem 2 does not require the absence of dividends. The proof is unchanged when, for all $\tau \in [t, T]$, the underlying pays a continuous proportional dividend at the rate $\delta^{\mathcal{U}}(\tau)$ and the contingent claim pays a continuous proportional dividend at the rate $\delta^{\mathcal{O}}(\tau)$.[11]

Proposition 2 (Call price convexity).

Under the conditions of Theorem 2, a call's price is everywhere convex in the value of the underlying: for all s and t, $c_{11}(s, y, t) \geq 0$. Further, for all s and t such that $\max[0, s - Ke^{-\int_t^T r(\tau)d\tau}] < c(s,t) < s$, we have strict convexity: $c_{11}(s, y, t) > 0$.

Weak convexity is a direct implication of Theorem 2. A proof of the strict convexity claim is contained in Appendix C. In the following sections III, IV, and V, we explore further properties of option prices for the one-dimensional diffusion case.

III. Properties of a Call Option's Price and its Replicating Portfolio

All the Propositions established in this section are predicated on the twin assumptions that (a) the underlying asset follows a one-dimensional diffusion, and (b) the call's

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10 Hull and White (1987) examine a model of stochastic volatility where the risk-neutralized process for s is a proportional stochastic process. Their closed-form expression for the call price is, therefore, everywhere convex in the value of the underlying.

11 Theorem 2 need not be applicable when dividends are non-proportional. For example, consider a zero exercise price European call written on a stock paying a continuous version of the non-proportional dividend discussed in footnote 16 of Chapter 4 of Cox and Rubinstein (1985).

price can be expressed as its discounted expected payoff under a risk-neutral probability measure. In the one-dimensional case, the value of a call, $c(s,t)$, is given by the solution of

$$r(t)c_1(s,t)s - r(t)c(s,t) + c_2(s,t) + \frac{1}{2}[\sigma(s,t)s]^2 c_{11}(s,t) = 0, \tag{7}$$

subject to the terminal condition $c(s,T) = \max[0, s-K]$.

Figure 3 illustrates the feasible shapes of the relation between $c(s,t)$ and s given Propositions 1 and 2. Figures 3a, 3b and 3c are familiar from a Black-Scholes world with infinite, finite and zero time to maturity respectively. Figures 3d, 3e and 3f illustrate the relation when the underlying asset has zero probability of finishing in-the-money (out-of-the-money) for low (high) stock prices. Interestingly, Propositions 1 and 2 do not seem to rule out the type of relation illustrated in Figure 3g and 3h. We will examine the possibility that $\lim_{s\to\infty} c(s,t) - \max[0, s - Ke^{-\int_t^T r(\tau)d\tau}] > 0$ in section V.

A. Bounds on A Call's Elasticity

Proposition 3. *For all s and t such that $0 < c(s,t) < s$, the call's elasticity, $\Omega(s,t)$, satisfies $1 < \Omega(s,t) \leq 1 + Ke^{-\int_t^T r(\tau)d\tau}/c(s,t)$.*

Proof: We first consider the lower bound.

$$c(s,t) = c(0,t) + \int_0^s c_1(y,t)dy = \int_0^s c_1(y,t)dy.$$

Proposition 2 implies that for $0 < c(s,t) < s$, we have

$$\int_0^s c_1(y,t)dy < \int_0^s c_1(s,t)dy = c_1(s,t)s.$$

Therefore $\Omega(s,t) = \frac{c_1(s,t)s}{c(s,t)} > 1$. Turning to the upper bound, we have from Proposition 1 and the no-arbitrage bounds on call prices that

$$\frac{c_1(s,t)s}{c(s,t)} \leq \frac{s}{c(s,t)} \leq \frac{c(s,t) + Ke^{-\int_t^T r(\tau)d\tau}}{c(s,t)} = 1 + \frac{Ke^{-\int_t^T r(\tau)d\tau}}{c(s,t)}.$$

QED

B. A Bound on a Call's Delta

An immediate implication of Proposition 3 is:

Proposition 4. *For all s and t such that $0 < c(s,t) < s$, the position in stock in a call's replicating portfolio, $c_1(s,t)$, satisfies $c_1(s,t) > \frac{c(s,t)}{s}$.*

C. A Call is a Wasting Asset

Proposition 5. *For all s and t, $c_2(s,t) \leq 0$. For all s and t such that either* (i) $\max[0, s - Ke^{-\int_t^T r(\tau)d\tau}] < c(s,t) < s$ *and* $\sigma(s,t) > 0$, *or* (ii) $0 < c(s,t) < s$ *and* $r(t) > 0$, *a call is a strictly wasting asset:* $c_2(s,t) < 0$.

Proof: Rewriting expression (7) gives

$$c_2(s,t) = -\frac{1}{2}[\sigma(s,t)s]^2 c_{11}(s,t) - r(t)c(s,t)\big(\Omega(s,t) - 1\big).$$

Consider the first term, $-\frac{1}{2}[\sigma(s,t)s]^2 c_{11}(s,t)$. From Proposition 2, we have that for all s and t, this term is non-positive. Now consider the second term, $-r(t)c(s,t)\big(\Omega(s,t) - 1\big)$. From Proposition 3 we have $\Omega(s,t) > 1$ for all s and t such that $0 < c(s,t) < s$. For $c(s,t) = s$, $\Omega(s,t) = 1$. For $c(s,t) = 0$, we have the immediate result that $c_2(s,t) = 0$. Thus, since $r(t) \geq 0$ for all t, the second term is non-positive for all s and t. Since for all s and t, both the first and second terms are non-positive, the weak inequality claim of the Proposition is established. Turning to the strong inequality claim, Proposition 2 implies that for all s and t such that $\max[0, s - Ke^{-\int_t^T r(\tau)d\tau}] < c(s,t) < s$, the first term is strictly negative provided $\sigma(s,t) > 0$. When $0 < c(s,t) < s$ and $r(t) > 0$, the second term is strictly negative.

QED

D. The Dynamic Behavior of a Call's Replicating Portfolio

A call is replicated by continuously adjusting a position in the stock, given by $c_1(s_t, t)$, and a position in bonds, given by $c(s_t, t) - c_1(s_t, t)s_t$. How does the position in stock change as the stock price changes for a given time to maturity? Proposition 2 gives the answer immediately.

Proposition 6. *For all s and t, the position in stock in a call's replicating portfolio, $c_1(s,t)$, is non-decreasing in the stock price, and is strictly increasing whenever $c(s,t)$ satisfies* $\max[0, s - Ke^{-\int_t^T r(\tau)d\tau}] < c(s,t) < s$.

Proof: Proposition 6 is simply a restatement of Proposition 2. The change in the position in stock as s changes, being the change in $c_1(s,t)$ as s changes, is simply $c_{11}(s,t)$.

QED

Does the position in bonds in the replicating portfolio, $c(s_t,t) - c_1(s_t,t)s_t$, involve borrowing or lending? Again, Proposition 2 gives the answer immediately as:

Proposition 7. *For all s and t such that $0 < c(s,t) < s$, a call's replicating portfolio consists of a levered position in stock, with the amount of borrowing being less than $Ke^{-\int_t^T r(\tau)d\tau}$.*

Proof: For all s and t such that $0 < c(s,t) < s$, we have from Proposition 3 that the call's elasticity with respect to the stock, $\Omega(s,t)$, satisfies $\Omega(s,t) > 1$.

$$\Omega(s,t) = \frac{c_1(s,t)s}{c(s,t)} > 1 \;\Rightarrow\; c(s,t) - c_1(s,t)s < 0.$$

Hence replication requires a short position in bonds. From Proposition 1 we have:

$$c(s,t) - c_1(s,t)s \geq c(s,t) - s \geq -Ke^{-\int_t^T r(\tau)d\tau}.$$

QED

Proposition 7 is a special case of the following Theorem (the proof of which is available upon request).

Theorem 3. *If the underlying follows a one-dimensional diffusion, then for any contingent claim, the bond position in a replicating portfolio is bounded below (above) by the product of $e^{-\int_t^T r(\tau)d\tau}$ and the* inf *(*sup*) of the bond position at maturity.*

Theorem 3 is a natural counterpart to Theorem 1. The number of bonds in a replicating portfolio is bounded by the number held at maturity (Theorem 3) just as the number of units of the underlying in a replicating portfolio is bounded by the number held at maturity (Theorem 1). One can think of Theorem 3 as using the price of the underlying as the numeraire, reversing the roles of the 'risky' and 'riskless' assets, and reapplying Theorem 1.

E. The Risk Premium on a Call Option

A call is equivalent to a levered position in the stock, with $\Omega(s,t)$ as the leverage ratio: 'total assets' in the portfolio of $c_1(s,t)s$ relative to 'owner's equity' in the portfolio of $c(s,t)$. Let $\mu^s(\cdot)$ and $\mu^c(\cdot)$ denote the instantaneous expected rates of return on the underlying stock and the call, respectively. Consider the relative size of the risk premium on the call (the levered position) versus the risk premium on the underlying.[12]

Proposition 8. *For all s and t such that $0 < c(s,t) < s$, the absolute value of the risk premium on a call exceeds that on the underlying stock.*

Proof: The arbitrage-free price of the option is such that the instantaneous returns on the option and stock are perfectly correlated, which implies

$$\mu^c(\cdot) - r(t) = \Omega(s,t)\big(\mu^s(\cdot) - r(t)\big).$$

From Proposition 3 we have $\Omega(s,t) > 1$ for $0 < c(s,t) < s$, and hence

$$\big|\mu^c(\cdot) - r(t)\big| = \Omega(s,t)\big|\mu^s(\cdot) - r(t)\big| > \big|\mu^s(\cdot) - r(t)\big|.$$

QED

[12] Grundy (1991) shows that option prices contain information not only about the risk-neutralized distribution of the underlying, but also about its *true* distribution, provided the underlying follows a one-dimensional diffusion and the risk premium on the option can be bounded. Proposition 8 establishes that the particular bound examined in that paper is always satisfied for a one-dimensional diffusion.

F. A Call's Elasticity is Increasing in its Exercise Price

Consider two otherwise equivalent calls on the same stock with exercise prices of $K^{\mathcal{A}}$ and $K^{\mathcal{B}}$, with $K^{\mathcal{B}} > K^{\mathcal{A}}$. Let $c^{\mathcal{A}}$ and $c^{\mathcal{B}}$ denote their respective prices at time t, and assume that $c^{\mathcal{B}} > 0$. Let $\Omega^{\mathcal{A}}$ and $\Omega^{\mathcal{B}}$ denote their respective elasticities with respect to the underlying stock.

Proposition 9. *For all s and $t < T$ such that $0 < c(s,t) < s$, the elasticity of a call is increasing in its exercise price.*

Proof: For $\Delta \equiv K^{\mathcal{B}} - K^{\mathcal{A}}$, we have $\max[0, s_T - K^{\mathcal{B}}] = \max\left[0, \max[0, s_T - K^{\mathcal{A}}] - \Delta\right]$. Call $\mathcal{B}$ is equivalent to a call option with an exercise price of Δ written on call $\mathcal{A}$. We wish to consider the properties of call $\mathcal{A}$ as an underlying asset. From Propositions 1 and 2, we have the result that for all s and t such that $c(s,t) > 0$, the function $c(s,t)$ is invertible in s. Thus there exists a function $\mathcal{V}(c^{\mathcal{A}}, t)$ such that, applying Ito's Lemma and simplifying, the diffusion describing changes in the value of $c^{\mathcal{A}}$ can be expressed as

$$dc_t^{\mathcal{A}} = \alpha^{\mathcal{A}}(\cdot)dt + \mathcal{V}(c_t^{\mathcal{A}}, t)c_t^{\mathcal{A}} dz_t. \tag{8}$$

When call $\mathcal{A}$ is viewed as the asset underlying call $\mathcal{B}$, we see from (8) that changes in this underlying asset take the form of a one-dimensional diffusion: the volatility of the underlying, $\mathcal{V}(c_t^{\mathcal{A}}, t)$, depends only on its contemporaneous value, $c_t^{\mathcal{A}}$, and time. From Proposition 3 we then have that the elasticity of call $\mathcal{B}$ with respect to call $\mathcal{A}$ must exceed unity; i.e., $c_1^{\mathcal{B}}(c^{\mathcal{A}}(s,t),t)c^{\mathcal{A}}(s,t)/c^{\mathcal{B}}(c^{\mathcal{A}}(s,t),t) > 1$. Applying the chain rule of differentiation, the elasticity of call $\mathcal{B}$ with respect to the underlying stock can be expressed in terms of its elasticity with respect to call $\mathcal{A}$ as

$$\Omega^{\mathcal{B}} = \frac{c_1^{\mathcal{B}}(c^{\mathcal{A}}(s,t),t)c^{\mathcal{A}}(s,t)}{c^{\mathcal{B}}(c^{\mathcal{A}}(s,t),t)} \; \frac{c_1^{\mathcal{A}}(s,t)s}{c^{\mathcal{A}}(s,t)} = \frac{c_1^{\mathcal{B}}(c^{\mathcal{A}}(s,t),t)c^{\mathcal{A}}(s,t)}{c^{\mathcal{B}}(c^{\mathcal{A}}(s,t),t)} \; \Omega^{\mathcal{A}} > \Omega^{\mathcal{A}}.$$

QED

G. The Relation Between Delta and a Call's Sensitivity to Changes in the Exercise Price

Our notation $c(s,t)$ subsumes the dependence of a call's price on its exercise price K. Breeden and Litzenberger (1978) have shown that the value of a state claim, defined as a contingent claim paying \$1 if and only if $s_T \geq K$, is given by $-\frac{\partial c(s,t)}{\partial K}$.[13] Note that the payoff to such a state claim is non-decreasing in s_T.

Proposition 10. *A call's delta is non-increasing in K.*

Proof: Given that a state claim's payoff is non-decreasing in s_T, Theorem 1 immediately implies that $-\frac{\partial c_1(s,t)}{\partial K}$, being the first partial of the value of this contingent claim with respect to the value of the underlying asset, must be non-negative. Thus $\frac{\partial c_1(s,t)}{\partial K}$, being the first partial of the call's delta with respect to the exercise price, is non-positive.

QED

Proposition 11. *The percent decrease in a call's delta as its exercise price increases is not greater than the percent decrease in the call's value.*

Proof: Proposition 9 gives:

$$\frac{\partial \Omega(s,t)}{\partial K} = \frac{s}{[c(s,t)]^2}\left(\frac{\partial c_1(s,t)}{\partial K}c(s,t) - c_1(s,t)\frac{\partial c(s,t)}{\partial K}\right) \geq 0; \text{ i.e., } \frac{\left|\frac{\partial c_1(s,t)}{\partial K}\right|}{c_1(s,t)} \leq \frac{\left|\frac{\partial c(s,t)}{\partial K}\right|}{c(s,t)}. \quad (9)$$

QED

The following Proposition provides a bound on a call's delta in terms of the call's sensitivity to K, $\frac{\partial c(s,t)}{\partial K}$, that is applicable whenever the underlying asset's *diffusion parameter* is non-decreasing in s; e.g., in a Black-Scholes world, or in a CEV diffusion world where the underlying asset's diffusion parameter takes the form $\hat{\sigma} S^\theta$ with $\theta > 0$. The proposition will be of most practical importance when one can use observed call prices to determine tight bounds on $\frac{\partial c(s,t)}{\partial K}$.

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[13] In the one-dimensional case, the partial $\frac{\partial c(s,t)}{\partial K}$ is equal to the position in bonds in a replicating portfolio if and only if the volatility of the underlying asset is deterministic.

Proposition 12. *If the underlying asset's diffusion parameter is non-decreasing in s, then a call's delta exceeds the negative of* $e^{\int_t^T r(\tau)d\tau}$ *times* $\frac{\partial c(s,t)}{\partial K}$.

Proof: Let $g(s) = \max[0, s - K]$ and $Z(s,t) = [\sigma(s,t)s]^2$. Z is the square of the underlying asset's diffusion parameter. The proof of Theorem 2 in Appendix A has established that the call's delta can be expressed as

$$c_1(s,t) = E\{g_1(\xi_T^{1\,s,t})\},$$

where $\xi^{1^{s,t}}$ solves the SDE

$$d\xi_\tau^1 = \left(r(\tau)\xi_\tau^1 + \frac{1}{2}Z_1(\xi_\tau^1, \tau)\right)d\tau + \sqrt{Z(\xi_\tau^1, \tau)}dB_\tau^1,$$

with initial condition s at time t. The superscript '1' on ξ and B is an index not a power. If the underlying's diffusion parameter is non-decreasing in s, it follows that

$$c_1(s,t) \geq E\{g_1(\xi_T^{2\,s,t})\},$$

where $\xi^{2^{s,t}}$ solves the SDE

$$d\xi_\tau^2 = r(\tau)\xi_\tau^2 d\tau + \sqrt{Z(\xi_\tau^2, \tau)}dB_\tau^2,$$

with initial condition s at time t. The superscript '2' on ξ and B is an index not a power. The inequality follows from two observations. First, under the conditions of the Proposition, the drift of the ξ_τ^2 process is always at least as great as the drift of the ξ_τ^1 process. Hence, the distribution of $\xi_T^{1\,s,t}$ first-order stochastically dominates the distribution of $\xi_T^{2\,s,t}$. (See Proposition 2.18 of Chapter 5 of Karatzas and Shreve (1991).) Second, the function $g_1(s)$ is non-decreasing in s:

$$g_1(s) = \begin{cases} 1, & \text{if } s > K; \\ 0, & \text{otherwise} \end{cases}$$

Given the form of the g_1 function, the expectation $E\{g_1(\xi_T^{2\,s,t})\}$ can be written as

$$E\{g_1(\xi_T^{2\,s,t})\} = \Pr(\xi_T^{2\,s,t} > K).$$

Breeden and Litzenberger (1978) have shown that

$$-\frac{\partial c(s,t)}{\partial K} = e^{-\int_t^T r(\tau)d\tau} \Pr\big(\xi_T^{2\,s,t} > K\big).$$

Hence

$$c_1(s,t) \geq -\,e^{\int_t^T r(\tau)d\tau}\,\frac{\partial c(s,t)}{\partial K}.$$

QED

H. The Relation Between Elasticity, Time and the Value of the Underlying

In the deterministic volatility world of Black-Scholes, it is well known that for all s and t, $\Omega_1(s,t) < 0$ and $\Omega_2(s,t) > 0$. These properties do not necessarily generalize to a one-dimensional diffusion with non-deterministic volatility. Still, they will be true for sufficiently large s and sufficiently large $(T-t)$. Consider first the relation between elasticity and the share price. From Proposition 3 we have that for all s and t, $\Omega(s,t) > 1$. Since

$$\lim_{s\to\infty} \Omega(s,t) = 1,$$

Ω must be decreasing in s for sufficiently large s. Now consider the relation between elasticity and the passage of time. Assuming that the time t value of a pure discount bond maturing at time T goes to zero as $t \to -\infty$,

$$\lim_{t\to-\infty} \Omega(s,t) = 1,$$

and Ω must be increasing with the passage of time for t sufficiently small relative to T.

To see that these properties need not be true in general, consider first $\Omega_1(s,t)$.

$$\Omega_1(s,t) = \frac{c_{11}(s,t)s}{c(s,t)} + \frac{c_1(s,t)}{c(s,t)}\Big(1 - \Omega(s,t)\Big).$$

The second term on the right-hand-side is non-positive and, from Propositions 2 and 3, is bounded below by $-Ke^{-\int_t^T r(\tau)d\tau}/[c(s,t)]^2$. The first term is non-negative and unbounded above. Whenever the relation is sufficiently locally convex, the positive first term will dominate and Ω_1 will be positive. Now consider $\Omega_2(s,t)$:

$$\Omega_2(s,t) = \frac{s}{[c(s,t)]^2}\Big(c_{12}(s,t)c(s,t) - c_1(s,t)c_2(s,t)\Big).$$

To see that $\Omega_2(s,t)$ can be negative, suppose that for some s and t, $c(s,t)$ is locally quadratic in s, and, for simplicity, assume that for all t, $r(t) = 0$. Differentiating the p.d.e. in (7) with respect to s, and assuming that $r(t) = 0$ for all t, gives

$$c_{12}(s,t) = -\sigma(s,t)s\big(\sigma_1(s,t)s + \sigma(s,t)\big)c_{11}(s,t) - \frac{1}{2}[\sigma(s,t)s]^2 c_{111}(s,t),$$

which for $c_{111}(s,t) = 0$ simplies to

$$c_{12}(s,t) = -\sigma(s,t)s\big(\sigma_1(s,t)s + \sigma(s,t)\big)c_{11}(s,t).$$

Substituting for $c_2(s,t)$ and $c_{12}(s,t)$ in $\Omega_2(s,t)$ gives

$$\begin{aligned}\Omega_2(s,t) &= \frac{s}{[c(s,t)]^2}\Big(c_1(s,t)\frac{1}{2}[\sigma(s,t)s]^2 c_{11}(s,t) - \sigma(s,t)s\big(\sigma_1(s,t)s + \sigma(s,t)\big)c_{11}(s,t)c(s,t)\Big) \\ &= \frac{c_{11}(s,t)\sigma(s,t)[s]^2}{[c(s,t)]^2}\Big(c_1(s,t)\frac{1}{2}\sigma(s,t)s - \big(\sigma_1(s,t)s + \sigma(s,t)\big)c(s,t)\Big).\end{aligned}$$

For $\sigma_1(s,t)$ sufficiently large and positive, $\Omega_2(s,t)$ will be negative.

