DYNAMIC CONSUMPTION-PORTFOLIO CHOICE AND ASSET PRICING WITH NON-PRICE-TAKING AGENTS

by

Süleyman Başak

7-95

RODNEY L. WHITE CENTER FOR FINANCIAL RESEARCH
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6367

Dynamic Consumption-Portfolio Choice and Asset Pricing with Non-Price-Taking Agents

Suleyman Basak*
Finance Department
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6367
(215) 898-6087

This Revision January 1995

*I am grateful to Domenico Cuoco, Bob Dammon, Rick Green, Richard Kihlstrom, Bart Lipman, Duane Seppi, Steve Shreve, Jean-Luc Vila, Jiang Wang, and seminar participants at Carnegie Mellon University, Temple University, and the Wharton School of the University of Pennsylvania for their helpful comments. All errors are solely my responsibility.

Dynamic Consumption-Portfolio Choice and Asset Pricing with Non-Price-Taking Agents

Abstract

This paper develops a continuous-time pure-exchange model to study the dynamic consumption-portfolio problem of an agent who acts as a non-price-taker, and to analyze the implications of his behavior on equilibrium security price dynamics. The non-price-taker is modeled as a price leader in all markets; his price impact is then recast as an effect of consumption on the Arrow-Debreu prices, allowing the use of martingale methods in a tractable way. Besides the aggregate consumption, the endowment stream of the non-price-taker appears as an additional factor in driving equilibrium allocations and prices. Comparisons of equilibria between a price-taking and a non-price-taking economy are carried out.

Journal of Economic Literature Classification Numbers: C60, D40, D50, D90, G11, G12.

1 Introduction

Central to the equilibrium-based asset pricing models in finance is the competitive agents paradigm: each agent is atomistic relative to the market, and hence takes prices to be unaffected by his or her actions. However, an observation of today's security markets (and especially government bond markets) reveals the ever-increasing importance of large pension funds and financial institutions in the marketplace. "Large" investors are particularly prevalent in smaller security markets outside the U.S.A. (e.g., Belgium, France, Hong Kong, Singapore, Sweden). Such a "large" investor may have a significant effect on prices, and hence may prefer to choose a strategy taking the price impact of his own behavior into account. It is well-known that large trades do have a permanent price impact. This is attributable partially to the information a large trade reveals about future cash flows (Kraus and Stoll [24], Holthausen, Leftwich, and Mayers [16, 17], Seppi [33]), but conceivably also to the effect of such a large position on security supply and demand. Internationally, there is certainly widespread anectodal evidence that large trades can affect prices independently of any information they may contain. It would be of interest to re-investigate the traditional equilibrium-based asset pricing models in the presence of "non-price-taking" investors, who take account of the price impact of the positions they take on (independent of any information revealed by their trading).

The objective of this paper is to study the dynamic optimization of a non-price-taking agent and to analyze the effect of the presence of a non-price-taker on dynamic market equilibrium. Part of this objective is for the formulation to be fully consistent with rational expectations, no arbitrage, and market clearing. Our approach is to introduce a non-price-taking agent into a standard asset pricing environment (Lucas [27]), while retaining the usual assumptions of complete and frictionless markets, and symmetric information. To this end, we develop a continuous-time equilibrium model of a pure-exchange economy consisting of $N-1$ price-taking agents and one non-price-taking agent. The presence of the non-price-taker is specified exogenously and not generated endogenously, for example, by his superior information over the other agents, (e.g., Kyle [25]) or by the size of his security holdings.[1]

Our notion of equilibrium is in the spirit of the price leadership model in the oligopoly literature (e.g., Varian [35, Chapter 16]). Accordingly, the non-price-taker chooses his consumption-portfolio process, aware that the price processes must adjust so that the remaining (price-taking) agents' optimal demands clear all security and consumption good markets.[2] We recast the non-price-taker's impact on the security market prices as an impact of his consumption choice on the

Arrow-Debreu prices. This "consumption-based" formulation of the problem allows us to adapt martingale methods (Cox and Huang [6, 7], Karatzas, Lehoczky and Shreve [21], Pliska [31]) in a natural way, making the analysis highly tractable. If the preferences of the price-takers are such that their representative agent's utility does not depend on their individual wealth allocations, the non-price-taker's consumption at any given time and state affects the state price (the price of one unit of consumption) only at that time and state; otherwise the whole state price process is affected. We focus on the former case for most of the paper.

Solving for the equilibrium consumption allocations reveals that the non-price-taking agent deviates from his price-taking behavior by tending to move his consumption towards his endowment stream. The extent of this deviation depends on how much of a net trader the non-price-taker is, and on the risk aversion of the other agents in the economy. In addition to the aggregate consumption stream, the non-price-taker's endowment stream appears as an extra factor in explaining the equilibrium asset and Arrow-Debreu prices, and their dynamics. This leads to a two-factor consumption-based CAPM, stating that an asset's risk premium depends on the covariance of its return with changes in the non-price-taking agent's endowment stream as well as with changes in the aggregate consumption.

To derive further implications of non-price-taking behavior, we specialize to the case of all agents' preferences exhibiting constant absolute risk aversion (CARA) and one risky asset. When the non-price-taker is initially wealthy relative to the rest of the market, he is found to react more to changes in the aggregate consumption than if he were a price-taker, and as a result his consumption drift and volatility increase. This consumption behavior leads to an increase in the volatility of the Arrow-Debreu price return (i.e., the market price of risk). The reverse is true when the non-price-taker has a low endowment compared with the rest of the market. Using tools from Malliavin calculus, in particular the Clark-Ocone formula, we derive representations for the agents' portfolio strategies and the equity market volatility and risk premium, providing insights into the effects of a non-price-taker. (For related applications of Malliavin calculus in finance, see Detemple and Zapatero [9]—interest rate and risk premium formulae—and Ocone and Karatzas [30]—optimal portfolio representations.)

The closest work related to this paper is Lindenberg [26]. In contrast to our dynamic formulation, Lindenberg works in a single-period mean-variance framework. Some of the agents in his model recognize that the market clearing security prices depend on their positions and formulate their optimization problems accordingly. He finds the non-price-taking investors to hold an optimal portfolio that is unbalanced (i.e., contains differing percentages of the supply of shares of each security). A two-factor CAPM results, in which an asset's risk premium is driven by the covariance of its return with the market return and with the return on the aggregate portfolio of the price-affecting investors. Taking a quite different starting point, Jarrow [20] exogenously specifies a dependence of asset prices on the non-price-taker's trading strategy, hence not working in an equilibrium framework. The main focus of his work is on the existence of market manipulation strategies which essentially generate arbitrage opportunities for the non-price-taker.

The remainder of our paper is organized as follows. Section 2 outlines the pure-exchange continuous-time framework of our model. In Section 3 we present the martingale formulation of the non-price-taking equilibrium and characterize the agents' equilibrium consumption allocations. Section 4 derives the modified consumption-based CAPM and interest rate formulae. In Section 5 we specialize to the CARA utility and one risky asset to derive further results. In Section 6 we summarize our conclusions and propose further work. The Appendices provide the proofs of all propositions and corollaries.

2 General Formulation

This section describes a continuous-time variation of the Lucas [27] pure-exchange economy. The formulation follows the continuous-time pure exchange general equilibrium models recently developed by Duffie and Huang [12], Duffie [11], Huang [18], Duffie and Zame [13], Karatzas, Lehoczky and Shreve [22] and Karatzas, Lakner, Lehoczky and Shreve [23]. We consider a finite horizon $[0,T]$ economy in which there is a single consumption good. All quantities (prices, endowments etc.) are expressed in units of this consumption good. We let $W = (W_1, ..., W_L)^\top$ be an L-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and let $\{\mathcal{F}_t; t \in [0,T]\}$ be the augmentation by null sets of the filtration generated by W. All the uncertainty in the economy is represented by this L-dimensional Brownian motion. All the stochastic processes appearing in the analysis are assumed to be adapted to $\{\mathcal{F}_t\}$. All the stated equalities involving random variables hold $\mathcal{P}$-almost surely.

2.1 Securities

We assume there are $L+1$ securities. One is an instantaneously riskless bond in zero net supply; L are risky stocks, each in constant net supply of 1 and paying out a dividend stream at rate $\delta_i(t)$

in $[0,T]$. We assume the following dynamics for the aggregate dividend process $\delta(t) \equiv \sum_{i=1}^{L} \delta_i(t)$:

$$d\delta(t) = \mu_\delta(t)dt + \sum_{j=1}^{L} \sigma_{\delta j}(t) dW_j(t) , \quad t \in [0,T] ,$$

where $\mu_\delta(\cdot)$ and $\sigma_{\delta j}(\cdot)$ are $\mathcal{F}_t$-measurable processes.

Each security price is modeled as a diffusion process relative to the Brownian filtration. The bond and risky stock price dynamics are

$$dP_0(t) = P_0(t) r(t) dt, \quad t \in [0,T] ,$$

$$dP_i(t) + \delta_i(t)dt = P_i(t) \left[\mu_i(t)dt + \sum_{j=1}^{L} \sigma_{ij}(t) dW_j(t) \right] , \quad i = 1, \ldots, L, \quad t \in [0,T] .$$

The vector of ex-dividend prices $P(\cdot) = (P_1(\cdot), \ldots, P_L(\cdot))^\top$ must satisfy $P_i(T) = 0$, $i = 1, \ldots, L$. The interest rate $r(\cdot)$ of the bond, the vector of drifts $\mu(\cdot) = (\mu_1(\cdot), \ldots, \mu_L(\cdot))^\top$ and the volatility matrix $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}$ are (possibly path-dependent) $\mathcal{F}_t$-measurable processes. The price system, represented by these coefficients, is to be determined endogenously in equilibrium. Assuming that $\sigma(\cdot)$ is invertible, the market in this set-up is dynamically complete since the number of risky securities is equal to the number of dimensions of uncertainty (L).

In our analysis, we use the martingale representation technology, which requires the construction of certain processes, related to the price dynamics (Cox and Huang [6, 7], Harrison and Kreps [15], Karatzas, Lehoczky and Shreve [21], Pliska [31]). We briefly present the required notions for our set-up and do not state all regularity conditions.

We define *the state price density process* $\xi(t)$ as a process with dynamics

$$d\xi(t) = -\xi(t) \left[r(t)dt + \theta(t)^\top dW(t) \right] , \tag{1}$$

where $\theta(t)$ is the L-dimensional $\mathcal{F}_t$-measurable *market price of risk process*, defined by $\theta(t) \equiv \sigma(t)^{-1}[\mu(t) - r(t)\mathbf{1}]$, where $\mathbf{1}$ is an L-dimensional vector with every component equal to 1. $\xi(t,\omega)$ is interpreted as the Arrow-Debreu price (per unit of probability $\mathcal{P}$) of one unit of consumption good in state $\omega \in \Omega$ at time t. Under mild regularity conditions, the state price density process provides the following relationship between the asset prices and their future dividends consistent with no arbitrage (e.g., see Duffie [10]):

$$P_i(t) = \frac{1}{\xi(t)} E\left[\int_t^T \xi(s)\delta_i(s) ds \mid \mathcal{F}_t \right] , \quad i = 1, \ldots, L, \quad t \in [0,T] . \tag{2}$$

2.2 Agents' Preferences and Endowments

We assume there are N agents in the economy. Agent N is a non-price-taker in the sense that he takes into account the fact that his dynamic consumption-portfolio choice affects the whole process of the price system $(r(t), \mu(t), \sigma(t))$, $t \in [0, T]$. We discuss the way agent N affects prices in Section 3. The other agents $n = 1, \ldots, N-1$ are price-takers. Each agent, n, is endowed at time zero with e_{ni} shares of risky security i. Then, define the initial wealth of agent n as

$$x_{no} \equiv \sum_{i=1}^{L} P_i(0) e_{ni} = \frac{1}{\xi(0)} E\left[\int_0^T \xi(t)\epsilon_n(t)dt\right], \quad n = 1, \ldots, N,$$

where $\epsilon_n(t) \equiv \sum_{i=1}^{L} e_{ni}\delta_i(t)$ and the second equality follows from equation (2). $\epsilon_n(t)$ can be interpreted as the endowment stream of agent n since it is the sum of dividend streams from the initial endowments of the agent.

For each agent n, we define a consumption process $c_n(t)$, and an L-dimensional portfolio process $\alpha_n(t) = (\alpha_{n1}(t), \ldots, \alpha_{nL}(t))^\top$, where $\alpha_{ni}(t)$ denotes the number of shares of asset i held by agent n at time t. Then the nth agent's wealth process follows

$$dX_n(t) = \tag{3}$$
$$r(t)X_n(t)dt - c_n(t)dt + \sum_{i=1}^{L} \alpha_{ni}(t)\, P_i(t)[\mu_i(t) - r(t)]dt + \sum_{i=1}^{L}\sum_{j=1}^{L} \alpha_{ni}(t)\, P_i(t)\, \sigma_{ij}(t)\, dW_j(t).$$

Each agent is assumed to derive time-additive, state-independent utility $u_n(c_n(t))$ from consumption at all times in $[0, T]$. We assume the utility functions are continuous with continuous first derivatives, strictly increasing and strictly concave. Throughout the paper, a symbol with a caret (^) denotes the optimal (expected utility maximizing) quantity, corresponding to $(\hat{c}_n(t), \hat{\alpha}_n(t))$. A symbol with an asterisk (*) denotes equilibrium in a non-price-taking economy; a symbol with an overbar (¯), equilibrium in a price-taking economy (where all agents are price-takers).

3 Agents' Optimization and Equilibrium

This section presents the agents' optimization problems and defines equilibrium in a multi-agent economy consisting of one non-price-taking agent and $N-1$ price-taking agents. We define equilibrium based on a price leadership model. In such a model, the non-price-taker observes the demands of the other agents as a function of the price processes, and then simultaneously chooses price processes and his own consumption-portfolio process so as to maximize his objective function subject to the condition that all markets must then clear. The non-price-taker solves both his optimality and for the equilibrium prices simultaneously.

As the first step, we solve for the price-takers' optimal demands. A price-taker maximizes his expected lifetime utility subject to his dynamic budget constraint (3), taking the price system as given. Using the martingale representation approach, each price-taker's dynamic optimization problem is converted into the following static variational problem (Cox and Huang [6], Karatzas, Lehoczky and Shreve [21]):

$$\max_{c_n(\cdot)} E\left[\int_0^T u_n(c_n(t))dt\right] \quad \text{subject to} \quad E\left[\int_0^T \xi(t)c_n(t)dt\right] \leq E\left[\int_0^T \xi(t)\epsilon_n(t)dt\right] .$$

Using the Lagrangian method, the first order conditions of the price-takers are

$$u_n'(\hat{c}_n(t)) = y_n\xi(t) , \quad t \in [0,T] , \quad n = 1,\ldots,N-1 , \tag{4}$$

where each y_n is the Lagrange multiplier such that agent n's static budget constraint holds with equality at the optimal, i.e., y_n satisfies

$$E\left[\int_0^T \xi(t)I_n(y_n\xi(t))dt\right] = E\left[\int_0^T \xi(t)\epsilon_n(t)dt\right], \quad n = 1,\ldots,N-1 , \tag{5}$$

where $I_n(\cdot)$ is the inverse of n's marginal utility. Equation (4) states that for a price-taker, the marginal benefit from an extra unit of consumption at time t and state ω is proportional to the cost $\xi(t,\omega)$ of that extra unit of consumption.

