OPTIMAL CONSUMPTION AND EQUILIBRIUM PRICES WITH PORTFOLIO CONE CONSTRAINTS AND STOCHASTIC LABOR INCOME

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Abstract

This paper examines the individual's consumption and investment problem when labor income follows a general bounded process and the dollar amounts invested in the risky assets are constrained to take values in a given nonempty, closed, convex cone. Short sale constraints, as well as incomplete markets, can be modeled as special cases of this setting. Existence of optimal policies is established using martingale and duality techniques under fairly general assumptions on the security price coefficients and the individual's utility function. This result is obtained by reformulating the individual's dynamic optimization problem as a dual static problem over a space of martingales. An explicit characterization of equilibrium risk premia in the presence of portfolio constraints is also provided. In the unconstrained case, this characterization reduces to Consumption-based Capital Asset Pricing Model.

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1. Introduction

This paper examines the individual's consumption and investment problem when labor income follows a general bounded process and the dollar amounts invested in the risky assets are constrained to take values in a given closed, convex cone. Short sale constraints, as well as incomplete markets, can be modeled as special cases of this setting.

Available results on the existence of optimal consumption policies in incomplete markets with stochastic income are surprisingly fragmentary. Merton (1971) used stochastic dynamic programming to solve the special case in which security prices follow a geometric Brownian motion, the investor has an infinite horizon and negative exponential utility, and the income follows a Poisson (jump) process. Svensson and Werner (1993) obtained solutions for the same setting, but under the assumption that that the income process is either locally riskless or an arithmetic Brownian motion. These explicit solutions exploit the fact that with negative exponential utility the agent's portfolio choices are independent of wealth. Moreover, they ignore the non-negativity constraint on consumption. Duffie, Fleming and Zariphopoulou (1991) and Koo (1991), also used stochastic dynamic programming to show existence of optimal policies in an incomplete market model in which security prices and labor income follow a geometric Brownian motion and the investor has an infinite horizon with preferences displaying constant relative risk aversion. For this case, the indirect utility function possesses a homogeneity property that makes it possible to reduce the Hamilton-Jacobi-Bellman (HJB) equation to an ordinary differential equation which can be solved explicitly. Duffie and Zariphopoulou (1993) recently employed the theory of viscosity solutions to the HJB equation to show existence of optimal policies in the infinite-horizon model with incomplete markets and constant price coefficients when labor income follows a more general Markovian Itô process.

Results on the characterization and existence of optimal consumption and investment policies with a finite horizon and general security price processes were obtained using martingale and duality techniques by He and Pearson (1991), Karatzas, Lehoczky, Shreve and Xu (1991) and Xu and Shreve (1992a)—who examine the optimal consumption/investment problem with incomplete markets and/or short-sale constraints—and by Cvitanić and Karatzas (1992)—who consider the more general case in which the *portfolio weights* are constrained to take values in a closed convex subset. However, all of these papers assume that individuals are only endowed with some nonnegative amount of wealth at the initial date and there is no labor income, so that they do not address the problem of a nontraded endowment. Their approach consists in transforming the primal constrained maximization problem into a dual unconstrained minimization problem that solves for the individual shadow state prices (intertemporal marginal rates of substitution). Given the general characterization of state prices as martingales due to Harrison and Kreps (1979) and Kreps (1981), it seems natural to conjecture that these shadow state prices should be martingales. On the other hand, the mentioned papers established existence of optimal policies by allowing the shadow state prices to be local martingales, rather than true martingales. As explicitly observed by Karatzas, Lehoczky, Shreve and Xu (1991, p. 705), while such generalization is sufficient for the treatment of models with no income streams, it also precludes the applicability of their results to models with stochastic income.

The only successful application of martingale techniques to establish existence of optimal policies in the presence of stochastic income and constraints on the investment policies is in a recent paper by He and Pagés (1993), who consider the case of limited borrowing (modeled as a non-negativity constraint on wealth). For this special case, the dual problem amounts to a minimization over a set of *absolutely continuous* processes, and thus the issue of the martingale property of shadow prices is sidestepped.

This paper generalizes the existing literature by allowing for the presence of nontraded stochastic income in finite-horizon models with general price coefficients and cone constraints. Moreover, we accommodate more general utility functions than those covered by existing work: in particular, our approach does not require the somewhat artificial (although customary) introduction of a bequest function for final wealth with infinite marginal utility at zero. Finally, we show that the individual shadow state prices should indeed be martingales (rather than just local martingales), and in fact we establish the existence of optimal constrained consumption/investment policies by formulating the dual problem directly over a space of (uniformly integrable) martingales. Since a uniformly integrable martingale can be uniquely identified with its terminal random variable, this allows us to reduce the dynamic optimization problem to a static problem formulated over a standard L^1 space. This reformulation constitutes the methodological innovation of this paper and should be usefully applicable to other optimal consumption problems. We emphasize that differently from Cvitanić and Karatzas (1992) we focus on the case of constraints on dollar amounts invested in risky assets, rather than on portfolio weights: this different formulation is required by the fact that with nontrivial income streams, and hence possibly negative wealth, portfolio weights are not defined.

Our main result not only provides conditions for the existence of an optimal consumption policy, but also shows that under the stated conditions the stochastic Euler equations will hold, even in the presence of the class of constraints we study: in other words, the marginal utility process of an optimizing agent is proportional to a state-price density. This result is important since the stochastic Euler equations are often assumed in applied and empirical work. In particular, we provide a characterization of the viable state-price densities. This characterization involves some subtleties. For example, in the case of incomplete markets, we find that state price densities correspond (after normalization by the bond price) to the densities of probability measures under which the gain processes for traded assets are local martingales: however, unless the individual preferences incorporate a bequest function for final wealth with infinite marginal utility at zero, these probability measures are not necessarily equivalent to the probability representing the agent's beliefs.

Finally, an explicit characterization of the equilibrium risk premia in the presence of portfolio cone constraints is also provided, leading to a constrained version of the consumption-based Capital Asset Pricing Model of Breeden (1979).

2. The Economic Setting

We consider a continuous-time economy on the finite time span $[0, T)$, in which an individual endowed with some initial wealth and a stochastic income stream chooses an optimal consumption and investment policy.

Information structure. The uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$, on which is defined a n-dimensional Brownian motion

$$w = \{(w_1(t), \ldots, w_n(t))^\top : t \in [0, T)\}.$$

The filtration $\mathbf{F} = \{\mathcal{F}_t\}$ is the augmentation under P of the filtration generated by w.[1] We assume that $\mathcal{F} = \sigma(\cup_{0 \le t < T} \mathcal{F}_t)$, or that the true state of nature is completely determined by the sample paths of w on $[0, T)$. We interpret the sigma-field $\mathcal{F}_t$ as representing the information of the individual at time t and the probability measure P as representing his beliefs. All the stochastic processes to appear in the sequel are assumed to be adapted to $\mathbf{F}$ and all the equalities involving random variables are understood to hold P-a.s..[2]

Consumption space. There is a single perishable good (the numeraire). The consumption space $\mathcal{C}$ is given by the set of adapted consumption rate processes c with $\int_0^t |c(\tau)|\, d\tau < \infty$ for all $t \in [0, T)$. The individual consumption set will be shortly specified as a subset of the non-negative orthant $\mathcal{C}_+$.

Securities market. The investment opportunities are represented by $n + 1$ long-lived securities. The first security is a locally riskless bond paying no dividends. Its price process, denoted by B, is given by

$$B(t, \omega) = \exp\left(\int_0^t r(\tau, \omega)\, d\tau\right). \tag{1}$$

for some interest rate process r.

Assumption 1. *The interest rate process r satisfies*

$$\int_0^T r(t)^-\, dt < K_r \tag{2}$$

for some $K_r > 0$, where $x^- = \max(0, -x)$ denotes the negative part of the real number x.

Clearly, the above assumption is in particular satisfied if the interest rate is nonnegative or bounded below.

The remaining n assets are risky. Letting $S = (S_1, \ldots, S_n)$ denote their price process and $D = (D_1, \ldots, D_n)$ their cumulative dividend process, we assume that $S + D$ is an Itô process:

$$S(t, \omega) + D(t, \omega) = S(0) + \int_0^t I_S(\tau, \omega)\mu(\tau, \omega)\, d\tau + \int_0^t I_S(\tau, \omega)\sigma(\tau, \omega)\, dw(\tau, \omega), \tag{3}$$

where $I_S(t)$ denotes the $n \times n$ diagonal matrix with elements $S(t)$.

[1] The augmented Brownian filtration $\mathbf{F} = \{\mathcal{F}_t\}$ is defined by $\mathcal{F}_t = \sigma(\mathcal{F}_t^w \cup \mathcal{N})$, where $\mathcal{F}_t^w = \sigma(w(\tau) : \tau \in [0, t])$ is the smallest sigma-field with respect to which $w(\tau)$ is measurable for every $\tau \in [0, t]$ and $\mathcal{N} = \{E \subset \Omega : \exists G \in \mathcal{F} \text{ with } E \subseteq G,\ P(G) = 0\}$ denotes the set of P-null events. It is well known that the augmented filtration is continuous and that w is still a Brownian motion with respect to it (Karatzas and Shreve (1988), Corollary 2.7.8 and Proposition 2.7.9).

[2] A process $X = \{X(t) : t \in [0, T)\}$ is *adapted* if $X(t)$ is measurable with respect to $\mathcal{F}_t$ for all t.