IV. The Comparative Statics of Changes in Interest Rates, Dividends, and Volatility

As in section III, the results of this section are predicated on the twin assumptions that (a) the underlying asset follows a one-dimensional diffusion, and (b) the call's price can be expressed as its discounted expected payoff under a risk-neutral probability measure. The demonstration of some of our results will be made more transparent by measuring the prices of stock and call options thereon relative to the price of a pure discount bond maturing at time T; i.e., using the bond as numeraire. Using upper (lower) case notation to denote relative (absolute) price levels, the normalized prices take the form:

$$S_t = e^{\int_t^T r(\tau)d\tau} s_t,$$

$$\text{and } C(S_t, t) = e^{\int_t^T r(\tau)d\tau} c(s_t, t) = e^{\int_t^T r(\tau)d\tau} c\Big(e^{\int_t^T -r(\tau)d\tau} S_t, t\Big).$$

The normalized call price has the following partial derivatives:

$$C_1(S,t) = c_1(s,t).$$

$$C_{11}(S,t) = c_{11}(s,t)\, e^{-\int_t^T r(\tau)d\tau}.$$

Note that convexity of $C(S,t)$ in S for all S and t implies the convexity of $c(s,t)$ in s for all s and t, and vice-versa. Using the normalized pricing system, the option's value satisfies the following p.d.e.,

$$C_2(S,t) + \frac{1}{2}[v(S,t)S]^2 C_{11}(S,t) = 0, \tag{10}$$

$$\text{where } v(S,t) = \sigma(e^{-\int_t^T r(\tau)d\tau} S, t) = \sigma(s,t).$$

The notation $v(\cdot)$ is used to emphasize that this is a different function from $\sigma(\cdot)$. The transformation from $\sigma(\cdot)$ to $v(\cdot)$ is non-trivial. Other than in a deterministic volatility (Black-Scholes) world, the transformation requires knowledge of $r(\tau)$ for *all* $\tau \in [t,T]$. The implications of this observation will be made clear in subsection $A.2$.

A. *The Comparative Statics of Interest Rate Changes*

We wish to compare the prices of options across two economies. In economy $\mathcal{A}$ the interest rate is $r^{\mathcal{A}}(t)$. The interest rate in an otherwise equivalent economy, economy $\mathcal{B}$, is $r^{\mathcal{B}}(t)$. In each economy the underlying asset pays no dividends prior to the option's maturity. Theorem 4 establishes that if $\delta(\tau) = r^{\mathcal{B}}(\tau) - r^{\mathcal{A}}(\tau) \geq 0$ for all $\tau \in [t,T]$ and $\int_t^T \delta(\tau)d\tau > 0$, then calls in economy $\mathcal{B}$ are at least as valuable as otherwise equivalent calls in economy $\mathcal{A}$.

A.1. An upward shift in the term structure increases call prices

Theorem 4. *Consider two otherwise equivalent economies, $\mathcal{A}$ and $\mathcal{B}$, differing only in their instantaneous forward rates, $r^{\mathcal{A}}(\tau)$ and $r^{\mathcal{B}}(\tau)$ respectively. Suppose that for all $\tau \in [t,T]$, $r^{\mathcal{B}}(\tau) \geq r^{\mathcal{A}}(\tau)$ and $\int_t^T r^{\mathcal{B}}(\tau)d\tau > \int_t^T r^{\mathcal{A}}(\tau)d\tau$. For all s and t the value of a call in economy $\mathcal{B}$ maturing at time T, $c^{\mathcal{B}}(s,t)$, is at least as great as the value of an otherwise equivalent call in economy $\mathcal{A}$, $c^{\mathcal{A}}(s,t)$. For all s and t such that $0 < c^{\mathcal{B}}(s,t) < s$, then $c^{\mathcal{B}}(s,t) > c^{\mathcal{A}}(s,t)$.*

Proof: Consider the following transformed price systems:

$$S_t = s_t e^{\int_t^T r^{\mathcal{A}}(\tau)d\tau},$$

$$C^{\mathcal{A}}(S_t,t) = c^{\mathcal{A}}\left(S_t e^{-\int_t^T r^{\mathcal{A}}(\tau)d\tau}, t\right) e^{\int_t^T r^{\mathcal{A}}(\tau)d\tau},$$

$$C^{\mathcal{B}}(S_t,t) = c^{\mathcal{B}}\left(S_t e^{-\int_t^T r^{\mathcal{A}}(\tau)d\tau}, t\right) e^{\int_t^T r^{\mathcal{A}}(\tau)d\tau}.$$

One can think of the transformation as setting the interest rate to zero in economy $\mathcal{A}$ and to $\delta(\tau) = r^{\mathcal{B}}(\tau) - r^{\mathcal{A}}(\tau)$ in economy $\mathcal{B}$.

$C^{\mathcal{A}}(S,t)$ solves the p.d.e.

$$C_2^{\mathcal{A}}(S,t) + \frac{1}{2}[v(S,t)S]^2 C_{11}^{\mathcal{A}}(S,t) = 0 \tag{11}$$

subject to $C^{\mathcal{A}}(S,T) = \max[0, S-K]$, where $v(S,t) = \sigma(e^{-\int_t^T r^{\mathcal{A}}(\tau)d\tau} S, t) = \sigma(s,t)$. $C^{\mathcal{B}}(S,t)$ solves the p.d.e.

$$C_2^{\mathcal{B}}(S,t) + \frac{1}{2}[v(S,t)S]^2 C_{11}^{\mathcal{B}}(S,t) = -\delta(t)\big(C_1^{\mathcal{B}}(S,t)S - C^{\mathcal{B}}(S,t)\big) \tag{12}$$

subject to $C^{\mathcal{B}}(S,T) = \max[0, S-K]$.

Let $X(S_t,t)$ denote the difference across economies in the transformed values of the two calls:

$$X(S_t,t) = C^{\mathcal{B}}(S_t,t) - C^{\mathcal{A}}(S_t,t).$$

Note that $X(S_t,t) > 0$ implies $c^{\mathcal{B}}(s_t,t) > c^{\mathcal{A}}(s_t,t)$.

From (11) and (12) we have

$$X_2(S,t)+\frac{1}{2}[v(S,t)S]^2X_{11}(S,t)+\delta(t)\big(C_1^{\mathcal{B}}(S,t)S-C^{\mathcal{B}}(S,t)\big)=0. \qquad (13)$$

For a given value of S_T at maturity, the time T prices of the two calls coincide:

$$X(S_T,T)=C^{\mathcal{B}}(S_T,T)-C^{\mathcal{A}}(S_T,T)=\max[0,S_T-K]-\max[0,S_T-K]=0.$$

Thus $X(S,t)$ is given by the solution to the p.d.e. in (13) subject to the terminal condition $X(S,T)=0$.

From the Feynman-Kac Theorem we have

$$X(S,t)=E\Big\{\int_t^T\delta(\tau)\big(C_1^{\mathcal{B}}(\xi_\tau^{S,t},\tau)\xi_\tau^{S,t}-C^{\mathcal{B}}(\xi_\tau^{S,t},\tau)\big)d\tau\Big\},$$

where $\xi_\tau^{S,t}$ solves the SDE

$$d\xi_\tau=v(\xi_\tau,\tau)\xi_\tau dB_\tau$$

with initial condition S at time t.

For $c^{\mathcal{B}}(s_t,t)=s_t$, the weak inequality is satisfied immediately since $c^{\mathcal{A}}(s_t,t)\le s_t$. For $c^{\mathcal{B}}(s_t,t)=0$, it follows that for all $\tau\in[t,T]$, $C^{\mathcal{B}}(\xi_\tau,\tau)=0$, and the integrand, $\delta(\tau)\big(C_1^{\mathcal{B}}(\xi_\tau,\tau)S_\tau-C^{\mathcal{B}}(\xi_\tau,\tau)\big)$, is zero for all $\tau\in[t,T]$. The weak inequality is then satisfied. Finally for $0<c^{\mathcal{B}}(s_t,t)<s_t$, it follows that for all $\tau\in[t,T]$, $0\le C^{\mathcal{B}}(\xi_\tau,\tau)<\xi_\tau$. For $C^{\mathcal{B}}(\xi_\tau,\tau)=0$ the integrand is zero. For $0<C^{\mathcal{B}}(\xi_\tau,\tau)<\xi_\tau$, we have from Proposition 3 that $C_1^{\mathcal{B}}(\xi_\tau,\tau)\xi_\tau-C^{\mathcal{B}}(\xi_\tau,\tau)>0$, and hence $\delta(\tau)\big(C_1^{\mathcal{B}}(\xi_\tau,\tau)\xi_\tau-C^{\mathcal{B}}(\xi_\tau,\tau)\big)$ is non-negative, and strictly positive for $\delta(\tau)>0$. Thus for $0<c^{\mathcal{B}}(s_t,t)<s_t$, since $\int_t^T\delta(\tau)d\tau>0$,

$$X(S,t)=E\Big\{\int_t^T\delta(\tau)\big(C_1^{\mathcal{B}}(\xi_\tau^{S,t},\tau)\xi_\tau^{S,t}-C^{\mathcal{B}}(\xi_\tau^{S,t},\tau)\big)d\tau\Big\}>0.$$

QED

Rather than apply the Feynman-Kac Theorem, the task of demonstrating that $X(S,t) > 0$ can be transformed into a familiar, and intuitively positive, valuation problem.

Suppose first that in the normalized (zero interest rate) economy $\mathcal{A}$, we wished to value a contingent claim, $Y(S_t,t)$, with the following contractual terms: The party long the contract will at all times $\tau \in [t,T]$ receive a continuous income stream from the short equal to $\delta(\tau)\big(C_1^{\mathcal{B}}(S_\tau,\tau)S_\tau - C^{\mathcal{B}}(S_\tau,\tau)\big)$, and nothing thereafter. Given the assumptions of Theorem 4 such an income stream is always non-negative. Further, when $0 < c^{\mathcal{B}}(s,t) < s$, the income stream will, with positive probability, be strictly positive over some time interval. Thus at time t this income stream contract has a strictly positive value to the long; i.e., $Y(S_t,t) > 0$. At its maturity, the income stream contract is valueless, and $Y(S_T,T) = 0$.

Now we wish to show that this income stream contingent claim has the same value as the across economy difference in value of the two calls. Consider a portfolio in the normalized economy $\mathcal{A}$ consisting of a long position in one income stream contract and a short position in $Y_1(S_t,t)$ shares. Changes in the normalized wealth of the holder of such a portfolio are non-stochastic, and given by

$$\begin{aligned}&dY(S_t,t) + \delta(t)\big(C_1^{\mathcal{B}}(S_t,t)S_t - C^{\mathcal{B}}(S_t,t)\big)dt - Y_1(S_t,t)dS_t\\ = &\Big(Y_2(S_t,t) + \frac{1}{2}[v(S_t,t)S_t]^2Y_{11}(S_t,t) + \delta(t)\big(C_1^{\mathcal{B}}(S_t,t)S_t - C^{\mathcal{B}}(S_t,t)\big)\Big)dt.\end{aligned}$$

Hence to preclude arbitrage, $Y(S,t)$ must solve

$$Y_2(S,t) + \frac{1}{2}[v(S,t)S]^2Y_{11}(S,t) + \delta(t)\big(C_1^{\mathcal{B}}(S,t)S - C^{\mathcal{B}}(S,t)\big) = 0$$

subject to $Y(S,T) = 0$. The p.d.e. and terminal condition for this income stream contingent claim are identical to the p.d.e. in (13) and the terminal condition whose solution determines $X(S,t)$. It follows immediately that $X(S,t) = Y(S,t) \geq 0$. Further, provided $0 < c^{\mathcal{B}}(s,t) < s$, $X(S,t) > 0$.

A.2. *A decrease in $PV(K)$ can decrease call prices*

It is important to recognize what we have **not** established in Theorem 4. It is **not** the case that $\int_t^T r^{\mathcal{B}}(\tau)d\tau > \int_t^T r^{\mathcal{A}}(\tau)d\tau$ and $0 < c^{\mathcal{B}}(s_t,t) < s_t$ are sufficient conditions to establish that $c^{\mathcal{B}}(s_t,t) > c^{\mathcal{A}}(s_t,t)$. In order to guarantee that $c^{\mathcal{B}}(s_t,t) > c^{\mathcal{A}}(s_t,t)$ it is **not** enough that the term structures differ across the two economies in such a way that the time t value of a riskless bond maturing at time T is smaller in economy $\mathcal{B}$. Theorem 4 requires in addition that $r^{\mathcal{B}}(\tau) \geq r^{\mathcal{A}}(\tau)$ for all $\tau \in [t,T]$. Note that the stronger comparative static result is, in fact, true in a world where the volatility parameter of the underlying asset's diffusion depends only on time and not on the value of the underlying; i.e., the stronger comparative static result is true in a Black-Scholes world. In fact, it is only in Black-Scholes world that the time t normalized call price that solves the p.d.e. in (10) does not depend on the set of $r(\tau)$ for $\tau \in [t,T]$.

A necessary condition for this stronger comparative static result to be true is that whenever $\int_t^T r^{\mathcal{A}}(\tau)d\tau = \int_t^T r^{\mathcal{B}}(\tau)d\tau$, then $c^{\mathcal{A}}(S_t,t) = c^{\mathcal{B}}(S_t,t)$.[14] Whenever this equality restriction is not satisfied, it will be possible to construct a counter-example to the stronger comparative static result. To see that this equality restriction can be violated, consider the following two otherwise equivalent economies. In economy $\mathcal{A}$ the interest rate is given by

$$r^{\mathcal{A}}(\tau) = \begin{cases} \mathcal{R}, & \text{for } \tau \in [t, T - \frac{1}{2}(T-t)]; \\ 0, & \text{for } \tau \in (T - \frac{1}{2}(T-t), T]. \end{cases}$$

In economy $\mathcal{B}$ the interest rate is given by

$$r^{\mathcal{B}}(\tau) = \begin{cases} 0, & \text{for } \tau \in [t, T - \frac{1}{2}(T-t)]; \\ \mathcal{R}, & \text{for } \tau \in (T - \frac{1}{2}(T-t), T]. \end{cases}$$

Interest rates in the two economies are depicted in Figure 4a. Note that

$$\int_t^T r^{\mathcal{A}}(\tau)d\tau = \frac{T-t}{2}\mathcal{R} = \int_t^T r^{\mathcal{B}}(\tau)d\tau.$$

[14] In the deterministic volatility world of Black-Scholes only the integral, $\int_t^T r(\tau)d\tau$, enters the Black-Scholes model.

Suppose that the underlying asset follows a diffusion of the form

$$ds_\tau = r(\tau)s_\tau d\tau + \sigma(s_\tau, \tau)s_\tau dB_\tau,$$

where the volatility function depends on both the underlying asset's contemporaneous value and time in the following way: For some $H > 0$ and some strictly non-zero $\eta(s, \tau)$,

$$\sigma(s, \tau) = \begin{cases} \eta(s, \tau), & \text{if } s > H \text{ and } \tau \in [t, T - \frac{T-t}{2}]; \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $s_t^{\mathcal{A}} = s_t^{\mathcal{B}} = s_t$ and that s_t, H and K are such that

$$s_t < H < e^{\mathcal{R}\frac{T-t}{2}} s_t < K.$$

In economy $\mathcal{A}$, $s_\tau^{\mathcal{A}}$ will grow deterministicly at the interest rate $\mathcal{R}$ until, at time $t + \ln(H/s_t)/\mathcal{R}$, the value of the underlying asset reaches the level H. After that time, $s_\tau^{\mathcal{A}}$ will follow the diffusion

$$ds_\tau^{\mathcal{A}} = \begin{cases} \mathcal{R}s_\tau^{\mathcal{A}}dt + \eta(s_\tau^{\mathcal{A}}, \tau)s_\tau^{\mathcal{A}}dB_\tau, & \text{for } \tau \in (t + \ln(H/s_t)/\mathcal{R}, T - \frac{T-t}{2}]; \\ 0, & \text{for } \tau \in (T - \frac{T-t}{2}, T]. \end{cases}$$

A possible sample path for $s_\tau^{\mathcal{A}}$ is depicted in Figure 4b. With positive probability $s_T^{\mathcal{A}} > K$, and hence $c^{\mathcal{A}}(s_t, t) > 0$.

In economy $\mathcal{B}$, $s_\tau^{\mathcal{B}}$ will remain equal to s_t until time $T - \frac{T-t}{2}$. After time $T - \frac{T-t}{2}$, $s_\tau^{\mathcal{B}}$ will grow deterministicly at the rate $\mathcal{R}$ until it reaches the level $s_t e^{\mathcal{R}\frac{T-t}{2}}$ at the option's maturity. The sample path for $s_\tau^{\mathcal{B}}$ is also depicted in Figure 4b. With certainty $s_T^{\mathcal{B}} < K$, and hence $c^{\mathcal{B}}(s_t, t) = 0$.

Thus, in this non-Black-Scholes world, shifts in the term structure that leave the current value of a bond maturing at the option's maturity unaffected can affect the option's value. In this example, simply 'reordering' the interest rates affects option prices.

B. The Comparative Statics of Dividend Rate Changes

Consider two assets $\mathcal{A}$ and $\mathcal{B}$. For all $\tau \in [t, T]$, asset $\mathcal{A}$ pays a continuous proportional dividend at the time-varying rate $\delta(\tau) \geq 0$, with $\int_t^T \delta(\tau)d\tau > 0$. Asset $\mathcal{B}$ will pay no dividends prior to time T. For all s and t, $\sigma^{\mathcal{A}}(s,t) = \sigma^{\mathcal{B}}(s,t)$. We wish to compare the prices of otherwise equivalent call options written on $\mathcal{A}$ and $\mathcal{B}$. Not surprisingly, given $s_t^{\mathcal{A}} = s_t^{\mathcal{B}}$, options on the dividend-paying asset are no more valuable than options on the non-dividend-paying asset. What is less obvious, is that one can place an upper bound on the relative values of the two options purely in terms of the fraction of the dividend-paying stock's price that is due to dividends to be paid beyond the option's maturity. When dividends are paid at the continuous rate $\delta(\tau)$, the present value at time t of the stock price at time T is $e^{-\int_t^T \delta(\tau)d\tau} s_t$; equivalently, the time t value of the distributions beyond time T is $e^{-\int_t^T \delta(\tau)d\tau} s_t$. The bound we will develop is familiar from a Black-Scholes world. In a Black-Scholes world, the value of a call on asset $\mathcal{A}$ when $s_t^{\mathcal{A}} = s'$ is equal to the value of an otherwise equivalent call written on the fraction $e^{-\int_t^T \delta(\tau)d\tau}$ of asset $\mathcal{B}$ when $s_t^{\mathcal{B}} = s'$. In turn, such a call has the same value as an otherwise equivalent call written on one complete unit of asset $\mathcal{B}$ when $s_t^{\mathcal{B}} = e^{-\int_t^T \delta(\tau)d\tau} s'$. From the strict convexity of call prices in a Black-Scholes world, we then have

$$c^{\mathcal{A}}(s,t) = c^{\mathcal{B}}(e^{-\int_t^T \delta(\tau)d\tau} \cdot s, t) < e^{-\int_t^T \delta(\tau)d\tau} c^{\mathcal{B}}(s,t).$$

$$\frac{c^{\mathcal{A}}(s,t)}{c^{\mathcal{B}}(s,t)} < e^{-\int_t^T \delta(\tau)d\tau}.$$

In a non-Black-Scholes world, when the volatility depends on the (contemporaneous) stock price, it is no longer necessarily the case that

$$c^{\mathcal{A}}(s,t) = c^{\mathcal{B}}(e^{-\int_t^T \delta(\tau)d\tau} \cdot s, t).$$

Still, as is shown in Theorem 5, it is always the case that

$$c^{\mathcal{A}}(s,t) \leq e^{-\int_t^T \delta(\tau)d\tau} c^{\mathcal{B}}(s,t),$$

with strict inequality for all s and t such that $0 < c^{\mathcal{B}}(s,t) < s$.