For notational and analytical convenience we introduce a representative agent formulation for the price-takers (following, for example, Huang [18]). We define the price-taker representative agent's utility function by

$$U(c;\Lambda) \equiv \max_{c_1,\ldots,c_{N-1}} \sum_{n=1}^{N-1} \lambda_n u_n(c_n)$$

subject to $\sum_{n=1}^{N-1} c_n = c$, where $\Lambda \equiv (\lambda_1,\ldots,\lambda_{N-1}) \in (0,\infty)^{N-1}$. It can be shown that the inverse of $U'(c;\Lambda)$ is given by $J(h;\Lambda) \equiv \sum_{n=1}^{N-1} I_n(h/\lambda_n)$. Identifying $\Lambda = (1/y_1,\ldots,1/y_{N-1})$ and summing (4) over all price-takers, the aggregate optimal consumption is given by

$$\sum_{n=1}^{N-1} \hat{c}_n(t) = J(\xi(t);1/y_1,\ldots,1/y_{N-1}) . \tag{6}$$

Formally, we define equilibrium as follows.

Definition. *Equilibrium* in an economy with one non-price-taker and $N-1$ price-takers is defined as a price system $(r(\cdot), \mu(\cdot), \sigma(\cdot))$ and consumption-portfolio processes $(c_n^*(\cdot), \alpha_n^*(\cdot))$ such that the price-takers choose their optimal consumption-portfolio strategy at the given prices, the non-price-taker chooses his optimal consumption-portfolio strategy taking account of the fact that the price system responds to clear the markets, and the price system is such that the good and the security markets do clear, i.e.,

$$\sum_{n=1}^{N} c_n^*(t) = \delta(t) \; ; \quad \sum_{n=1}^{N} \alpha_{ni}^*(t) = 1 \; , \quad i = 1, \ldots, L \; ; \quad \sum_{n=1}^{N} X_n^*(t) = \sum_{i=1}^{L} P_i(t) \; , \quad t \in [0, T] \; . \quad (7)$$

We now formulate the non-price-taker's optimization problem. According to the definition of equilibrium, the non-price-taker acts as a price leader in all (consumption good and security) markets. However, it is well-known that in dynamically complete markets to ensure clearing in all markets at agents' optima, it suffices to simply clear the consumption good market (e.g., Karatzas, Lehoczky and Shreve [22], Basak [1]). Hence, we need only focus on the price leadership in the consumption good market. Recalling the representative price-taker's demand in equation (6), clearing in the consumption good (7) implies

$$c_N(t) = \delta(t) - J(\xi(t); 1/y_1, \ldots, 1/y_{N-1}) \; . \quad (8)$$

This expression can be interpreted as the "residual supply curve", analogous to the notion of residual demand in the price leadership model of oligopoly theory (e.g., Varian [35, Chapter 16]). Hence we have the following relationship between $\xi(t)$ and the non-price-taker's consumption:

$$\xi(t) = U'(\delta(t) - c_N(t); 1/y_1, \ldots, 1/y_{N-1}) \; . \quad (9)$$

So, as a price leader, the non-price-taker's influence on prices manifests itself, via (9), as the consumption process affecting the Arrow-Debreu prices. Recasting the non-price-taker's effect in this way combines naturally with the martingale method of solution of the agents' optimization problems. However, since the agent finances his consumption process through his portfolio strategy, and since asset prices are determined from the state price process via (2), this non-price-taking price dependence is equivalent to a dependence of the asset prices on the trading strategy.[3]

The non-price-taker, then, solves the following dynamic optimization problem:

$$\max_{(c_N(\cdot), \alpha_N(\cdot)), (r(\cdot), \mu(\cdot), \sigma(\cdot))} E\left[\int_0^T u_N(c_N(t))dt\right]$$

subject to the dynamic budget constraint (3) and equation (9). Similarly to the case of the price-takers, we may convert this problem into a static variational problem. In the well-known case of price-taking agents, the dynamic budget constraint (3) is equivalent to the static budget constraint (10) for all price systems (e.g., Karatzas, Lehoczky and Shreve [21]). This result under the restricted price system obeying (9) applies to our non-price-taker, and so we can rewrite his optimization problem as

$$\max_{c_N(\cdot),\xi(\cdot)} E\left[\int_0^T u_N(c_N(t))dt\right]$$

$$\text{subject to} \quad E\left[\int_0^T \xi(t)\, c_N(t)\, dt\right] \leq E\left[\int_0^T \xi(t)\, \epsilon_N(t)\, dt\right] \tag{10}$$

$$\text{and} \quad \xi(t) = U'(\delta(t) - c_N(t); 1/y_1, \ldots 1/y_{N-1}) \, ,$$

where $(y_1, \ldots, y_{N-1})$ satisfy (5). The remainder of the analysis depends upon the nature of the representative price-taker's utility function, which can be divided into two cases.

3.1 The Case of Price-Taker Representative Agent Independent of Individual Weights

In this subsection and throughout most of the paper we take the case where the representative agent's utility function can be written as

$$U(c; \Lambda) = h(\Lambda)U(c).$$

Examples of this include the case of only one price-taker, or all price-takers having the same log or power or negative exponential utility function. In general, it can be shown that in equilibrium the vector $(y_1, \ldots, y_{N-1})$ is only determined up to a multiplicative constant, since it is only the relative weights of agents that matter. So without loss of generality we let $h(\Lambda) = 1$, meaning we express all agents' weights relative to this aggregate weight, implying $U(c; \Lambda) = U(c)$ and $J(h; \Lambda) = J(h)$. Hence, from (9) we obtain the mapping

$$\xi(t, \omega) = U'(\delta(t, \omega), -c_N(t, \omega)) \, . \tag{11}$$

In this case the non-price-taker's time t, state ω consumption only affects the state price at that time and state, with $\xi(t, \omega)$ increasing in $c_N(t, \omega)$. The following proposition characterizes the non-price-taker's solution to his dynamic optimization problem, and hence the equilibrium consumption allocations and state prices.

Proposition 1. *Assume $U(c;\Lambda) = h(\Lambda)U(c)$. If an equilibrium exists, then the non-price-taker's equilibrium consumption and weight, $c_N^*(t)$ and y_N, satisfy*

$$u_N'(c_N^*(t)) = y_N \left[U'(\delta(t) - c_N^*(t)) - U''(\delta(t) - c_N^*(t))(c_N^*(t) - \epsilon_N(t))\right], \quad t \in [0,T], \tag{12}$$

and

$$E\left[\int_0^T U'(\delta(t) - c_N^*(t))c_N^*(t)dt\right] = E\left[\int_0^T U'(\delta(t) - c_N^*(t))\epsilon_N(t)dt\right]. \tag{13}$$

Subsequently, the equilibrium state price density is determined from (11) and the price-takers' equilibrium consumption and weights from (4) and (5).

We assume from now on that equilibrium exists, and proceed to characterize the equilibrium behavior implied by (12) and (13). The effect of agent N being a non-price-taker is an extra term on the right-hand-side of his first order condition (12). His marginal benefit from an extra unit of consumption at time t and state ω is proportional to the cost $\xi(t,\omega) = U'(\delta(t,\omega) - c_N(t,\omega))$ of that extra unit of consumption, plus an additional "cost" term, $-U''(\delta(t) - c_N^*(t))(c_N^*(t) - \epsilon_N(t))$, due to the direct effect of this incremental change in consumption on the price of consumption. Since $U''(\cdot) < 0$, when $(c_N^*(t) - \epsilon_N(t))$ is positive the additional term in (12) is positive, hence increasing $u_N'(c_N^*(t))$ or decreasing $c_N^*(t)$; and vice versa when $(c_N^*(t) - \epsilon_N(t))$ is negative. So the presence of the additional term tends to induce the non-price-taker to deviate towards his "own" dividend, $\epsilon_N(t)$. The intuition is that when $c_N^*(t)$ is greater than $\epsilon_N(t)$, say, the non-price-taker is a net "buyer" of consumption in that state; he is consuming more than the dividend from his initial endowment. So it is in his interest to reduce the price of consumption in that state, and he recognizes that he can do this by decreasing his consumption.

So, the non-price-taker's consumption tends to move towards $\epsilon_N(t)$ compared with his price-taking equilibrium consumption. (Since y_N and $U'(\delta(t) - c_N^*(t))$ differ across economies we cannot say that $c_N^*(t)$ is always closer to $\epsilon_N(t)$ than is N's consumption in the price-taking economy, but it is reasonable to assume that this will be the case when agent N is enough of a net buyer or seller of consumption good.) This behavior leads us to the intuition that the instantaneous volatility of the difference between $c_N(t)$ and $\epsilon_N(t)$ is lower in the non-price-taking economy. We formalize this idea in Section 5 when we put more structure into the model. Rewriting equation (12) as

$$u_N'(c_N^*(t)) = y_N U'(\delta(t) - c_N^*(t)) \left[1 - \frac{U''(\sum_{n=1}^{N-1} c_n^*(t))}{U'(\sum_{n=1}^{N-1} c_n^*(t))}(c_N^*(t) - \epsilon_N(t))\right], \quad t \in [0,T], \tag{14}$$

we see that the extent of agent N's deviation towards $\epsilon_N(t)$ depends both on how much of a net consumer he is $(c^*_N(t) - \epsilon_N(t))$ and on the absolute risk aversion of the representative price-taker $(-U''/U')$. The more risk averse the price-takers, the less their consumption reacts to changes in the state price or, conversely, the more the state price reacts to their (and hence also the non-price-taker's) consumption and so the more it is in the non-price-taker's interest to deviate. In the limit of risk neutral price-takers, the non-price-taker cannot affect the state price at all and so he does not deviate from his price-taking behavior.

The following proposition establishes formally that in the limit when agent N does not trade then there is no effect on the equilibrium of making him a non-price-taker. On the other hand if he does trade as a price-taker, then there is an effect of making him a non-price-taker.

Proposition 2.

(a) Suppose the agents' initial endowments are such that agent N does not trade in equilibrium in the economy where all are price-takers. Then in equilibrium in the economy where agent N is a non-price-taker, he does not deviate from his price-taking consumption-portfolio strategy.

(b) Similarly, if equilibrium in the non-price-taking economy is such that agent N does not trade, then he must not be deviating from his equilibrium price-taking strategy.

(c) If the initial endowments are such that agent N does trade in equilibrium in the economy where all are price-takers, in the non-price-taking equilibrium the non-price-taker N does deviate from his price-taking strategy.

If the price-taking optimal behavior of agent N is to not trade and consume his "own" dividend, it turns out that he always makes himself worse off by deviating from the price-taking case. Suppose in some state he increases his consumption a little so that $c_N(t)$ is greater than $\epsilon_N(t)$. Now he is a net buyer of consumption good in that state. Unfortunately, by increasing his consumption he simultaneously raises the price of consumption in that state, which would affect him adversely. If he decreases his consumption, he is a net seller of consumption but at the same time reduces its price; deviation in either direction has an adverse effect.

We note that the non-price-taker's influence on the equilibrium does not depend on how wealthy he is, but on how much of a trader he is. Even if agent N has no initial wealth $(e_N = 0)$ it may still be the case that he changes the equilibrium by being a non-price-taker. For example, for the CARA utility and one risky asset case considered in Section 5, it is not $e_N = 0$ but $e_N = 1/2$ for which there is (no trading and hence) no deviation from the price-taking equilibrium. In the case of CRRA (constant relative risk aversion) utility agents, however, if agent N has no initial wealth ($e_N = 0$) he never consumes nor trades in the price-taking economy and so his non-price-taking strategy does not deviate from his price-taking one. The reader should keep in mind, though, that the reason he has no effect is not that he has no wealth, but that he does not trade at all.

An appropriate question is whether the non-price-taker gains any advantage through taking into account his effect on prices. Proposition 3 states that indeed the non-price-taker is at least as well off as if he were a price-taker. This is because when solving his optimization problem the non-price-taking agent always has the option of choosing his price-taking equilibrium consumption.

Proposition 3. *In equilibrium, in the non-price-taking economy, agent N's expected lifetime utility from consumption is greater than or equal to that in equilibrium in the price-taking economy, i.e.,*

$$E\left[\int_0^T u_N(c_N^*(t))dt\right] \geq E\left[\int_0^T u_N(\bar{c}_N(t))dt\right].$$

We have seen in this section that, unlike for the price-takers, the non-price-taker's equilibrium consumption is not simply proportional to the state price. Hence his marginal rate of substitution between consumption in different states and times is not necessarily equal to the ratio of the state prices, whereas each price-taker's is. As stated below in Proposition 4, this discrepancy between agents' marginal rates of substitution, implies that the non-price-taking equilibrium consumption allocations are not pareto optimal.

Proposition 4. *If the non-price-taking equilibrium differs (with probability greater than zero) from the price-taking equilibrium, then the non-price-taking equilibrium allocations are not pareto optimal.*

Remark (Time-Consistency). In this paper we solve the problem where the agents choose their plans at time 0 and do not subsequently deviate from these plans, i.e., we characterize a self-commitment solution. A natural question to ask in a model with non-price-taking behavior is whether the agents have any incentive to deviate at a later date, in other words, whether their strategies are time-inconsistent (e.g., Sargent [32, p. 11], Merton [28, p. 177]).