Assumption 2. *The diffusion matrix $\sigma(t)$ satisfies the nondegeneracy condition*

$$x^\top \sigma(t)\sigma(t)^\top x \geq \varepsilon |x|^2 \tag{4}$$

almost surely for all $(x,t) \in \mathbb{R}^n \times [0,T)$ and some $\varepsilon > 0$. Moreover, letting

$$\kappa_0 = -\sigma^{-1}(\mu - r\bar{1}) \tag{5}$$

where $\bar{1} = (1,\ldots,1)^\top \in \mathbb{R}^n$, we have

$$\mathrm{E}\left[\exp\left(\frac{1}{2}\int_0^T |\kappa_0(t)|^2\,dt\right)\right] < \infty. \tag{6}$$

Condition (4) implies in particular that $\sigma(t)$ has full rank a.s. for all $t \in [0,T)$, so that in the absence of portfolio constraints markets are dynamically complete, and that $\sigma(t,\omega)^{-1}$ has essentially bounded matrix norm, uniformly in $(t,\omega) \in [0,T) \times \Omega$ (Karatzas and Shreve (1988), Problem 5.8.1). Condition (6) could be relaxed and is used to guarantee the existence of an equivalent martingale measure.

Trading strategies. Trading takes place continuously and there are no market frictions. An admissible trading strategy is a n-dimensional vector process $\theta = (\theta_1,\ldots,\theta_n)$—where $\theta_k(t)$ denotes the dollar amount invested at time t in the k-th risky asset—satisfying

$$\int_0^t |\theta(\tau)^\top(\mu(\tau) - r(\tau)\bar{1})|\,d\tau + \int_0^t |\theta(\tau)^\top \sigma(\tau)|^2\,d\tau < \infty \tag{7}$$

for all $t \in [0,T)$. The set of admissible trading strategies is denoted by Θ.

Preferences and endowments. Preferences for the individual are represented by a time-additive utility function[3]

$$U(c) = \mathrm{E}\left[\int_0^T u(c(t),t)\,dt\right], \tag{8}$$

which is well defined for all consumption processes $c \in \mathcal{C}_+$ satisfying

$$\mathrm{E}\left[\int_0^T u(c(t),t)^-\,dt\right] < \infty. \tag{9}$$

The individual consumption set, denoted by $\mathcal{C}_+^*$, will be accordingly given by the consumption processes $c \in \mathcal{C}_+$ satisfying (9).

Assumption 3. *The function $u(\cdot,t)$ is increasing, strictly concave and continuously differentiable on $(0,\infty)$ for all $t \in [0,T)$. Moreover, it satisfies the Inada conditions*

$$\lim_{c\downarrow 0} u_c(c,t) = \infty \quad \text{and} \quad \lim_{c\uparrow\infty} u_c(c,t) = 0, \tag{10}$$

and there exist constants $\delta \in (0,1)$ and $\gamma \in (0,\infty)$ such that

$$\delta u_c(c,t) \geq u_c(\gamma c,t) \qquad \forall (c,t) \in (0,\infty) \times [0,T). \tag{11}$$

Finally, $u(c,\cdot)$ is continuous on $[0,T)$ and integrable for all $c > 0$.

[3]The case in which the economy is extended to $[0,T]$ and preferences include a bequest function for final wealth is not substantially different and will be briefly discussed after the statement of the main theorem.

Remark. Condition (10) is well understood and it implies in particular that the derivative function $u_c(\cdot,t)$ has a continuous and strictly decreasing inverse $f(\cdot,t)$ mapping $(0,\infty)$ onto itself. Condition (11) has the purpose of guaranteeing that certain functionals to be introduced in the sequel can be differentiated under the integral sign. It is easily verified that this condition holds for the utility functions $u(c,t) = \rho(t)\log c$ or $u(c,t) = \rho(t)\frac{c^{1-b}}{1-b}$, $b > 0$, $b \neq 1$. Also, taking $c = f(y,t)$ in (11), applying $f(\cdot,t)$ to both sides and iterating shows that the following property holds

$$\forall\delta \in (0,\infty),\ \exists\gamma \in (0,\infty) \ \text{ such that } \ f(\delta y,t) \leq \gamma f(y,t),\ \forall(y,t) \in (0,\infty)\times[0,T). \tag{12}$$

The individual is endowed with some initial wealth $W_0 \geq 0$ and a *bounded* stochastic income stream $y \in C_+^*$.

3. Portfolio Constraint Sets

We fix from now on a nonempty, closed, convex cone $A \subseteq \mathbb{R}^n$ and denote by

$$\tilde{A} = \{y \in \mathbb{R}^n : x\cdot y \leq 0,\ \forall x \in A\}$$

the *polar* of A, which is also a nonempty, closed, convex cone (Rockafellar (1970), Theorem 14.1). We assume that the portfolio of the individual is constrained to lie in A at all times t. The following are examples of interesting constraints that can be modeled with the above setup.

(a) *No constraints:* $A = \mathbb{R}^n$, $\tilde{A} = \{0\}$.

(b) *Nontradeable assets* (incomplete markets): $A = \{x \in \mathbb{R}^n : x_k = 0,\ k = m+1,\ldots,n\}$, $\tilde{A} = \{y \in \mathbb{R}^n : y_k = 0,\ k = 0,\ldots,m\}$.

(c) *Short-sale constraints:* $A = \{x \in \mathbb{R}^n : x_k \geq 0,\ k = m+1,\ldots,n\}$, $\tilde{A} = \{y \in \mathbb{R}^n : y_k = 0,\ k = 1,\ldots,m;\ y_k \leq 0,\ k = m+1,\ldots,n\}$.

(d) *Buying constraints:* $A = \{x \in \mathbb{R}^n : x_k \leq 0,\ k = m+1,\ldots,n\}$, $\tilde{A} = \{y \in \mathbb{R}^n : y_k = 0,\ k = 1,\ldots,m;\ y_k \geq 0,\ k = m+1,\ldots,n\}$.

(e) *Portfolio mix constraints:* $A = \{x \in \mathbb{R}^n : \sum_{k=1}^n x_k \geq 0,\ x \in \hat{A}(\sum_{k=1}^n x_k)\}$, where $\hat{A}$ is any closed, convex subset of $\mathbb{R}^n$, $\tilde{A} = \{y \in \mathbb{R}^n : x\cdot y \leq 0,\ \forall x \in A\}$.

(f) *Any combination of the above.*

4. Feasible consumption processes

Given the price coefficients $\mathcal{P} = (r,\mu,\sigma)$, a consumption process $c \in C_+^*$ is said to be *feasible* if there exists an admissible trading strategy $\theta \in \Theta$ and a wealth process W such that

$$\begin{aligned} W(t) = W_0 + \int_0^t r(\tau)W(\tau)\,d\tau + \int_0^t \theta(\tau)^\top(\mu(\tau) - r(\tau)\bar{1})\,d\tau \\ + \int_0^t \theta(\tau)^\top\sigma(\tau)\,dw(\tau) - \int_0^t (c(\tau) - y(\tau))\,d\tau \end{aligned} \tag{13}$$

$$W(t) \geq -KB(t) \tag{14}$$

$$W(T^-) \geq 0 \tag{15}$$

for all $t \in [0,T)$ and some $K \in \mathbb{R}_+$, where $W(T^-) = \lim_{t \uparrow T} W(t)$. The consumption process c is said to be *A-feasible* if the above conditions are satisfied and in addition $\theta(t) \in A$ for all $t \in [0,T)$. In either case, the trading strategy θ is said to *finance c*. We will let $\mathcal{B}(\mathcal{P}, A)$ denote the set of A-feasible consumption processes given the price system $\mathcal{P}$.

Equation (13) is the usual dynamic budget constraint: it states that the wealth at any time $t \in [0,T)$ equals the initial wealth, plus the trading gains, minus the cumulative net consumption. Equations (14) and (15) state that, while the investor is allowed to borrow against future income and thus to have short-term deficits, the final wealth must be non-negative: in other words, final wealth must be sufficient to cover any amount borrowed. Moreover, the discounted wealth process is required to admit a uniform lower bound: this is necessary to rule out arbitrage opportunities, such as the doubling strategies discussed by Harrison and Kreps (1979).[4]

5. Solution of the individual consumption problem

Since security prices and the individual income stream are allowed to be possibly non-Markovian processes, dynamic programming techniques cannot be applied to analyze the individual consumption problem. Therefore we will derive a martingale characterization of the optimal policies: such a characterization has been developed by Karatzas, Lehoczky and Shreve (1987) and Cox and Huang (1989, 1991) for the unconstrained case and by He and Pearson (1991), Karatzas, Lehoczky, Shreve and Xu (1991) and Cvitanič and Karatzas (1992), among others, for the constrained case. In all of these papers it is assumed that $y = 0$, i.e., that agents are only endowed with some amount of wealth at date 0. In this section we therefore briefly review the results of those papers that will be needed in the sequel, with the necessary modifications to account for stochastic income.[5] We start by introducing a set of probabilities measures that will play a key role in characterizing optimal consumption policies.

Definition. A probability measure Q on $(\Omega, \mathcal{F})$ is *locally equivalent* to P if the restriction of Q to $\mathcal{F}_t$ is equivalent to the restriction of P to $\mathcal{F}_t$ for all $t \in [0,T)$.[6]

For $t \in [0,T)$, let Q_t denote restriction of the probability measure Q to $\mathcal{F}_t$. Also, let $\mathcal{Q}$ denote the set of locally equivalent probability measures Q that are absolutely continuous with respect to P on $(\Omega, \mathcal{F})$. We then have the following characterization of the set $\mathcal{Q}$.