Theorem 5. *Suppose that for all s and t, $\sigma^{\mathcal{A}}(s,t) = \sigma^{\mathcal{B}}(s,t)$. Suppose further that assets $\mathcal{A}$ and $\mathcal{B}$ differ in the following way: For all $\tau \in [t,T]$, asset $\mathcal{A}$ will pay a continuous proportional dividend at the rate $\delta^{\mathcal{A}}(\tau) \geq 0$, with $\int_t^T \delta^{\mathcal{A}}(\tau)d\tau > 0$; asset $\mathcal{B}$ pays no dividends prior to time T. Then, for all s and t, $c^{\mathcal{A}}(s,t) \leq e^{-\int_t^T \delta^{\mathcal{A}}(\tau)d\tau} c^{\mathcal{B}}(s,t)$, with the inequality being strict whenever $0 < c^{\mathcal{B}}(s,t) < s$.*

Proof: Suppose that changes in the value of a third underlying asset, $\mathcal{U}$, are also described by a one-dimensional diffusion, with $\sigma^{\mathcal{U}}(s,t) = \sigma^{\mathcal{A}}(s,t) = \sigma^{\mathcal{B}}(s,t) = \sigma(s,t)$. Suppose further that asset $\mathcal{U}$ pays a continuous proportional dividend at the rate $\delta^{\mathcal{U}}(\tau)$ for all $\tau \in [t,T]$. The superscript '$\mathcal{U}$' is a mnemonic for underlying. Consider a call option written on asset $\mathcal{U}$, $c^{\mathcal{U}}(s,t)$, with the usual payoff at maturity of $\max[0, s_T^{\mathcal{U}} - K]$, **and** the additional contractual feature that at all times $\tau \in [t,T]$, the short pays the long a continuous proportional dividend, proportional to the value of the call, at the rate $\delta^{\mathcal{O}}(\tau)$. The superscript '$\mathcal{O}$' is a mnemonic for option. We introduce the following notation to describe this call: $c(s,t,\delta^{\mathcal{U}},\delta^{\mathcal{O}})$. Using this notation we have

$$c^{\mathcal{U}}(s,t) = c(s,t,\delta^{\mathcal{U}},\delta^{\mathcal{O}}),$$

$$c^{\mathcal{A}}(s,t) = c(s,t,\delta^{\mathcal{A}},0),$$

and

$$c^{\mathcal{B}}(s,t) = c(s,t,0,0).$$

To preclude arbitrage it must be that

$$c^{\mathcal{U}}(s,t) = e^{\int_t^T \delta^{\mathcal{O}}(\tau)d\tau} c(s,t,\delta^{\mathcal{U}},0). \tag{14}$$

The value $c^{\mathcal{U}}(s,t)$ is given by the solution of the following p.d.e.

$$c_2^{\mathcal{U}}(s,t) + \frac{1}{2}[\sigma(s,t)s]^2 c_{11}^{\mathcal{U}}(s,t) + r(t)c^{\mathcal{U}}(s,t)\big(\Omega^{\mathcal{U}}(s,t) - 1\big) + \delta^{\mathcal{O}}(t)c(s,t) - c_1(s,t)\delta^{\mathcal{U}}(t)s = 0, \tag{15}$$

subject to the terminal condition $c^{\mathcal{U}}(s,T) = \max[0, s - K]$. Now suppose further that for all τ, $\delta^{\mathcal{U}}(\tau) = \delta^{\mathcal{O}}(\tau) = \delta^{\mathcal{A}}(\tau)$. Substituting into the p.d.e. in (15) gives

$$\begin{aligned} &c_2^{\mathcal{U}}(s,t) + \frac{1}{2}[\sigma(s,t)s]^2 c_{11}^{\mathcal{U}}(s,t) + r(t)c^{\mathcal{U}}(s,t)\big(\Omega^{\mathcal{U}}(s,t) - 1\big) + \delta^{\mathcal{A}}(t)c^{\mathcal{U}}(s,t) - c_1^{\mathcal{U}}(s,t)\delta^{\mathcal{A}}(t)s \\ &= c_2^{\mathcal{U}}(s,t) + \frac{1}{2}[\sigma(s,t)s]^2 c_{11}^{\mathcal{U}}(s,t) + (r(t) - \delta^{\mathcal{A}}(t))c^{\mathcal{U}}(s,t)\big(\Omega^{\mathcal{U}}(s,t) - 1\big) = 0. \end{aligned} \tag{16}$$

Thus, the p.d.e. in (16), whose solution determines the value of $c^{\mathcal{U}}(s,t)$, is identical to the p.d.e. that would determine the value of $c^{\mathcal{B}}(s,t)$ **in an otherwise equivalent economy in which, at all times $\tau \in [t,T]$, the interest rate was lower than $r(\tau)$ by the amount $\delta^{\mathcal{A}}(\tau)$**. It then follows from Theorem 4 that, for all s and t,

$$c^{\mathcal{B}}(s,t) \geq c^{\mathcal{U}}(s,t) \tag{17a}$$

and, for all s and t such that $0 < c^{\mathcal{B}}(s,t) < s$,

$$c^{\mathcal{B}}(s,t) > c^{\mathcal{U}}(s,t). \tag{17b}$$

Further, substituting $\delta^{\mathcal{U}}(\tau) = \delta^{\mathcal{O}}(\tau) = \delta^{\mathcal{A}}(\tau)$ into expression (14) gives

$$c^{\mathcal{U}}(s,t) = e^{\int_t^T \delta^{\mathcal{A}}(\tau)d\tau} c(s,t,\delta^{\mathcal{A}},0) = e^{\int_t^T \delta^{\mathcal{A}}(\tau)d\tau} c^{\mathcal{A}}(s,t). \tag{18}$$

Combining (17) and (18) gives, for all s and t,

$$c^{\mathcal{A}}(s,t) \leq e^{-\int_t^T \delta^{\mathcal{A}}(\tau)d\tau} c^{\mathcal{B}}(s,t),$$

and, for all s and t such that $0 < c^{\mathcal{B}}(s,t) < s$,

$$c^{\mathcal{A}}(s,t) < e^{-\int_t^T \delta^{\mathcal{A}}(\tau)d\tau} c^{\mathcal{B}}(s,t).$$

QED

C. *The Comparative Statics of Volatility Changes*

Jagannathan (1984) clarifies the relation between the value of a call and the riskiness of the underlying stock. Consider two stocks $\mathcal{A}$ and $\mathcal{B}$, but this time $\sigma^{\mathcal{A}}(s,\tau) \neq \sigma^{\mathcal{B}}(s,\tau)$ while, for all τ, $\delta^{\mathcal{A}}(\tau) = \delta^{\mathcal{B}}(\tau) = 0$. Given $s_t^{\mathcal{A}} = s_t^{\mathcal{B}}$, a sufficient condition for calls on stock $\mathcal{B}$ maturing at time T to be at least as valuable as otherwise equivalent calls on stock $\mathcal{A}$, is that the *risk-neutral* probability distribution of $s_T^{\mathcal{B}}$ is more risky than the *risk-neutral* probability distribution of $s_T^{\mathcal{A}}$ in the Rothschild-Stiglitz sense.[15] In this section we show that if changes in the value of $\mathcal{A}$ and $\mathcal{B}$ are described by one-dimensional diffusions with $\sigma^{\mathcal{B}}(s,t) \geq \sigma^{\mathcal{A}}(s,t)$ for all s and t, then given $s_t^{\mathcal{A}} = s_t^{\mathcal{B}}$, the *risk-neutral* probability distribution of $s_T^{\mathcal{B}}$ is more risky than the *risk-neutral* probability distribution of $s_T^{\mathcal{A}}$ in the Rothschild-Stiglitz sense.

Theorem 6. *Suppose that for all s and t, $\sigma^{\mathcal{B}}(s,t) \geq \sigma^{\mathcal{A}}(s,t)$, and $\sigma^{\mathcal{B}}(s,t) > \sigma^{\mathcal{A}}(s,t)$ for s and t in some region. For all s and t, $c^{\mathcal{B}}(s,t) \geq c^{\mathcal{A}}(s,t)$, with the inequality being strict whenever $0 < c^{\mathcal{B}}(s,t) < s$.*

Proof: Let $S_t^{\mathcal{A}}$ and $S_t^{\mathcal{B}}$ denote the time t normalized prices of the two assets.

$$dS_\tau^{\mathcal{A}} = \alpha^{\mathcal{A}}(\cdot)d\tau + v^{\mathcal{A}}(S_\tau^{\mathcal{A}},\tau)S_\tau^{\mathcal{A}}dB_\tau.$$

$$dS_\tau^{\mathcal{B}} = \alpha^{\mathcal{B}}(\cdot)d\tau + v^{\mathcal{B}}(S_\tau^{\mathcal{B}},\tau)S_\tau^{\mathcal{B}}dB_\tau.$$

For all S and τ, $v^{\mathcal{B}}(S,\tau) \geq v^{\mathcal{A}}(S,\tau)$, and, for S and τ in some region, $v^{\mathcal{B}}(S,\tau) > v^{\mathcal{A}}(S,\tau)$.

Let $X(S,t)$ denote the difference in values of the normalized call prices:

$$X(S,t) = C^{\mathcal{B}}(S,t) - C^{\mathcal{A}}(S,t).$$

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[15] $\widetilde{X}$ is more risky than $\widetilde{Y}$ in the Rothschild-Stiglitz sense if

$$\widetilde{X} \stackrel{d}{=} \widetilde{Y} + \widetilde{\varepsilon}, \quad \text{and} \quad E\{\widetilde{\varepsilon}|X\} = 0 \text{ for all } X.$$

$C^{\mathcal{A}}(S,t)$ solves

$$C_2^{\mathcal{A}}(S,t) + \frac{1}{2}[v^{\mathcal{A}}(S,t)S]^2 C_{11}^{\mathcal{A}}(S,t) = 0, \tag{19}$$

subject to $C^{\mathcal{A}}(S,T) = \max[0, S-K]$. $C^{\mathcal{B}}(S,t)$ solves

$$C_2^{\mathcal{B}}(S,t) + \frac{1}{2}[v^{\mathcal{B}}(S,t)S]^2 C_{11}^{\mathcal{B}}(S,t) = 0, \tag{20}$$

subject to $C^{\mathcal{A}}(S,T) = \max[0, S-K]$. $X(S,t)$ solves

$$X_2(S,t) + \frac{1}{2}[v^{\mathcal{A}}(S,t)S]^2 X_{11}(S,t) + + \frac{1}{2}\left[[v^{\mathcal{B}}(S,t)]^2 - [v^{\mathcal{A}}(S,t)]^2\right] S^2 C_{11}^{\mathcal{B}}(S,t) = 0, \tag{21}$$

subject to $X(S,T) = 0$.

From Theorem 2, the term

$$\frac{1}{2}\left[[v^{\mathcal{B}}(S,t)]^2 - [v^{\mathcal{A}}(S,t)]^2\right] S^2 C_{11}^{\mathcal{B}}(S,t) \tag{22}$$

in the p.d.e. in (21) is non-negative, and strictly positive for S and t in some region. The remainder of the proof parallels that of Theorem 4.

QED

Reviewing the proof of Theorem 6 we see that the critical element is that in expression (22), $C_{11}(S,t) \geq 0$ for all S and t. Theorem 6 applies, not just to regular call options, but to any contingent claim whose value is always convex in the value of the underlying. It is this observation that allows us, conditional on $s_t^{\mathcal{B}} = s_t^{\mathcal{A}}$, to characterize the *risk-neutral* probability distribution of $s_T^{\mathcal{B}}$ as a mean preserving spread of the *risk-neutral* probability distribution of $s_T^{\mathcal{A}}$.

Theorem 7. *Suppose that for all s and t, $\sigma^{\mathcal{B}}(s,t) \geq \sigma^{\mathcal{A}}(s,t)$. For $s_t^{\mathcal{A}} = s_t^{\mathcal{B}}$, the risk-neutral distribution of $s_T^{\mathcal{B}}$ is more risky in the Rothschild-Stiglitz sense than the risk-neutral distribution of $s_T^{\mathcal{A}}$.*

Proof: Rothschild and Stiglitz (1970) have shown that the claim in Theorem 7 is equivalent to the claim that for every convex function $g(\cdot)$, $E\{g(\xi_T^{\mathcal{B}^{s,t}})\} \geq E\{g(\xi_T^{\mathcal{A}^{s,t}})\}$. Here $\xi_\tau^{\mathcal{A}^{s,t}}$ and $\xi_\tau^{\mathcal{B}^{s,t}}$ are the time τ risk-neutralized prices of assets $\mathcal{A}$ and $\mathcal{B}$, with changes therein given by

$$d\xi_\tau^{\mathcal{A}} = r(\tau)\xi_\tau^{\mathcal{A}} d\tau + \sigma^{\mathcal{A}}(\xi^{\mathcal{A}}, \tau)\xi^{\mathcal{A}} dB_\tau$$

and

$$d\xi_\tau^{\mathcal{B}} = r(\tau)\xi_\tau^{\mathcal{B}} d\tau + \sigma^{\mathcal{B}}(\xi^{\mathcal{B}}, \tau)\xi^{\mathcal{B}} dB_\tau,$$

respectively, and a common initial condition of s at time t.

The proof then involves three observations. First, from the Feynman-Kac Theorem, $E\{g(\xi_T^{s,t})\}$ is $e^{\int_t^T r(\tau)d\tau}$ times the value of a contingent claim on an underlying asset worth s at time t, provided that (i) changes in the underlying's value are described by a one-dimensional diffusion and (ii) the claim's payoff at its maturity date T is $g(s_T)$. Second, Theorem 6 has established that, for $s_t^{\mathcal{B}} = s_t^{\mathcal{A}}$, the value of any contingent claim written on asset $\mathcal{B}$ is always at least as great as the value of an otherwise equivalent claim written on $\mathcal{A}$, provided the value of the claim is convex in the value of the underlying asset; i.e., provided $c_{11}(s,t) \geq 0$ for all s and t. Finally, from Theorem 2, we have the result that, when $c(s,t)$ denotes the value of a contingent claim whose payoff at maturity, $g(\cdot)$ is a convex function of the value of the underlying at maturity, $c_{11}(s,t) \geq 0$ for all s and t.

QED

D. *Conditions under which Black-Scholes provides Bounds on Option Prices*

Interesting special cases of Theorem 6 occur when, for all s and t, either $\sigma^{\mathcal{B}}(s,t) = \bar{\sigma}(t) \geq \sigma^{\mathcal{A}}(s,t)$, or $\sigma^{\mathcal{B}}(s,t) \geq \sigma^{\mathcal{A}}(s,t) = \underline{\sigma}(t)$. Let $c^{bs(\sigma)}(s,t)$ denote the Black-Scholes value of a call on a stock with deterministic volatility $\sigma(s,t) = \sigma(t)$ for all s and t.

Theorem 8. *If for all s and t, $\sigma^{\mathcal{B}}(s,t) = \bar{\sigma}(t) \geq \sigma^{\mathcal{A}}(s,t)$, then $c^{\mathcal{A}}(s,t) \leq c^{bs(\bar{\sigma})}(s,t)$. If for all s and t, $\sigma^{\mathcal{B}}(s,t) \geq \sigma^{\mathcal{A}}(s,t) = \underline{\sigma}(t)$, then $c^{\mathcal{B}}(s,t) \geq c^{bs(\underline{\sigma})}(s,t)$.*

Proof: Theorem 8 is a special case of Theorem 6.[16]

Q.E.D.

Of major practical relevance to anyone charged with hedging an option position is that, despite a lack of knowledge of the functional form of the relation $\sigma(s,t)$, knowledge of bounds on that relation over the option's life, $\bar{\sigma}$ and $\underline{\sigma}$, provides bounds on the option's delta for any s and t. These bounds are an immediate implication of Theorem 8 and the convexity property of option prices.

Proposition 13. *If for all s and t, $\underline{\sigma}(t) \leq \sigma(s,t) \leq \bar{\sigma}(t)$, $c_1^{bs(\bar{\sigma})}(s'',t) \leq c_1(s,t) \leq c_1^{bs(\bar{\sigma})}(s',t)$, where s'' solves $c^{bs(\underline{\sigma})}(s,t) = c^{bs(\bar{\sigma})}(s'',t) + c_1^{bs(\bar{\sigma})}(s'',t)(s''-s)$ and s' solves $c^{bs(\underline{\sigma})}(s,t) = c^{bs(\bar{\sigma})}(s',t) - c_1^{bs(\bar{\sigma})}(s',t)(s'-s)$.*

Proof: As depicted in Figure 5, if the lower bound on delta were violated, then, even if the option took on its minimal possible value, convexity would imply that for some $s < s''$, the option's value would violate its upper bound. Similarly, if the upper bound on delta were violated then, even if the option took on its minimal possible value, convexity would imply that for some $s > s'$, the option's value would violate its upper bound.

QED

If the value of the option is known, the bounds on its delta can be strengthened to

Proposition 14. *If for all s and t, $\underline{\sigma}(t) \leq \sigma(s,t) \leq \bar{\sigma}(t)$, then for any s and t such that one knows $c(s,t)$, $c_1^{bs(\bar{\sigma})}(s'',t) \leq c_1(s,t) \leq c_1^{bs(\bar{\sigma})}(s',t)$, where s'' solves $c(s,t) = c^{bs(\bar{\sigma})}(s'',t) + c_1^{bs(\bar{\sigma})}(s'',t)(s''-s)$ and s' solves $c(s,t) = c^{bs(\bar{\sigma})}(s',t) - c_1^{bs(\bar{\sigma})}(s',t)(s'-s)$.*

Proof: The logic is the same as that of Proposition 13.

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[16] El Karoui, Jeanblanc-Picque and Visvanathan provide an alternate proof of the special case result in Theorem 8. We thank Darrell Duffie for bringing this paper to our attention.

V. The Asymptotic Behavior of Option Prices

We return to our observation that, as illustrated by Figures 3g and 3h, a one-dimensional diffusion does not seem to rule out the possibility that

$$\lim_{s\to\infty}\left(c(s,t) - \max[0, s - Ke^{-\int_t^T r(\tau)d\tau}]\right) > 0;$$

i.e., Theorems 1 and 2 do not seem to imply that the difference between the value of a call and the lower no-arbitrage bound thereon must go to zero as the underlying asset becomes infinitely valuable. It may be that this possibility is ruled out by one of our assumptions underlying Theorems 1 and 2, namely that the option price can be expressed as its discounted expected payoff under a risk neutral distribution. In this section we will briefly consider sufficient conditions to guarantee that the above limit is zero. These conditions will be recognized as conditions often invoked as part of a set of conditions sufficient for the Feynman-Kac Theorem to be applicable in our setting.

Given put-call parity, our exploration of the properties of call prices, $c(s,t)$, can be easily translated into a study of the properties of European put prices, $p(s,t)$:

$$p(s,t) = s + Ke^{-\int_t^T r(\tau)d\tau} - c(s,t).$$

If

$$\lim_{s\to\infty}\left(c(s,t) - \max[0, s - Ke^{-\int_t^T r(\tau)d\tau}]\right) > 0,$$

then

$$\lim_{s\to\infty} p(s,t) = \lim_{s\to\infty}\left(c(s,t) + Ke^{-\int_t^T r(\tau)d\tau} - s\right) > 0.$$

If a put is to retain a strictly positive value even as the underlying becomes infinitely valuable, it must be that

$$\lim_{s_t\to\infty} \Pr(s_T < K) > 0.$$

To get an intuitive sense of how this might occur, consider the following two limits in the familiar Black-Scholes world.[17] Let $p^{bs(\sigma)}(s,t)$ denote the value of a put option on a stock with constant volatility, $\sigma(s,t) = \sigma$, for all s and t. First, consider the effect of an increase in the value of the underlying asset on the value of a put thereon:

$$p_1^{bs(\sigma)}(s,t) < 0, \tag{23a}$$

and

$$\lim_{s\to\infty} p^{bs(\sigma)}(s,t) = 0. \tag{23b}$$

Now consider the effect on the value of a put of an increase in the underlying asset's volatility:

$$\frac{\partial p^{bs(\sigma)}(s,t)}{\partial\sigma} > 0, \tag{24a}$$

and

$$\lim_{\sigma\to\infty} p^{bs(\sigma)}(s,t) = Ke^{-\int_t^T r(\tau)d\tau} > 0. \tag{24b}$$

Now suppose, that rather than being deterministic, the stock's volatility is an increasing function of the level of the stock price. It may be, that in such a non-Black-Scholes world, put options become 'suspended' somewhere between the two limits in (23b) and (24b), held there by the two opposing forces in (23a) and (24a). Whether this can happen despite the satisfaction of the conditions underlying the applicability of the Feynman-Kac Theorem remains an open research question. In this section we examine conditions under which, although the volatility may increase with the level of the stock price, the volatility does *not* increase sufficiently for the force in (24a) to offset the force in (23a).

Suppose that for all t, $\sigma(s,t) \leq \bar{\sigma}(t)$ for all s. Theorem 8 then states that

$$c(s,t) \leq c^{bs(\bar{\sigma})}(s,t). \tag{25}$$

[17] Note that in a Black-Scholes world, $\lim_{\sigma\to\infty} \Pr(s_T < K) = 1$.

Thus

$$\begin{aligned}\lim_{s\to\infty} p(s,t) &= \lim_{s\to\infty}\left(c(s,t) - s + Ke^{-\int_t^T r(\tau)d\tau}\right)\\ &\le \lim_{s\to\infty}\left(c^{bs(\bar\sigma)}(s,t) - s + Ke^{-\int_t^T r(\tau)d\tau}\right)\\ &= 0.\end{aligned}$$

Since for all s and t, $p(s,t) \ge 0$, it follows that

$$\lim_{s\to\infty} p(s,t) = 0.$$

Thus an upper bound on volatility implies asymptotic behavior familiar from the Black-Scholes world.

It is also interesting to consider a second property of option prices well established in a Black-Scholes world: When the volatility function is deterministic, $c_1(0,t) = 0$. This property of a zero partial at $s = 0$ also remains true so long as the volatility function is bounded above. Since we are considering only limited liability underlying assets, we have that

$$c(0,t) = 0. \tag{26}$$

Together (25) and (26) imply that

$$c_1(0,t) \le c_1^{bs(\bar\sigma)}(0,t) = 0. \tag{27}$$

Proposition 1 and (27) imply that

$$c_1(0,t) = 0.$$

When might volatility be bounded above? Suppose the product $\sigma(s,t)s$ in the diffusion in (1) satisfies a *global* Lipschitz condition. This would imply that the product satisfied both a local Lipschitz and a growth condition.[18] Given appropriate restrictions

[18] See Duffie (1992), page 240.

on $\alpha(\cdot)$, the existence and uniqueness of a solution to the stochastic differential equation in (1) would then be guaranteed. In addition though, much structure would be imposed on the values of options written on the underlying asset. If $\sigma(s,t)s$ is globally Lipschitz, then there exists a constant k, such that for any s and s' and any time t,

$$|\sigma(s,t)s - \sigma(s',t)s'| \leq k|s - s'|. \tag{28}$$

For our limited liability underlying asset we have

$$\sigma(s',t)s'\big|_{s'=0} = 0$$

and hence (28) implies that for all s and t

$$\sigma(s,t) \leq k.$$

For k' equal to the smallest value of k satisfying (28), we have from Theorem 8 the immediate result that

$$c(s,t) \leq c^{bs(k')}(s,t).$$

VI. Option Prices when the Underlying does not follow a One-Dimensional Diffusion

The preceding sections have examined the pricing of options in a diffusion world and have established that much of the intuition familiar from the Black-Scholes model carries over to the case where the underlying asset's volatility depends on time and the contemporaneous value of the underlying asset. In this section we show that there are many interesting settings not captured by a one-dimensional diffusion world, settings in which option prices need not possess any of their familiar properties. In particular, we will demonstrate that if the process describing changes in the value of the underlying asset is a multi-dimensional diffusion, or is non-Markovian, or discontinuous, then it can be that, for s in some range, $c_1(s,t) < 0$ and $c_{11}(s,t) < 0$.