It is well-known that the standard price-taking ($n = 1, \ldots, N-1$) strategy is time-consistent. The non-price-taker's optimization problem from some intermediate time $s \in (0,T)$ onwards is:

$$\max_{c_N(\cdot)} E\left[\int_s^T u_N(c_N(t))dt \mid \mathcal{F}_s\right]$$

$$\text{subject to} \quad E\left[\int_s^T \xi(t)c_N(t)dt \mid \mathcal{F}_s\right] \leq \xi(s)\alpha_{N0}(s)P_0(s) + E\left[\int_s^T \xi(t)\epsilon_N^s(t)dt \mid \mathcal{F}_s\right]$$

and equation (11), $t \in [s,T]$, where $\epsilon_N^s(t) \equiv \sum_{i=1}^L \alpha_{Ni}(s)\delta_i(t)$. The first order conditions become

$$u_N'(c_N(t)) = y_N^s \left[U'(\delta(t) - c_N(t)) - U''(\delta(t) - c_N(t))(c_N(t) - \epsilon_N^s(t))\right], \quad t \in [s,T],$$

where y_N^s is such that the non-price-taker's budget constraint at time s holds with equality. Generically the solution differs from the commitment solution, since $\epsilon_N^s(t)$ differs from $\epsilon_N(t)$, implying that the non-price-taker's strategy is not time-consistent. The intuition for this time-inconsistency is similar to the familiar situation of time-inconsistency arising in the analysis of pricing by a durable-good monopolist (e.g., Tirole [34, Chapter 1] and references therein) and related to the Coase Conjecture (1972). The financial assets in our model are similar to durable goods in that their value is durable over many periods. The reason for the time-inconsistency is that at time 0, say, the non-price-taker takes account of the effect of his future (say time s) actions on the current asset prices (through equation (2)). However, when agent N gets to time s he no longer cares about his effect on the past prices and so he changes his optimal strategy. We note that time-inconsistency also arises in a deterministic version of our model, with an asset paying out deterministic dividends, for the same reason.[4]

Since the non-price-taker has an incentive to deviate from his optimal strategy as time unfolds, and given that the price-takers should be aware of this, the question arises as to how meaningful our equilibrium is. However, given that it is optimal for the non-price-taker to follow a "self-commitment" strategy, it is reasonable to assume that he will create some mechanism to force himself to commit, and so the equilibrium is not unreasonable. For example, the non-price-taker could hire an agent or an institution at time 0 to carry out the optimal strategy for him, and part of the contract would be for the agent to agree to not let him come back and change his mind later. In this paper we do not discuss further the means by which the non-price-taker might force himself to commit, instead we focus on the dynamic consumption and price behavior.

It might be valuable for comparison to also solve for the non-price-taker's subgame perfect strategy, by backward induction. This can be thought of as the "short-sighted" strategy, since the non-price-taker acts in a short-sighted way by reoptimizing every period (Blanchard and Fisher [2, p. 595]). Our preliminary analysis suggests that this problem is intractable in our general framework for the type of non-price-taking agents we are considering. One obvious comparison is that the non-price-taker is better off to follow his commitment strategy than his short-sighted strategy, becasue when looking for the subgame perfect strategy he is restricted to only follow strategies which are optimal in all subgames. We finally note that, due to the time-inconsistency of the non-price-taker's strategy, this problem provides us with an example where dynamic programming cannot be used to solve for the commitment solution. In this case, to our knowledge, the martingale method is the only way to solve this problem.

3.2 Extension to Representative Price-Taking Agent Not Independent of Individual Weights

Here we extend the analysis of the previous subsection to the case of one non-price-taker and multiple price-takers whose representative agent utility function is not independent of the individual price-takers' weights. According to (9), now agent N's consumption at time t in state ω, depends not only on $\xi(t,\omega)$ but also on the weights Λ. Since these weights are determined from the budget constraints (5) of the price-taking agents which are driven by the whole process of $\xi(\cdot)$, $c_N(t,\omega)$ depends on the whole process $\xi(\cdot)$. Hence $c_N(t,\omega)$ affects the whole state price density process $\xi(\cdot)$; there is no longer a one-to-one mapping between $c_N(t,\omega)$ and $\xi(t,\omega)$. Now, when the non-price-taker chooses his optimal consumption he has to worry about the externalities he imposes on the other agents by his choice of consumption (and hence state price process), which determines the distribution of wealth across these agents. As a result the analysis of his optimization problem becomes much more complicated, as can be seen from his first order condition, presented in Proposition 5.

Proposition 5. *If an equilibrium exists, then the non-price-taker's consumption and all agents' weights, $c_N^*(t)$ and $(y_1,...,y_N)$, satisfy*

$$\begin{aligned} u_N'(c_N^*(t)) &= y_N\Big[U'\Big(\delta(t)-c_N^*(t);\Lambda\Big) - U''\Big(\delta(t)-c_N^*(t);\Lambda\Big)\Big(c_N^*(t)-\epsilon_N(t)\Big)\Big] \qquad (15)\\ &\quad - \sum_{n=1}^{N-1} K_n \frac{U''(\delta(t)-c_N^*(t);\Lambda)}{u_n''\Big(I_n(y_nU'(\delta(t)-c_N^*(t);\Lambda))\Big)} * \\ &\quad \Big[y_nU'\Big(\delta(t)-c_N^*(t);\Lambda\Big) + u_n''\Big(I_n(y_nU'(\delta(t)-c_N^*(t);\Lambda))\Big)\Big(I_n(y_nU'(\delta(t)-c_N^*(t);\Lambda))-\epsilon_n(t)\Big)\Big], \end{aligned}$$

$$E\left[\int_0^T U'(\delta(t)-c_N^*(t);\Lambda)c_N^*(t)dt\right] = E\left[\int_0^T U'(\delta(t)-c_N^*(t);\Lambda)\epsilon_N(t)dt\right], \qquad (16)$$

and

$$E\left[\int_0^T U'(\delta(t) - c_N^*(t); \Lambda) I_n(y_n U'(\delta(t) - c_N^*(t); \Lambda))dt\right] =$$
$$E\left[\int_0^T U'(\delta(t) - c_N^*(t); \Lambda)\epsilon_n(t)dt\right], \quad n = 1, \ldots, N-1 \, . \tag{17}$$

Subsequently, the equilibrium state price density is determined from (9), and the price-takers' equilibrium consumption from (4). If $U(c;\Lambda) = h(\Lambda)U(c)$, then $K_n = 0$ for all $n = 1, \ldots, N-1$.

Equation (15) is similar to equation (12) but with $N-1$ extra terms on the right-hand-side. Again, the marginal benefit the non-price-taker gets from an extra unit of consumption at time t and state ω must be equal to the total "costliness" to him of that extra unit of consumption. As in Section 3.1, the first and second terms on the right-hand-side of (15) are the cost $\xi(t,\omega) = U'(\delta(t,\omega) - c_N^*(t,\omega); \Lambda)$ of that extra unit of consumption, and the costliness to him due to the direct effect of $c_N(t;\omega)$ on $\xi(t,\omega)$. We note that the second term again has the effect of making $c_N^*(t)$ tend towards $\epsilon_N(t)$ as compared with a price-taking economy.

In this case, however, an extra unit of consumption is costly to agent N in a third way, represented by the extra N terms in (15). The non-price-taker also realizes now that an extra unit of $c_N(t;\omega)$ can affect the effective wealths of the other agents which in turn affect the whole state price process $\xi(\cdot)$, and hence his satisfaction at all other times and states. We argue in the proof of Proposition 5 that the extra terms in (15) are indeed the indirect incremental change in agent N's expected lifetime utility $E\left[\int_0^T u_N(c_N(t))dt\right]$ via the effect of an extra unit of $c_N(t;\omega)$ on each of the other agent's budget constraints.

As a final note, the last statement of Proposition 5 shows that expression (15) indeed collapses to our previous expression (12) in the case when the price-taker representative agent utility function is independent of the individual weights.

4 The Equilibrium Interest Rate and the Consumption-Based CAPM

Here we look more closely at the effect of the presence of a non-price-taking agent on equilibrium asset and state prices, and on the consumption-based CAPM. We discuss only the simpler case of representative price-taking agent's utility independent of individual weights, as in Subsection 3.1.

It is well-known that in the economy with agent N also a price-taking agent, the equilibrium consumption allocations, $\bar{c}_n(\delta(t);\bar{y}_N,\bar{y}_n)$, $n=1,\ldots,N-1$ and $\bar{c}_N(\delta(t);\bar{y}_N)$, are only a function of the aggregate consumption $\delta(t)$. Hence, all agents' consumption processes are perfectly correlated. Furthermore, the equilibrium state price density process $\bar{\xi}(\delta(t);\bar{y}_N)$ and its dynamics ($\bar{r}(t)$ and $\bar{\theta}(t)$) are also driven only by the aggregate consumption. The equilibrium interest rate $\bar{r}(t)$ is given by

$$\bar{r}(t)=\bar{\lambda}_\delta(t)\mu_\delta(t)+\bar{\nu}_\delta(t)\|\sigma_\delta(t)\|^2;\quad \bar{\lambda}_\delta(t)>0$$

where $\bar{\lambda}_\delta(t)\equiv -V''(\delta(t);\bar{y}_N)/V'(\delta(t);\bar{y}_N)$, $\bar{\nu}_\delta(t)\equiv -V'''(\delta(t);\bar{y}_N)/(2V'(\delta(t);\bar{y}_N))$. Here $V(\cdot;\bar{y}_N)$ is the utility function of the representative agent of all N agents. (This representation is just the pure-exchange, multi-agent version of Cox, Ingersoll and Ross [8].) Since $\bar{\lambda}_\delta(t)>0$ for concave utility functions, the interest rate is positively related to $\mu_\delta(t)$. Furthermore, if $V'''(\delta(t);\bar{y}_N)>0$ then the interest rate process is negatively related to the variance of the aggregate dividend $\|\sigma_\delta(t)\|^2$.

The pure-exchange version of Breeden's CCAPM [3, 4] in this price-taking economy is (see Duffie and Zame [13] or Karatzas, Lehoczky and Shreve [22])

$$\bar{\mu}(t)-\bar{r}(t)\mathbf{1}=\bar{\lambda}_\delta(t)\mathrm{cov}\left(\frac{d\bar{P}(t)}{\bar{P}(t)},d\delta(t)\right);\quad \bar{\lambda}_\delta(t)>0\ .$$

The risk premium of an asset is positively related to the (instantaneous, conditional) covariance of its return with $d\delta(t)$, the change in the aggregate consumption.

In our non-price-taking economy, the solutions (if they exist) for the equilibrium consumption allocations from Proposition 1, $c_n^*(\delta(t),\epsilon_N(t);y_N,y_n)$, $n=1,\ldots,N-1$ and $c_N^*(\delta(t),\epsilon_N(t);y_N)$, are driven by two factors, the aggregate consumption $\delta(t)$ and the dividend stream from N's initial endowment, $\epsilon_N(t)$. Consequently, the agents' consumption streams are no longer (instantaneously) perfectly correlated with each other, nor with the aggregate consumption $\delta(t)$. Furthermore, the equilibrium state prices process $\xi(\delta(t),\epsilon_N(t);y_N)$ is also driven by the two factors, as summarized in Proposition 6.

Proposition 6. *Assume an equilibrium exists in an economy with $N-1$ price-taking and one non-price-taking agent, where $U(c;\Lambda)=h(\Lambda)\,u(c)$. The equilibrium interest rate is given by:*

$$r^*(t)=\eta_\delta^*(t)\mu_\delta(t)+\eta_{\epsilon_N}^*(t)\mu_{\epsilon_N}(t)+\nu_\delta^*(t)\|\sigma_\delta(t)\|^2+\nu_{\epsilon_N}^*(t)\|\sigma_{\epsilon_N}(t)\|^2+\kappa^*(t)\sigma_\delta(t)^\top\sigma_{\epsilon_N}(t)\ ,\quad (18)$$

and the risk premia of risky securities are given by:

$$\mu^*(t)-r^*(t)1=\lambda_\delta^*(t)cov\left(\frac{dP(t)}{P(t)},d\delta(t)\right)+\lambda_{\epsilon_N}^*(t)cov\left(\frac{dP(t)}{P(t)},d\epsilon_N(t)\right),\quad (19)$$

where $\mu_{\epsilon_N}(t)$ and $\|\sigma_{\epsilon_N}(t)\|$ are the drift and volatility of $\epsilon_N(t)$, and

$$\lambda_\delta^*(t) \equiv -\frac{U''(\delta(t)-c_N^*(t))}{U'(\delta(t)-c_N^*(t))}\left\{\frac{u_N''(c_N^*(t))+y_N U''(\delta(t)-c_N^*(t))}{u_N''(c_N^*(t))-y_N U'''(\delta(t)-c_N^*(t))(c_N^*(t)-\epsilon_N(t))+2y_N U''(\delta(t)-c_N^*(t))}\right\}$$

$$\lambda_{\epsilon_N}^*(t) \equiv \frac{y_N U''(\delta(t)-c_N^*(t))}{u_N''(c_N^*(t))+y_N U''(\delta(t)-c_N^*(t))}\lambda_\delta^*(t)\ .$$

For general utility functions, the coefficients (η, ν, κ) in the interest rate formula are rather complicated, and not especially illuminating. So, we do not present them here. The signs of the dependence of $r^*(t)$ on $\mu_\delta(t)$, $\mu_{\epsilon_N}(t)$, $\|\sigma_\delta(t)\|^2$, $\|\sigma_{\epsilon_N}(t)\|^2$ and $\sigma_\delta(t)^\top\sigma_{\epsilon_N}(t)$ are in general ambiguous. In Section 5, we look more closely at the interest rate under a specific utility function.

The CCAPM is now a two-beta CCAPM, driven by the covariance of the price return with both $d\delta(t)$ and $d\epsilon_N(t)$. Since $U(\cdot)'' < 0$, both λ's are nonzero and have the same sign, so the dependences of an asset's risk premium on the covariance of its return with the non-price-taker's dividend and with the aggregate consumption are of the same sign. It can be shown that at the equilibrium, λ_δ^* and $\lambda_{\epsilon_N}^*$ are positive for price-taker representative agent with HARA utility and any non-price-taker's utility function defined over some domain (c_∞, ∞), where $c_\infty \geq -\infty$, and satisfying $\lim_{c\to\infty} u_N'(c) = 0$ and $\lim_{c\to c_\infty} u_N'(c) = \infty$.[5] It is an open question whether for other utility functions the λ's can go negative, depending on $U'''(\cdot)$ and $(c_N^*(t) - \epsilon_N(t))$. If so, the usual implication of the consumption CAPM would fail; the risk premium would be negatively related to the covariance of the asset return with changes in the aggregate consumption. A traditional one-factor consumption CAPM arises when $\delta(t)$ and $\epsilon_N(t)$ are (instantaneously) perfectly correlated, collapsing the two terms together. For the case of one risky asset, for example, $\delta(t)$ and $\epsilon_N(t)$ are perfectly correlated.

5 The CARA Utility and One Risky Asset Case

In order to derive further implications of non-price-taking behavior, we now specialize our general set-up to the case of the equity market consisting of only one risky asset (one dimension of uncertainty) and where both the representative price-taker (agent R) and the non-price-taker (agent N) exhibit CARA preferences, i.e., the utility function of both agents is of the form $u(c) = -\exp\{-ac\}/a;\quad a > 0$.[6] We frequently assume that $\mu_\delta(t)$ is positive, i.e., that the economy is expanding. We sometimes assume that the risky asset's dividend process follows an arithmetic Brownian motion.[7]

Assumption A1. *$\mu_\delta(t) \equiv \mu_\delta$, $\sigma_\delta(t) \equiv \sigma_\delta$ are constants.*

In this section our focus is on comparisons of equilibria between a benchmark price-taking economy (where both agents N and R are price-takers) and the non-price-taking economy. We first analyze equilibrium consumption allocations and then compare state and market price dynamics, as well as the equilibrium portfolio strategies and wealths of the agents. The results for the benchmark economy are straightforward to derive and are often quoted without proof.