[4]The fact that a uniform negative lower bound on discounted wealth is sufficient to rule out free lunches was proved by Dybvig and Huang (1989). This is a natural requirement in our setting, since in the absence of arbitrage opportunities discounted wealth must be greater than minus the shadow value of the endowment, and the latter will be shown to be bounded given our assumptions.

[5]He and Pagès (1991) have examined the individual consumption problem in the presence of complete markets, stochastic income and borrowing constraints. Our analysis of the individual consumption problem differs from theirs, as the only constraints on borrowing are given by (14)–(15), but we impose the cone constraint on portfolio holdings.

[6]A probability measure Q is *absolutely continuous* with respect to a probability measure P if $Q(E) = 0$ whenever $E \in \mathcal{F}$ and $P(E) = 0$. Q is *equivalent* to P if Q is absolutely continuous with respect to P and P is absolutely continuous with respect to Q (i.e., if P and Q have the same null events).

Proposition 1. *For any $Q_\nu \in \mathcal{Q}$, let ξ_ν denote the corresponding density process with respect to P (i.e., $\xi_\nu(t) = dQ_{\nu t}/dP_t$, $t \in [0,T)$). Then*

$$\left\{\xi_\nu : Q_\nu \in \mathcal{Q}\right\} = \left\{\xi_\nu : \xi_\nu \text{ is a uniformly integrable } P\text{-martingale with } \xi_\nu(t) > 0\ \forall t \in [0,T)\right\}.$$

Moreover, the limit $\xi_\nu(T) = \lim_{t\uparrow T} \xi_\nu(t)$ exists, and we have $\xi_\nu(T) = dQ_\nu/dP$. Therefore, Q_ν is equivalent to P on (Ω, F) if and only if $\xi_\nu(T) > 0$, P-a.s..

PROOF. This follows immediately from Propositions 7.2 and 7.11 in Jacod (1979). □

Finally, we recall that a stochastic process $\{\xi(t) : t \in [0,T)\}$ is said to be *of class D* if the family of random variables $\{\xi(\tau) : \tau \in \mathcal{T}\}$ is uniformly integrable, where $\mathcal{T}$ denotes the set of stopping times τ with $\tau < T$ almost surely. The following Proposition complements Proposition 1 by giving some alternative characterizations of the set of uniformly integrable martingales on $[0,T)$.

Proposition 2. *Let $\xi = \{\xi(t) : t \in [0,T)\}$ be a local martingale. Then the following conditions are equivalent:*

(a) ξ is of class D;

(b) ξ is a uniformly integrable martingale;

(c) $\{\xi(t) : t \in [0,T]\}$ is a martingale, where $\xi(T) = \lim_{t\uparrow T} \xi(t)$.

PROOF. The assertion follows from Proposition I.1.47 and Theorem I.1.42 in Jacod and Shiryaev (1987).

□

5.1. The unconstrained case

Consider first the unconstrained consumption problem ($A = \mathbb{R}^n$). Define the exponential local martingale

$$\xi_0(t) = \exp\left(\int_0^t \kappa_0(\tau)^\top dw(\tau) - \frac{1}{2}\int_0^t |\kappa_0(\tau)|^2\, dt\right). \tag{16}$$

By (6), ξ_0 is in fact a uniformly integrable martingale, so that Proposition 1 implies that the probability measure Q_0 defined by $dQ_0/dP = \xi_0(T)$ belongs to the set $\mathcal{Q}$. Also, it is well known (and easily verified by Girsanov's theorem) that Q_0 has the property that the discounted gain process

$$G(t) = B(t)^{-1}S(t) + \int_0^t B(s)^{-1}\, dD(s)$$

becomes a local martingale under it, and Q_0 is in fact the unique probability measure in the set $\mathcal{Q}$ with this property: it is alternatively known in the finance literature as the *risk-neutral probability* or the *equivalent martingale measure*. Moreover, the process $\pi_0 = B^{-1}\xi_0$ can be interpreted as the *state-price density* for the economy, in the sense that the value at time 0 of any consumption process c satisfying an integrability condition is given by $\mathrm{E}[\int_0^T \pi_0(t)c(t)\, dt]$. In particular, we have:

Lemma 1. *If $c \in \mathcal{B}(\mathcal{P}, \mathbb{R}^n)$ is a feasible consumption process, then*

$$\mathrm{E}\Big[\int_0^T \pi_0(t)(c(t) - y(t))\,dt\Big] \le W_0. \tag{17}$$

PROOF. Using Itô's lemma, it is easy to show that (13) holds if and only if

$$\pi_0(t)W(t) + \int_0^t \pi_0(\tau)(c(\tau) - y(\tau))\,d\tau = W_0 + \int_0^t \pi_0(\tau)(\theta(\tau)^\top \sigma(\tau) + W(\tau)\kappa_0(\tau)^\top)\,dw(\tau). \tag{18}$$

For each positive integer n, let

$$\tau_n = T \wedge \inf\Big\{t \in [0, T) : \int_0^t |\pi_0(\tau)(\theta(\tau)^\top \sigma(\tau) + W(\tau)\kappa_0(\tau)^\top)|^2\,d\tau \ge n\Big\},$$

with the usual convention, maintained for the remainder of the paper, that $\inf(\emptyset) = \infty$. Since the stochastic integral on the right-hand side of (18) is a martingale on $[0, \tau_n]$, taking expectations gives

$$\mathrm{E}[\pi_0(\tau_n)W(\tau_n)] + \mathrm{E}\Big[\int_0^{\tau_n} \pi_0(s)(c(s) - y(s))\,ds\Big] = W_0. \tag{19}$$

Letting $n \uparrow \infty$, we have $\tau_n \uparrow T$ (because of (6), (7) and the continuity of π_0 and W). Applying the monotone convergence theorem twice and using the fact that $\mathrm{E}[\int_0^T \pi_0(t)y(t)dt]$ is finite under our assumptions, shows that

$$\mathrm{E}\Big[\int_0^{\tau_n} \pi_0(s)(c(s)\,ds - y(s))\,ds\Big] \to \mathrm{E}\Big[\int_0^T \pi_0(s)(c(s) - y(s))\,ds\Big].$$

As for the first term in (19), we have from (14)

$$(\pi_0(\tau_n)W(\tau_n))^- \le K\xi_0(\tau_n).$$

Since ξ_0 is of class D by Proposition 2, Fatou's lemma for random variables uniformly integrable from below (Chow and Teicher (1988), Theorem 4.2.2) gives

$$\liminf_{n\to\infty} \mathrm{E}[\pi_0(\tau_n)W(\tau_n)] \ge \mathrm{E}[\pi_0(T)W(T^-)] \ge 0,$$

where the last inequality follows from (15). This establishes (17). □

We will refer to (17) as the *static budget constraint.* Karatzas, Lehoczky and Shreve (1987) and Cox and Huang (1991) have shown that, with no income stream and a nonnegative wealth constraint, a consumption process is feasible if *and only if* it satisfies the static budget constraint. Theorem 1 below shows in particular that, with complete markets, the same result holds for our economy.[7]

[7] This is of course hardly surprising, as the presence of an endowment stream is irrelevant with complete markets and unrestricted borrowing against future income, and Dybvig and Huang (1989) have shown that a uniform lower bound on discounted wealth is equivalent to a nonnegative wealth constraint.

Assume that there exists a $\psi^* > 0$ such that

$$\mathrm{E}\left[\int_0^T \pi_0(t)\Big(f(\psi^*\pi_0(t),t) - y(t)\Big)\,dt\right] = W_0, \tag{20}$$

where f denotes the inverse of the marginal utility function u_c. It then immediately follows from the Lagrangian theory of optimization that the consumption policy

$$c_0(t) = f\Big(\psi^*\pi_0(t),t\Big), \tag{21}$$

is optimal for the problem of maximizing expected utility subject to the static budget constraint in (17). Moreover, this policy is also optimal in the original program.

Theorem 1. *Suppose that $A = \mathbb{R}^n$ and that there exists a $\psi^* > 0$ solving (20). Then the optimal consumption policy is given by (21) and the optimal wealth process is given by*

$$W_0(t) = \pi_0(t)^{-1}\,\mathrm{E}\left[\int_t^T \pi_0(\tau)(c_0(\tau) - y(\tau))\,d\tau \;\Big|\; \mathcal{F}_t\right]. \tag{22}$$

PROOF. By the continuity of f and ξ_0, it is clear that $\int_0^t c_0(\tau)\,d\tau < \infty$ holds for all $t \in [0,T)$. Also, from the inequality

$$u(1,t) - y \le \max_{c>0}[u(c,t) - yc] = u(f(y,t),t) - yf(y,t), \tag{23}$$

we have

$$\mathrm{E}\left[\int_0^T u(c_0(t),t)^-\,dt\right] \le \int_0^T u(1,t)^-\,dt + \psi^*\,\mathrm{E}\left[\int_0^T B(t)^{-1}\xi_0(t)\,dt\right] < \infty$$

(where the last inequality follows from Assumption 3, (2) and the martingale property of ξ_0). Hence, $c_0 \in \mathcal{C}_+^*$.

Next, let $c \in \mathcal{B}(\mathcal{P}, \mathbb{R}^n)$ be arbitrary. Since by concavity

$$u(f(y,t),t) - u(c,t) \ge y[f(y,t) - c] \qquad \forall c > 0, y > 0, \tag{24}$$

we have

$$U(c_0) - U(c) = \mathrm{E}\left[\int_0^T (u(c_0(t),t) - u(c(t),t))\,dt\right] \ge \psi^*\,\mathrm{E}\left[\int_0^T \pi_0(t)(c_0(t) - c(t))\,dt\right] \ge 0$$

where the last inequality follows from the fact that

$$\mathrm{E}\left[\int_0^T \pi_0(t)(c_0(t) - y(t))\,dt\right] = W_0$$

by the definition of ψ^*, while (17) holds for all $c \in \mathcal{B}(\mathcal{P}, \mathbb{R}^n)$. Therefore, c_0 is certainly optimal if it is feasible.