A. Option Prices and Stochastic Volatility

We wish to show that if the underlying asset's price is described by a multi-dimensional diffusion, a call option need not be everywhere increasing and convex in the value of the underlying. Suppose that interest rates are zero and the risk-neutralized processes for s and y are given by

$$ds_t = \sigma(s_t, y_t, t) s_t dB_t^1.$$

and

$$dy_t = \theta(s_t, y_t, t) dB_t^2.$$

As depicted in Figure 6, if s is high and y is low, then $\sigma(s,y,t) = \theta(s,y,t) = 0$. If s is low and y is low, then $\sigma(s,y,t) = 0$ and $\theta(s,y,t) > 0$. If y is high, then $\sigma(s,y,t) > 0$ and $\theta(s,y,t) > 0$. With initial condition $\{s', y'\}$, $s_T = s' < K$ and $c(s', y', t) = 0$. With initial condition $\{s'', y'\}$, there is a positive probability that $s_T > K$ and $c(s'', y', t) > 0$. Thus over a range of s values, the option price is decreasing in s and can not then be everywhere convex in s. Note that our example does *not* satisfy condition (ii) of Theorems 1 and 2.

B. Option Prices in a Discontinuous Markovian World

Suppose that there are jumps in the value of the underlying and that the probability of a jump is related to the level of the process. As a very simple example, consider the following stock price process:

For all t and all $s_t \geq H > 0$, $\quad ds_t = r(t) s_t dt$; i.e., for s_t high enough, the underlying grows deterministicly at the rate of interest.

For all t and all $0 < s_t < H$, $ds_t = \big(r(t) - \lambda(J-1)\big)s_t dt + s_t dq_t$; i.e., for stock prices below H, the underlying follows a simple version of a mixed diffusion-jump process.[19] Here q_t is a Poisson process governing jumps in the stock price, the mean number of jumps per unit time is λ, and $(J-1)$ is the percentage price increase in the stock if the Poisson event occurs.

Possible stock price paths are depicted in Figure 7. Now consider a call option on this asset with an exercise price of K. Consider two possible time t stock prices, s' and s'', with $0 < s'' < H < s'$, $s'e^{\int_t^T r(\tau)d\tau} < K$, and $s''J > K$. Conditional on $s_t = s'$, the option will, with certainty, finish out-of-the-money. Conditional on $s_t = s''$, the option has a positive probability of finishing in-the-money. Thus we have

$$0 = c(0,t) < c(s'',t) > c(s',t) = 0;$$

i.e., the option price is not everywhere increasing and convex in the value of the underlying.

A second example of a Markovian world in which call prices are not increasing convex functions of the value of the underlying is depicted in Figure 8.[20] The underlying asset's price can be represented as a non-recombining binomial tree. Such a tree may be the outcome when the management of the underlying firm faces the following incentive problem. Suppose that management will be evaluated on the basis of the stock price at date T relative to a goal, $\mathcal{G}$. Failure to meet the benchmark level, $\mathcal{G}$, will result in termination. Exceeding the goal will bring forth a bonus. If at date $T-1$ the firm has done poorly and the stock price is low, say $s_{T-1} = s''$, the firm must switch to high variance projects in order for there to be any chance of meeting the benchmark necessary for management to retain their posts. Alternately, if the stock price is high at date $T-1$, say $s_{T-1} = s'$,

[19] See Merton (1976) for the development of an option pricing model applicable when jumps in the value of an underlying asset are diversifiable.

[20] For a further example, see footnote 14 of Chapter 4 of Cox and Rubinstein (1985).

management can, and will, effectively lock in their future bonuses by switching to a low risk investment strategy. Now consider the date $T-1$ value of a call option with a date T maturity written on the stock of this company. For the level of K depicted in Figure 8 we again have that

$$0 = c(0, T-1) < c(s'', T-1) > c(s', T-1) = 0.$$

One implication is that if the stock price back at time $T-2$ is equal to s''', then the replicating strategy at that time involves *shorting* the underlying stock and *lending*.

Figures 7 and 8 depict settings where, for a given realization of the random component of the processes, the stock price path starting at $s_t = s''$ can, in effect, jump through the path starting at $s_t = s' > s''$. Continuity precludes this in a one-dimensional diffusion world. We now turn to non-Markovian worlds in which something like this happens quite naturally. As an introduction to the properties of option prices in a non-Markovian world, it is interesting to consider a thought experiment under which Figure 8 can be viewed as depicting a continuous, but non-Markovian process. Suppose that between trading dates one could observe (but not trade along) the trajectory of prices. Now suppose that at time $\check{t}$, one observed a trajectory level of $\check{s}$. One could *not* then characterize the distribution of s_T given only the knowledge that $s_{\check{t}} = \check{s}$. One would also need to know a *past* stock price as well; e.g., whether s_{T-1} was equal to s' or s''.

C. *Option Prices in a Continuous Non-Markovian World*

We are interested in the properties of option prices when the underlying asset follows a *retarded* process[21] such that the instantaneous volatility depends not only on the contemporaneous price and time, but also on past prices. To motivate such a characterization of volatility we first ask why it might be that volatility depends on the contemporaneous

[21] For a discussion of such processes, see Mohammed (1978).

price level. A natural answer is that the volatility of a stock reflects the underlying firm's investment policy and capital structure decisions.

Consider first a potential link between firm investment policy and stock volatility. Suppose that an *unlevered* firm has two divisions. One division undertakes an effectively riskless project and the other undertakes a risky project. The variance of the firm's stock will then change through time as the proportion of the firm's asset portfolio devoted to the risky division changes. Assuming that assets are **not** reallocated across the two divisions, the proportion will change as a function of how well the risky division has performed relative to the riskless division. The asset mix at any time t can then be proxied by the contemporaneous stock price, s_t. Rubinstein (1983) has developed the *Displaced Diffusion Option Pricing Model* to price options written on the stock of such a firm.

Now consider a potential link between firm capital structure and stock volatility. Assume the firm is levered and, for simplicity, assume that the volatility of the firm's portfolio of assets is constant through time. The equity of a levered firm is analogous to a call option on those assets. If the firm does **not** recapitalize during the option's life, then, following the steps in section IIIF, the volatility of the equity can be expressed as a function of the contemporaneous s_t and time. When the firm does well (poorly), the volatility of the stock will decrease (increase). Geske (1979) has developed the *Compound Option Pricing Model* to price options on the stock of such a firm.

In both the investment policy and capital structure settings considered above, it is theoretically possible that the firm might continuously and appropriately adjust its investment and financing decisions so as to offset any changes in the stock's volatility associated with changes in firm value. In fact, if, in these settings, there were an optimal asset mix, an optimal capital structure, and zero costs of reallocating and recapitalizing, then the volatility of the stock of an optimizing firm could be a constant. But in the presence of adjustment costs, the optimal controls will not be continuous function of the underlying firm value; instead, they would exhibit hysteresis.[22,23]

Thus the volatility function at time τ can take the form $\sigma(s_\tau, \tau)$, when

i) a stock's volatility reflect's the underlying firm's investment and capital structure decisions, and

iia) those decisions will remain unchanged over the life of the option.

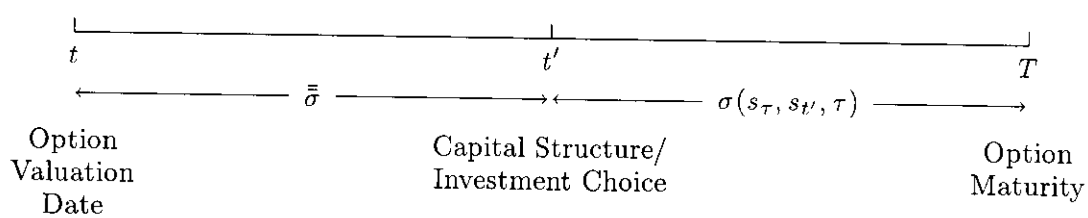
Our twist is to change iia) to iib):

iib) at time t' prior to the option's maturity, the underlying firm will make an investment and/or financing decision that will remain unchanged over the option's remaining life,

and to add

iii) the decision made at time t' will depend on the value of the firm at t' as proxied by $s_{t'}$.

The time line below depicts the setting we have in mind. For simplicity, we assume that prior to time t', the stock's volatility is a constant $\bar{\bar{\sigma}}$.



It is true that, *ex-post*, it will be possible to represent the volatility of the stock at all times $\tau \in [t', T]$ as some function $\Sigma(s_\tau, \tau)$. But the functional form of $\Sigma(\cdot)$ can not be

[22] Dixit and Pindyck (1984) model optimal investment policy given the irreversibility of investment. Fischer, Heinkel and Zechner (1989) model a firm's optimal dynamic capital structure choice given recapitalization costs.

[23] That the firm faces adjustment costs is not inconsistent with our implicit assumption that the securities issued by the firm, and contingent claims thereon, are traded in frictionless capital markets.

determined exante. *Exante*, the volatility at all times $\tau \in [t', T]$ takes the form $\sigma(s_\tau, s_{t'}, \tau)$. Exante, the process is non-Markovian. Our setting will be most relevant for the valuation of long-dated options such as warrants.[24]

C.1. Firm Investment Decisions prior to the Option's Maturity

Consider an unlevered firm that will pay no dividends prior to time T. At time $t' < T$ the firm will replace its assets – franchises may have expired, existing assets may fully economically depreciated. Management will choose the replacement assets in a manner that reflects an incentive problem similar to that underlying Figure 8: The lower the value of $s_{t'}$, the higher the volatility of the replacement assets. The following example is chosen only for its analytical tractability. Assume that for $\tau \in [t', T]$,

$$\sigma(s_\tau, s_{t'}, \tau) = \sigma(s_{t'}) = \begin{cases} -\ln\left(s_{t'}/Be^{-\int_{t'}^{T} r(w)dw}\right)/\sqrt{T-t'}, & \text{for } s_{t'} \leq Be^{-\int_{t'}^{T} r(w)dw}; \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

For $s_{t'} \leq Be^{-\int_{t'}^{T} r(w)dw}$, the firm chooses replacement assets that subsequently have constant volatility. The level of that volatility is zero when $s_{t'} = Be^{-\int_{t'}^{T} r(w)dw}$, and is increasing as $s_{t'}$ decreases. For $s_{t'} > Be^{-\int_{t'}^{T} r(w)dw}$, the firm chooses zero risk replacement assets.

Now consider the time t' value of a call option on this stock with a time T maturity date and an exercise price equal to B. Since subsequent to t' the stock will have a constant volatility, the level of that volatility being determined by $s_{t'}$ as in (29),

$$c(s, t') = c^{bs(\sigma(s_{t'}))}(s, t'). \tag{30}$$

[24] Lauterbach and Schultz (1990) conclude that a square root CEV model, where volatility depends on the contemporaneous s, outperforms a deterministic volatility (Black-Scholes) model in explaining observed warrant prices. We suggest that allowing volatility to reflect the stock price history may yield significantly better predictions still.

The value $c(s,t')$ is plotted in Figure 9a for $T - t' = 1$ year, $r(\tau) = 10\%$ for all τ, and $K = B = \$3$. The plot consists of the locus of points where a vertical line drawn from the x-axis at, say, the point s', intersects the dashed convex curve which plots $c^{bs(\sigma(s'))}(s,t')$. Notice how the vertical drawn from $s'' < s'$ simultaneously intersects the locus and a second dashed convex curve that lies everywhere above $c^{bs(\sigma(s'))}(s_{t'},t')$. This higher convex curve plots $c^{bs(\sigma(s''))}(s,t')$. From the volatility specification in (29) we have $\sigma(s'') > \sigma(s')$ since $s'' < s'$. The humped shape of the plot is determined by two opposing forces. As $s_{t'}$ increases, the underlying stock becomes more valuable; but the underlying stock also becomes less volatile over the remaining life of the option. For all $s_{t'} > Be^{-\int_{t'}^{T} r(w)dw}$ the locus of points is very easy to construct. With zero volatility in the future, the call is certain to finish-in-the-money, and is worth $s_{t'} - Be^{-\int_{t'}^{T} r(w)dw}$ at time t'.

It is worth considering how $c(s,t')$ would be determined in a Markovian world. The humped shape could not survive. As an example, suppose that for **all** $\tau \in [t',T]$,

$$\sigma(s_\tau, s_{t'}, \tau) = \sigma(s_\tau) \begin{cases} > 0, & \text{for } s_\tau \leq Be^{-\int_{t'}^{T} r(w)dw}; \\ = 0, & \text{otherwise.} \end{cases} \tag{31}$$

Comparing (29) and (31) we see that in the non-Markovian case, (29), the volatility at any time $\tau > t'$ is positive so long as $s_{t'}$ was less than $Be^{-\int_{t'}^{T} r(w)dw}$. In the Markovian case, (31), the volatility at any time $\tau > t'$ is only positive so long as the contemporaneous s_τ is less than $Be^{-\int_{t'}^{T} r(w)dw}$.

Because the stock price process is continuous, it cannot be that the stock price grows from a value below $Be^{-\int_{t'}^{T} r(w)dw}$ at time t' to a level above $B = K$ at time T, without at some interim date being equal to $Be^{-\int_{t'}^{T} r(w)dw}$. But at that date, the underlying stock will become riskless, and will subsequently grow deterministicly at the rate $r(\tau)$. There is then zero probability that $s_T > K = B$. Hence for all $s \leq Be^{-\int_{t'}^{T} r(w)dw}$, we have $c(s,t') = 0$ and the hump vanishes. Thus under the Markovian counterpart to (29), given by (31),

$$c(s,t') = \max[0, s - Ke^{-\int_{t'}^{T} r(w)dw}],$$

and the option price is once again an increasing, convex function of the value of the underlying stock.

We now turn to the valuation of the option at some earlier date $t < t'$. We assume that for all $\tau \in [t, t')$ the stock's volatility is a constant $\bar{\bar{\sigma}}$. Thus during the interval from t to t' we are in a Markovian world. Rearranging expression (7) gives

$$c_2(s,t) = -\frac{1}{2}[\sigma(s,t)s]^2 c_{11}(s,t) - r(t)c(s,t)\big(\Omega(s,t) - 1\big). \qquad (32)$$

When the call is convex in s for all s and t, the two terms on the right-hand-side of (32) are non-positive: $c_{11}(s,t) \geq 0$ and $\Omega(s,t) > 1$. When, as here, the call is strictly concave in s over some region (and $\sigma(s,t) \equiv \bar{\bar{\sigma}} > 0$), then the first term is strictly positive in that region, and the second term can also be strictly positive in some region. Thus the call can be a "bloating" asset. Figure 9b illustrates the "non-wasting" nature of the call over some range of s, by depicting both $c(s,t)$ and $c(s,t')$ when $t' - t = 1$ year, $T - t' = 1$ year, $\bar{\bar{\sigma}} = 30\%$ per annum, $K = \$3$, and $r(\tau) = 10\%$ per annum for all $\tau \in [t, T]$.

When $s = \$1$, $c(s,t) = \$0.106885$. Over the region in which the call's value is decreasing in s it appears to act more like a put in a standard Black-Scholes world. The value of a put in a Black-Scholes world is a decreasing function of the interest rate. A value for s of \$1 is in the put-like region of the above example. Now suppose that the interest rate during the first year, the period from t to t', were to be higher and equal to 11% not 10%. This upward shift *in the entire term structure* will cause a *decline* in the call's value to $c(s,t) = \$0.103806$.

C.2. Firm Financing Decisions prior to the Option's Maturity

Consider a currently unlevered, non-dividend-paying firm. The value of the firm's assets are assumed to follow a diffusion as in (1) with a constant expected rate of return, $\bar{\bar{\mu}}$, and constant volatility, $\bar{\bar{\sigma}}$. At time t' the firm will issue pure discount bonds promising $\mathcal{F}$ at time $\mathcal{M}$. Let $V_{t'}$ denote the post-issue value of the firm's assets. The firm has an *optimal* debt to asset ratio such that $\mathcal{F}$ will be chosen as $\mathcal{F} = \gamma V_{t'}$. If $V_{\mathcal{M}} < \mathcal{F}$ the firm will be bankrupt at time $\mathcal{M}$, and, in that event, $s_\tau = 0$ for all $\tau \geq \mathcal{M}$. At all times $\tau > t'$, the volatility of the firm's stock will depend upon the amount of debt outstanding, which will depend upon the past value $V_{t'}$ as proxied by $s_{t'}$; i.e., s_τ will follow a non-Markovian process.

Now consider the properties of options on this firm's stock. In particular, consider a very long-dated option with $T > \mathcal{M}$. What is the likelihood that a very long-dated put option, with an exercise price of $K > 0$, will finish in-the-money?

$$\begin{aligned}
\Pr(s_T < K) \geq \Pr(s_T = 0) &\geq \Pr(s_{\mathcal{M}} = 0) \\
&= \Pr(V_{\mathcal{M}} \leq \mathcal{F}) \\
&= 1 - N\Big(\frac{\ln\big(V_{t'}/\mathcal{F}\big) + \big(\bar{\bar{\mu}} - \frac{1}{2}\bar{\bar{\sigma}}^2\big)(\mathcal{M} - t')}{\bar{\bar{\sigma}}\sqrt{\mathcal{M} - t'}}\Big) \\
&= 1 - N\Big(\frac{\ln\big(1/\gamma\big) + \big(\bar{\bar{\mu}} - \frac{1}{2}\bar{\bar{\sigma}}^2\big)(\mathcal{M} - t')}{\bar{\bar{\sigma}}\sqrt{\mathcal{M} - t'}}\Big).
\end{aligned}$$

Now consider that likelihood when the underlying stock becomes infinitely valuable today.

$$\begin{aligned}
\lim_{s_t \to \infty} \Pr(s_T < K) &\geq \lim_{s_t \to \infty}\Big[1 - N\Big(\frac{\ln\big(1/\gamma\big) + \big(\bar{\bar{\mu}} - \frac{1}{2}\bar{\bar{\sigma}}^2\big)(\mathcal{M} - t')}{\bar{\bar{\sigma}}\sqrt{\mathcal{M} - t'}}\Big)\Big] \\
&= 1 - N\Big(\frac{\ln\big(1/\gamma\big) + \big(\bar{\bar{\mu}} - \frac{1}{2}\bar{\bar{\sigma}}^2\big)(\mathcal{M} - t')}{\bar{\bar{\sigma}}\sqrt{\mathcal{M} - t'}}\Big) \; > \; 0.
\end{aligned}$$

Thus, no matter how valuable the stock is today, there is always a positive probability that it will be valueless at time T. The more valuable the stock today, the more debt the firm is likely to issue at time t'. Hence, we have an illustration of the type of asymptotics portrayed in Figures 3g and 3h.

$$\lim_{s_t \to \infty} p(s_t, t) \geq e^{-\int_t^T r(w)dw} K \cdot \left[1 - N\left(\frac{\ln\left(e^{\int_{t'}^{\mathcal{M}} r(w)dw}/\gamma\right) - \frac{1}{2}\bar{\bar{\sigma}}^2(\mathcal{M} - t')}{\bar{\bar{\sigma}}\sqrt{\mathcal{M} - t'}}\right)\right] > 0.$$

VII. Conclusions

This paper has established that the bulk of the option pricing properties established in Merton's classic *Theory* for the case when the option price is homogeneous of degree one in the value of the underlying and the exercise price, extend to any one-dimensional diffusion world. We show that, for any contingent claim, a replicating portfolio's positions in the underlying asset and in riskless bonds are bounded by the inf and the sup of the positions at maturity. This means that a call's delta is bounded between zero and one; equivalently, a call's price is increasing in the value of the underlying. We further establish that, for any contingent claim, if the claim's payoff at maturity is convex (concave) in the value of the underlying at expiration, then the current value of the claim is convex (concave) in the current value of the underlying. We show that the bounds on a contingent claim's delta also apply in a world of stochastic volatility provided the drift and diffusion parameters of the risk-neutralized version of the process driving changes in volatility are independent of the value of the underlying. The convexity (concavity) result applies in a world of stochastic volatility if, in addition, the instantaneous covariance between percent changes in the underlying and the process driving volatility is independent of the value of the underlying.

As a consequence of the bounds on delta and inherited convexity, we are able to undertake a comparative static analysis of the effect of changes in interest rates, dividend rates and volatility on the value of options. Two of our comparative static results are particularly striking. First, it is only in a Black-Scholes world that a decrease in the present value of the exercise price necessarily leads to an increase in call prices. Second, when the underlying asset's volatility is bounded above (below), then, whatever the functional form of the relation between volatility, time and the contemporaneous stock price, the option's price is bounded above (below) by its Black-Scholes value calculated at the bounding volatility levels. Further, the option's delta can be bounded above and below.

We also examine the properties of option prices in worlds where the price of the underlying asset follows a general multi-dimensional diffusion (as in the case of stochastic volatility) and in worlds where we relax either the continuity or the Markovian properties inherent in a diffusion. We show that in such worlds, call options can be 'bloating' assets, whose value over some range is a decreasing, concave function of the value of the underlying. We argue that when considering the valuation of long-dated options on the stock of a firm, it is intuitive to view the dynamics of the underlying stock price as non-Markovian.