5.1 The Equilibrium Consumption Allocations

In the benchmark price-taking economy, the equilibrium consumption processes are

$$\bar{c}_N(t) = \frac{1}{2}\delta(t) - \frac{1}{2a}\ln(\bar{y}_N)\,, \quad \bar{c}_R(t) = \frac{1}{2}\delta(t) + \frac{1}{2a}\ln(\bar{y}_N)\,, \tag{20}$$

where $\bar{y}_N$ is given by

$$\frac{1}{2a}\ln(\bar{y}_N) = \left(\frac{1}{2} - e_N\right)\frac{E\left[\int_0^T \delta(t)\exp\left\{-\frac{1}{2}a\delta(t)\right\}dt\right]}{E\left[\int_0^T \exp\left\{-\frac{1}{2}a\delta(t)\right\}dt\right]}\,. \tag{21}$$

Each agent shares the risk equally, hence consuming half of the dividend plus a constant depending on how wealthy that agent initially is. When $e_N > 1/2$, agent N consumes more than half of the dividend $\delta(t)$, and when $e_N < 1/2$ he consumes less than half of $\delta(t)$.

In the non-price-taking economy the equilibrium consumptions of the two agents are given by Proposition 1. Hence for exponential utility, $c_N^*(t)$ and $c_R^*(t)$ solve

$$c_N^*(t) = \frac{1}{2}\delta(t) - \frac{1}{2a}\ln\Big[1 + a\Big(c_N^*(t) - \epsilon_N(t)\Big)\Big] - \frac{1}{2a}\ln(y_N)\,, \tag{22}$$

$$c_R^*(t) = \frac{1}{2}\delta(t) + \frac{1}{2a}\ln\Big[1 + a\Big(c_N^*(t) - \epsilon_N(t)\Big)\Big] + \frac{1}{2a}\ln(y_N)\,, \tag{23}$$

where y_N is such that the process $c_N^*(t)$ satisfies

$$E\left[\int_0^T c_N^*(t)\exp\Big\{-a\Big(\delta(t) - c_N^*(t)\Big)\Big\}dt\right] = e_N E\left[\int_0^T \delta(t)\exp\Big\{-a\Big(\delta(t) - c_N^*(t)\Big)\Big\}dt\right]\,.$$

In the non-price-taking economy, agents consume half of the dividends plus additional stochastic terms. The following proposition compares the levels of consumption of agent N across economies, and the levels of consumption of agents N and R within economies.

Proposition 7.

(a) For $e_N > 1/2$: $\begin{cases} c_N^*(t) > \bar{c}_N(t) & \textit{when } \delta(t) > \delta_{crit} \\ c_N^*(t) = \bar{c}_N(t) & \textit{when } \delta(t) = \delta_{crit} \\ c_N^*(t) < \bar{c}_N(t) & \textit{when } \delta(t) < \delta_{crit}\,; \end{cases}$

$$\textit{for } e_N < 1/2: \begin{cases} c_N^*(t) < \bar{c}_N(t) & \textit{when } \delta(t) > \delta_{crit} \\ c_N^*(t) = \bar{c}_N(t) & \textit{when } \delta(t) = \delta_{crit} \\ c_N^*(t) > \bar{c}_N(t) & \textit{when } \delta(t) < \delta_{crit}\ ; \end{cases}$$

for $e_N = 1/2$: $c_N^* = \bar{c}_N(t)$,

where δ_{crit} *is the constant given by* $\delta_{crit} = \dfrac{1}{a(e_N - 1/2)} \left[1 - \dfrac{\bar{y}_N}{y_N} - \dfrac{1}{2}\ln(\bar{y}_N)\right]$.

(b) *When* $e_N > 1/2$: $\bar{c}_N(t) > \bar{c}_R(t)$ *a.s., but in the non-price-taking economy we may have* $c_N^*(t) < c_R^*(t)$. *When* $e_N < 1/2$: $\bar{c}_N(t) < \bar{c}_R(t)$ *a.s., but in the non-price-taking economy we may have* $c_N^*(t) > c_R^*(t)$.

The comparison in part (a) depends on whether the non-price-taker initially owns more or less than half of the market. Furthermore, this comparison can be divided into two regions of the dividend. When the non-price-taking agent initially owns more than half of the market, as the dividend increases above a critical level he consumes more than when he is a price-taker and as it decreases below the critical level he consumes less; in a sense he is reacting more to changes in dividend. The intuition for this can be seen by considering the price-taking economy, where for $e_N > 1/2$, when the dividend is relatively high, agent N is a net seller of consumption. Hence as a non-price-taker, agent N increases his consumption, thereby increasing the price of consumption, in that state. He does the opposite when the aggregate dividend is low. For the case when the non-price-taker is initially less wealthy, the opposite happens. When $e_N = 1/2$, in the benchmark economy, no trade takes place. Then as a special case of Proposition 2 there is no effect of the presence of one non-price-taking agent.

In the case of an expanding economy where δ grows over time on average, Proposition 7 implies that, when initially endowed with more than half of the market, the non-price-taker on average postpones consumption to later in his lifetime relative to if he were a price-taker. When initially endowed with less than half of the market, he on average consumes more towards the beginning of his lifetime, i.e., is more impatient, than if he were a price-taker.

Part (b) of Proposition 7 compares agents N and R within economies; we note, for example, that even if the non-price-taker initially owns more than half of the market, there are states in which he chooses to consume less than the less-endowed price-taker.

Next we compare the dynamics of the agents' consumption streams across economies. We define the drift $\mu_{c_n}(t)$ and volatility $\sigma_{c_n}(t)$ of agent n's consumption by

$$dc_n(t) = \mu_{c_n}(t)dt + \sigma_{c_n}(t)dW(t)\ , \quad n = N, R\ .$$

In the benchmark economy the equilibrium drift and volatility of both agents' consumption are given by

$$\bar{\mu}_{c_n}(t) = \frac{1}{2}\mu_\delta(t) , \quad \bar{\sigma}_{c_n}(t) = \frac{1}{2}\sigma_\delta(t) , \quad n = N, R .$$

Proposition 8. *In the non-price-taking economy, the equilibrium drift and volatility of the non-price-taking agent's consumption are given by*

$$\mu^*_{c_N}(t) = g(t)\mu_\delta(t) + \frac{2f(t)}{1+f(t)} a \left[g(t)^2 + g(t) - \frac{3}{4} \right] \sigma_\delta(t)^2 , \tag{24}$$

and

$$\sigma^*_{c_N}(t) = g(t)\sigma_\delta(t) , \tag{25}$$

where

$$f(t) \equiv \frac{1}{y_N} \exp\left\{ -2a(c^*_N(t) - \frac{1}{2}\delta(t)) \right\} ; \quad f(t) > 0 , \tag{26}$$

$$g(t) \equiv \frac{e_N + f(t)}{1 + 2f(t)} = \frac{\partial c^*_N(t)}{\partial \delta(t)} ; \quad g(t) \in (0,1) ; \quad \textit{for} \quad \begin{cases} e_N > 1/2 , & g(t) > 1/2 , \\ e_N < 1/2 , & g(t) < 1/2 . \end{cases} \tag{27}$$

As a consequence:

(i) For $\begin{cases} e_N > 1/2: & |\sigma^*_{c_N}(t)| > |\bar{\sigma}_{c_N}(t)| , \quad \textit{and} \quad \mu^*_{c_N}(t) > \bar{\mu}_{c_n}(t) \quad \textit{if} \quad \mu_\delta(t) > 0 \\ e_N < 1/2: & |\sigma^*_{c_N}(t)| < |\bar{\sigma}_{c_N}(t)| , \quad \textit{and} \quad \mu^*_{c_N}(t) < \bar{\mu}_{c_n}(t) \quad \textit{if} \quad \mu_\delta(t) > 0 . \end{cases}$

(ii) $\mathrm{var}\Big(d(c^*_N(t) - \epsilon_N(t))\Big) < \mathrm{var}\Big(d(\bar{c}_N(t) - \epsilon_N(t))\Big)$.

The comparative statics results of Proposition 8 are derived from the properties of the process $g(t)$, which captures how the non-price-taker reacts to changes in the aggregate dividend. In Result (i) we see that if the non-price-taking agent initially owns more than half of the market, he increases the drift and volatility (riskiness) of his consumption compared with the price-taking case; otherwise he decreases the drift and volatility. To see why, suppose that the non-price-taker initially owns, say, 3/4 of the market and consider two consumption strategies: (1) he holds on to his initial endowment at all times (absorbing 3/4 of the dividend risk and growth), or (2) he follows his price-taking strategy (sharing the dividend risk and growth equally with the other agent). Recall from Section 3 that the non-price-taker deviates from his price-taking consumption (strategy 2) towards his endowment (strategy 1), hence absorbing more of the dividend risk and growth. Result (ii) of Proposition 8 states that the instantaneous volatility of the difference between $c_N(t)$ and $\epsilon_N(t)$, is lower in the economy where N is a non-price-taker than in the benchmark economy, as anticipated in Subsection 3.1.

5.2 The Equilibrium State Prices, Interest Rate and Market Price of Risk

We now begin to investigate the effect of the presence of the non-price-taker on the equilibrium prices. In the benchmark economy, the equilibrium state price density process is $\bar{\xi}(t) = \exp\{-a\delta(t)/2\}\bar{y}_N^{-1/2}$. Comparisons of the absolute level of ξ are not meaningful, but an application of Itô's Lemma yields its dynamics, the interest rate and the market price of risk, as

$$\bar{r}(t) = \frac{a}{2}\mu_\delta(t) - \frac{a^2}{8}\sigma_\delta(t)^2 , \quad \bar{\theta}(t) = \frac{a}{2}\sigma_\delta(t) .$$

In the non-price-taker economy, $\xi^*(t)$ satisfies equation (9). By applying Itô's Lemma for the case of exponential utility to (9) and making use of (24)–(25), we derive the following results.

Proposition 9. *In the non-price-taking economy, the equilibrium interest rate and market price of risk are given by*

$$r^*(t) = a(1 - g(t))\mu_\delta(t) - \frac{2f(t)}{1+f(t)}a\left[g(t)^2 + g(t) - \frac{3}{4}\right]\sigma_\delta(t)^2 - \frac{a^2}{2}(1 - g(t))^2\sigma_\delta(t)^2 , \qquad (28)$$

$$\theta^*(t) = a(1 - g(t))\sigma_\delta(t) , \qquad (29)$$

where $f(t)$ and $g(t)$ are defined as in equations (26) and (27).

As a consequence, if $e_N > 1/2$, $|\theta^*(t)| > |\bar{\theta}(t)|$; if $e_N < 1/2$, $|\theta^*(t)| < |\bar{\theta}(t)|$.

As in a price-taking economy, the market price of risk is positively related to the aggregate consumption risk in the economy. When the non-price-taker initially owns more than half of the market, he increases the market price of risk. In other words he makes the Arrow-Debreu state prices (the price of consumption) riskier. This follows from the result of the previous section that he chooses a riskier consumption stream. The opposite holds for $e_N < 1/2$.

As in a price-taking economy, the interest rate is positively related to the growth in aggregate consumption, and negatively related to the aggregate consumption risk. Comparisons between $r^*(t)$ and $\bar{r}(t)$ are not unambiguous as they are for $\theta(t)$. This is because (when $\mu_\delta(t) > 0$) there are competing effects of the non-price-taker's higher (or lower) consumption drift and his higher (or lower) consumption volatility deduced in Proposition 8. Which of these effects dominates depends on the ratio $2\mu_\delta/a\sigma_\delta^2$.[8]

We may investigate the expected growth of state prices across economies by appealing to results for mean comparison of solutions to stochastic differential equations (Ikeda and Watanabe [19], Hajek [14]).[9] Existing mean comparison theorems do not accommodate comparisons between two processes both of whose drifts and volatilities are stochastic (unless one process is Markov), so in Corollary 1 we make assumption A1. We see later that Corollary 1 is relevant for asset price volatility and portfolio strategy comparisons across economies.

Corollary 1. *Assume A1 with $\mu_\delta > 0$. Then for $e_N < 1/2$*

$$E\left[\frac{\xi^*(s)}{\xi^*(t)} \mid \mathcal{F}_t\right] \leq E\left[\frac{\bar{\xi}(s)}{\bar{\xi}(t)} \mid \mathcal{F}_t\right], \quad s > t\,,$$

and so

$$E\left[\int_t^T \frac{\xi^*(s)}{\xi^*(t)} ds \mid \mathcal{F}_t\right] \leq E\left[\int_t^T \frac{\bar{\xi}(s)}{\bar{\xi}(t)} ds \mid \mathcal{F}_t\right]\,.$$

The first terms compared correspond to the price at time t of a bond paying a certain payout of one unit at time s; the second terms compared correspond to the price at time t of an annuity paying a certain payout of one unit at all times until the end of the horizon. We mentioned before that, with $\delta(t)$ drifting upwards, when $e_N < 1/2$ the non-price-taker on average consumes more earlier on in his lifetime compared with when he is a price-taker. Hence he puts a lower value on a bond providing sure future consumption than if he were a price-taker.

5.3 Agents' Portfolio Strategies

For the remainder of this section, we no longer obtain unambiguous comparisons. Instead we merely offer insight and intuition into the effects of the non-price-taker, as well as representations for various quantities. We have discussed the intuition that the non-price-taker's consumption tends to deviate towards his endowment, $\epsilon_N(t)$. This might seem to suggest that the non-price-taker deviates his portfolio strategy in the risky security towards not trading at all. This in turn would lead to less net riskless lending and borrowing in the non-price-taking equilibrium. Here we look at the trading strategies in detail to see if this is indeed the case.

In the benchmark economy, the equilibrium portfolio strategies are

$$\bar{\alpha}_N(t) = \frac{1}{2}\,; \quad \bar{\alpha}_R(t) = \frac{1}{2}\,.$$

Regardless of their initial endowments, identical CARA utility agents share the risk equally in the economy, and hence each agent holds half of the risky asset. As long as $e_N \neq 1/2$ there is net riskless lending and borrowing in equilibrium. If $e_N > 1/2$, the initially wealthier agent N is a lender; if $e_N < 1/2$, agent N is a borrower.

Proposition 10 provides explicit representations for the trading strategies of the agents in the non-price-taking economy. These expressions are derived by appealing to the Clark formula and some properties of Malliavin derivatives. This use of Malliavin calculus has also been employed by Ocone and Karatzas [30]. We say more about Malliavin calculus in the next subsection.