Define the process $W_0(t)$ by (22). Then (15) is clearly satisfied, and we have (from (2) and the martingale property of ξ_0)

$$W_0(t) \ge -\pi_0(t)^{-1}\,\mathrm{E}\left[\int_t^T \pi_0(\tau)y(\tau)\,d\tau \;\Big|\; \mathcal{F}_t\right] \ge -Te^{K_r}\bar{y}B(t),$$

where $\bar{y} < \infty$ is an upper bound on y, so that (14) is also satisfied, and we are only left to show that there exists a process θ such that (13), or equivalently (18), holds. Since

$$\pi_0(t)W_0(t) + \int_0^t \pi_0(\tau)(c_0(\tau) - y(\tau))\, d\tau = \mathrm{E}\Big[\int_0^T \pi_0(t)(c_0(t) - y(t))\, dt \,\Big|\, \mathcal{F}_t\Big]$$

it follows from the martingale representation theorem (Jacod and Shiryaev (1987), Theorem III.4.33) that there exists an adapted process φ with $\int_0^T |\varphi(t)|^2 dt < \infty$ such that

$$\pi_0(t)W_0(t) + \int_0^t \pi_0(\tau)(c_0(\tau) - y(\tau))\, d\tau = W_0 + \int_0^t \varphi(\tau)^\top dw(\tau).$$

Comparing the above equality with (18) then shows that

$$\theta(t) = \left(\sigma(t)^\top\right)^{-1}\left(\pi_0(t)^{-1}\varphi(t) - W_0(t)\kappa_0(t)\right)$$

finances c_0, so that c_0 is marketed and the wealth process is given by (22). □

5.2. The constrained case

The consumption policy c_0 of the previous subsection satisfies the first-order condition

$$u_c(c_0(t), t) = u_c(c_0(0), 0)\pi_0(t),$$

which states that the individual's intertemporal marginal rates of substitution equal the state-price density π_0. In fact, it follows from the work of Harrison and Kreps (1979) and Huang (1985) that, when the portfolio policies are unconstrained ($A = \mathbb{R}^n$), π_0 is the unique state-price density consistent with the absence of arbitrage opportunities. On the contrary, with constrained portfolios ($A \subset \mathbb{R}^n$), there exist infinitely many state-price densities that are consistent with no arbitrage, and we have the following characterization.

Let $\mathcal{N}$ denote the set of n-dimensional adapted processes ν with values in $\tilde{A}$ satisfying

$$\int_0^t |\sigma(\tau)^{-1}\nu(\tau)|^2\, d\tau < \infty \tag{25}$$

for all $t \in [0, T)$. For each $\nu \in \mathcal{N}$ the processes

$$\kappa_\nu(t) = -\sigma(t)^{-1}(\mu(t) - \nu(t) - r(t)\bar{1}) = \kappa_0 + \sigma(t)^{-1}\nu(t),$$
$$\xi_\nu(t) = \exp\left(\int_0^t \kappa_\nu(\tau)^\top dw(\tau) - \frac{1}{2}\int_0^t |\kappa_\nu(\tau)|^2\, d\tau\right),$$
$$\pi_\nu(t) = B(t)^{-1}\xi_\nu(t)$$

are well defined on $[0, T)$ and ξ_ν is a local martingale. Let $\mathcal{N}^*$ denote the subset of elements $\nu \in \mathcal{N}$ for which ξ_ν is of class D. We remark that $\mathcal{N}^*$ is non-empty, since (6) ensures that we always have $0 \in \mathcal{N}^*$.

It follows from Propositions 1 and 2 that each ξ_ν with $\nu \in \mathcal{N}^*$ can be interpreted as the density process corresponding to some probability measure $Q_\nu \in \mathcal{Q}$. It is easily verified by Itô's lemma that in the case of incomplete markets (example (b) of section 3) the set

$\{Q_\nu : \nu \in \mathcal{N}^*\}$ corresponds to the set of probability measures $Q_\nu \in \mathcal{Q}$ under which the discounted gain process

$$G_k(t) = B(t)^{-1}S_k(t) + \int_0^t B(s)^{-1}\, dD_k(s)$$

for the tradeable assets is a local martingale. Similarly, in the case of short-sale (buying) constraints (examples (c) and (d) of section 3) the set $\{Q_\nu : \nu \in \mathcal{N}^*\}$ corresponds to set of probability measures $Q_\nu \in \mathcal{Q}$ under which the discounted gain process for the unconstrained assets is a local martingale, while the discounted gain process for the constrained assets is a local supermartingale (submartingale). Of course, in the unconstrained case $\mathcal{N}^* = \{0\}$, and we recover the unique state-price density π_0 of the previous subsection.

Also, it is clear that each π_ν with $\nu \in \mathcal{N}^*$ can be interpreted as the unique state-price density in a fictious unconstrained economy with price coefficients $\mathcal{P} = (r, \mu - \nu, \sigma)$. More generally, the following analogue of Lemma 1 shows that each π_ν with $\nu \in \mathcal{N}^*$ constitutes an arbitrage-free state-price density in the original economy when the portfolio policies are constrained to lie in A.[8]

Lemma 2. *If a consumption policy $c \in C_+^*$ is A-feasible, then*

$$\mathrm{E}\Big[\int_0^T \pi_\nu(t)(c(t) - y(t))\, dt\Big] \le W_0 \tag{26}$$

holds for all $\nu \in \mathcal{N}^*$.

Proof. Using (13) and Itô's lemma shows that the equivalent

$$\begin{aligned} \pi_\nu(t)W(t) + \int_0^t \pi_\nu(\tau)(c(\tau) - y(\tau))\, d\tau - \int_0^t \pi_\nu(\tau)\theta(\tau)^\top \nu(\tau)\, d\tau \\ = W_0 + \int_0^t \pi_\nu(\tau)\Big(\theta(\tau)^\top\sigma(\tau) + W(\tau)\kappa_\nu(\tau)^\top\Big)\, dw(\tau). \end{aligned} \tag{27}$$

of (18) holds for all $\nu \in \mathcal{N}^*$. The claim now follows from a localization argument similar to that in the proof of Lemma 1, using the fact that $\theta(t)^\top\nu(t) \le 0$ for $\theta(t) \in A$, $\nu(t) \in \tilde{A}$. □

The previous lemma shows that an A-feasible consumption process has to satisfy an infinite number of static budget constraints of the form (26), one for each $\nu \in \mathcal{N}^*$. The following theorem gives a converse to this result by showing that the satisfaction of these budget constraints is also sufficient for A-feasibility.

Theorem 2. *Let $c \in C_+^*$ be a consumption process and suppose that there exists a process $\nu^* \in \mathcal{N}^*$ such that for all $\nu \in \mathcal{N}^*$:*

$$\mathrm{E}\Big[\int_0^T \pi_\nu(t)(c(t) - y(t))\, dt\Big] \;\le\; \mathrm{E}\Big[\int_0^T \pi_{\nu^*}(t)(c(t) - y(t))\, dt\Big] \;=\; W_0. \tag{28}$$

Then $c \in \mathcal{B}(\mathcal{P}, A)$.

Proof. See Appendix A.

[8] In our setting, an *arbitrage opportunity* is a nonzero consumption process $c \in C_+^*$ that is A-feasible for zero initial wealth and zero income. Equation (26) shows that there are no arbitrage opportunities if the shadow state-price-density in the economy is given by π_ν for some $\nu \in \mathcal{N}^*$.

Now suppose that the consumption plan c in the above Theorem is the optimal policy c^*. If the associated state-price density π_{ν^*} satisfying the condition of Theorem 2 existed and were known, then c^* would be the solution to the problem of maximizing utility subject to a single budget constraint of the form (26). This suggests, heuristically, that there should exist a Lagrangian multiplier $\psi^* > 0$ such that (c^*, ψ^*, ν^*) is a saddle point of the map

$$\mathcal{L}(c, \psi, \nu) = U(c) - \psi\Big(\mathrm{E}\int_0^T \pi_\nu(t)(c(t) - y(t))\,dt - W_0\Big), \tag{29}$$

where we maximize with respect to c and minimize with respect to (ψ, ν).

Let

$$\tilde{u}(y, t) \equiv \max_{c \geq 0}[u(c, t) - yc] = u(f(y, t), t) - yf(y, t) \tag{30}$$

denote the convex conjugate of $-u(-c, t)$. The following lemma collects some properties of the function $\tilde{u}$ that will be used repeatedly in the sequel.