It is worth reflecting that the Black (1976) and Nelson (1991) empirical documentation that volatility tends to increase (decrease) following negative (positive) returns strongly suggests that history matters. Perhaps modeling stochastic volatility as driven by both the current and past levels of the market will prove a fruitful direction for research into the pricing of index options.

Appendix A

The Cauchy Problem

For given $T > 0$, find $f \in C^{2,1}(\mathbb{R}^N \times [0,T))$ solving

$$\mathcal{D}f(x,t) - R(x,t)f(x,t) + h(x,t) = 0, \quad (x,t) \in \mathbb{R}^N \times [0,T), \tag{A1}$$

with the terminal condition

$$f(x,T) = g(x), \quad x \in \mathbb{R}^N, \tag{A2}$$

where

$$\mathcal{D}f(x,t) = f_{N+1}(x,t) + \sum_{i=1}^{N} f_i(x,t)\mu^i(x,t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{N}\gamma^i(x,t)\gamma^j(x,t)\eta^{ij}(x,t)f_{ij}(x,t), \tag{A3}$$

where $f_{N+1}(x,t)$ denotes the partial of f with respect to t, and, for $i = 1,\dots,N$, $f_i(x,t)$ denotes the partial of f with respect to the i'th element of the vector x, $R : \mathbb{R}^N \times [0,T] \to \mathbb{R}$, $h : \mathbb{R}^N \times [0,T] \to \mathbb{R}$, $g : \mathbb{R}^N \to \mathbb{R}$, and $\mu(x,t)$ is an $N \times 1$ vector whose i'th element $\mu^i(x,t)$ is such that $\mu^i : \mathbb{R}^N \times [0,T] \to \mathbb{R}$, and the superscripts on $\gamma^i(x,t)\gamma^j(x,t)\eta^{ij}(x,t)$ do not denote powers but are instead indices. $\gamma(x,t)$ is an $N \times 1$ vector whose i'th element $\gamma^i(x,t)$: $\mathbb{R}^N \times [0,T] \to \mathbb{R}$, for $i = 1,\dots,N$. Each function $\eta^{ij}(x,t) = \eta^{ji}(x,t)$: $\mathbb{R}^N \times [0,T] \to \mathbb{R}$, for all $i,j = 1,\dots,N$.

The Feynman-Kac solution to (A1)–(A3), *when it exists*, is given by

$$f(x,t) = E\Big\{\int_t^T \varphi_{t,\tau} h(\xi_\tau^{x,t},\tau)d\tau + \varphi_{t,T}\, g\big(\xi_T^{x,t}\big)\Big\}, \tag{A4}$$

where

$$\varphi_{t,\tau} = e^{-\int_t^\tau R(\xi_w^{x,t},w)dw},$$

and the elements, $\xi^{i\,x,t}$, of the $N \times 1$ vector $\xi^{x,t}$ solve the system of SDE's

$$d\xi_\tau^i = \mu^i(\xi_\tau,\tau)d\tau + \gamma^i(\xi_\tau,\tau)dB_\tau^i,$$

with initial condition x at time t, and with $dB_\tau^i dB_\tau^j = \eta^{ij}(\xi_\tau,\tau)d\tau$.

Example 1:

Suppose the underlying asset follows the two dimensional diffusion given in (2), interest rates are deterministic, and the price of volatility risk is given by $\lambda(s,y,t)$.

As shown in appendix 12B of Hull (1993), the value of a contingent claim is given by the solution of the p.d.e.,

$$c_3(s,y,t)+c_1(s,y,t)r(t)s+c_2(s,y,t)\big(\beta(s,y,t)-\lambda(s,y,t)\theta(s,y,t)\big)+\frac{1}{2}c_{11}(s,y,t)[\sigma(s,y,t)s]^2$$
$$+\frac{1}{2}c_{22}(s,y,t)[\theta(s,y,t)]^2+c_{12}(s,y,t)\sigma(s,y,t)s\theta(s,y,t)\rho(s,y,t)-r(t)c(s,y,t)=0, \qquad (A5)$$

subject to the terminal condition $c(s,y,T)=g(s)$, where, for example, if the contingent claim is a call option, $g(s)=\max[0,s-K]$.

Thus $f=c$ solves the p.d.e. in (A1) when

$$N=2. \qquad \mu(x,t)=\begin{pmatrix} r(t)s \\ \beta(s,y,t)-\lambda(s,y,t)\theta(s,y,t)\end{pmatrix}.$$
$$x=\begin{pmatrix} s \\ y\end{pmatrix}. \qquad \gamma(x,t)=\begin{pmatrix}\sigma(s,y,t)s \\ \theta(s,y,t)\end{pmatrix}.$$
$$h(x,t)=0.$$
$$R(s,t)=r(t). \qquad \eta^{ij}(x,t)=\begin{cases}\rho(s,y,t), & \text{if } i\neq j; \\ 1, & \text{otherwise.}\end{cases}$$

The Feynman-Kac solution for the value of the contingent claim, when it exists, is then given by expressions (5) and (6).

Example 2:

Let the superscripts on the functions $Y^1,\ldots,Y^4$ denote indices not powers, and define the functions as:

$$Y^1(s,y,t)=\beta(s,y,t)-\lambda(s,y,t)\theta(s,y,t). \qquad Y^3(s,y,t)=[\theta(s,y,t)]^2$$
$$Y^2(s,y,t)=[\sigma(s,y,t)s]^2. \qquad Y^4(s,y,t)=\sigma(s,y,t)s\theta(s,y,t)\rho(s,y,t).$$

We can then rewrite (A5) as

$$c_3(s,t) + c_1(s,y,t)r(t)s + c_2(s,y,t)Y^1(s,y,t) + \frac{1}{2}c_{11}(s,Y,t)Y^2(s,y,t)$$
$$+\frac{1}{2}c_{22}(s,y,t)Y^3(s,y,t) + c_{12}(s,y,t)Y^4(s,y,t) - r(t)c(s,y,t) = 0. \qquad (A6)$$

Taking the partial of (A6) with respect to s gives

$$c_{13}(s,y,t) + c_{11}(s,y,t)r(t)s + c_1(s,y,t)r(t) + c_{12}(s,y,t)Y^1(s,y,t) + c_2(s,y,t)Y_1^1(s,y,t)+$$
$$\frac{1}{2}c_{111}(s,y,t)Y^2(s,y,t) + \frac{1}{2}c_{11}(s,y,t)Y_1^2(s,y,t) + \frac{1}{2}c_{122}(s,y,t)Y^3(s,y,t)+$$
$$\frac{1}{2}c_{22}(s,y,t)Y_1^3(s,y,t) + c_{112}(s,y,t)Y^4(s,y,t) + c_{12}(s,y,t)Y_1^4(s,y,t) - r(t)c_1(s,y,t) = 0. \ (A7)$$

Let f be the value of the first partial of a contingent claim's value with respect to the value of the underlying. The p.d.e. in (A7) can then be rewritten as

$$f_3(s,y,t) + f_1(s,y,t)[r(t)s + \frac{1}{2}Y_1^2(s,y,t)] + f_2(s,y,t)[Y^1(s,y,t) + Y_1^4(s,y,t)]$$
$$+\frac{1}{2}f_{11}(s,y,t)Y^2(s,y,t) + \frac{1}{2}f_{22}(s,y,t)Y^3(s,y,t) + f_{12}(s,y,t)Y^4(s,y,t)$$
$$+c_2(s,y,t)Y_1^1(s,y,t) + \frac{1}{2}c_{22}(s,y,t)Y_1^3(s,y,t) = 0. \qquad (A8)$$

Expression (A8) is in the same form as (A1) with

$$N = 2. \qquad h(x,t) = c_2(s,y,t)Y_1^1(s,y,t) + \frac{1}{2}c_{22}(s,y,t)Y_1^3(s,y,t).$$
$$x = \begin{pmatrix} s \\ y \end{pmatrix}. \qquad \mu(x,t) = \begin{pmatrix} r(t)s + \frac{1}{2}Y_1^2(s,y,t) \\ Y^1(s,y,t) + Y_1^4(s,y,t)) \end{pmatrix}.$$
$$R(s,t) = 0. \qquad \gamma(x,t) = \begin{pmatrix} \sqrt{Y^2(s,y,t)} \\ \sqrt{Y^3(s,y,t)} \end{pmatrix}.$$
$$\eta^{ij}(x,t) = \begin{cases} \rho(s,y,t), & \text{if } i \neq j; \\ 1, & \text{otherwise.} \end{cases}$$

Thus the Feynman-Kac solution for $c_1(s,y,t)$, when it exists, is then given by

$$c_1(s,y,t) = E\Big\{\int_t^T (c_2(\xi_\tau^{1^{s,y,t}}, \xi_\tau^{2^{s,y,t}}, \tau)Y_1^1(\xi_\tau^{1^{s,y,t}}, \xi_\tau^{2^{s,y,t}}, \tau)$$
$$+\frac{1}{2}c_{22}(\xi_\tau^{1^{s,y,t}}, \xi_\tau^{2^{s,y,t}}, \tau)Y_1^3(\xi_\tau^{1^{s,y,t}}, \xi_\tau^{2^{s,y,t}}, \tau))d\tau + g_1(\xi_T^{1^{s,y,t}})\Big\},$$

where $\xi^{1^{s,y,t}}$ and $\xi^{2^{s,y,t}}$ solve the system of SDE's

$$d\xi_\tau^1 = \big(r(\tau)\xi_\tau^1 + \frac{1}{2}Y_1^2(\xi_\tau^1, \xi_\tau^2, \tau)\big)d\tau + \sqrt{Y^2(\xi_\tau^1, \xi_\tau^2, \tau)}dB_\tau^1,$$

$$d\xi_\tau^2 = \big(Y^1(\xi_\tau^1, \xi_\tau^2, \tau) + Y_1^4(\xi_\tau^1, \xi_\tau^2, \tau)\big)d\tau + \sqrt{Y^3(\xi_\tau^1, \xi_\tau^2, \tau)}dB_\tau^2,$$

with initial condition $\{s, y\}$ at time t. In addition, $dB_\tau^1 dB_\tau^2 = \rho(\xi^1 tau, \xi_\tau^2, \tau)d\tau$.

Proof of Theorem 2

Condition (i) of the theorem implies that there exists a function Z such that

$$Z(s,t) = [\sigma(s,y,t)s]^2.$$

The Feynman-Kac solution for the first partial of the value of the contingent claim with respect to the value of the underlying given in Example 2 then simplifies to

$$c_1(s,t) = E\{g_1(\xi_T^{s,t})\},$$

where $\xi^{s,t}$ solves the SDE

$$d\xi_\tau = \big(r(\tau)\xi_\tau + \frac{1}{2}Z_1(\xi_\tau, \tau)\big)d\tau + \sqrt{Z(\xi_\tau, \tau)}dB_\tau,$$

with initial condition s at time t.

Suppose $s' > s''$. The distribution of $\xi_T^{s',t}$ first-order stochastically dominates the distribution of $\xi_T^{s'',t}$. If for all s, $g_{11} \geq 0$ then g_1 is non-decreasing in s, and hence

$$c_1(s',t) = E\{g_1(\xi_T^{s',t})\} \geq E\{g_1(\xi_T^{s'',t})\} = c_1(s'',t).$$

Similarly, if for all s, $g_{11} \leq 0$, $c_1(s',t) \leq c_1(s'',t)$.

Now consider condition (ii) of the theorem. Condition (ii) implies that there exists functions G^1, G^2 and G^3, where superscripts denote indices not powers, such that

$$G^1(y,t) = Y^1(s,y,t),$$

$$G^2(y,t) = Y^3(s,y,t),$$

$$G^3(y,t)s = Y^4(s,y,t).$$

The Feynman-Kac solution for the value of the first partial of the contingent claim with respect to the underlying given in Example 2 then simplifies to

$$c_1(s,y,t) = E\Big\{g_1\big(\xi_T^{1\,s,y,t}\big)\Big\},$$

where $\xi^{1\,s,y,t}$ and $\xi^{2\,s,y,t}$ solve the system of SDE's

$$d\xi_\tau^1 = \big(r(\tau)\xi_\tau^1 + \frac{1}{2}Y_1^2(\xi_\tau^1,\xi_\tau^2,\tau)\big)d\tau + \sqrt{Y^2(\xi_\tau^1,\xi_\tau^2,\tau)}dB_\tau^1,$$
$$d\xi_\tau^2 = (G^1(\xi_\tau^2,\tau) + G^3(\xi_\tau^2,\tau))d\tau + \sqrt{G^2(\xi_\tau^2,\tau)}dB_\tau^2,$$

with initial condition $\{s,y\}$ at time t. In addition, $dB_\tau^1 dB_\tau^2 = \rho(\xi_\tau^1,\xi_\tau^2,\tau)$. Suppose $s' > s''$. The distribution of $\xi_T^{1\,s',y,t}$ first-order stochastically dominates the distribution of $\xi_T^{1\,s'',y,t}$. If for all s, $g_{11} \geq 0$, then g_1 is non-decreasing in s. Hence

$$c_1(s',y,t) = E\Big\{g_1\big(\xi_T^{1\,s',y,t}\big)\Big\} \geq E\Big\{g_1\big(\xi_T^{1\,s'',y,t}\big)\Big\} = c_1(s'',y,t).$$

Similarly, if for all s, $g_{11} \leq 0$, then g_1 is non-increasing in s. Hence $c_1(s',y,t) \leq c_1(s'',y,t)$.

QED

Appendix B

Properties of Solutions to P.D.E.'s With Convex Terminal Conditions

Suppose $C(S,t)$ is the solution to

$$C_2(S,t) + \frac{1}{2}[v(S,t)S]^2 C_{11}(S,t) = 0, \tag{B1}$$

subject to

$$C(S,T) = g(S) \text{ for all } S.$$

$C(S_t,t)$ can be thought of as the time t normalized value of a contingent claim on an underlying asset worth S_t, where the payoff to the claim at its maturity date T is given by $g(S_T)$.

Further suppose that $g(\cdot)$ has the following properties:

1.) $g(0) = 0$,

2.) $g(S)$ in non-decreasing in S,

3.) $g(S)$ is convex in S,

4.) $\lim_{S\to\infty} \frac{dg}{dS} = k < \infty$, and

5.) there exists an $H < \infty$ such that for all S, $g(S) \geq kS - H$.

The relation between k, H and $g(S)$ is as depicted in Figure B1. Note that from the characterization of $g(S)$ it follows immediately that

$$\max[0, kS - H] \leq g(S) \leq kS.$$

In this appendix we show that for all S and for all $t < T$, $C_{11}(S,t) \geq 0$. We first establish some bounds on $C(S,t)$ and $C_1(S,t)$.

Lemma 1. *For all* $t \leq T$, $\max[0, kS - H] \leq C(S,t) \leq kS$.

From the Feynman-Kac Theorem, $C(S,t)$ can be expressed as

$$C(S,t) = E\{g(\xi_T^{S,t})\},$$

$$\text{where} \;\; d\xi_\tau = v(\xi_\tau, \tau)\xi_\tau dz_\tau. \qquad (B2)$$

The distribution of ξ_T is the distribution of S_T under the equivalent martingale measure. First consider the upper bound.

$$C(S,t) = E\{g(\xi_T^{S,t})\} \leq E\{k\xi_T^{S,t}\} = kS,$$

where the second equality follows from the martingale property of ξ_τ. Now consider the lower bound.

$$C(S,t) = E\{g(\xi_T^{S,t})\} \geq E\{\max[0, k\xi_T^{S,t} - H]\}$$
$$\geq \max[0, kE\{\xi_T^{S,t}\} - H] = \max[0, kS - H].$$

QED

Lemma 2. *For all* $t \leq T$, $0 \leq C_1(S,t) \leq k$.

Proof: Lemma 2 is an immediate implication of Theorem 1.

Armed with Lemmas 1 and 2, the proof of convexity proceeds by contradiction. We initially assume that there exists at least one region within which $C(S,t)$ is strictly concave in S. Let $a(t)$ and $b(t)$ denote the boundaries of any one such region, with $b(t) > a(t)$.

When $b(t)$ is finite, $b(t)$ is the time t value of S such that the relation is **strictly** concave for $S = b(t) - \epsilon$ and convex for $S = b(t) + \epsilon$ for arbitrarily small positive ϵ. Thus $C_{11}(b(t), t) = 0$.

When $a(t) > 0$, $a(t)$ is the time t value of S such that the relation is convex for $S = a(t) - \epsilon$ and **strictly** concave for $S = a(t) + \epsilon$ for arbitrarily small positive ϵ. Thus $C_{11}(a(t), t) = 0$.

We first introduce some properties of the option price at the end points of this strictly concave region.

Lemma 3. *For $b(t)$ finite,* $C(b(t),t) > C(a(t),t)$.

Proof: The inequality is strict since $C_1(S,t) \geq 0$, $b(t) > a(t)$ and the relation between $C(S,t)$ and S is strictly concave for $S \in (a(t), b(t))$.

QED

Lemma 4. *For $b(t)$ finite,* $C_{11}(b(t),t) = 0 = C_2(b(t),t)$.

Proof: The zero value for $C_{11}(b(t),t)$ follows from its definition. The zero value for $C_2(b(t),t)$ then follows from the p.d.e. in $(A1)$.

QED

Lemma 5. *For $a(t) > 0$,* $C_{11}(a(t),t) = 0 = C_2(a(t),t)$.

Proof: The proof parallels the proof of Lemma 4.

QED

Lemma 6.[B1] *For $b(t)$ finite,*

$$\frac{dC_1(b(t),t)}{dt} = C_{12}(b(t),t) = -\frac{1}{2}[v(b(t),t)b(t)]^2 C_{111}(b(t),t) \leq 0.$$

Proof:

$$\begin{aligned}\frac{dC_1(b(t),t)}{dt} &= C_{11}(b(t),t)\frac{db(t)}{dt} + C_{12}(b(t),t)\\ &= C_{12}(b(t),t) \quad \text{since } C_{11}(b(t),t) = 0.\end{aligned}$$

Differentiating the p.d.e. in $(A1)$ with respect to S we have

$$\frac{1}{2}\frac{\partial v^2(S,t)}{\partial S}S^2 C_{11}(S,t) + v^2(S,t)SC_{11}(S,t) + \frac{1}{2}[v(S,t)S]^2 C_{111}(S,t) + C_{21}(S,t) = 0.$$

[B1] Note the implicit assumptions that C_{111}, C_{12} and $\frac{db(t)}{dt}$ exist. These assumptions can be relaxed at the expense of vast tracts of forest reserves. We subsequently use only the result that $C_1(b(t),t)$ is non-increasing with the passage of time.

Evaluating this equality at $S = b(t)$ gives

$$\frac{1}{2}[v(b(t),t)b(t)]^2 C_{111}(b(t),t) + C_{21}(b(t),t) = 0;$$

$$\text{i.e.,} \quad C_{21}(b(t),t) = -\frac{1}{2}[v(b(t),t)b(t)]^2 C_{111}(b(t),t).$$

C_{11} switches from negative to non-negative at $b(t)$ and hence

$$C_{111}(b(t),t) \geq 0.$$

QED

Lemma 7. *For $a(t) > 0$,*

$$\frac{dC_1(a(t),t)}{dt} = C_{12}(a(t),t) = -\frac{1}{2}v^2(a(t),t)a^2(t)C_{111}(a(t),t) \geq 0.$$

Proof: The proof parallels the proof of Lemma 6.

QED

Lemma 8 *For $b(t)$ finite, $C_1(b(t),t) < k$.*

Proof: The result follows from the combination of the bound $C_1(S,t) \leq k$ of Lemma 2 and the definition of the value $b(t)$, namely that there exists a value $\epsilon > 0$ such that $C(S,t)$ is strictly concave in S for $S \in (b(t) - \epsilon, b(t))$.

QED

Lemmas 6 and 7 provide the heart of the proof by contradiction. The following is an intuitive, geometric presentation of the reasoning. Consider the angle formed by the intersection of a line through the point $[a(t), C(a(t),t)]$ with a slope of $C_1(a(t),t)$ and a line through the point $[b(t), C(b(t),t)]$ with a slope of $C_1(b(t),t)$. This angle is depicted in Figure B2 as θ. At the terminal date T, the relation between $C(S,T)$ and S is everywhere convex. Thus on or before the terminal date the angle must go to either 180° or 0°. The angle can't become less than an angle of size ϕ as depicted in Figure B2 without Lemma 2 being violated. Can the angle increase from θ to 180°? The slope of the line tangential at the point $(b(t), C(b(t), t)$ is always non-increasing (Lemma (6)), while the slope of the line tangential at the point $(a(t), C(a(t), t)$ is always non-decreasing (Lemma (7)). Hence the angle θ is non-increasing with the passage of time. If we think of a measure of the degree of concavity as the deviation of θ from 180°, then the concavity does not disappear with the passage of time – rather it becomes more pronounced.

As further aids in the formal proof, we define a number of deterministic auxiliary functions and explore their properties. When $b(t)$ is finite, define the deterministic functions $g(t)$ and $h(t)$ as

$$g(t) = k\frac{C(b(t), t) - C_1(b(t), t)b(t)}{k - C_1(b(t), t)}$$

and

$$h(t) = k\frac{C(b(t), t) - C_1(b(t), t)(b(t) - H/k)}{k - C_1(b(t), t)},$$

respectively. Note that from Lemma 8 the denominator common to the definitions of both $g(t)$ and $h(t)$ is strictly positive. While $h(t)$ is always non-negative, $g(t)$ can take on either sign.

When $b(t)$ is finite and $a(t) > 0$, define another deterministic function $m(t)$ as

$$m(t) = C_1(a(t), t)(b(t) - a(t)) - \big(C(b(t), t) - C(a(t), t)\big) > 0.$$

The relation between $a(t)$, $b(t)$, $g(t)$, $h(t)$ and $m(t)$ is as depicted in Figure B3.