Proposition 10. *Assume A1. The equilibrium portfolio strategies in the non-price-taking economy can be expressed as*

$$\alpha_N^*(t) = \frac{1}{2} + \frac{\sigma_\delta}{P^*(t)^2} E\left[\int_t^T \frac{\xi^*(s)}{\xi^*(t)}[g(s) - 1/2]ds \mid \mathcal{F}_t\right]$$
$$+ \frac{\sigma_\delta}{P^*(t)^2} E\left[\int_t^T \frac{\xi^*(s)}{\xi^*(t)}[g(s) - g(t)](c_N^*(s) - \delta(s)/2)ds \mid \mathcal{F}_t\right],$$
$$\alpha_R^*(t) = \frac{1}{2} - \frac{\sigma_\delta}{P^*(t)^2} E\left[\int_t^T \frac{\xi^*(s)}{\xi^*(t)}[g(s) - 1/2]ds \mid \mathcal{F}_t\right]$$
$$- \frac{\sigma_\delta}{P^*(t)^2} E\left[\int_t^T \frac{\xi^*(s)}{\xi^*(t)}[g(s) - g(t)](c_N^*(s) - \delta(s)/2)ds \mid \mathcal{F}_t\right].$$

Agents no longer hold half each of the risky asset as they do in the benchmark economy; there are additional terms in their portfolio strategies. If $e_N > 1/2$, we have $g(s) > 1/2$, so the second term in the non-price-taker's portfolio strategy is positive, making the non-price-taker tend to hold more than half of the risky asset and hence supporting our intuition of less net trading. For $e_N < 1/2$, the second term is negative, again supporting our intuition. The sign of the third term on the other hand is not unambiguous and prevents our conjecture of less trading from being exact. This term is driven by the changes in the non-price-taker's reaction to dividend stream changes $(g(s) - g(t))$, and by how much the non-price-taker is deviating from consuming one half of the aggregate dividend $(c_N^*(s) - \delta(s)/2)$.

5.4 Market Price, Volatility and Risk Premium in Equilibrium

When the non-price-taker initially owns (sufficiently) more than half of the market, he is a net seller of the market, and so we would expect him to raise the market level in the non-price-taking economy as compared with the price-taking economy (i.e., we expect $P^*(t) > \bar{P}(t)$ for $e_N > 1/2$). Similarly, we expect $P^*(t) < \bar{P}(t)$ for $e_N < 1/2$. We have not proved these results though.[10]

Our intuition in Subsection 5.3 is that the non-price-taker should tend to hold more than half the market when $e_N > 1/2$. Hence to clear the markets he must make the risky asset less attractive to the price-taker. One way to achieve this would be to increase the market price volatility, so we might conjecture a higher volatility when $e_N > 1/2$. Similarly, we might conjecture a lower volatility when $e_N < 1/2$. To compare market volatility across economies, we make use of some techniques of Malliavin calculus (Ikeda and Watanabe [19], Ocone [29]), in particular the Clark-Ocone formula. These techniques can be used to write representations of, and in some cases evaluate, the dynamics of conditional expectations. In particular we may derive the following representation for the market volatility when $\xi(t)$ and $\delta(t)$ are driven by general Itô processes.

Lemma 1. *When $\xi(t)$ and $\delta(t)$ follow the processes*

$$d\xi(t) = -\xi(t)\left[r(t)dt + \theta(t)dW(t)\right] ,$$

and

$$d\delta(t) = \mu_\delta(t)dt + \sigma_\delta(t)dW(t) ,$$

the market price volatility in a one risky asset economy may be expressed as

$$P(t)\sigma(t) = \sigma_\delta(t)E\left[\int_t^T \frac{\xi(s)}{\xi(t)}ds \mid \mathcal{F}_t\right] + E\left[\int_t^T \frac{\xi(s)}{\xi(t)}\left\{\int_t^s \mathcal{D}_t\mu_\delta(u)du + \int_t^s \mathcal{D}_t\sigma_\delta(u)dW(u)\right\} \mid \mathcal{F}_t\right]$$

$$- E\left[\int_t^T \frac{\xi(s)}{\xi(t)}\delta(s)\left\{\int_t^s \mathcal{D}_t r(u)du + \int_t^s \mathcal{D}_t\theta(u)dW(u) + \frac{1}{2}\int_t^s \mathcal{D}_t\theta(u)^2 du\right\} \mid \mathcal{F}_t\right], \qquad (30)$$

where $\mathcal{D}_tF$ is the Malliavin derivative of the functional F, as described in the Appendix.

The Malliavin derivative of a Brownian functional represents the change in that functional due to a perturbation in the path of $W(t)$. Equation (30) shows that in an arbitrage-free economy, the market price volatility, $P(t)\sigma(t)$, is equal to the aggregate dividend risk times the price of an annuity paying one unit of sure consumption till the terminal date, plus two additional terms arising from the stochastic nature of the coefficients of the processes in the conditional expectation. Individual contributions arise due to shocks in the market price of risk, the interest rate, and the drift and volatility of the aggregate dividend process. We apply Lemma 1 to our two economies, for the case when $\delta(t)$ is driven by an arithmetic Brownian motion.

Proposition 11. *Assume A1. The equilibrium market volatility and risk premia in the two economies are given by*

$$\bar{P}(t)\bar{\sigma}(t) = \sigma_\delta E\left[\int_t^T \frac{\bar{\xi}(s)}{\bar{\xi}(t)}ds \mid \mathcal{F}_t\right],$$

$$\bar{P}(t)(\bar{\mu}(t) - \bar{r}) = a\sigma_\delta^2 E\left[\int_t^T \frac{\bar{\xi}(s)}{\bar{\xi}(t)}ds \mid \mathcal{F}_t\right],$$

and

$$P^*(t)\sigma^*(t) = \sigma_\delta E\left[\int_t^T \frac{\xi^*(s)}{\xi^*(t)} ds \mid \mathcal{F}_t\right] + a\sigma_\delta E\left[\int_t^T \frac{\xi^*(s)}{\xi^*(t)}\delta(s)[g(s) - g(t)]ds \mid \mathcal{F}_t\right], \qquad (31)$$

$$\begin{aligned} &P^*(t)(\mu^*(t) - r^*(t)) \qquad (32)\\ &= a(1-g(t))\sigma_\delta^2 E\left[\int_t^T \frac{\xi^*(s)}{\xi^*(t)} ds \mid \mathcal{F}_t\right] + a^2(1-g(t))\sigma_\delta^2 E\left[\int_t^T \frac{\xi^*(s)}{\xi^*(t)}\delta(s)[g(s)-g(t)]ds \mid \mathcal{F}_t\right]. \end{aligned}$$

There are extra terms in the volatility of the non-price-taking economy introduced because $\xi^*(t)$ is no longer driven by a geometric Brownian motion, i.e., due to the stochastic nature of the market price of risk and the interest rate.

We argued previously from Corollary 1 that if $e_N < 1/2$ and $\mu_\delta > 0$, the price of the annuity in the first term of the market volatility expressions is lower in a non-price-taking economy than in a price-taking economy. In other cases we cannot make such unambiguous statements. The second term in equation (31) does not have an unambiguous sign. This term is driven by the changes in the non-price-taker's reaction to dividend stream changes, $g(s) - g(t)$. Similar comments can be made about equation (32). We cannot draw unambiguous conclusions about the effect of the non-price-taker on the market price volatility, or the excess drift of the market price.

6 Conclusion

In this paper we develop a continuous-time, pure-exchange, general equilibrium model to include an agent who acts as a price leader in the security and good markets. We analyze the equilibrium consumption-portfolio choice of this non-price-taking agent for general time-additive, state-independent utility functions and in more detail for the special case of CARA utility. We also investigate the effect of the presence of the non-price-taker on the asset and state price dynamics.

A major methodological contribution of this paper is to demonstrate that the non-price-taker's dynamic price-impact manifests itself through an impact of his consumption choice on the state price density process. The advantage of this formulation is that the problem can be analyzed using martingale techniques, making its analysis highly tractable. A main conclusion of this work is that, in addition to the aggregate consumption, the non-price-taking-taker's endowment stream is an extra factor driving the equilibrium allocations and prices. This leads to modified formulae for the consumption CAPM and the interest rate. Further comparisons of the price-taking and non-price-taking equilibria are carried out.

Further work related to this paper may include the following. An extension to multiple non-price-taking agents would be of interest. One could initially formulate this extension as a one-shot Cournot game played in the Arrow-Debreu securities market. The main results of the single non-price-taker economy would extend qualitatively, with multiple additional factors now driving the economy, the endowment streams of each non-price-taking agent. Furthermore, one could extend the analysis in Section 5 as a starting point towards investigating the case of multiple risky assets. More than one risky asset is needed for the non-price-taker's endowment to not be perfectly correlated with the aggregate dividend, and for the two-beta consumption CAPM to apply. This extension should yield further insights, such as the effect of the non-price-taker on individual asset prices and on the correlations between state prices, consumption allocations and dividends. Finally, an application of this paper would be to study non-price-taking behavior in the currency markets, a natural non-price-taking environment. Here, one could have a central bank representing a country, and model each central bank as a non-price-taker in its own currency.

APPENDICES

A Agents' Optimization and Equilibrium

Proof of Proposition 1: Applying the Lagrangian method to agent N's static variational problem implies (12) and (13). *Q.E.D.*

Proof of Proposition 2:

(a) By assumption, $\bar{c}_N(t) = \epsilon_N(t)$ is a solution to equilibrium in the price-taking economy. Hence there exists a constant $\bar{y}_N$ such that $u'_N(\epsilon_N(t)) = \bar{y}_N \xi(t) = \bar{y}_N U'(\delta(t) - \epsilon_N(t))$. Adding a term equaling zero to the right hand side of this equation, we obtain:

$$u'_N(\epsilon_N(t)) = \bar{y}_N \Big[U'\Big(\delta(t) - \epsilon_N(t)\Big) - U''\Big(\delta(t) - \epsilon_N(t)\Big)\Big(\epsilon_N(t) - \epsilon_N(t)\Big)\Big] ,$$

the sufficient condition for equilibrium in the non-price-taking economy, equation (12), for $c^*_N(t) = \epsilon_N(t)$ and $y_N = \bar{y}_N$. Clearly agent N's budget constraint holds with equality for $c^*_N(t) = \epsilon_N(t)$. So $c^*_N(t) = \epsilon_N(t)$, $t \in [0, T]$, is also an equilibrium in the non-price-taking economy.

(b) By assumption, $c^*_N(t) = \epsilon_N(t)$ is a solution to equilibrium in the non-price-taking economy. So, there exists y_N such that

$$u'_N(\epsilon_N(t)) = y_N \Big[U'\Big(\delta(t) - \epsilon_N(t)\Big) - U''\Big(\delta(t) - \epsilon_N(t)\Big)\Big(\epsilon_N(t) - \epsilon_N(t)\Big)\Big] = y_N U'\Big(\delta(t) - \epsilon_N(t)\Big) .$$

Together with N's budget constraint (which must obviously hold at $\bar{c}_N(t) = \epsilon_N(t)$), this condition is sufficient for $\bar{c}_N(t) = \epsilon_N(t) = c^*_N(t)$, $t \in [0, T]$ to be an equilibrium in the price-taking economy.

(c) By assumption, $\bar{c}_N(t)$ is a solution to equilibrium in the price-taking economy. Hence there exists a $\bar{y}_N$ such that $u'_N(\bar{c}_N(t)) = \bar{y}_N U'(\delta(t) - \bar{c}_N(t))$. Assume $\bar{c}_N(t)$ is also a solution to equilibrium in the non-price taking economy. Then there exists a constant y_N such that

$$u'_N(\bar{c}_N(t)) = y_N \Big[U'\Big(\delta(t) - \bar{c}_N(t)\Big) - U''\Big(\delta(t) - \bar{c}_N(t)\Big)\Big(\bar{c}_N(t) - \epsilon_N(t)\Big)\Big] , \quad t \in [0, T]$$

and $\bar{c}_N(t)$ satisfies N's budget constraint with equality. The above expressions imply

$$\frac{\bar{y}_N - y_N}{\bar{y}_N} = \frac{U''(\delta(t) - \bar{c}_N(t))(\bar{c}_N(t) - \epsilon_N(t))}{U'(\delta(t) - \bar{c}_N(t))} , \quad t \in [0, T] .$$

Since $U''(\cdot) < 0$ and $U'(\cdot) > 0$, and by assumption $\bar{c}_N(t) - \epsilon_N(t) \neq 0$ for some interval of t with probability > 0, for the right hand side of the above expression to be a constant, we must have either (i) $\bar{c}_N(t) - \epsilon_N(t) > 0$, $t \in [0, T]$ or (ii) $\bar{c}_N(t) - \epsilon_N(t) < 0$, $t \in [0, T]$, either of which contradicts N's budget constraint holding with equality. *Q.E.D.*

Proof of Proposition 3: Define the set of price-taking equilibrium agent-N consumption processes:

$$\mathcal{A}_N \equiv \{c_N(\cdot); \quad \text{there exists } \bar{y}_1,\ldots,\bar{y}_N,\bar{\xi}(\cdot) \quad \text{such that} \quad \sum_{n=1}^{N} I_n(\bar{y}_n\bar{\xi}(t)) = \delta(t)\,, \quad t\in[0,T]\,,$$

$$E\left[\int_0^T \Big(I_n(\bar{y}_n\bar{\xi}(t)) - \epsilon_n(t)\Big)\bar{\xi}(t)dt\right] = 0\,, \quad n=1,\ldots,N\,, \quad \text{and} \quad c_N(t) = I_N(\bar{y}_N\bar{\xi}(t))\,, \quad t\in[0,T]\}\,.$$

In a non-price-taking economy, agent N solves

$$\max_{c_N(\cdot)\in\mathcal{B}_N} E\left[\int_0^T u_N(c_N(t))dt\right],$$

where

$$\mathcal{B}_N \equiv \{c_N(\cdot); \quad \text{there exists } \tilde{y}_1,\ldots,\tilde{y}_N,\tilde{\xi}(\cdot) \quad \text{such that} \quad c_N(t) = \delta(t) - \sum_{n=1}^{N-1} I_n(\tilde{y}_n\tilde{\xi}(t))\,, \quad t\in[0,T]\,,$$

$$E\left[\int_0^T \Big(I_n(\tilde{y}_n\tilde{\xi}(t)) - \epsilon_n(t)\Big)\tilde{\xi}(t)dt\right] = 0\,, \quad n=1,\ldots,N-1\,, \quad \text{and} \quad E\left[\int_0^T \Big(c_N(t) - \epsilon_N(t)\Big)\tilde{\xi}(t)dt\right] = 0\}\,.$$

We need to show that $\mathcal{A}_N$ is a subset of $\mathcal{B}_N$ so that the non-price-taker could always have chosen any price-taking equilibrium consumption. Take any $\bar{c}_N(\cdot) \in \mathcal{A}_N$. Then there exists $\bar{y}_1,\ldots,\bar{y}_N,\bar{\xi}(\cdot)$ such that $\bar{c}_N(t) = \delta(t) - \sum_{n=1}^{N-1} I_n(\bar{y}_n\bar{\xi}(t)), t\in[0,T]$, and $E\left[\int_0^T(\bar{c}_N(t)-\epsilon_N(t))\bar{\xi}(t)dt\right] = 0$. So $\bar{c}_N(\cdot)\in\mathcal{B}_N$ with $\tilde{y}_n = \bar{y}_n, n=1,\ldots,N, \tilde{\xi}(\cdot) = \bar{\xi}(\cdot)$. *Q.E.D.*

Proof of Proposition 4: We first argue that if the non-price-taking equilibrium differs with probability > 0 from the price-taking equilibrium, then there exist subsets $A, B \subset \Omega$ and time intervals (t_{a1}, t_{a2}), $(t_{b1}, t_{b2}) \subset [0,T]$ such that $c_N^*(t,\omega) > \epsilon_N(t,\omega)$, $t\in(t_{a1},t_{a2})$; $\omega\in A$ and $c_N^*(t,\omega) < \epsilon_N(t,\omega)$, $t\in(t_{b1},t_{b2})$; $\omega\in B$. Assume not, and that $c_N^*(t) = \epsilon_N(t)$ a.s., $t\in[0,T]$. Then, from (12), $u_N'(\epsilon_N(t)) = y_N U'(\delta(t)-\epsilon_N(t))$, $t\in[0,T]$, and $c_N^*(t)$ clearly satisfies N's budget constraint, implying that $c_N^*(t)$ is also the solution to equilibrium in the price-taking economy. Hence, by the contrapositive there must exist a finite time and probability interval such that $c_N^*(t)\neq\epsilon_N(t)$. By the continuity of $c_N^*(t)$ and $\epsilon_N(t)$ and by (13), there must exist finite time and probability intervals in which $c_N^*(t) > \epsilon_N(t)$ and in which $c_N^*(t) < \epsilon_N(t)$.