Lemma 3. *The function $\tilde{u}(\cdot, t) : (0, \infty) \to \mathbb{R}$ is strictly decreasing and strictly convex for all $t \in [0, T)$, with $\frac{\partial}{\partial y}\tilde{u}(y, t) = -f(y, t)$. Moreover*

$$\tilde{u}(0+, t) = u(\infty, t), \qquad \tilde{u}(\infty, t) = u(0+, t).$$

PROOF. See, e.g., Karatzas, Lehoczky, Shreve and Xu (1991), p. 707. □

Maximization of (29) with respect to c gives

$$J(\psi, \nu) = \mathrm{E}\Big[\int_0^T \tilde{u}(\psi\pi_\nu(t), t)\,dt + \psi\int_0^T \pi_\nu(t)y(t)\,dt\Big] + \psi W_0, \tag{31}$$

where we remark that the above expectation is well defined for all $(\psi, \nu) \in (0, \infty) \times \mathcal{N}^*$, since we have from (23)

$$\mathrm{E}\Big[\int_0^T \tilde{u}(\psi\pi_\nu(t), t)^-\,dt\Big] \leq \int_0^T u(1, t)\,dt + \psi\,\mathrm{E}\Big[\int_0^T \pi_\nu(t)\,dt\Big] < \infty.$$

Therefore $J : (0, \infty) \times \mathcal{N}^* \mapsto \mathbb{R} \cup \{\infty\}$ and we are left with the shadow state-price problem

$$\min_{\psi > 0}\min_{\nu \in \mathcal{N}^*} J(\psi, \nu). \tag{P^*}$$

The following theorem establishes the duality between the individual's constrained optimization problem and (P^*).

Theorem 3. *Assume that $(\psi^*, \nu^*) \in (0, \infty) \times \mathcal{N}^*$ solves the shadow state-price problem (P^*) and*

$$\mathrm{E}\Big[\int_0^T \pi_{\nu^*}(t)\Big(f(\psi^*\pi_{\nu^*}(t), t) - y(t)\Big)\,dt\Big] < \infty. \tag{32}$$

Then the policy

$$c_{\nu^*}(t) = f(\psi^*\pi_{\nu^*}(t), t) \tag{33}$$

is optimal in the constrained problem and the optimal wealth process is given by

$$W_{\nu^*}(t) = \pi_{\nu^*}(t)^{-1}\,\mathrm{E}\Big[\int_t^T \pi_{\nu^*}(\tau)(c_{\nu^*}(\tau) - y(\tau))\,dt \;\Big|\; \mathcal{F}_t\Big]. \tag{34}$$

PROOF. See Appendix A. □

We state as corollaries two special cases of Theorem 4.

Corollary 1. *Assume that* $u(\infty, t) = \infty$ *for all* $t \in [0, T)$ *and* $u(c, t)$ *is bounded below on* $(0, \infty) \times [0, T)$. *If conditions (b) and (c) of Theorem 4 hold, then a minimax state-price density and a constrained optimal consumption/investment policy exist.*

PROOF. We only need to verify that conditions (a) and (d) of Theorem 4 are satisfied. Condition (a) is obvious, as u is bounded below. For $\delta \in (1, \infty)$ arbitrary and some $\gamma \in (0, \infty)$, (12) and the properties of $\tilde{u}$ in Lemma 3 imply

$$\tilde{u}(y, t) - \tilde{u}(\infty, t) \geq \tilde{u}(y, t) - \tilde{u}(\delta y, t) \geq (\delta - 1) y f(\delta y, t) \geq \frac{\delta - 1}{\gamma} y f(y, t).$$

Taking $y = u_c(c, t)$ in the above inequality and recalling (30) gives

$$u(c, t) - c u_c(c, t) - u(0, t) \geq \frac{\delta - 1}{\gamma} c u_c(c, t).$$

Therefore

$$\left(1 + \frac{\delta - 1}{\gamma}\right) c u_c(c, t) \leq \sup_{t \in [0, T)} u(0+, t)^- + u(c, t),$$

and hence condition (d) also holds.

□

Among the isoelastic utility functions with $u(c, t) = \rho(t) \frac{c^{1-b}}{1-b}$ for some $b > 0$, $b \neq 1$ and some bounded measurable function $\rho : [0, T) \mapsto (0, \bar{\rho}]$, the above corollary covers the cases in which $b \in (0, 1)$. The next corollary treats the important special case of logarithmic utilities.

Corollary 2. *Assume that* $u(c, t) = \rho(t) \log c$ *for some bounded measurable function* $\rho : [0, T) \mapsto (0, \bar{\rho}]$. *If conditions (b) and (c) of Theorem 4 hold, then a minimax state-price density and a constrained optimal consumption/investment policy exist.*

PROOF. By concavity, we have $\log(1) - \log(c) \leq c^{-1}(1 - c)$, so that $u(c, t) \geq -\bar{\rho} c^{-1}$ and condition (a) of Theorem 4 holds. Also, in this case we have $c u_c(c, t) = \rho(t) \leq \bar{\rho}$, so that condition (d) is satisfied as well.

□

Remark. Since $0 \in \mathcal{N}^*$, a sufficient condition for assumption (c) of Theorem 4 is that $J(\psi, 0) < \infty$ for all $\psi \in (0, \infty)$. In particular, if $u(c, t)$ is nonnegative and satisfies the growth condition $u(c, t) \leq k(1 + c^{1-b})$ for some $k > 0$, $b \in (0, 1)$, then (c) will also hold provided that (6) is strengthened to

$$\int_0^T |\kappa_0(t)|^2 \, dt \leq K$$

for some $K > 0$ (cf. Karatzas, Lehoczky, Shreve and Xu (1991), Remark 11.9).

We can interpret the constrained-optimal policy c_{ν^*} as the unconstrained-optimal policy in an economy with price coefficients $\mathcal{P}_{\nu^*} = (r, \mu - \nu^*, \sigma)$, as c_{ν^*} also solves the problem

$$\max_{c \in \mathcal{C}^*} U(c)$$
$$\text{s.t. } \mathrm{E}\left[\int_0^T \pi_{\nu^*}(t)(c(t) - y(t))\, dt\right] \leq W_0.$$

In other words, ν^* represents the change in the assets' expected instantaneous returns such that the individual would optimally choose an investment policy with $\theta(t) \in A$ for all t. Following the terminology introduced by He and Pearson, we will refer to ξ_{ν^*} as the *minimax martingale measure* and to π_{ν^*} as the *minimax state-price density* for the individual. They must clearly be unique when they exists.

The next theorem, which constitutes the main result of the paper, provides sufficient conditions for the existence of a minimax state-price density and of a constrained optimal consumption/investment policy.

Theorem 4. *Assume that*

(a) $u(\infty, t) = \infty$ *for all* $t \in [0, T)$ *and* $u(c,t)^- \leq k(1 + c^{1-b})$ *on* $(0, \infty) \times [0, T)$ *for some* $k \geq 0$, $b \geq 1$*;*

(b) *either* $W_0 > 0$ *or* $y/B > \varepsilon$ $(\lambda \times P) - a.e.$ *for some* $\varepsilon > 0$*;*

(c) $\forall \psi \in (0, \infty)$, $\exists \nu \in \mathcal{N}^*$ *such that* $J(\psi, \nu) < \infty$.

Then the minimum in (P^*) *is attained and hence a minimax state-price density exists. If in addition*

(d) $cu_c(c,t) \leq a + (1-b)u(c,t)$ *on* $(0, \infty) \times [0, T)$ *for some* $a \geq 0$, $b > 0$,

then condition (32) of Theorem 3 is also satisfied, and hence there exists a constrained optimal consumption/investment policy.

PROOF. See Appendix B. □

Remark. Since the space $\mathcal{N}^*$ does not have any nice topological structure, existence of a solution to the dual problem is proved in the Appendix by reformulating (P^*) as a minimization problem directly over a space of uniformly integrable martingales. In fact, since a uniformly integrable martingale can be uniquely identified with its terminal random variable, we reduce the dual minimization problem to a standard problem over a closed convex subset of a L^1 space, and then use the conjugate duality theory of Rockafellar (1974, 1975) to establish existence. This reformulation of the problem constitutes a methodological innovation of this paper and should be usefully applicable to other optimal consumption problems. By contrast, the papers by He and Pearson (1991), Karatzas, Lehoczky, Shreve and Xu (1991), Xu and Shreve (1992a) and Cvitanić and Karatzas (1992) attacked the corresponding dual problems directly, and this allowed them to only prove existence by allowing the shadow prices to be local martingales rather than true martingales.

For the important special case of incomplete markets and/or short-sale constraints, the above theorem generalizes the similar existence results in He and Pearson (1991), Karatzas, Lehoczky, Shreve and Xu (1991) and Xu and Shreve (1992a) by allowing for a stochastic income stream. Moreover, we verify a conjecture by Karatzas, Lehoczky, Shreve and Xu (1991, Remark 10.3) by showing that the minimax martingale measures are indeed (uniformly integrable) martingales, rather than just local martingales, a result that is critical in establishing the supermartingale property of wealth without a uniform nonnegativity constraint. We point out that our assumptions on the utility functions are more general than those in the cited papers, since we require neither that the Arrow-Pratt coefficient of relative risk aversion to be everywhere strictly less than one, nor that the utility function be bounded below. Also, we do not need to introduce a bequest function with infinite marginal utility at zero. However, it should be easy to see that exactly the same argument used in the proof of Theorem 4 would apply to the case in which the economy is extended to $[0,T]$ and a bequest function $V(W)$ is added to (8), provided that V satisfies the same conditions placed on $u(\cdot,t)$. Moreover, in this case we necessarily have $\xi_{\nu^*}(T) > 0$, so that the minimax martingale measure is equivalent to P on $(\Omega,\mathcal{F})$.