Lemma 9. $g(t) \leq C(b(t), t)$.

Proof:

$$\begin{aligned} g(t) &= k\frac{C(b(t), t) - C_1(b(t), t)b(t)}{k - C_1(b(t), t)} = C(b(t), t)\frac{k - C_1(b(t), t)\frac{b(t)k}{C(b(t),t)}}{k - C_1(b(t), t)} \\ &\leq C(b(t), t)\frac{1 - C_1(b(t), t)}{1 - C_1(b(t), t)} = C(b(t), t). \end{aligned}$$

QED

Lemma 10. $\frac{dg(t)}{dt} \geq 0$.

Proof:

$$\begin{aligned}
\frac{dg(t)}{dt} &= k\frac{C_1(b(t),t)\frac{db(t)}{dt} + C_2(b(t),t) - C_{11}(b(t),t)\frac{db(t)}{dt}b(t)}{k - C_1(b(t),t)} \\
&\quad - k\frac{C_{12}(b(t),t)b(t) + C_1(b(t),t)\frac{db(t)}{dt}}{k - C_1(b(t),t)} \\
&\quad - k\frac{\Big(C(b(t),t) - C_1(b(t),t)b(t)\Big)\Big(-C_{11}(b(t),t)\frac{db(t)}{dt} - C_{12}(b(t),t)\Big)}{\Big(k - C(b(t),t)\Big)^2} \\
&= -k\frac{C_{12}(b(t),t)b(t)}{k - C_1(b(t),t)} + k\frac{\Big(C(b(t),t) - C_1(b(t),t)b(t)\Big)C_{12}(b(t),t)}{\Big(k - C_1(b(t),t)\Big)^2} \\
&= k\frac{C_{12}(b(t),t)\Big(C(b(t),t) - kb(t)\Big)}{\Big(k - C_1(b(t),t)\Big)^2} \geq 0.
\end{aligned}$$

The inequality follows from Lemmas 1, 6 and 8.

QED

Lemma 11. $h(t) \geq C(b(t),t)$.

Proof:

$$\begin{aligned}
h(t) &= k\frac{C(b(t),t) - C_1(b(t),t)\big(b(t) - H/k\big)}{k - C_1(b(t),t)} = k\frac{C(b(t),t)\Big(1 - C_1(b(t),t)\frac{kb(t)-H}{C(b(t),t)}\frac{1}{k}\Big)}{k - C_1(b(t),t)} \\
&\geq k\frac{C(b(t),t)\big(1 - C_1(b(t),t)/k\big)}{k - C_1(b(t),t)} = \frac{C(b(t),t)\big(k - C_1(b(t),t)\big)}{k - C_1(b(t),t)} \\
&= C(b(t),t).
\end{aligned}$$

The inequality follows from Lemma 1.

QED

Lemma 12. $\frac{dh(t)}{dt} \leq 0$.

Proof:

$$\frac{dh(t)}{dt} = k\frac{C_{12}(b(t),t)\big(C(b(t),t) - (kb(t) - H)\big)}{\Big(k - C_1(b(t),t)\Big)^2} \leq 0.$$

The inequality follows from Lemmas 1, 6 and 8.

QED

Lemma 13. $\frac{dm(t)}{dt} \geq 0$.

Proof:

$$\frac{dm(t)}{dt} = C_{12}(a(t),t)(b(t) - a(t)) \geq 0.$$

QED

Armed with Lemmas 1 through 13 we can now prove that the convexity of the terminal condition is inherited by the solution $C(S,t)$ at all earlier dates.

Inherited Convexity Theorem. *Let $C(S,t)$ denote the solution to the p.d.e. in (A1) subject to the terminal condition $C(S,T) = g(S,T)$, where $g(\cdot)$ possesses the five properties listed at the start of Appendix B. For all $t < T$, $C(S,t)$ is convex in S.*

Proof: Suppose otherwise, and that for some t, $C(S,t)$ has one or more strictly concave regions. Any strictly concave region can be characterized as (I) $C(S,t)$ strictly concave in S for all $S > a(t) \geq 0$; i.e., no finite value for $b(t)$ exists, or (II) $C(S,t)$ strictly concave in S for all $S < b(t)$ finite; i.e., no nonzero value for $a(t)$ exists, or (III) $C(S,t)$ strictly concave in S for all $0 < a(t) < S < b(t)$ finite.

Case (I). *$C(S,t)$ is strictly concave in S for all $S \geq a(t) \geq 0$.*

Strict concavity in S for all $S \geq a(t)$ implies that for $\lambda > 1$ and $S \geq a(t)$,

$$C(\lambda S,t) < C(S,t) + (\lambda - 1)SC_1(S,t),$$

i.e., for all $S > a(t)$ the graph of the function lies strictly below a line tangential to the function at the point S. For $\epsilon > 0$, strict concavity and Lemma 2 imply $C_1(a(t)+\epsilon,t) < k$. Consider a value for λ such that

$$\begin{aligned}\lambda &> \frac{C(a(t)+\epsilon,t)+H-(a(t)+\epsilon)C_1(a(t)+\epsilon,t)}{(a(t)+\epsilon)\big(k-C_1(a(t)+\epsilon,t)\big)} \\ &\geq \frac{k(a(t)+\epsilon)-H+H-(a(t)+\epsilon)C_1(a(t)+\epsilon,t)}{(a(t)+\epsilon)\big(k-C_1(a(t)+\epsilon,t)\big)} = 1.\end{aligned}$$

For such a value of λ we have

$$\begin{aligned}C(\lambda(a(t)+\epsilon),t) &< C(a(t)+\epsilon,t)+(\lambda-1)(a(t)+\epsilon)C_1(a(t)+\epsilon,t) \\ &< C(a(t)+\epsilon,t)+\big(\lambda k(a(t)+\epsilon)-C(a(t)+\epsilon,t)-H\big) \\ &< \lambda k(a(t)+\epsilon)-H\end{aligned}$$

which violates Lemma 1. Thus Case (I) and Lemma 1 are inconsistent.

Case (II). *$C(S,t)$ is strictly concave in S for all $S < b(t)$ finite.*

At maturity the relation between C and S is convex. Thus the strict concavity at time t must disappear at some time $\mathcal{T} \leq T$. The event that at some earlier time τ, prior to the disappearance of the concavity at time $\mathcal{T}$, the relation between $C(S,\tau)$ and S becomes convex for values of $S \in (a(\tau)-\epsilon, a(\tau))$ with $\epsilon > 0$ and $a(\tau) < b(\tau)$ finite is considered under Case (III). We consider here only the event that at all times $\tau < \mathcal{T}$, $C(S,\tau)$ remains strictly concave in S for all $S < b(\tau)$.

Define $b^{\mathcal{T}}$ as $\lim_{\tau\to\mathcal{T}} b(\tau)$. We first establish that whenever a finite value for $b(t)$ exists and the region of concavity disappears at time $\mathcal{T}$, then $b^{\mathcal{T}}$ is finite. It follows from Lemmas 1 and 11 that for every $\tau \in (t,\mathcal{T})$,

$$kb(\tau) \leq H + h(\tau).$$

Lemma 12 then implies

$$kb^{\mathcal{T}} \leq H + \lim_{\tau \to \mathcal{T}} h(\tau) \leq H + h(t).$$

So $b^{\mathcal{T}}$ is bounded from above and cannot go to infinity as $\tau \to \mathcal{T}$. Thus and the region of strict concavity cannot disappear through a process whereby Case (II) first merges into Case (I).

Having established for Case (II) that $b^{\mathcal{T}} < \infty$, we now also establish for Case (II) that $b^{\mathcal{T}} > 0$. Since, under Case (II), $C(S,t)$ is strictly concave in S for $S \in (0, b(t))$,

$$C_1(b(t),t) < \frac{C(b(t),t)}{b(t)}$$

and hence

$$g(t) = k\frac{C(b(t),t) - C_1(b(t),t)b(t)}{k - C_1(b(t),t)} > 0. \qquad (B3)$$

From Lemmas 1, 9 and 10 and relation $(A3)$, we have for every $\tau \in [t, \mathcal{T})$

$$kb(\tau) \geq C(b(\tau),\tau) \geq g(\tau) \geq g(t) > 0$$

and therefore

$$\lim_{\tau \to \mathcal{T}} b(\tau) > 0.$$

With these bounds on $b^{\mathcal{T}}$ determined, it follows that at time $\mathcal{T}$ when the region of strict concavity disappears, $C(S, \mathcal{T})$ becomes affine in S for $S \in (0, b^{\mathcal{T}})$. The existence of $C_{11}(S_\tau, \tau)$ for all $\tau < T$, and the condition that $C(0,\tau) = 0$ for all τ, then imply that

$$\lim_{\tau \to \mathcal{T}} C_1(b(\tau),\tau) = \frac{\lim_{\tau \to \mathcal{T}} C(b(\tau),\tau)}{b^{\mathcal{T}}}.$$

Thus

$$\lim_{\tau \to \mathcal{T}} g(\tau) = k\frac{\lim_{\tau \to \mathcal{T}} C(b(\tau),\tau) - \lim_{\tau \to \mathcal{T}} C_1(b(\tau),\tau)b^{\mathcal{T}}}{k - \lim_{\tau \to \mathcal{T}} C_1(b(\tau),\tau)} = 0. \qquad (B4)$$

The quantity $\lim_{\tau \to \mathcal{T}} g(\tau)$ is well defined since, from Lemmas 6 and 8,

$$\lim_{\tau \to \mathcal{T}} C_1(b(\tau),\tau) \leq C_1(b(t),t) < k.$$

But relations $(A3)$, $(A4)$ and Lemma 10 are contradictory. It cannot be that for $t < \mathcal{T}$, $g(t) > 0$ (relation $(A3)$), yet $\frac{dg(\tau)}{d\tau} \geq 0$ (Lemma 10) and $\lim_{\tau \to \mathcal{T}} g(\tau) = 0$ (relation $(A4)$). Thus the option price cannot exhibit the strict concavity considered under Case (II).

Case (III). $C(S,t)$ *is strictly concave in* S *for all* $0 < a(t) < S < b(t) < \infty$.

Again the region of strict concavity must disappear at some time $\mathcal{T} \leq T$. Recall that $\lim_{\tau \to \mathcal{T}} b(\tau)$ is finite. The disappearance of the region of strict concavity requires either that (i) $\lim_{\tau \to \mathcal{T}} a(\tau) = \lim_{\tau \to \mathcal{T}} b(\tau) \geq 0$, or (ii) $\lim_{\tau \to \mathcal{T}} C_1(a(\tau), \tau) = \lim_{\tau \to \mathcal{T}} C_1(b(\tau), \tau)$, or (iii) $\lim_{\tau \to \mathcal{T}} b(\tau) = 0$ and for some time $t'' < \mathcal{T}$, $\lim_{\tau \to t''} a(\tau) = 0$.

In the event of either (i) or (ii),

$$\lim_{\tau \to \mathcal{T}} m(\tau) = 0.$$

But this contradicts $m(t) > 0$ and Lemma 13.

In the event of (iii), $C(S, t'')$ is strictly concave in S for $S \in (0, b(t''))$ and hence $g(t'') > 0$. Combining Lemmas 1, 9 and 10 gives $kb(\tau) \geq C(b(\tau), \tau) \geq g(\tau) \geq g(t'')$ for all $\tau \in (t'', \mathcal{T})$. Hence

$$\lim_{\tau \to \mathcal{T}} kb(\tau) \geq g(t'') > 0$$

and therefore

$$\lim_{\tau \to \mathcal{T}} b(\tau) > 0$$

which is inconsistent with the occurrence of event (iii) itself. Thus the option price cannot exhibit the strict concavity considered under Case (III).

QED

Appendix C

Proof of strict convexity of call prices for all s and t such that $\max[0, s - Ke^{-\int_t^T r(\tau)d\tau}] < c(s,t) < s$.

S and C are the normalized prices of the underlying asset and the call as defined in the introductory paragraph of section IV. Suppose that at time t' the strict convexity claim is violated for prices in some region. Let $a(t')$ and $b(t') \geq a(t')$ denote the normalized prices marking the end points of that region; i.e., for all $S \in (a(t'), b(t')]$, $\max[0, S-K] < C(S,t') < S$ yet $C_{11}(S,t') = 0$.

Suppose $a(t') > 0$. Since $C(a(t'),t') > \max[0, a(t') - K]$, yet $C(a(t'),T) = \max[0, a(t') - K]$, there must exist a set of times $\tau \in (t', T]$ at which $C_2(a(t'),\tau) < 0$. Let us then consider the particular value of t' such that not only is $\max[0, a(t') - K] < C(a(t'),t') < a(t')$ and $C_{11}(a(t'),t') = 0$, but for some $t'' > t'$ we have that for all $\tau \in (t', t'')$, $C_2(a(t'),\tau) < 0$. Given the p.d.e. in (7) we see immediately that for all $\tau \in (t', t'')$, $C_{11}(a(t'),\tau) > 0$ and $v(a(t'),\tau) > 0$. Assume that $v(a(t'),t') > 0$.[C1]

Since $C_2(a(t'),t') = 0$ and $v(a(t'),t') > 0$, there then exists an arbitrarily small positive $\bar{\varepsilon}$ such that for all $\varepsilon \in (0,\bar{\varepsilon})$, not only is $C_{11}(a(t')-\varepsilon,t') > 0$ and $v(a(t')-\varepsilon,t') > 0$, but $C_{12}(a(t')-\varepsilon,t') > 0$. Further, since $C_2(a(t'),t') = 0$ and for all S, $C_2(S,t') \leq 0$, it follows that $C_{12}(a(t'),t') = 0$.

Strict convexity for all $\tau \in (t', t'')$ requires that for time t'^+,

$$\lim_{\varepsilon \to 0} \frac{C_1(a(t'),t'^+) - C_1(a(t')-\varepsilon,t'^+)}{\varepsilon} > 0.$$

[C1] The condition $v(a(t'),t') > 0$ can be relaxed. But this leads to the significant use of mathematical tools like Newton diagrams and the analysis of singularities. More about the techniques used to analyze dynamic behavior near local minimum can be found in Wiener (1993). Observe that $[v(S,t)S]^2$ is a non-negative function which can therefore have zero values only at local minimums. In any particular case, one should differentiate the p.d.e. in (7) a sufficient number of times until the singularity is resolved.

But since $C_{12}(a(t'), t') = 0$,

$$\lim_{\varepsilon \to 0} \frac{C_1(a(t'), t'^+) - C_1(a(t') - \varepsilon, t'^+)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{C_1(a(t'), t') - C_1(a(t') - \varepsilon, t'^+)}{\varepsilon},$$

which, since $C_{12}(a(t') - \varepsilon, t') > 0$,

$$\leq \lim_{\varepsilon \to 0} \frac{C_1(a(t'), t') - C_1(a(t') - \varepsilon, t')}{\varepsilon}$$

$$= C_{11}(a(t'), t') = 0,$$

and we have a contradiction.

Analogous arguments rule out a finite value for $b(t')$. Finally, the possibility that $a(t') = 0$ and $b(t')$ is infinite is equivalent to the internally contradictory claim that for all $S > 0$, $\max[0, S - K] < C(S, t') < S$, yet $C_{11}(S, t') = 0$.

QED

References

Arnold, Ludwig, 1992, *Stochastic Differental Equations: Theory and Evidence* (Krieger Publishing Company, Malabar, Fla.).

Black, Fisher, 1976, Studies of stock market volatility changes, *Proceedings of the American Statistical Association, Business and Economic Statistics Section*, 177–181.

Black, Fisher, and Myron Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* 81, 637–659.

Breeden , Douglas T., and Robert H. Litzenberger, 1978, Prices of state contingent claims implicit in option prices, *Journal of Business* 51, 621–652.

Cox, John C., and Stephen A. Ross, 1976, A survey of some new results in financial option pricing theory, *Journal of Finance* 31, 383–402.

Cox, John C., and Mark Rubinstein, 1985, *Options Markets* (Prentice Hall, Inc., Englewood Cliffs, N.J.).

Duffie, Darrell, 1992, *Dynamic Asset Pricing Theory* (Princeton University Press, Princeton, N.J.).

Dixit, Avinash K., and Robert S. Pindyck, 1994, *Investment under Uncertainty* (Princeton University Press, Princeton, N.J.).

El Karoui, Nicole, Monique Jeanblanc-Picque and Ravi Visvanathan, On the robustness of Black-Scholes equation, Undated Laboratoire de Probabilités Paris VI manuscript.

Fischer, Edwin O., Robert Heinkel and Josef Zechner, 1989, Dynamic capital structure choice: Theory and tests, *Journal of Finance* 44, 19–40.

Gordon, R.D., 1941, Values of Mill's ratio of area to bounding ordinate and of the normal probability integral for large values of the arguments, *Annals of Mathematical Statistics* 12, 364-366.

Geske, Robert, 1979, The valuation of compound options, *Journal of Financial Economics* 7, 63–82.

Grundy, Bruce D., 1991, Option prices and the underlying asset's return distribution, *Journal of Finance* 46, 1045-1069.

Hull, John C., 1993, *Options, Futures, and other Derivative Securities* (Prentice Hall, Inc., Englewood Cliffs, N.J.).

Hull, John C., and Alan White, 1988, The pricing of options on assets with stochastic volatilities, *Journal of Finance* 42, 281–300.

Jagannathan, Ravi K., 1984, Call options and the risk of underlying securities, *Journal of Financial Economics* 13, 425–434.

Jarrow, Robert A., and Andrew Rudd, 1983, *Option Pricing* (Richard D. Irwin, Inc., Homewood, Il.).

Karlin, Samuel and Howard M. Taylor, 1981,*A Second Course in Stochastic Processes* (Academic Press, San Diego, Ca.).

Karatzas, Ioannis and Steven E. Shreve, 1991, Brownian Motion and Stochastic Calculus (Springer-Verlag, New York, N.Y.).

Lauterbach, Beni and Paul Schultz, 1990, Pricing warrants: An empirical study of the Black-Scholes model and its alternatives, *Journal of Finance* 45, 1181-1209.

Merton, Robert C., 1973, Theory of rational option pricing, *Bell Journal of Economics and Management Science* 4, 141–183.

Merton, Robert C., 1976, Option pricing when underlying stock returns are discontinuous, *Journal of Financial Economics* 3, 125–144.

Mohammed, S.E.A., 1978, Retarded Functional Differential Equations: A Global Point of View (Pitman Publishing, London, UK.).

Nelson, Daniel B., 1991, Conditional heteroskedasticity in asset returns: A new approach, *Econometrica* 59, 347–370.

Rothschild, Michael and Joseph E. Stiglitz, 1970, Increasing risk I: A definition, *Journal of Economic Theory* 2, 225–243.

Rubinstein, Mark, 1983, Displaced diffusion option pricing, *Journal of Finance* 38, 213–217.

Wiener, Zvi, 1993, Instability with two zero frequencies, *Journal of Differential Equations* 103, 58-68.

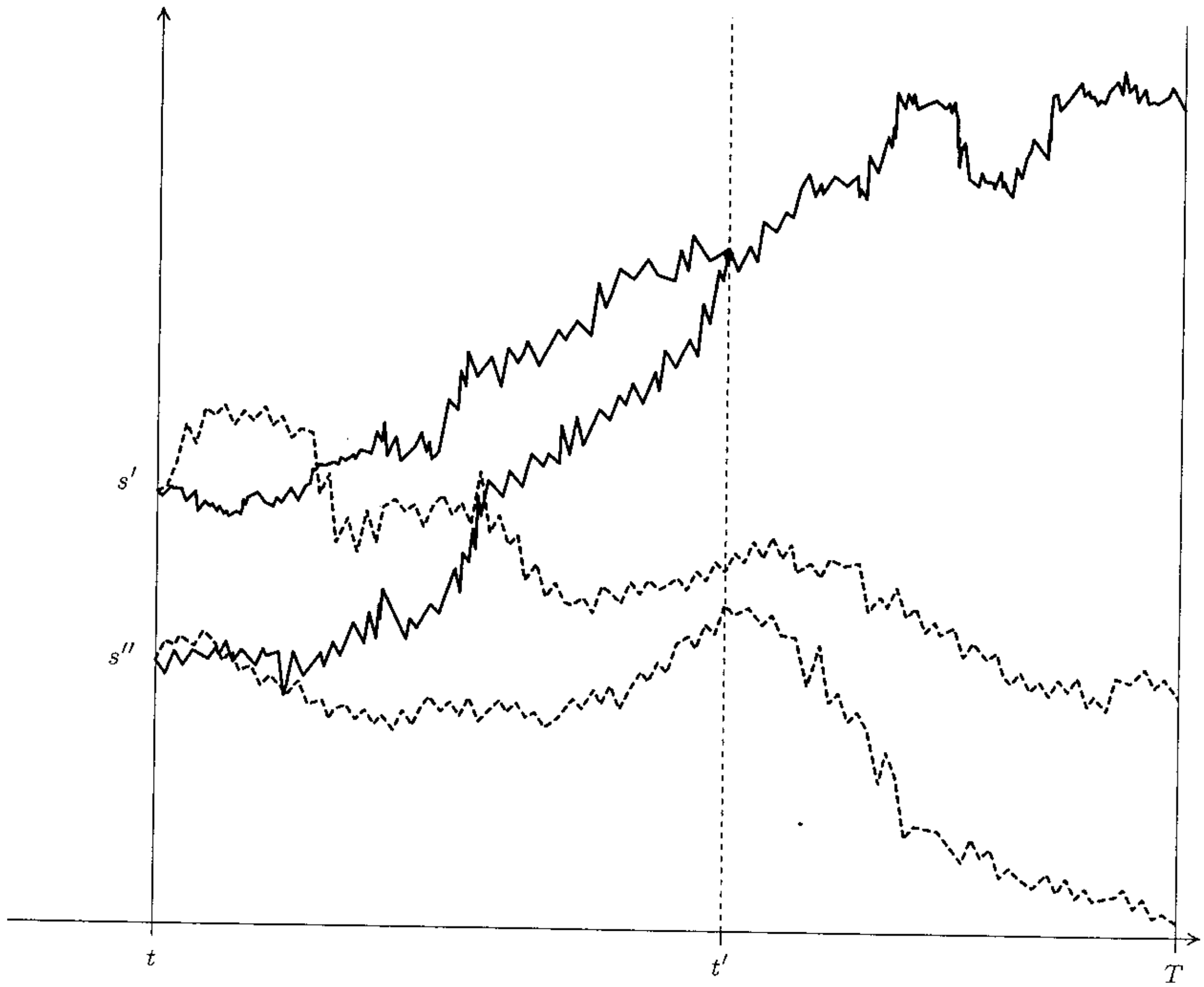


Figure 1. Illustration of feasible sets of sample paths for ξ_τ given $\xi_t = s''$ and $\xi_t = s' > s''$, $d\xi_\tau = r(\tau)\,\xi_\tau\,d\tau + \sigma(\xi_\tau,\tau)\xi_\tau\,dB_\tau$.