Define a process $\phi(t)$ by

$$\phi(t) \equiv \frac{1}{2}\left(\frac{u_N'(c_N^*(t))}{y_N} + \xi^*(t)\right).$$

Then by concavity of $u_N(\cdot)$ and $U(\cdot)$, and continuity of $u_N'(\cdot)$ and $U'(\cdot)$, and by (9) and (12) there exists $\Upsilon > 0$ such that, for all $v\in(0,\Upsilon)$,

$$\frac{u_N'(c_N^*(t,\omega))}{y_N} > \frac{u_N'(c_N^*(t,\omega)+v)}{y_N} > \phi(t,\omega) \;>\; U'(\delta(t,\omega)-c_N^*(t,\omega)-v)$$
$$> U'(\delta(t,\omega)-c_N^*(t,\omega))\,, \quad t\in(t_{a1},t_{a2})\,, \quad \omega\in\mathcal{A} \quad (33)$$
$$\frac{u_N'(c_N^*(t,\omega))}{y_N} < \frac{u_N'(c_N^*(t,\omega)-v)}{y_N} < \phi(t,\omega) \;<\; U'(\delta(t,\omega)-c_N^*(t,\omega)+v)$$
$$< U'(\delta(t,\omega)-c_N^*(t,\omega))\,, \quad t\in(t_{b1},t_{b2})\,, \quad \omega\in\mathcal{B}\,. \quad (34)$$

Next define a process $\psi(t)$ where $\psi(t,\omega)>0$ for $t\in(t_{a1},t_{a2})$ and $\omega\in A$; $\psi(t,\omega)<0$ for $t\in(t_{b1},t_{b2})$ and $\omega\in B$; $\psi(t,\omega)=0$ otherwise; and $\psi(t)$ satisfies $E\left[\int_0^T \phi(t)\psi(t)dt\right]=0$, implying

$$E_B\left[\int_{t_{b1}}^{t_{b2}} \phi(t)\psi(t)dt\right] = E_A\left[\int_{t_{a1}}^{t_{a2}} \phi(t)\psi(t)dt\right]\,, \tag{35}$$

where E_A and E_B denote expectations over the subsets A and B, respectively. Then we choose $\epsilon>0$ such that $\epsilon|\psi(t)|<\Upsilon \qquad t\in[0,T]$.

We perturb $c_N^*(t)$ to $\tilde{c}_N(t)\equiv c_N^*(t)+\epsilon\,\psi(t)$ and $\sum_{n=1}^{N-1} c_n^*(t)$ to $\delta(t)-\tilde{c}_N(t)$, which is feasible. Agent N's and the price-taker representative agent's expected lifetime utility become

$$E\left[\int_0^T u_N(\tilde{c}_N(t))dt\right] = E\left[\int_0^T \left\{u_N(c_N^*(t)) + \int_{c_N^*(t)}^{c_N^*(t)+\epsilon\psi(t)} u_N'(c)dc\right\}dt\right]$$
$$= E\left[\int_0^T u_N(c_N^*(t))dt\right] + E_A\left[\int_{t_{a1}}^{t_{a2}}\left\{\int_{c_N^*(t)}^{c_N^*(t)+\epsilon\psi(t)} u_N'(c)dc\right\}dt\right] + E_B\left[\int_{t_{b1}}^{t_{b2}}\left\{-\int_{c_N^*(t)+\epsilon\psi(t)}^{c_N^*(t)} u_N'(c)dc\right\}dt\right]$$
$$> E\left[\int_0^T u_N(c_N^*(t))dt\right] + E_A\left[\int_{t_{a1}}^{t_{a2}} \epsilon\phi(t)\psi(t)dt\right] - E_B\left[\int_{t_{b1}}^{t_{b2}} \epsilon\phi(t)\psi(t)dt\right] > E\left[\int_0^T u_N(c_N^*(t))dt\right],$$
$$E\left[\int_0^T U(\delta-\tilde{c}_N(t))dt\right] = E\left[\int_0^T \left\{U(\delta(t)-c_N^*(t)) - \int_{c_N^*(t)}^{c_N^*(t)+\epsilon\psi(t)} U'(\delta(t)-c)dc\right\}dt\right]$$
$$= E\left[\int_0^T U(\delta(t)-c_N^*(t))dt\right] - E_A\left[\int_{t_{a1}}^{t_{a2}}\int_{c_N^*(t)}^{c_N^*(t)+\epsilon\psi(t)} U'(\delta(t)-c)dcdt\right] + E_B\left[\int_{t_{b1}}^{t_{b2}}\int_{c_N^*(t)+\epsilon\psi(t)}^{c_N^*(t)} U'(\delta(t)-c)dcdt\right]$$
$$> E\left[\int_0^T U(\delta(t)-c_N^*(t))dt\right] - E_A\left[\int_{t_{a1}}^{t_{a2}} \epsilon\phi(t)\psi(t)dt\right] + E_B\left[\int_{t_{b1}}^{t_{b2}} \epsilon\phi(t)\psi(t)dt\right] > E\left[\int_0^T U(\delta(t)-c_N^*(t))dt\right]\,,$$

using (33)–(35). Hence the non-price-taking equilibrium is not pareto optimal. *Q.E.D.*

Proof of Proposition 5: The optimality of $c_n^*(t)$, $n=1,\ldots,N-1$, given by (4), the expression for $\xi(t)$ in (9), and the N agents' budget constraints holding with equality imply (16)–(17).

Substituting for $\xi(t)$ using (9), agent N solves

$$\max_{c_N(\cdot)} E\left[\int_0^T u_N(c_N(t))dt\right]$$

$$\text{subject to}\quad E\left[\int_0^T g(t)dt\right]=0 \quad\text{and}\quad E\left[\int_0^T h_n(t)dt\right]=0\,,\quad n=1,\ldots,N-1\,,$$

where by $g(t)\equiv U'(\delta(t)-c_N(t);\Lambda)(c_N(t)-\epsilon_N(t))$ and $h_n(t)\equiv U'(\delta(t)-c_N(t);\Lambda)(I_n(y_nU'(\delta(t)-c_N(t);\Lambda))-\epsilon_n(t))$ representing the cost of N's and n's "net" consumption at time t.

We may define the mappings G, G^{-1} and H_n, $n=1,\ldots,N-1$, as follows:

$$G(c,y_1,\ldots,y_{N-1};t,\omega)=U'(\delta(t,\omega)-c;(1/y_1,\ldots,1/y_{N-1}))(c-\epsilon_N(t,\omega))\,,$$

$$g=U'\left(\delta(t,\omega)-G^{-1}(g,y_1,\ldots,y_{N-1};t,\omega);1/y_1,\ldots,1/y_{N-1}\right)\left(G^{-1}(g,y_1,\ldots,y_{N-1};t,\omega)-\epsilon_N(t,\omega)\right)\,,$$

$$H_n(g,y_1,\ldots,y_{N-1};t,\omega)=U'\left(\delta(t,\omega)-G^{-1}(g,y_1,\ldots,y_{N-1};t,\omega);(1/y_1,\ldots,1/y_{N-1})\right)*$$
$$\left(I_n\left(y_nU'(\delta(t,\omega)-G^{-1}(g,y_1,\ldots,y_{N-1};t,\omega);(1/y_1,\ldots,1/y_{N-1}))\right)-\epsilon_n(t,\omega)\right).$$

We will use the notation G_n, G_n^{-1} and H_{nk} to denote the derivatives of these mappings with respect to their kth argument. Then agent N's optimization problem can be written as

$$\max_{g(\cdot)}E\left[\int_0^T u_N\left(G^{-1}(g(t),y_1,\ldots,y_{N-1};t)\right)dt\right]$$

$$\text{subject to}\quad E\left[\int_0^T g(t)dt\right]=0\quad\text{and}\quad E\left[\int_0^T H_n(g(t),y_1,\ldots,y_{N-1};t)dt\right]=0,\quad n=1,\ldots,N-1\,.$$

Suppose that $c_N^*(\cdot)$ is an equilibrium agent-N consumption process with associated weights $y_1^*,\ldots,y_{N-1}^*$ and that $g^*(t)=G(c_N^*(t),y_1^*,\ldots,y_{N-1}^*;t)$, $t\in[0,T]$. Let us use the notation $G(*,t)$, $G^{-1}(*,t)$, etc., to denote the various mappings evaluated at this equilibrium at time t. Then perturb the process $g^*(\cdot)$ to $g^\epsilon(t)=g^*(t)+\epsilon\eta(t)$. Since by assumption $g^*(\cdot)$ is the solution to the above optimization problem we must have

$$\lim_{\epsilon\to0}\frac{d}{d\epsilon}E\left[\int_0^T u_N\left(G^{-1}(g^\epsilon(t),y_1(\epsilon),\ldots,y_{N-1}(\epsilon);t,\omega)\right)dt\right] \tag{36}$$
$$=E\left[\int_0^T u_N'(c_N^*(t))G_1^{-1}(*,t)\eta(t)dt\right]+\sum_{j=1}^{N-1}y_j'(0)E\left[\int_0^T u_N'(c_N^*(t))G_{j+1}^{-1}(*,t)dt\right]=0$$

for all processes $\eta(t)$ satisfying $E\left[\int_0^T\eta(t)dt\right]=0$, where the functions $y_j(\epsilon)$ are determined from

$$E\left[\int_0^T H_n(g^\epsilon(t),y_1(\epsilon),\ldots,y_{N-1}(\epsilon);t,\omega)dt\right]=0\,,\quad n=1,\ldots,N-1\,. \tag{37}$$

If we define a matrix $\mathcal{H}$ by

$$\mathcal{H}_{nj}\equiv E\left[\int_0^T H_{jn}(g^*(t),y_1^*,\ldots,y_{N-1}^*;t,\omega)dt\right],\quad n,j=1,\ldots,N-1\,,$$

then by taking derivatives of (37), evaluating at $\epsilon = 0$ and rearranging we solve for the $y_j'(0)$ as

$$y_j'(0) = -\sum_{n=1}^{N-1} \left(\mathcal{H}^{-1}\right)_{jn} E\left[\int_0^T H_{n1}(g^*(t), y_1^*, \ldots, y_{N-1}^*; t, \omega)\eta(t)dt\right], \quad j = 1, \ldots, N-1 .$$

Substituting into (36) we obtain the condition

$$E\left[\int_0^T \left\{u_N'(c_N^*(t))G_1^{-1}(*,t) - \sum_{j=1}^{N-1} E\left[\int_0^T u_N'(c_N^*(s))G_{j+1}^{-1}(*,s)ds\right] \sum_{n=1}^{N-1} \left(\mathcal{H}^{-1}\right)_{jn} H_{n1}(*,t)\right\} \eta(t)dt\right] = 0 ,$$

for all $\eta(\cdot)$ satisfying $E\left[\int_0^T \eta(t)dt\right] = 0$. Hence we must have

$$u_N'(c_N^*(t))G_1^{-1}(*,t) - \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} E\left[\int_0^T u_N'(c_N^*(s))G_{j+1}^{-1}(*,s)ds\right] \left(\mathcal{H}^{-1}\right)_{jn} H_{n1}(*,t) = y_N .$$

Evaluating the G_1^{-1} and H_{n1} terms and rearranging we arrive at (15) with

$$K_n \equiv \sum_{j=1}^{N-1} E\left[\int_0^T u_N'(c_N^*(s))G_{j+1}^{-1}(g^*(s), y_1^*, \ldots, y_{N-1}^*; s, \tilde{\omega})ds\right] \left(\mathcal{H}^{-1}\right)_{jn} .$$

The extra terms compared with equation (12) are $-K_n H_{n1}/G_1^{-1}$. Each expectation term in K_n is the marginal utility to N due to a change in agent j's weight y_j. The elements $\mathcal{H}_{nj}$ represent the sensitivity of agent n's budget constraint to agent j's weight y_j. Hence K_n is the sensitivity of agent N's expected lifetime utility to agent n's budget constraint. Then H_{n1} is the sensitivity of agent n's budget constraint to agent N's cost of "net" consumption, and G_1^{-1} is the sensitivity of agent N's current consumption to his cost of net consumption. Hence H_{n1}/G_1^{-1} is the sensitivity of agent n's budget constraint to agent N's time t, state ω consumption. Therefore each extra term $-K_n H_{n1}/G_1^{-1}$ represents the indirect marginal disutility to N of an extra unit of $c_N(t,\omega)$ via agent n's budget constraint.