6. The Markov Case

We now specialize our model by assuming that the price coefficients (r,μ,σ), as well as the income stream y, are deterministic functions of time and some s-dimensional vector of state variables Y, whose stochastic process is Markovian and satisfies

$$Y(t,\omega) = Y(0) + \int_0^t \alpha(Y(\tau,\omega),\tau)\,d\tau + \int_0^t \beta(Y(\tau,\omega),\tau)\,dw(\tau,\omega).$$

Introducing the additional state variable $Z_\nu = \psi\pi_\nu$, it follows from Itô's lemma that Z satisfies the stochastic integral equation

$$Z(t) = \psi - \int_0^t r(\tau)Z(\tau)\,d\tau + \int_0^t \kappa_\nu(\tau)Z(\tau)\,dw(\tau). \tag{35}$$

Let $\mathcal{D}_\nu$ denote the differential generator of (Z_ν, Y):

$$\mathcal{D}_\nu V = \frac{1}{2}|\kappa_\nu|^2 Z^2 V_{ZZ} + \frac{1}{2}\operatorname{tr}(\beta\beta^\top V_{YY}) + \kappa_\nu^\top\beta^\top Z V_{ZY} - rZV_Z + \alpha^\top V_Y,$$

and define the *dual indirect utility function*[9] V by

$$V(Z,Y,t) = \inf_{\nu\in\mathcal{N}^*} \mathrm{E}\Big[\int_t^T \tilde{u}(Z_\nu(\tau),\tau)\,d\tau + \int_t^T Z_\nu(\tau)y(\tau)\,d\tau \;\Big|\; Z_\nu(t) = Z,\ Y(t) = Y\Big]. \tag{36}$$

It is easily verified that $V(\cdot,Y,t)$ is decreasing and convex (because of the convexity of $\tilde{u}(\cdot,t)$ and of the set $\{Z_\nu : \nu\in\mathcal{N}^*\}$). Moreover, it follows from the standard argument of dynamic programming that, under some regularity conditions, V must solve the Hamilton-Jacobi-Bellman equation of optimality

$$0 = \min_{x\in\tilde{A}}\Big[Z^2V_{ZZ}\Big(\frac{1}{2}|\sigma^{-1}x|^2 + \kappa_0^\top\sigma^{-1}x\Big) + ZV_{ZY}^\top\beta\sigma^{-1}x\Big] + \mathcal{D}_0V + V_t + \tilde{u} + Zy \tag{37}$$

[9]The notion of a dual indirect utility function was introduced by He and Pagès (1993).

with the boundary condition

$$V(Z, Y, T) = 0. \tag{38}$$

Conversely, if there exists a solution V to the above partial differential equation which also satisfies the conditions of a verification theorem from stochastic dynamic programming, then V coincides with the indirect utility function of (36), and we have the following result on the existence of constrained-optimal consumption and investment policies.

Proposition 3. *Suppose that there exists a solution V to the partial differential equation (37) with the associated boundary condition (38) and that V satisfies the conditions of a verification theorem of stochastic dynamic programming. If the conditions of Theorem 4 hold, then there exists a solution ψ^* to the problem*

$$\min_{\psi \in (0,\infty)} V(\psi, Y(0), 0) + \psi W_0$$

and the constrained-optimal consumption policy is given by

$$c_{\nu^*}(t) = f(Z_{\nu^*}(t), t),$$

where Z_{ν^*} denotes the optimally controlled process with initial condition $Z_{\nu^*}(0) = \psi^*$. The optimal wealth process can be recovered as in (34).

PROOF. The argument used at the end of the proof of Theorem 4 ensures that (32) is satisfied under the given assumptions. The claim then follows immediately from Lemma B1 in Appendix B and Theorem 3. □

In the special case in which the security price coefficients and the endowment are deterministic functions of time only, it follows from the convexity of V that the differential equation (37) reduces to

$$0 = \frac{1}{2} \min_{x \in \tilde{A}} [|\kappa_0 + \sigma^{-1} x|^2] Z^2 V_{ZZ} - rZV_Z + V_t + \tilde{u} + Zy, \tag{39}$$

so that ν is independent of the state variable Z. Standard results (e.g., Theorem 3.12.1 in Luenberger (1969)) guarantee that the minimum norm problem in (39) always has a solution and that it is unique. Moreover, for this case we can prove directly a simpler verification result.

Proposition 4. *Assume that the security price coefficients (r, μ, σ) and the endowment stream y are deterministic functions of time and suppose that there exists a solution V to the partial differential equation (39) with the associated boundary condition $V(Z, T) = 0$, such that V is decreasing, convex, three times continuously differentiable with respect to Z and continuously differentiable with respect to t, and that V_Z is continuously differentiable with respect to t. Suppose further that the following conditions are satisfied:*

(a) there exists a process $\nu^* \in \mathcal{N}^*$ such that

$$\nu^*(t) = \operatorname*{argmin}_{x \in \tilde{A}} |\kappa_0(t) + \sigma(t)^{-1} x|^2 \quad (\lambda \times P)\text{-a.e.},$$

where λ denotes the Lebesgue measure on $[0, T)$;

(b) $-V_Z(\psi^*, 0) = W_0$ *for some* $\psi^* > 0$;

(c) *the stochastic integral*

$$\int Z_{\nu^*}(V_Z + Z_{\nu^*}V_{ZZ})\kappa_{\nu^*}^\top dw$$

is a martingale, where Z_{ν^*} *is the process of (35) with initial condition* $Z_{\nu^*}(0) = \psi^*$.

Then the constrained-optimal consumption, wealth, and portfolio processes are given by

$$c_{\nu^*}(t) = f(Z_{\nu^*}(t), t)$$
$$W_{\nu^*}(t) = -V_Z(Z_{\nu^*}(t), t)$$
$$\theta(t) = (\sigma(t)\sigma(t)^\top)^{-1}(\mu(t) - r(t)\bar{1})Z_{\nu^*}(t)V_{ZZ}(t) - (\sigma(t)\sigma(t)^\top)^{-1}\nu^*(t)Z_{\nu^*}(t)V_{ZZ}(t).$$

PROOF. We start by showing that $\theta(t) \in A$ and $\theta(t)^\top\nu^*(t) = 0$ $(\lambda \times P)$-a.e.. Let ν be an arbitrary process taking values in $\tilde{A}$. By the convexity of $\tilde{A}$ we have $\nu^*(t)+\varepsilon(\nu(t)-\nu^*(t)) \in \tilde{A}$ for all $\varepsilon \in [0, 1]$, and hence, since $\nu^*(t) \in \text{argmin}_{x\in\tilde{A}} |\kappa_{\nu^*}(t)|^2$ $(\lambda \times P)$-a.e.,

$$0 \leq \frac{\partial}{\partial\varepsilon}|\kappa_0(t)+\sigma(t)^{-1}[\nu^*(t)+\varepsilon(\nu(t)-\nu^*(t))]|^2\Big|_{\varepsilon=0} = 2\kappa_{\nu^*}(t)^\top\sigma(t)^{-1}(\nu(t)-\nu^*(t)) \quad (\lambda \times P)\text{-a.e.}$$

Since $\theta(t) = -(\sigma(t)^{-1})^\top\kappa_{\nu^*}(t)Z_{\nu^*}(t)V_{ZZ}(t)$ and V is convex, this implies

$$\theta(t)^\top(\nu(t) - \nu^*(t)) \leq 0 \quad (\lambda \times P)\text{-a.e..}$$

Taking $\nu(t) = \nu^*(t) + x$, $x \in \tilde{A}$, shows that $\theta(t)^\top x \leq 0$ $(\lambda \times P)$-a.e. for all $x \in \tilde{A}$. By Theorem 14.1 in Rockafellar (1970), this implies $\theta(t) \in A$. On the other hand, taking $\nu \equiv 0$ shows that $\theta(t)^\top\nu^*(t) \geq 0$ $(\lambda \times P)$-a.e.. Since $\nu^*(t) \in \tilde{A}$, this implies $\theta(t)^\top\nu^*(t) = 0$ $(\lambda \times P)$-a.e..

Next, we show that θ finances c_{ν^*}. By Itô's lemma and the fact that $-V_Z(0) = W_0$

$$\begin{aligned}
&Z_{\nu^*}(t)W_{\nu^*}(t) + \int_0^t Z_{\nu^*}(\tau)(c_{\nu^*}(\tau) - y(\tau))\, d\tau \\
&= \psi^*W_0 - \int_0^t Z_{\nu^*}(\tau)\Big(\frac{1}{2}|\kappa_{\nu^*}(\tau)|^2 Z_{\nu^*}(\tau)^2 V_{ZZZ}(\tau) - r(\tau)Z_{\nu^*}(\tau)V_{ZZ}(\tau) + V_{Zt}(\tau)\Big)\, d\tau \\
&\quad + \int_0^t Z_{\nu^*}(\tau)\Big(r(\tau)V_Z(\tau) - |\kappa_{\nu^*}(\tau)|^2 Z_{\nu^*}(\tau)V_{ZZ}(\tau)\Big)\, d\tau + \int_0^t Z_{\nu^*}(\tau)\Big(c_{\nu^*}(\tau) - y(\tau)\Big)\, d\tau \\
&\quad - \int_0^t Z_{\nu^*}(\tau)\Big(V_Z(\tau) + Z_{\nu^*}(\tau)V_{ZZ}(\tau)\Big)\kappa_{\nu^*}(\tau)^\top dw(\tau) \\
&= \psi^*W_0 - \int_0^t Z_{\nu^*}(\tau)\Big(V_Z(\tau) + Z_{\nu^*}(\tau)V_{ZZ}(\tau)\Big)\kappa_{\nu^*}(\tau)^\top dw(\tau) \\
&= \psi^*W_0 + \int_0^t Z_{\nu^*}(\tau)\Big(\theta(\tau)^\top\sigma(\tau) + W_{\nu^*}(\tau)\kappa_{\nu^*}(\tau)^\top\Big)\, dw(\tau),
\end{aligned}$$

where the second equality follows from differentiating (39) with respect to Z and the last is immediate from the definitions. Since $\theta^\top\nu^* = 0$ $(\lambda \times P)$-a.e. and $Z_{\nu^*}(t) = \psi^*\pi_{\nu^*}(t)$, this shows that the budget constraint in (27) is satisfied. Hence, (13) also holds. Finally, dividing both terms in the last equation by ψ^* and using the fact that $V_Z(Z,T) = 0$ and the stochastic integral is a martingale, gives

$$\pi_{\nu^*} W_{\nu^*}(t) + \int_0^t \pi_{\nu^*}(\tau)(c_{\nu^*}(\tau) - y(\tau))\, d\tau = \mathrm{E}\left[\int_0^T \pi_{\nu^*}(\tau)(c_{\nu^*}(\tau) - y(\tau))\, d\tau \,\Big|\, \mathcal{F}_t\right],$$

or

$$W_{\nu^*}(t) = \pi_{\nu^*}(t)^{-1}\, \mathrm{E}\left[\int_t^T \pi_{\nu^*}(\tau)(c_{\nu^*}(\tau) - y(\tau))\, d\tau \,\Big|\, \mathcal{F}_t\right].$$

Therefore, (14) and (15) are also satisfied and hence $c_{\nu^*} \in \mathcal{B}(\mathcal{P}, A)$.