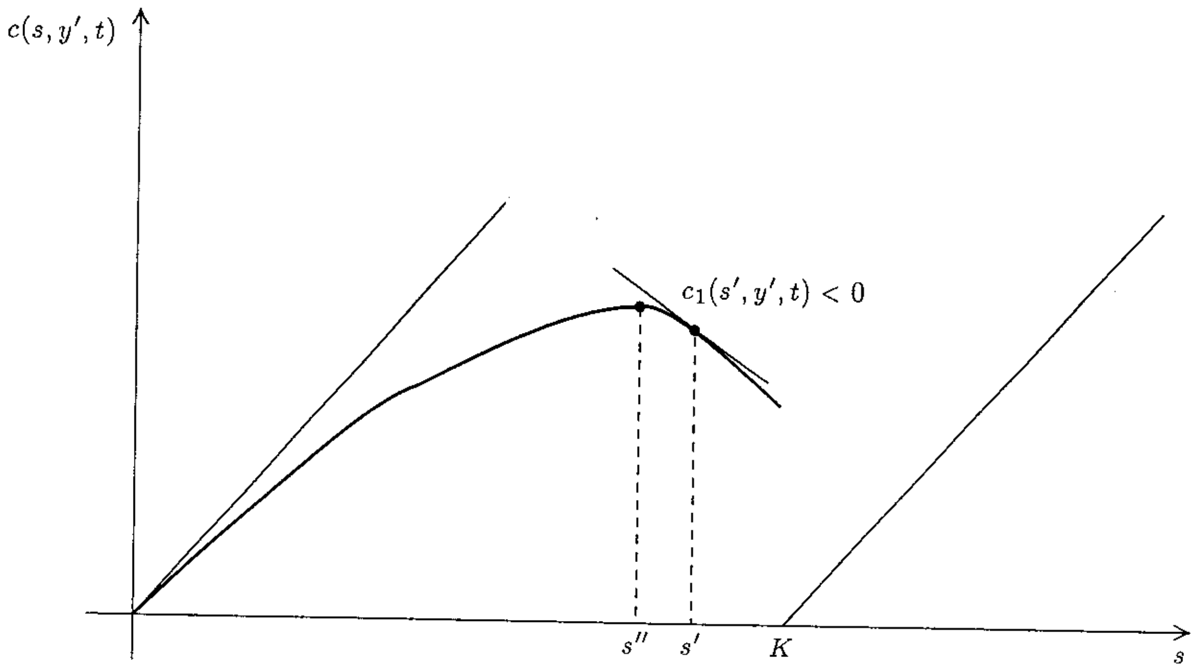


Figure 2a.

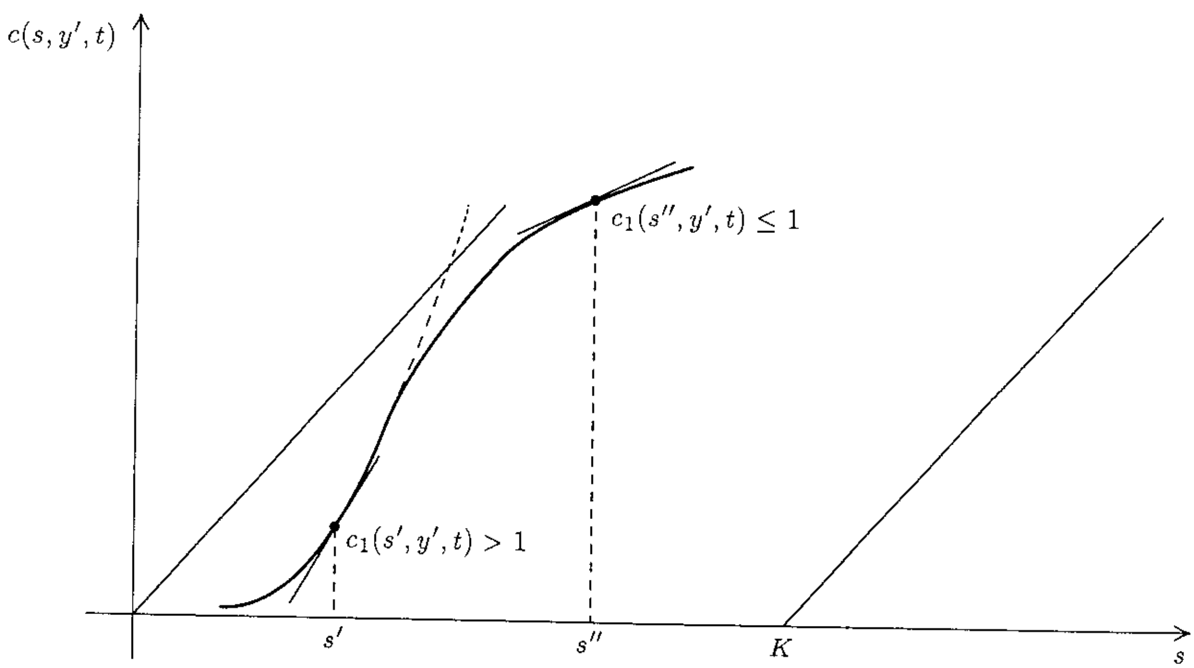


Figure 2b.

Figure 2. Demonstration that unless $0 \leq c_1(s, y, t) \leq 1$ for all s, y, and t, $c(s, y, t)$ is concave in s over some region.

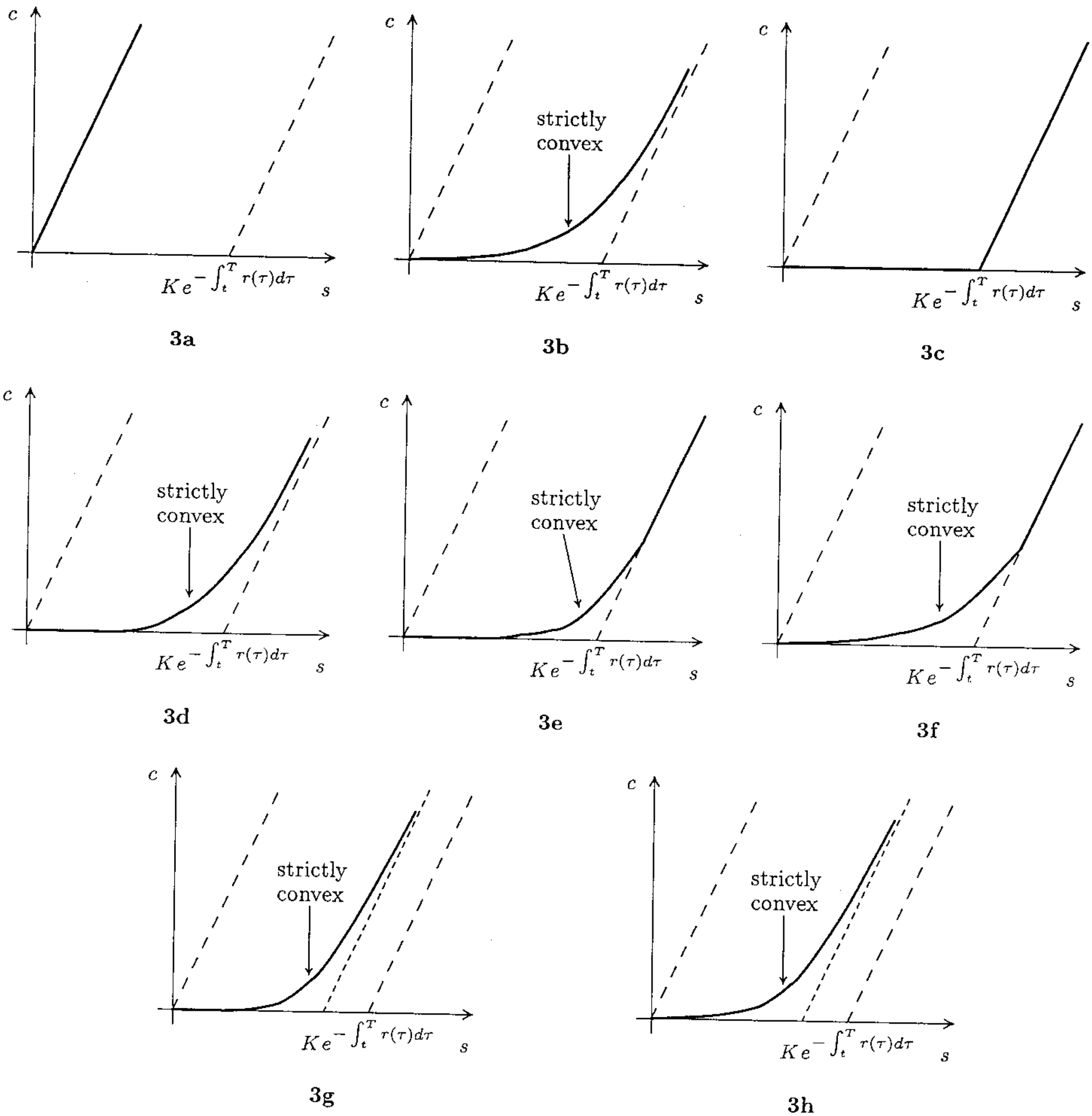


Figure 3. **Illustration of the possible relations between $c(s,t)$ and s consistent with $0 \leq c_1(s,t) \leq 1$ (Proposition 1) and $c_{11}(s,t) \geq 0$ for all s and t, and $c_{11}(s,t) > 0$ for all s and t such that $0 < c(s,t) < s$ (Proposition 2).**

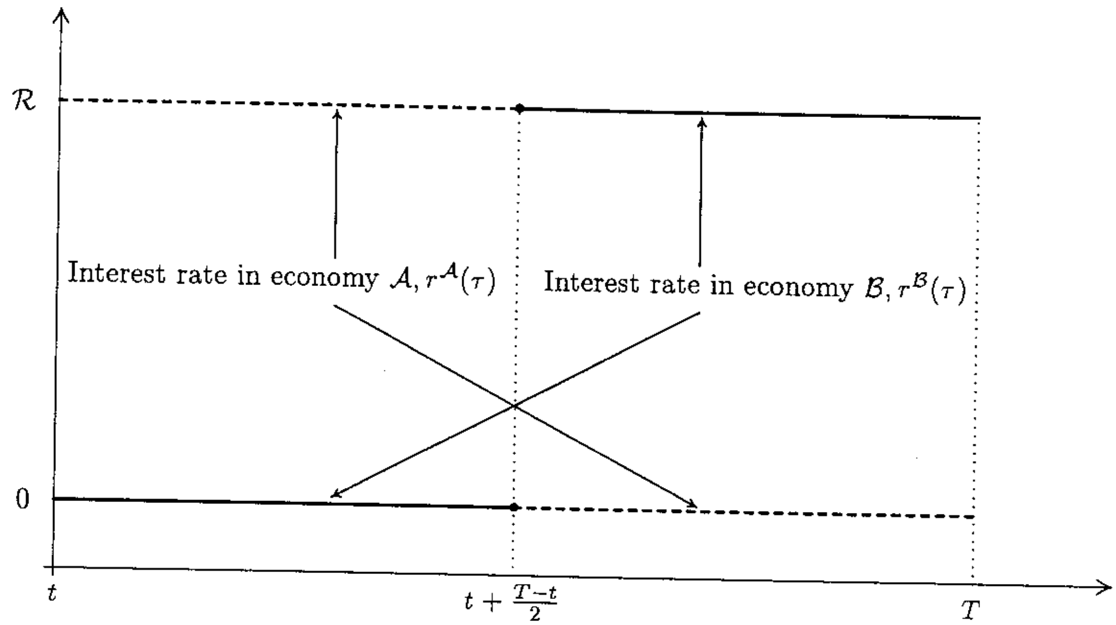


Figure 4a. Interest rates in the otherwise equivalent economies $\mathcal{A}$ and $\mathcal{B}$.

$$\int_t^T r^{\mathcal{A}}(\tau)d\tau = \frac{T-t}{2}\mathcal{R} = \int_t^T r^{\mathcal{B}}(\tau)d\tau$$

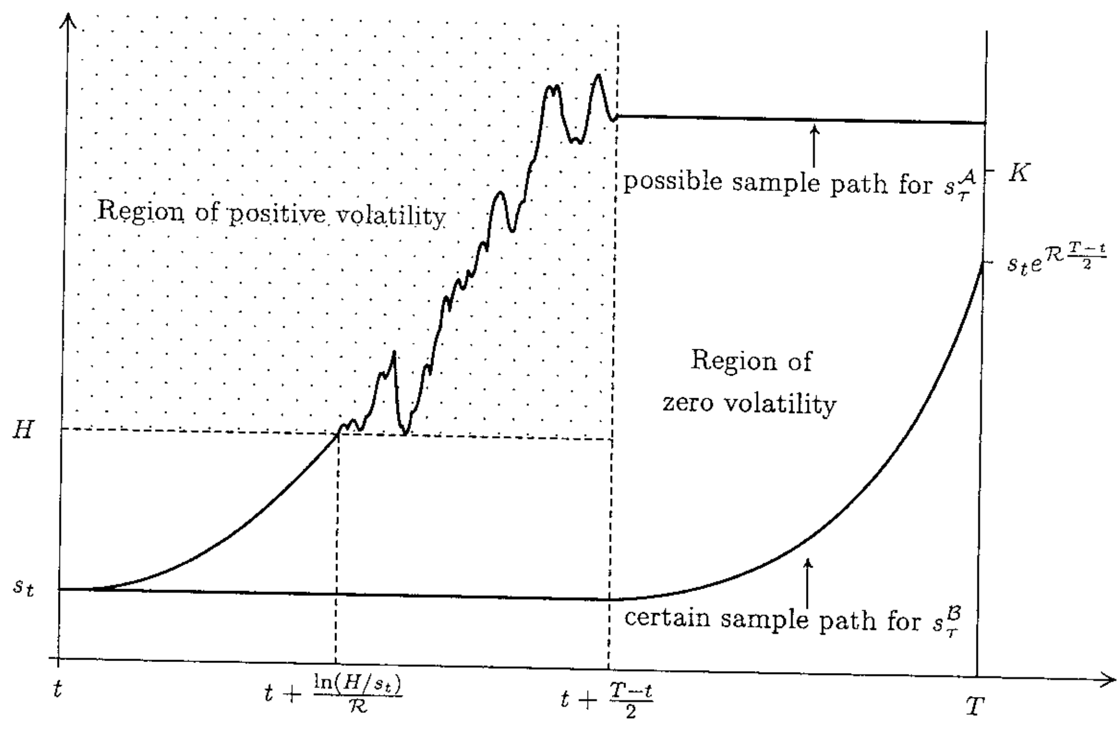


Figure 4b. Possible sample path for $s_T^{\mathcal{A}}$.
Certain sample path for $s_T^{\mathcal{B}}$. $s_T^{\mathcal{B}} < K < s_T^{\mathcal{A}}$.

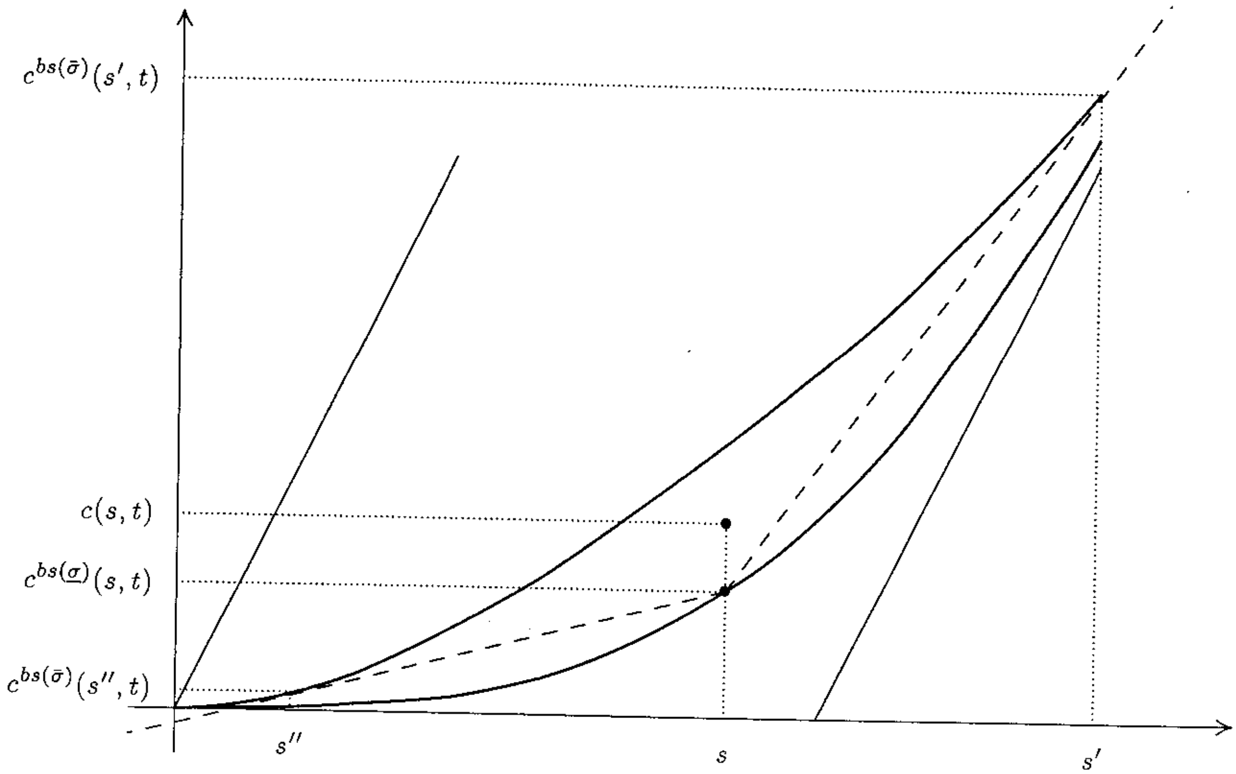


Figure 5. Illustration of the bounds on a call's delta when for all s and t,

$$\underline{\sigma} \leq \sigma(s, t) \leq \bar{\sigma}.$$

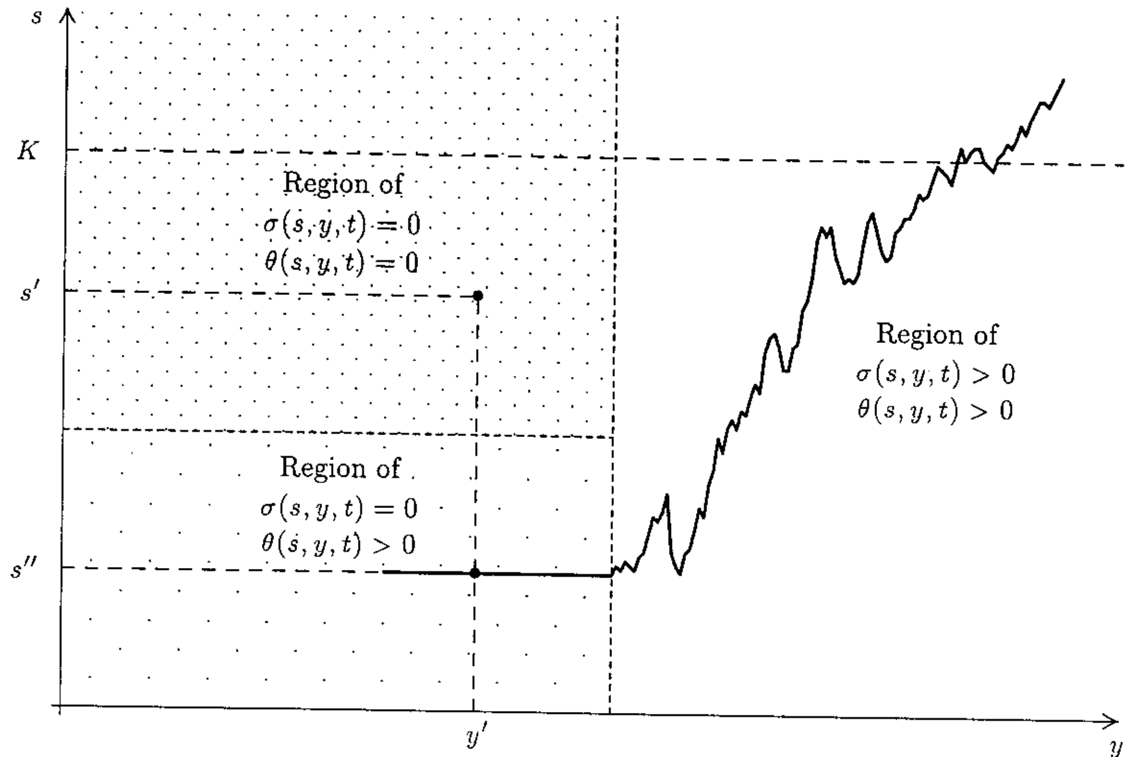


Figure 6. Two-dimensional diffusion. Certain sample path given initial condition $\{s', y'\}$. Possible sample path given initial condition $\{s'', y'\}$.