Finally, we show that (15) collapses to (12) when $U(c;\Lambda) = h(\Lambda)U(c)$. For $j = 1, \ldots, N-1$,

$$\begin{aligned} E\left[\int_0^T u_N'(c_N^*(t))G_{j+1}^{-1}(*,t)dt\right] &= E\left[\int_0^T \frac{u_N'(c_N^*(t))U_{yj}'(\delta(t) - c_N(t);\Lambda)(c_N(t) - \epsilon_N(t))}{U'(\delta(t) - c_N(t);\Lambda) - U''(\delta(t) - c_N(t);\Lambda)(c_N(t) - \epsilon_N(t))}dt\right] \\ &= y_N \frac{\partial h(\Lambda)}{\partial y_j} E\left[\int_0^T U'(\delta(t) - c_N(t);\Lambda)(c_N(t) - \epsilon_N(t))dt\right] = 0 , \end{aligned}$$

where we have used $U_{yj}'(c;\Lambda) = (\partial h(\Lambda))/(\partial y_j)U'(c)$; have supposed that (12) does hold, and finally used agent N's budget constraint. Hence $K_n = 0$. *Q.E.D.*

B The Equilibrium Interest Rate and the Consumption-Based CAPM

Proof of Proposition 6: Apply Itô's Lemma to (9), use (1), and equate terms to derive

$$r(t) = -\frac{U''(\delta(t) - c_N^*(t))}{U'(\delta(t) - c_N^*(t))}\left(\mu_\delta(t) - \mu_{c_N}^*(t)\right) - \frac{U'''(\delta(t) - c_N^*(t))}{2U'(\delta(t) - c_N^*(t))}\|\sigma_\delta(t) - \sigma_{c_N}^*(t)\|^2 , \tag{38}$$

$$\theta(t) = -\frac{U''(\delta(t) - c_N^*(t))}{U'(\delta(t) - c_N^*(t))}\left(\sigma_\delta(t) - \sigma_{c_N}^*(t)\right) . \tag{39}$$

To express the endogenous parameters $\mu_{c_N}^*(t)$ and $\sigma_{c_N}^*(t)$ in terms of the exogenous parameters $\delta(t)$ and $\epsilon_N(t)$, apply Itô's Lemma to both sides of equation (12) and equate terms. Substitution into (39) and using $\mu(t) - r(t)1 = \sigma(t)\theta(t)$ yields (19), where we write $\sigma(t)\sigma_\delta(t)$ and $\sigma(t)\sigma_{\epsilon_N}(t)$ as $\mathrm{cov}(dP(t)/P(t), d\delta(t))$ and $\mathrm{cov}(dP(t)/P(t), d\epsilon_N(t))$. Similar substitution into (38) would yield an expression for $r(t)$ in terms of $\mu_\delta(t)$, $\mu_{\epsilon_N}(t)$, $\sigma_\delta(t)$, and $\sigma_{\epsilon_N}(t)$ as shown. *Q.E.D.*

C The CARA Utility and One Risky Asset Case

Proof of Proposition 7: From (20) $\bar{c}_N(t) - \delta(t)/2$ is a constant. (22) can be rearranged as

$$\exp\{-2a(c_N^*(t) - \delta(t)/2)\} = y_N\left[1 + a(c_N^*(t) - \delta(t)/2) - a\delta(t)(e_N - 1/2)\right] \tag{40}$$

and differentiated implicitly (state by state) with respect to $\delta(t,\omega)$ to give

$$\frac{\partial(c_N^*(t,\omega) - \delta(t,\omega)/2)}{\partial\delta(t,\omega)} = \frac{ay_N}{ay_N + 2a\exp\{-2a(c_N^*(t,\omega) - \delta(t,\omega)/2)\}}(e_N - 1/2) .$$

We conclude that $(c_N^*(t,\omega) - \delta(t,\omega)/2)$ is strictly monotonically increasing in $\delta(t,\omega)$ if $e_N > 1/2$; strictly monotonically decreasing in $\delta(t,\omega)$ if $e_N < 1/2$, and constant if $e_N = 1/2$.

Let us now show there exists a δ_{crit} such that $c_N^*(t) - \delta(t)/2 = \bar{c}_N(t) - \delta(t)/2$. At δ_{crit} we have $c_N^*(t) = \bar{c}_N(t)$, so we substitute (20) into (22) and rearrange to yield

$$\delta_{crit} = \frac{1}{a(e_N - 1/2)}\left[1 - \frac{\bar{y}_N}{y_N} - \frac{1}{2}\ln(\bar{y}_N)\right] .$$

Then, for $e_N > 1/2$, since $(\bar{c}_N(t) - \delta(t)/2)$ is constant and $(c_N^*(t) - \delta(t)/2)$ is monotonically increasing in $\delta(t)$, for $\delta(t) > \delta_{crit}$, $c_N^*(t) - \delta(t)/2 > \bar{c}_N(t) - \delta(t)/2$ and for $\delta(t) < \delta_{crit}$, $c_N^*(t) - \delta(t)/2 < \bar{c}_N(t) - \delta(t)/2$. The analogous results for $e_N < 1/2$ or $e_N = 1/2$ provide part (a).

For part (b), again use the fact that $(c_N^*(t) - \delta(t)/2)$ is monotonically increasing $(e_N > 1/2)$, decreasing $(e_N < 1/2)$ or constant $(e_N = 1/2)$. Define $\bar{\delta} = \frac{y_N - 1}{ay_N(e_N - 1/2)}$ where, from (40), $c_N^*(t) - \delta(t)/2 = 0$. Hence for $\delta(t) < \bar{\delta}$, $c_N^*(t) < \delta(t)/2$ for $e_N > 1/2$, $c_N^*(t) > \delta(t)/2$ for $e_N < 1/2$, and $c_N^*(t) = \delta(t)/2$ for $e_N = 1/2$. Since $c_R^*(t) = \delta(t) - c_N^*(t)$, we deduce part (b). *Q.E.D.*

Proof of Proposition 8: Equation (22) can be rewritten as

$$\frac{1}{y_N}\exp\{-2ac_N^*(t)\}\exp\{a\delta(t)\} = [1 + a(c_N^*(t) - \epsilon_N(t))] \equiv f(t) \;. \tag{41}$$

Applying Itô's Lemma to both sides and matching coefficients yields (24)–(27).

When $e_N > 1/2$, $g(t) > (1/2 + f(t))/(1 + 2f(t)) > 1/2$, when $e_N < 1/2$, $g(t) < 1/2$, yielding the comparisons between $\bar{\sigma}_{c_N}(t)$ and $\sigma^*_{c_N}(t)$. When $g(t) > 1/2$, $g(t)^2 + g(t) - 3/4 > 0$; when $g(t) < 1/2$, $g(t)^2 + g(t) - 3/4 < 0$, yielding the comparisons between $\bar{\mu}_{c_N}(t)$ and $\mu^*_{c_N}(t)$. Finally, $\mid \bar{\sigma}_{c_N}(t) - \sigma_{\epsilon_N}(t) \mid = |1/2 - e_N||\sigma_\delta(t)| <\mid \sigma^*_{c_N}(t) - \sigma_{\epsilon_N}(t) \mid = |g(t) - e_N||\sigma_\delta(t)|$, by the conclusions about $g(t)$. *Q.E.D.*

Proof of Proposition 9: Substituting for $U(c) = -\exp\{-ac\}/a$ into equations (38) and (39) in the proof of Proposition 6, we obtain

$$r^*(t) = a\left[\mu_\delta(t) - \mu^*_{c_N}(t)\right] - \frac{a^2}{2}|\sigma_\delta(t) - \sigma^*_{c_N}(t)|^2 \,, \quad \theta^*(t) = a\left(\sigma_\delta(t) - \sigma^*_{c_N}(t)\right) \,.$$

Substitute for $\mu^*_{c_N}(t)$ and $\sigma^*_{c_N}(t)$ from (24) and (25) to obtain (28) and (29). The properties of $g(t)$ in Proposition 8 imply the comparisons between $\theta^*(t)$ and $\bar{\theta}(t)$. *Q.E.D.*

Proof of Corollary 1: We will need the following Lemma.

Lemma 2. (Hajek 1985) *Let x and y be semimartingales with representations $dx(s) = \mu(s)ds + \sigma(s)dw(s)$, $dy(s) = mds + \rho dv(s)$, where w and v are Wiener processes and m and ρ are constants. Suppose that $\mu(s) \le m$ and $|\sigma(s)| \le \rho$ and that $x(0) \le y(0)$. Then for any nondecreasing convex function ϕ on $\mathcal{R}$*

$$E\phi(x(s)) \le E\phi(y(s)) \,.$$

Applying Itô's Lemma at s to the process $\ln(\xi(s)/\xi(t))$, $s > t$, yields

$$d\left[\ln\left(\frac{\xi(s)}{\xi(t)}\right)\right] = d(\ln(\xi(s))) = -\left(r(s) + \frac{1}{2}\theta(s)^2\right)ds - \theta(s)dW(s) \,, \quad s > t \,.$$

We have $\bar{r} + \bar{\theta}^2/2 = a\mu_\delta/2$ and from Proposition 9,

$$r^*(s) + \frac{1}{2}\theta^*(s)^2 = a(1 - g(s))\mu_\delta - \frac{2f(s)}{1 + f(s)}a\left[g(s)^2 + g(s) - \frac{3}{4}\right]\sigma_\delta^2 \,.$$

So if $e_N < 1/2$, $g(t) < 1/2$, $g(t)^2 + g(t) - 3/4 < 0$ with $\mu_\delta > 0$ implying $-\left(r^*(s) + \frac{1}{2}\theta^*(s)^2\right) < -\left(\bar{r} + \frac{1}{2}\bar{\theta}^2\right)$. We also have $|-\theta^*(s)| < |-\bar{\theta}| = a\sigma_\delta/2$. We apply Hajek's Lemma to the processes

$x(s) \equiv \ln(\xi^*(s+t)/\xi^*(t))$ and $y(s) \equiv \ln(\bar{\xi}(s+t)/\bar{\xi}(t))$, with $x(0) = y(0) = 0$. Now define the function $\phi(x) \equiv \exp(x)$ and we conclude

$$E\left[\frac{\xi^*(s+t)}{\xi^*(t)} \mid \mathcal{F}_t\right] = E\phi(x(s)) \leq E\phi(y(s)) = E\left[\frac{\bar{\xi}(s+t)}{\bar{\xi}(t)} \mid \mathcal{F}_t\right], \quad s > 0,$$

obtaining the desired result. *Q.E.D.*

Proof of Proposition 10: First we briefly state required notions and results from Malliavin calculus. For more details see Ikeda and Watanabe [19] or Ocone [29].

Malliavin Calculus of Smooth Brownian Functionals

Suppose $F = F(W(t_1), \ldots, W(t_n))$ is a smooth Brownian functional, i.e., a functional of a finite dimensional Brownian motion W at a number of points in time such that the function F is bounded and has bounded derivatives of all orders. Then the Malliavin derivative of the functional F is defined by

$$\mathcal{D}_t^i F \equiv \sum_{j=1}^{n} \frac{\partial}{\partial W_i(t_j)} F(W(t_1), \ldots, W(t_n)) 1_{[0,t_j]}(t), \quad i = 1, \ldots, d,$$

and can be interpreted as the change in F due to a perturbation in the path of $W_i(t)$. The Malliavin derivative of a continuously differentiable function $\phi\left(F^1, \ldots, F^M\right)$ of a finite number of Brownian functionals, with bounded partial derivatives is given by[11]

$$\mathcal{D}\phi\left(F^1, \ldots, F^M\right) = \sum_{i=1}^{M} \frac{\partial \phi(\cdot)}{\partial F^i} \mathcal{D}F^i.$$

The Malliavin derivative of an integral and a stochastic integral are given by

$$\mathcal{D}_t \int_0^T \psi(s)ds = \int_t^T \mathcal{D}_t\psi(s)ds, \quad \mathcal{D}_t \int_0^T \psi(s)dW(s) = \int_t^T \mathcal{D}_t\psi(s)dW(s) + \psi(t).$$

The Clark-Ocone Formula: *A Brownian functional F can be represented by*

$$F = E[F] + \int_0^T E\left[\mathcal{D}_s F \mid \mathcal{F}_s\right] dW(s),$$

and hence

$$E[F \mid \mathcal{F}_t] = E[F] + \int_0^t E\left[\mathcal{D}_s F \mid \mathcal{F}_s\right] dW(s),$$

or

$$dE[F \mid \mathcal{F}_t] = E\left[\mathcal{D}_t F \mid \mathcal{F}_t\right] dW(t). \tag{42}$$

It is well-known (e.g., Cox and Huang [6]) that an agent's wealth can be expressed as

$$X_n(t) = \xi(t)\, E\left[\int_t^T \xi(s)c_n(s)ds \mid \mathcal{F}_t\right] .$$

We rearrange as

$$d\left(\xi(t)X_n(t)\right) = dE\left[\int_0^T \xi(s)c_n(s)ds \mid \mathcal{F}_t\right] - \xi(t)c_n(t)dt .$$

Then by the Clark-Ocone formula, equation (42), we have

$$d\left(\xi(t)X_n(t)\right) = E\left[\mathcal{D}_t \int_t^T \xi(s)c_n(s)ds \mid \mathcal{F}_t\right] dW(t) - \xi(t)c_n(t)dt.$$

We can also apply Itô's Lemma directly to $\xi(t)X_n(t)$ and use (1) and (3) to obtain

$$d(\xi(t)X_n(t)) = \xi(t)\left[\alpha_n(t)P(t)\sigma(t) - X_n(t)\theta(t)\right] dW(t) - \xi(t)c_n(t)dt .$$

Equating the "$dW(t)$" terms on the right hand sides of the two last equations yields

$$\alpha_n(t) = X_n(t)\frac{\theta(t)}{P(t)\sigma(t)} + \frac{1}{P(t)\sigma(t)\xi(t)} E\left[\mathcal{D}_t \int_t^T \xi(s)c_n(s)ds \mid \mathcal{F}_t\right].$$

Using the above stated properties of Malliavin derivatives of integrals we may expand this as

$$\begin{aligned}\alpha_n(t) &= \frac{X_n(t)\theta(t)}{P(t)\sigma(t)} + \frac{1}{P(t)\sigma(t)\xi(t)} \\ &* E\left[\int_t^T \xi(s)c_n(s)\left\{-\int_s^t \mathcal{D}_t(r(u) + \frac{1}{2}\theta(u)^2)du - \int_s^t \mathcal{D}_t\theta(u)dW(u) - \theta(t)\right\} ds \mid \mathcal{F}_t\right] \\ &+ \frac{1}{P(t)\sigma(t)\xi(t)} E\left[\int_t^T \xi(s)\left\{\int_s^t \mathcal{D}_t\mu_{c_n}(u)du + \int_s^t \mathcal{D}_t\sigma_{c_n}(u)dW(u) + \sigma_{c_n}(t)\right\} ds \mid \mathcal{F}_t\right] .\end{aligned}$$

Using Propositions 8 and 9, we obtain

$$\int_t^s \mathcal{D}_t\left(r^*(u) + \frac{1}{2}\theta^*(u)^2\right) du + \int_t^s \mathcal{D}_t\theta^*(u)dW(u) = -a\left\{\int_t^s \mathcal{D}_t\mu^*_{c_N}(u)du + \int_t^s \mathcal{D}_t\sigma^*_{c_N}(u)dW(u)\right\}.$$