The optimality of c_{ν^*} now follows from the fact that

$$\mathrm{E}\left[\int_0^T \pi_{\nu^*}(t)(c_{\nu^*}(t) - y(t))\, dt\right] = W_{\nu^*}(0) = W_0,$$

so that for any $c \in \mathcal{B}(\mathcal{P}, A)$ we have

$$U(c_{\nu^*}) - U(c) = \mathrm{E}\left[\int_0^T (u(c_{\nu^*}(t), t) - u(c(t), t))\, dt\right] \geq \psi^*\, \mathrm{E}\left[\int_0^T \pi_{\nu^*}(t)(c_{\nu^*}(t) - c(t))\, dt\right] \geq 0$$

because of (24) and (26). □

7. Equilibrium Asset Prices

In this section we briefly discuss, in a quite general setting, the implications of portfolio cone constraints on equilibrium risk premia. In particular, we show that the risk premium on any security can be separated into two parts: a component that is proportional to the asset covariance with the aggregate consumption process, and a component that depends on the type of constraints imposed on the individual investors. The first component coincides with the risk premium implied by the Consumption-based Capital Asset Pricing Model (CCAPM) of Breeden (1979).

We consider an economy with I individuals indexed by $i = 1, \ldots, I$. Each individual has a time-additive utility function and we denote with $u_i(c,t)$ the utility function of individual i for instantaneous consumption at time t. We assume that these functions are three times continuously differentiable in their first argument and continuously differentiable in their second argument. The portfolio policy of individual i is constrained to lie in a closed convex cone $A_i \subseteq \mathbb{R}^n$ at all times. We assume that the equilibrium prices are Itô processes and that a minimax martingale measure ξ_{ν_i} exists at the equilibrium prices for all agents i (sufficient conditions for this to happen are given by Theorem 4): in other words, letting c_i denote the optimal consumption choice for investor i, we have $u_{ic}(c_i(t), t) = \psi_i B(t)^{-1} \xi_{\nu_i}(t)$ for all i. Let $C = \sum_{i=1}^I c_i$ denote the aggregate consumption process and

$$\alpha_i(t) = -\frac{u_{ic}(c_i(t), t)}{u_{icc}(c_i(t), t)}$$

the Arrow-Pratt coefficient of absolute risk tolerance for individual i at time t.

We recall that if X and Y are continuous semimartingales, their *quadratic covariation* is the finite-variation process defined by

$$[X,Y](t) = X(t)Y(t) - X(0)Y(0) - \int_0^t X(\tau)\,dY(\tau) - \int_0^t Y(\tau)\,dX(\tau).$$

In particular, if X and Y are Itô processes with diffusion vectors σ_X and σ_Y, respectively, then

$$[X,Y](t) = \int_0^t \sigma_X(\tau)^\top \sigma_Y(\tau)\,d\tau,$$

so that $\frac{d}{dt}[X,Y](t) = \sigma_X(t)^\top \sigma_Y(t)$, which is usually written as $\operatorname{cov}(dX(t), dY(t))$.

Proposition 5. *Under the stated assumptions, the equilibrium risk premia are determined by*

$$\mu(t) - r(t)\bar{1} = \frac{I_S(t)^{-1}\operatorname{cov}(dS(t), dC(t))}{\sum_{i=1}^I \alpha_i(t)} + \frac{\sum_{i=1}^I \alpha_i(t)\nu_i(t)}{\sum_{i=1}^I \alpha_i(t)}. \tag{40}$$

Proof. Let $u_{ic}(t) = u_{ic}(c_i(t), t)$. Under the stated assumptions, c_i is an Itô process for all i, so that Itô's lemma gives

$$u_{ic}(t) = u_{ic}(0) + \int_0^t u_{icc}(c_i(\tau),\tau)\,dc_i(\tau) + \int_0^t u_{ict}(c_i(\tau),\tau)\,d\tau + \frac{1}{2}\int_0^t u_{iccc}(c_i(\tau),\tau)\,d[c_i,c_i](\tau).$$

Since the quadratic covariation between a continuous semimartingale and a process of finite variation is zero (Jacod and Shiryaev (1987), Proposition I.4.49), we have

$$[u_{ic}, S](t) = \Big[\int u_{icc}\,dc_i, S\Big](t) = \int_0^t u_{icc}(c_i(\tau),\tau)\,d[c_i,S](\tau),$$

where the last equality follows from Theorems I.4.40 and I.4.52 in Jacod and Shiryaev (1987). On the other hand, since $u_{ic} = \psi_i \pi_{\nu_i}$, we also have

$$\begin{aligned}[u_{ic}, S](t) &= [\psi_i\pi_{\nu_i}, S](t) = \Big[\int \psi_i \pi_{\nu_i}\kappa_{\nu_i}^\top dw, \int I_S\sigma\,dw\Big](t)\\ &= \psi_i \int_0^t \pi_{\nu_i}(\tau) I_S(\tau)\sigma(\tau)\kappa_{\nu_i}(\tau)\,d\tau = -\int_0^t u_{ic}(\tau)I_S(\tau)(\mu(\tau) - r(\tau)\bar{1} - \nu_i(\tau))\,d\tau\end{aligned}$$

The last two equations imply

$$\alpha_i(t)(\mu(t) - r(t)\bar{1} - \nu_i(t))\,dt = I_S(t)^{-1}\,d[c_i, S](t).$$

Summing over i gives

$$\sum_{i=1}^I \alpha_i(t)(\mu(t) - r(t)\bar{1} - \nu_i(t))\,dt = I_S(t)^{-1}\,d[C,S](t),$$

and hence

$$\mu(t) - r(t)\bar{1} = \Big(\sum_{i=1}^I \alpha_i(t)\Big)^{-1}\Big(I_S(t)^{-1}\frac{d}{dt}[C,S](t) + \sum_{i=1}^I \alpha_i(t)\nu_i(t)\Big).$$ □

Equation (40) gives a constrained CCAPM. It shows that the assets' risk premia can be decomposed into two components: the first corresponds to the risk premium implied by the standard CCAPM, while the second is a weighted average of the processes ν_i identifying the minimax martingale measures for the individuals in the economy. Clearly, in the unconstrained case ($A_i = \mathbb{R}^n$) we have $\nu_i \equiv 0$ for all i and the above relationship reduces to the CCAPM. More generally, in the case in which the portfolio of each individual is either unconstrained or constrained to take values in some convex cone A (independent of i), it follows from the convexity of $\tilde{A}$ that the last term in (40) must take values in $\tilde{A}$. Therefore, knowledge of the set $\tilde{A}$ allows to draw immediate conclusions on the type of possible deviations of equilibrium risk premia from the CCAPM.

For example, it is immediate to see that the CCAPM must hold—as observed by Grossman and Shiller (1982)—in the case of incomplete markets (example (b) of Section 3), while the CCAPM will overpredict (underpredict) risk premia in the presence of short sale (buying) constraints (examples (c) and (d) of Section 3). As a less obvious example, consider the case in which some agents (e.g., mutual funds) are required to keep a nonnegative amount invested in risky assets: in other words $A_i = \mathbb{R}^n$ for $i = 1, \ldots, m$, and $A_i = A$ for $i = m+1, \ldots, I$, where $A = \{x \in \mathbb{R}^n : \sum_{k=1}^n x_k \geq 0\}$. It is easily verified that we have $\tilde{A} = \{y \in \mathbb{R}^n : y_1 = \ldots = y_n,\ y_1 \geq 0\}$. Therefore, (40) implies that in this case excess returns will be linearly related to their covariances with aggregate consumption: however, contrary to what implied by the CCAPM, this linear relationship has a positive intercept.

8. Concluding Remarks

This paper has examined the individual's optimal consumption and investment problem with portfolio cone constraints and stochastic labor income. The main result is related to the existence of optimal policies under fairly general assumptions on the security price coefficients and on the income process. As pointed out in the remarks following Theorem 4, even if we have assumed no bequest function for final wealth, the introduction of such a function can be easily accommodated, and in fact would simplify the statement of some results. Of course, the case in which the agent is maximizing the expected utility from final wealth only could be treated similarly.