$$d\xi_t^1 = \sigma(\xi_t^1, \xi_t^2, t)\xi_t^1 dB_t^1.$$
$$d\xi_t^2 = \theta(\xi_t^1, \xi_t^2, t) dB_t^2.$$
$$0 = c(0, y', t) < c(s'', y', t) > c(s', y', t) = 0.$$

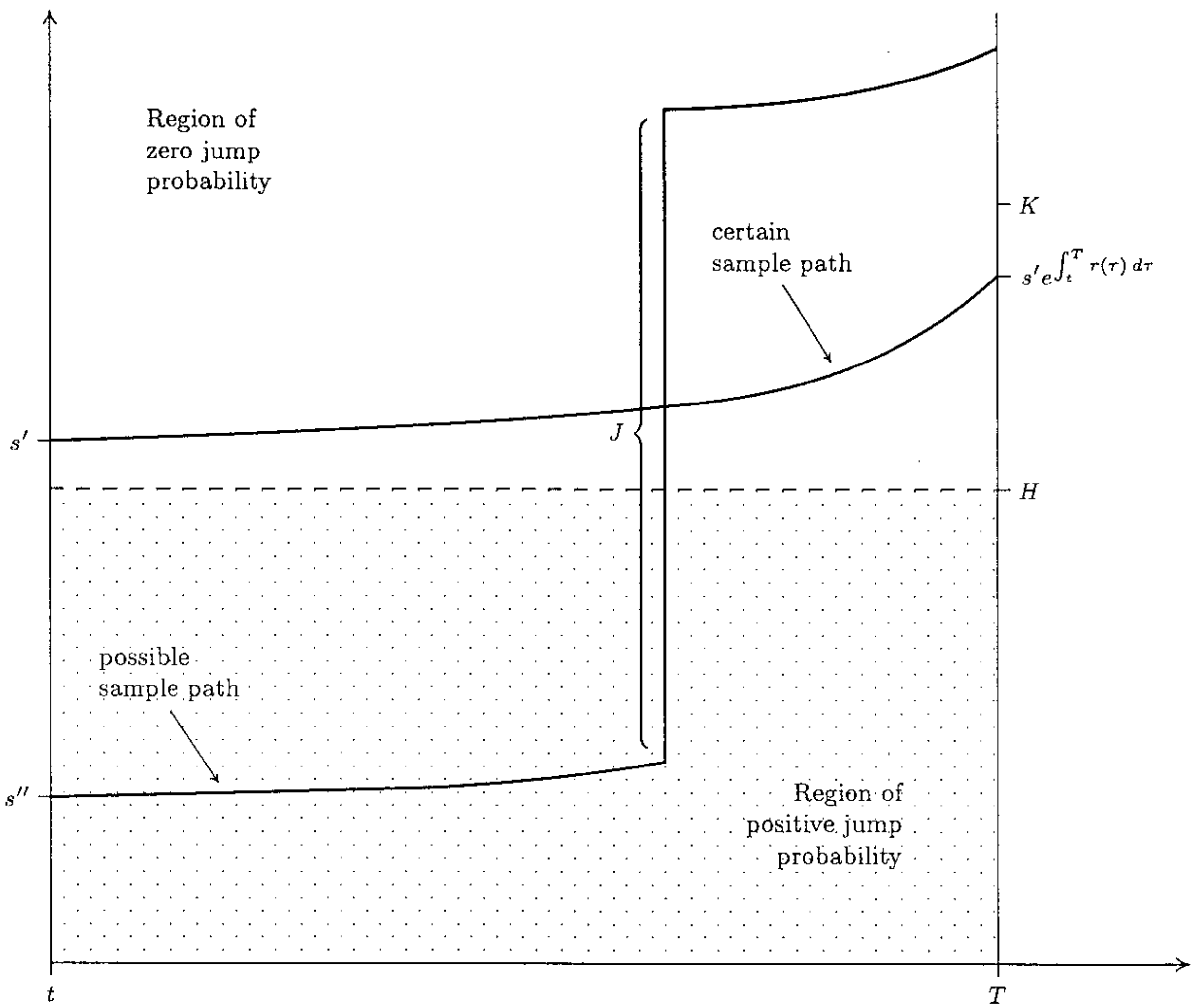


Figure 7. A Markovian world with a mixed diffusion-jump process for $s_\tau < H$ and a deterministic process for $s_\tau \geq H$. Possible sample path given $s_t = s'' < H$. Certain sample path given $s_t = s' > H$.

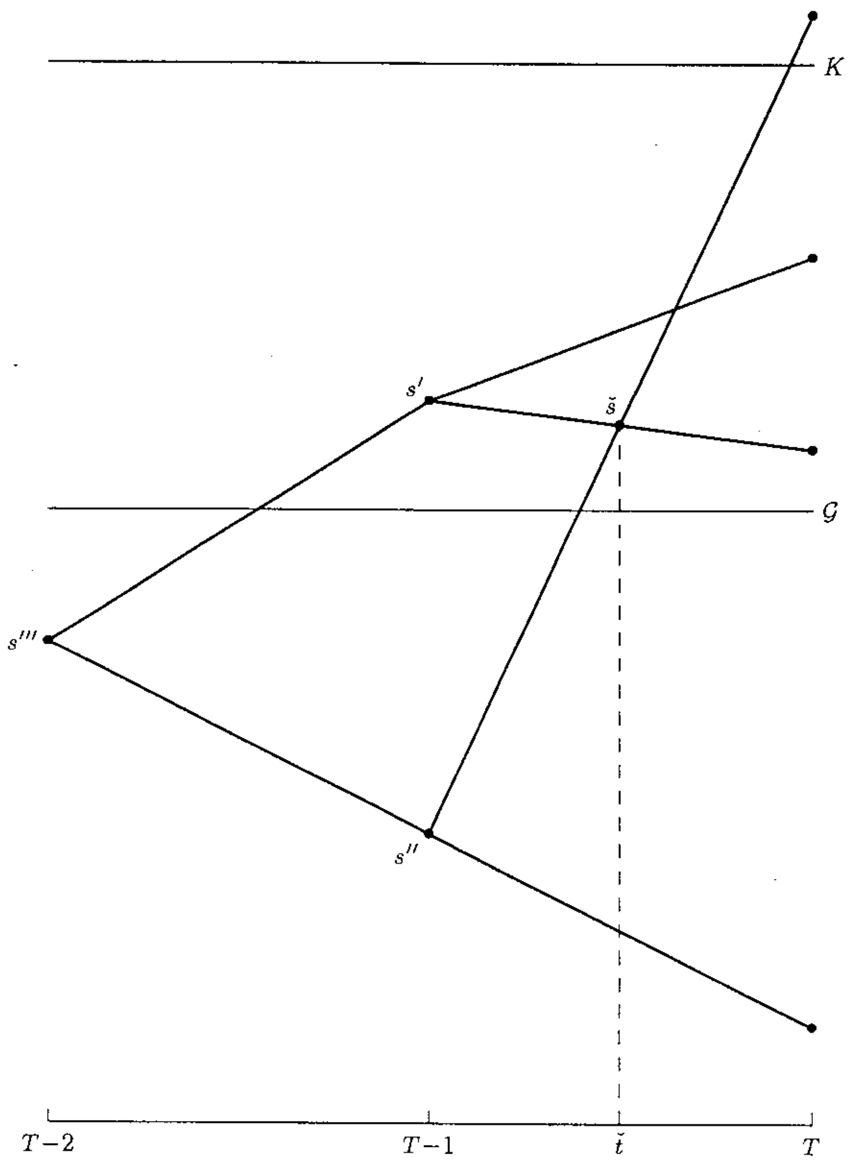


Figure 8. Non-recombining binomial tree.

$$0 = c(0, T-1) < c(s'', T-1) > c(s', T-1) = 0 \,.$$

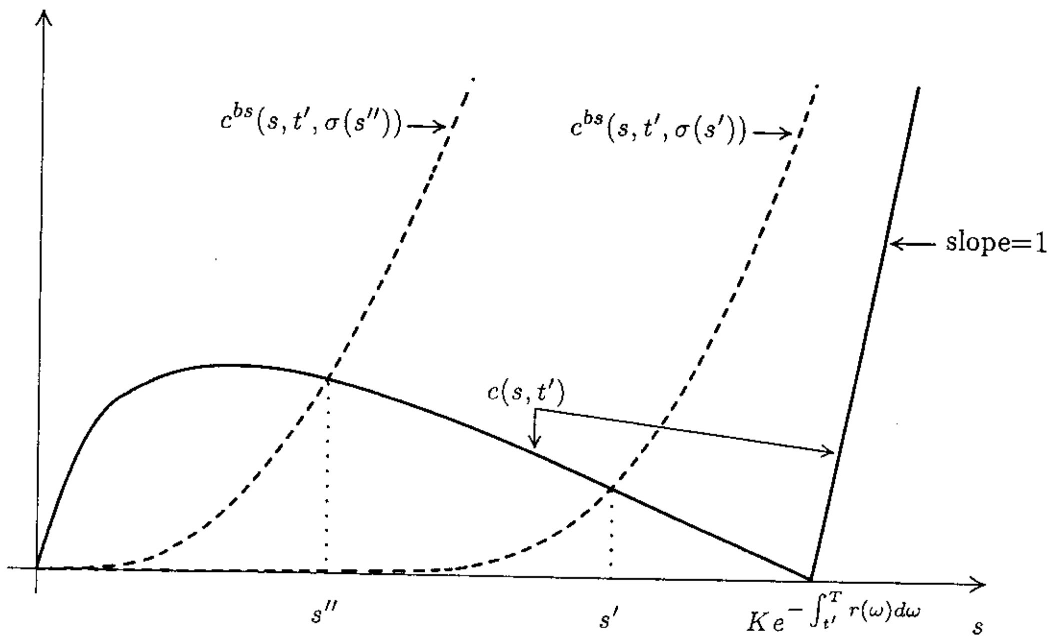


Figure 9a. The relation between $c(s,t')$ and s in a non-Markovian world. The call is not everywhere non-decreasing and convex in s.

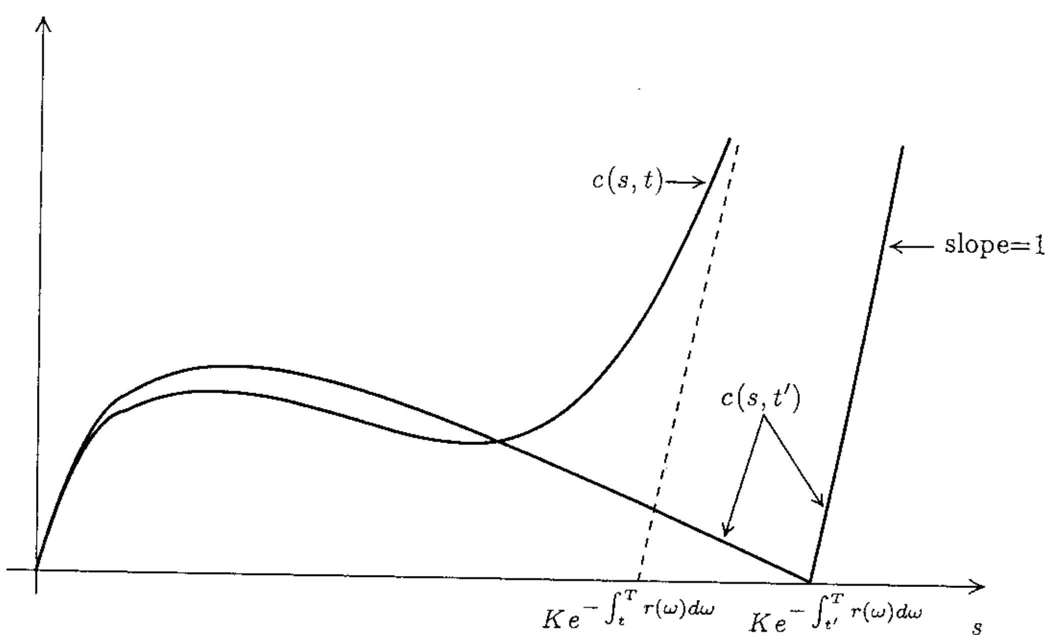


Figure 9b. The relation between $c(s,\tau)$ and τ in a non-Markovian world. $t < t'$. The call is not, for all s and t, a "wasting" asset.

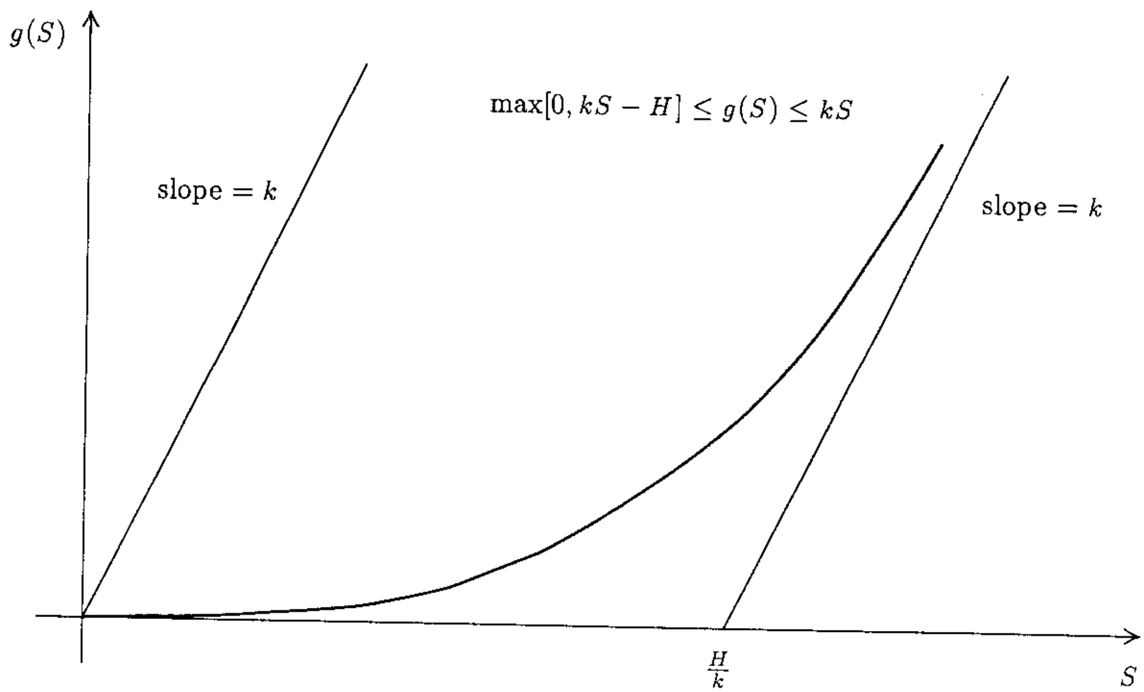


Figure B1. The relation between k, H, and $g(S)$.

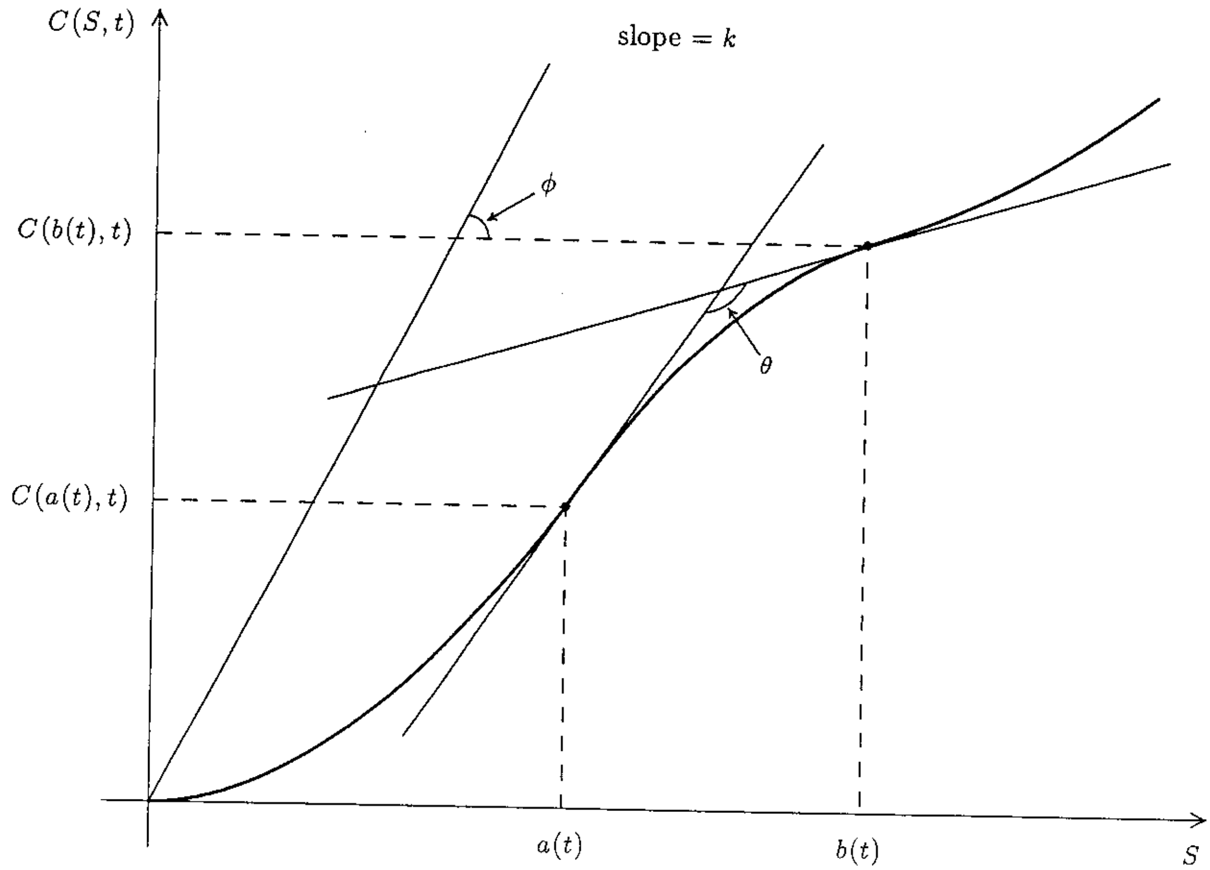


Figure B2. The angle θ as a measure of the concavity in the relation between $C(S,t)$ and S.

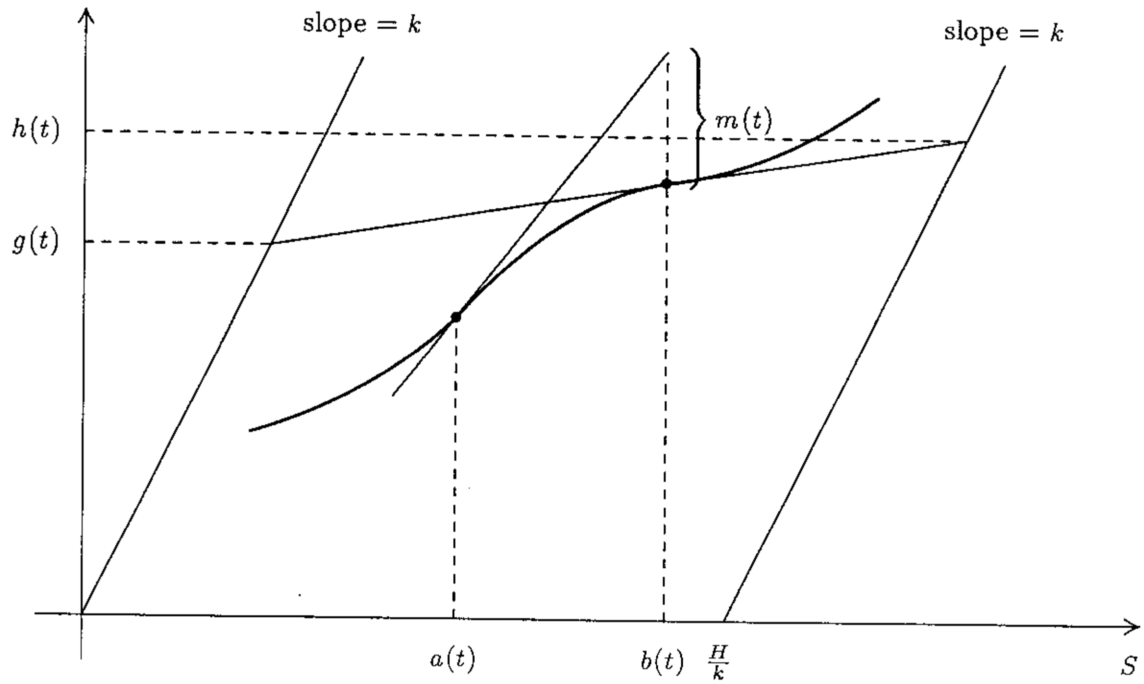


Figure B3. The relation between $a(t)$, $b(t)$, $g(t)$, $h(t)$, and $m(t)$.