Now let us look at the Malliavin derivative of $c_N^*(s)$. Note from equation (22) that $c_N^*(s,\omega) = c_N^*(\delta(s,\omega); e_N, y_N)$, so we have

$$\begin{aligned}\mathcal{D}_t c_N^*(s,\omega) &= \frac{\partial c_N^*(s,\omega)}{\partial\delta(s,\omega)}\mathcal{D}_t\delta(s,\omega) = \frac{\partial c_N^*(s,\omega)}{\partial\delta(s,\omega)}\sigma_\delta \equiv g(s,\omega)\sigma_\delta = \sigma^*_{c_N}(s,\omega) \\ &= \int_t^s \mathcal{D}_t\mu^*_{c_N}(u)du + \int_t^s \mathcal{D}_t\sigma^*_{c_N}(u)dW(u) + \sigma^*_{c_N}(t) .\end{aligned}$$

Using these results the above expression for $\alpha_n(t)$ becomes, for the non-price-taker

$$\alpha_N^*(t) = \frac{\sigma_\delta}{\xi^*(t)P^*(t)^2} E\left[\int_t^T \xi^*(s)\Big\{g(t) + \Big(1 + ac_N^*(s)\Big)\Big(g(s) - g(t)\Big)\Big\}ds \mid \mathcal{F}_t\right] .$$

Anticipating and rearranging the result of Proposition 11 (32) yields

$$P^*(t) = \frac{\sigma_\delta}{\sigma^*(t)\xi^*(t)} E\left[\int_t^T \xi^*(s)ds \mid \mathcal{F}_t\right] + \frac{a\sigma_\delta}{\sigma^*(t)\xi^*(t)} E\left[\int_t^T \xi^*(s)\delta(s)[g(s)-g(t)]ds \mid \mathcal{F}_t\right] ,$$

which when substituted into the previous expression and rearranged gives the required result for $\alpha_N^*(t)$. Then $\alpha_R^*(t)$ is found from $\alpha_R^*(t) = 1/2 - \alpha_N^*(t)$. *Q.E.D.*

Proof of Lemma 1: From equation (2) and the Clark-Ocone formula we have

$$\xi(t)P_i(t) = E\left[\int_0^T \xi(s)\delta_i(s)ds\right] + \int_0^t E\left[\mathcal{D}_s \int_0^T \xi(u)\delta_i(u)du \mid \mathcal{F}_s\right] dW(s) - \int_0^t \xi(s)\delta_i(s)ds .$$

Then

$$d(\xi(t)P_i(t)) = E\left[\int_t^T \mathcal{D}_t(\xi(s)\delta_i(s))ds \mid \mathcal{F}_t\right] dW(t) - \xi(t)\delta_i(t)dt .$$

Applying Itô's Lemma directly to $\xi(t)P_i(t)$ we obtain

$$d(\xi(t)P_i(t)) = \xi(t)P_i(t)\left[\sigma_i(t) - \theta(t)^\top\right] dW(t) - \xi(t)\delta_i(t)dt ,$$

and equating coefficients with the Clark-Ocone representation yields the ith row of $\sigma(\cdot)$ as

$$\sigma_i(t) = \theta(t)^\top + \frac{1}{\xi(t)P_i(t)} E\left[\int_t^T \mathcal{D}_t(\xi(s)\delta_i(s))ds \mid \mathcal{F}_t\right] . \tag{43}$$

By the properties of the Malliavin derivatives (stated earlier), we have

$$\mathcal{D}_t(\xi(s)\delta_i(s)) = \xi(s)\mathcal{D}_t\delta_i(s) + \delta_i(s)\mathcal{D}_t\xi(s) ,$$

$$\begin{aligned}
\mathcal{D}_t^j\delta_i(s) &= \mathcal{D}_t^j\left\{\delta_i(0) + \int_0^s \mu_{\delta_i}(u)du + \int_0^s \sigma_{\delta_i}(u)dW(u)\right\} \\
&= \int_t^s \mathcal{D}_t^j\mu_{\delta_i}(u)du + \int_t^s \sum_l \mathcal{D}_t^j\sigma_{\delta_i l}(u)dW_l(u) + \sigma_{\delta_{ij}}(t) ,
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_t^j\xi(s) &= \mathcal{D}_t^j\left\{\delta(0)\exp\left\{-\int_0^s r(u)du - \int_0^s \sum_l \theta_l(u)dW_l(u) - \frac{1}{2}\int_0^s \|\theta(u)\|^2 du\right\}\right\} \\
&= \xi(s)\left\{-\int_t^s \mathcal{D}_t^j r(u)du - \theta_j(u) - \int_t^s \sum_l \mathcal{D}_t^j\theta_l(u)dW_l(u) - \frac{1}{2}\int_t^s \mathcal{D}_t^j\|\theta(u)\|^2 du\right\} .
\end{aligned}$$

Substituting into (43) yields the desired result. *Q.E.D.*

Proof of Proposition 11: We use Lemma 1. By assumption A1, $\mathcal{D}_t\mu_\delta(u) = \mathcal{D}_t\sigma_\delta(u) = 0$ so the second term in (30) is zero in both economies. In the price-taking economy, $\bar{r}(t)$ and $\bar{\theta}(t)$ are constants and hence the third term in (30) is zero and the volatility is as quoted. In the proof of Proposition 10, we evaluated the {} term in the third term of (30) for the non-price-taking economy. Substitution in gives the required result for the market volatility. To find the risk premia we use $\mu(t) - r(t) = \theta(t)\sigma(t)$ and substitute for $\theta(t)$ from Section 5.2. *Q.E.D.*

FOOTNOTES

[1]In a frictionless economy such as ours, the size of agents' security holdings is not restricted by their wealth, so it should not be argued that an investor with large net wealth can affect prices more than a small investor; however, in practice given the presence of short sales constraints and transaction costs in the marketplace, it is plausible that some investors are able to make larger trades and hence affect prices more than others.

[2]The non-price-taking strategy we solve for is a "self-commitment" strategy in which the non-price-taker chooses a plan initially and then does not deviate from that plan.

[3]We could alternatively work from the price leadership in the securities markets by calculating the price-taker's asset demands as a function of the entire process of the price system, writing the residual supplies as

$$\alpha_{Ni}(t) = 1 - \sum_{n=1}^{N-1} \hat{\alpha}_{nit}\Big(r(\cdot), \mu(\cdot), \sigma(\cdot)\Big) , \quad i = 1, \ldots, L ; \quad \alpha_{N0}(t) = - \sum_{n=1}^{N-1} \hat{\alpha}_{not}\Big(r(\cdot), \mu(\cdot), \sigma(\cdot)\Big) ,$$

and then attempting to deduce the effect of $\alpha_N(t)$ on the price system. However, a derivation of the price-takers' asset demand functions in our general set-up (by, for example, Malliavin calculus as in Ocone and Karatzas [30]), reveals very complicated expressions in terms of the whole price process (in $[0, T]$). In general, inverting such expressions does not yield a simple non-price-taking effect such as increased asset holdings leading to a higher asset price. On the other hand, if instead of following a price leadership model, we were to specify a particular price-dependence in an asset market (as, for example, in Jarrow [20]), for an equilibrium analysis it would be necessary to ensure consistency of this specification with market clearing and optimality. This is not a straightforward exercise and in general may not be possible.

[4]If instead we had a deterministic model with deterministic endowment streams and a zero net supply (riskless) security, this security is no longer like a durable good and its price today ($P_0(t) = 1/\xi(t)$) depends only on today's actions of the non-price-taker. Hence in this case the strategy is time-consistent.

[5]The HARA form of utility is $U(c) = \frac{1-\gamma}{\gamma}\left(\frac{\beta c}{1-\gamma} + \eta\right)^{\gamma}$, defined over the domain where $\left(\frac{\beta c}{1-\gamma} + \eta\right) > 0$, where $\beta > 0$, $\eta \geq 0$, $\gamma \neq 1$; for $\gamma = -\infty$, $\eta = 1$. The HARA family includes power utility ($\beta = 1$, $\eta = 0$, $\gamma < 1$), log utility ($\beta = 1$, $\eta = 0$, $\gamma = 0$) and negative

exponential utility ($\gamma = -\infty$, $\eta = 1$). For $\lambda_\delta^*, \lambda_{\epsilon_N}^* > 0$, we also need to assume that agents' "own" dividend processes, $\epsilon_N(t)$ and $\sum_{n=1}^{N-1} \epsilon_n(t)$ lie within the domains of their respective utility functions, i.e., $\epsilon_N(t) > c_\infty$ and $\left(\frac{\beta \sum_{n=1}^{N-1} \epsilon_n(t)}{1-\gamma} + \eta\right) > 0, \quad t \in [0,T]$.

[6]With power or log utility and one risky asset, in the price-taking equilibrium (when N is also a price-taker) each agent holds on to his initial asset endowment and no trading occurs. Hence, according to Proposition 2, a power or log utility non-price-taking agent does not deviate from his price-taking behavior and there is no effect on the optimal strategies or equilibrium prices. So instead we choose the case of exponential utility in which trading does take place in the price-taking equilibrium even in one risky asset.

[7]This process has the undesirable feature that the dividend can go negative since it is normally distributed at each point in time. However, the probability of this happening can be made arbitrarily small, and hence we do not concern ourselves with this inconvenience.

[8]Our numerical analysis reveals that when $2\mu_\delta / a\sigma_\delta^2 > 1$, for $e_N > 1/2$, $r^*(t) < \bar{r}(t)$; for $e_N < 1/2$, $r^*(t) > \bar{r}(t)$. Calibrating the price-taking economy to "normal" market conditions of $r = 8\%, \sigma = 20\%, \mu = 15\%$, yields $2\mu_\delta / a\sigma_\delta^2 = 1.15$.

[9]I thank Steve Shreve for bringing to my attention these mean comparison theorems of solutions to stochastic differential equations.

[10]A modified mean comparison theorem applied to the price formula of equation (2), might prove applicable to this comparison.

[11]The Malliavin derivative defined here is a special case of the more general Malliavin derivative defined on Brownian functionals satisfying certain smoothness and integrability conditions.

REFERENCES

1. S. Basak, "A General Equilibrium Model of Portfolio Insurance," mimeo, The Wharton School, University of Pennsylvania, 1994.

2. O. J. Blanchard and S. Fisher, "Lectures on Macroeconomics," The MIT Press, Cambridge, Massachusetts, 1989.

3. D. T. Breeden, An intertemporal asset pricing model with stochastic consumption and investment opportunities, *J. Financial Economics* 7 (1979), 265–296.

4. D. T. Breeden, Consumption, production, inflation and interest rates: a synthesis, *J. Financial Economics* 16 (1986), 3-39.

5. R. H. Coase, Durability and Monopoly, *J. Law and Economics* 15 (1972), 143–149.

6. J. C. Cox and C. F. Huang, Optimal consumption and portfolio policies when asset prices follow a diffusion process, *J. Economic Theory* 49 (1989), 33–83.

7. J. C. Cox and C. F. Huang, A variational problem arising in financial economics, *J. Mathematical Economics* 20 (1991), 465–487.

8. J. C. Cox, J. E. Ingersoll and S. A. Ross, An intertemporal general equilibrium model of asset prices, *Econometrica* 53 (1985), 363–384.

9. J. B. Detemple and F. Zapatero, Asset prices in an exchange economy with habit formation, *Econometrica* 59 (1991), 1633–1657.

10. D. Duffie, "Dynamic Asset Pricing Theory," Princeton University Press, Princeton, New Jersey, 1992.

11. D. Duffie, Stochastic equilibria: existence, spanning number, and the 'no expected financial gain from trade' hypothesis, *Econometrica* 54 (1986), 1161–1383.

12. D. Duffie and C. F. Huang, Implementing Arrow-Debreu equilibria by continuous trading of a few long-lived securities, *Econometrica* 53 (1985), 1337–1356.

13. D. Duffie and W. Zame, The consumption-based capital asset pricing model, *Econometrica* 57 (1989), 1279–1297.

14. B. Hajek, Mean comparison of diffusions, *Z. Wahrscheinlichkeits-theorie verw. Gebiete* 68 (1985), 315–329.

15. J. M. Harrison and D. M. Kreps, Martingales and multiperiod securities markets, *J. Economic Theory* 20 (1979), 381–408.

16. R. Holthausen, R. Leftwich, and D. Mayers, Large block transactions, the speed of response, and temporary and permanent stock-price effects, *J. Financial Economics* 26 (1987), 71–95.

17. R. Holthausen, R. Leftwich, and D. Mayers, The effect of large block transactions on security prices, *J. Financial Economics* 19 (1990), 237–267.

18. C. F. Huang, An intertemporal general equilibrium asset pricing model: the case of diffusion information, *Econometrica* 55 (1987), 117–142.

19. N. Ikeda and S. Watanabe, "Stochastic Differential Equations and Diffusion Processes," North Holland, New York, 1989.

20. R. A. Jarrow, Market manipulation, bubbles, corners, and short squeezes, *J. Financial and Quantitative Analysis* 27 (1992), 311–336.

21. I. Karatzas, J. P. Lehoczky and S. E. Shreve, Optimal portfolio and consumption decisions for a 'small investor' on a finite horizon, *SIAM J. Control and Optimization* 25 (1987), 1557–1586.

22. I. Karatzas, J. P. Lehoczky and S. E. Shreve, Existence and uniqueness of multi-agent equilibrium in a stochastic, dynamic consumption/investment model, *Mathematics of Operations Research* 15 (1990), 80–128.

23. I. Karatzas, P. Lakner, J. P. Lehoczky and S. E. Shreve, Equilibrium in a simplified dynamic, stochastic economy with heterogeneous agents, in "Stochastic Analysis," Academic Press, New York, 1991.

24. A. Kraus and H. Stoll, Price impacts of block trading on the New York Stock Exchange, *J. Finance* 27 (1972), 569–588.

25. A. S. Kyle, Continuous auctions and insider trading, *Econometrica* 53 (1985), 1315–1335.

26. E. B. Lindenberg, Capital market equilibrium with price affecting institutional investors", in Elton and Gruber, "Portfolio Theory 25 Years Later," Amsterdam, North Holland, 1979, 109–124.

27. R. Lucas, Asset prices in an exchange economy, *Econometrica* 46 (1978), 1429–1445.

28. R. C. Merton, "Continuous-Time Finance," Blackwell, Oxford, 1990.

29. D. Ocone, A guide to the stochastic calculus of variations, *Lecture Notes in Mathematics* 1316 (1988), 1–79.

30. D. Ocone and I. Karatzas, A generalized Clark representation formula, with application to optimal portfolios, *Stochastics and Stochastics Reports* 34 (1991), 187–220.

31. S. R. Pliska, A stochastic calculus model of continuous trading: optimal portfolios, *Mathematics of Operations Research* 11 (1986), 371–382.

32. T. Sargent, "Dynamic Macroeconomic Theory," Harvard University Press, Cambridge, Massachusetts, 1987.

33. D. J. Seppi, Block trading and information revelation around quarterly earnings announcements, *R. Financial Studies* 5 (1992), 281–305.

34. J. Tirole, "The Theory of Industrial Organization," MIT Press, Cambridge, Massachusetts, 1988.

35. H. R. Varian, "Microeconomic Analysis," Norton, New York, 1992.