We also point out that while we have assumed that the set A is the same at all times, all the theory would go through, under a regularity condition, if we are given a family $\{A(t,\omega) : (t,\omega) \in [0,T) \times \Omega\}$ of nonempty, closed, convex cones in $\mathbb{R}^n$ and require that $\theta(t,\omega) \in A(t,\omega)$.[10] Finally, the assumption that the constraint sets A be cones could also be somewhat relaxed to their being convex sets with the property that the support function $\delta(y|A) = \sup_{x \in A} x^\top y$ is constant on its effective domain $\tilde{A} = \{y \in \mathbb{R}^n : \delta(y|A) < \infty\}$ (if A is a cone, then $\tilde{A}$ is the polar cone and $\delta(\cdot|A) \equiv 0$ on $\tilde{A}$). However, the treatment of more general convex subset constraints in the presence of stochastic labor income is beyond the scope of this paper and is left for future research.

[10] The regularity condition is the same as in Cvitanić and Karatzas (1992, p. 804): letting $\mathcal{N}$ denote the set of processes satisfying $\nu(t,\omega) \in \tilde{A}(t,\omega)$ and $\int_0^t |\nu(\tau)|^2 \, d\tau < \infty$ for all t, there exists a sequence $\{\nu_n\} \subset \mathcal{N}$ such that for every $x \in \tilde{A}(t,\omega)$ it is possible to find a subsequence $\{\nu_{n_k}\}$ with $\lim_{k \uparrow \infty} \nu_{n_k}(t,\omega) = x$.

Appendix A

This Appendix is devoted to the proof of Theorems 2 and 3. The argument of the proof follows Cvitanić and Karatzas (1992), the main difference being the presence of an income stream.

PROOF OF THEOREM 2. For $\nu \in \mathcal{N}^*$, define the process W_ν by

$$W_\nu(t) = \pi_\nu(t)^{-1} \mathrm{E}\Big[\int_t^T \pi_\nu(\tau)(c(\tau) - y(\tau))\, dt \,\Big|\, \mathcal{F}_t\Big].$$

We intend to show that W_{ν^*} is the optimal wealth process associated with c. Clearly, $W_{\nu^*}(T^-) = 0$ and hence (15) is satisfied. Also, the same argument used in the proof of Theorem 1 shows that $W_{\nu^*}(t) \geq -Te^{K_r}\bar{y}B(t)$, where $\bar{y} < \infty$ is an upper bound on y, so that (14) is also satisfied. Finally, since the process

$$M(t) = \pi_{\nu^*}(t)W_{\nu^*}(t) + \int_0^t \pi_{\nu^*}(\tau)(c(\tau) - y(\tau))\, d\tau$$

is a P-martingale with $M(0) = W_0$, it follows from the martingale representation theorem that there exists an adapted process φ with $\int_0^T |\varphi(t)|^2\, dt < \infty$ a.s. such that

$$\pi_{\nu^*}(t)W_{\nu^*}(t) + \int_0^t \pi_{\nu^*}(\tau)(c(\tau) - y(\tau))\, d\tau = M(t) = W_0 + \int_0^t \varphi(\tau)^\top dw(\tau). \qquad (41)$$

Define the portfolio strategy $\theta \in \Theta$ by

$$\pi_{\nu^*}(t)(\sigma(t)^\top \theta(t) + W_{\nu^*}(t)\kappa_{\nu^*}(t)) = \varphi(t). \qquad (42)$$

A comparison with (27) then reveals that in order to prove that $c \in \mathcal{B}(\mathcal{P}, A)$ we are only left to show that

$$\theta(t,\omega) \in A, \qquad (\lambda \times P)\text{-a.e.} \qquad (43)$$

(where λ denotes the Lebesgue measure on $[0,T)$) and that

$$\theta(t,\omega)^\top \nu^*(t,\omega) = 0, \qquad (\lambda \times P)\text{-a.e..} \qquad (44)$$

We intend to show that (43) and (44) are implied by (28).

Fix an arbitrary $\nu \in \mathcal{N}$ and define the process

$$\zeta(t) = \int_0^t \Big(\sigma(\tau)^{-1}(\nu(\tau) - \nu^*(\tau))\Big)^\top \Big(dw(\tau) - \kappa_{\nu^*}(\tau)\, d\tau\Big),$$

as well as the sequence of stopping times

$$\tau_n = T \wedge \inf\Big\{ t \in [0,T) : |\zeta(t)| + \int_0^t |\sigma(\tau)^{-1}(\nu(\tau) - \nu^*(\tau))|^2\, d\tau \geq n,$$
$$\text{or } |\pi_{\nu^*}(t)| + |W_{\nu^*}| \geq n, \text{ or } \int_0^t |\sigma(\tau)^\top \theta(\tau) + W_{\nu^*}(\tau)\kappa_{\nu^*}(\tau)|^2\, d\tau \geq n \Big\}.$$

Then $\tau_n \uparrow T$ a.s.. Also, letting

$$\nu_{\varepsilon,n}(t) = \nu^*(t) + \varepsilon[\nu(t) - \nu^*(t)]1_{\{t \le \tau_n\}},$$

for $\varepsilon \in (0,1)$, we have $\nu_{\varepsilon,n} \in \mathcal{N}$ (because of the convexity of $\tilde{A}$) and

$$\xi_{\nu_{\varepsilon,n}}(t) = \xi_{\nu^*}(t) \exp\left(\varepsilon\zeta(t \wedge \tau_n) - \frac{\varepsilon^2}{2} \int_0^{t\wedge\tau_n} |\sigma(t)^{-1}(\nu(\tau) - \nu^*(\tau))|^2 \, d\tau\right).$$

It then follows from the definition of the stopping times τ_n that

$$e^{-\varepsilon n}\xi_{\nu^*}(t) \le \xi_{\nu_{\varepsilon,n}}(t) \le e^{\varepsilon n}\xi_{\nu^*}(t), \tag{45}$$

so that $\xi_{\nu_{\varepsilon,n}}$ is of class D and hence $\nu_{\varepsilon,n} \in \mathcal{N}^*$.

We will show below that for any $\nu \in \mathcal{N}$ we have:

$$\lim_{\varepsilon \downarrow 0} \frac{W_{\nu_{\varepsilon,n}}(0) - W_{\nu^*}(0)}{\varepsilon} = \mathrm{E}\left[\int_0^{\tau_n} \pi_{\nu^*}(t)\theta(t)^\top(\nu(t) - \nu^*(t))\, dt\right]. \tag{46}$$

By (28), the left-hand side of (46) is nonpositive, and thus so is the right-hand side.

Taking $\nu = \nu^* + \rho$, $\rho \in \mathcal{N}$, it follows from the fact that $\tilde{A}$ is a convex cone that $\nu \in \mathcal{N}$, and hence (46) gives

$$\mathrm{E}\left[\int_0^{\tau_n} \pi_{\nu^*}(t)\theta(t)^\top\rho(t)\, dt\right] \le 0.$$

Since $\rho \in \mathcal{N}$ was arbitrary, this implies the existence of a set E having full $(\lambda \times P)$ measure such that

$$\theta(t,\omega)^\top x \le 0, \qquad \forall (t,\omega) \in E,\ x \in \tilde{A}$$

By Theorem 14.1 in Rockafellar (1970) the above implies (43).

On the other hand, for $\nu \equiv 0$ (46) gives

$$\mathrm{E}\left[\int_0^{\tau_n} \pi_{\nu^*}(t)\theta(t)^\top\nu^*(t)\, dt\right] \ge 0$$

and it then follows from the fact that $\theta(t)\nu^*(t) \le 0$ for $\theta(t) \in A$, $\nu^*(t) \in \tilde{A}$, that (44) also holds.

To show the inequality in (46), we start by observing that for all $\varepsilon \in (0,1)$

$$\frac{W_{\nu_{\varepsilon,n}}(0) - W_{\nu^*}(0)}{\varepsilon} = \mathrm{E}\left[\int_0^T \pi_{\nu^*}(t)\frac{c(t) - y(t)}{\varepsilon}\left(\frac{\pi_{\nu_{\varepsilon,n}}(t)}{\pi_{\nu^*}(t)} - 1\right) dt\right]$$

and (from (45))

$$\left|\frac{\pi_{\nu_{\varepsilon,n}}(t)}{\pi_{\nu^*}(t)} - 1\right| \le e^{\varepsilon n} - 1, \tag{47}$$

so that

$$\left|\pi_{\nu^*}(t)\frac{c(t) - y(t)}{\varepsilon}\left(\frac{\pi_{\nu_{\varepsilon,n}}(t)}{\pi_{\nu^*}(t)} - 1\right)\right| \le K_n\pi_{\nu^*}(t)(c(t) + y(t)),$$

where

$$K_n = \sup_{\varepsilon \in (0,1)} \frac{e^{\varepsilon n} - 1}{\varepsilon} < \infty. \tag{48}$$

Since the above expression is integrable, it follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\frac{W_{\nu_{\varepsilon,n}}(0)-W_{\nu^*}(0)}{\varepsilon} &= \mathrm{E}\Big[\int_0^T \pi_{\nu^*}(t)\big(c(t)-y(t)\big)\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\Big(\frac{\pi_{\nu_{\varepsilon,n}}(t)}{\pi_{\nu^*}(t)}-1\Big)\,dt\Big] \\ &= \mathrm{E}\Big[\int_0^T \pi_{\nu^*}(t)\zeta(t\wedge\tau_n)\big(c(t)-y(t)\big)\,dt\Big] \\ &= \mathrm{E}\Big[\int_0^{\tau_n} \pi_{\nu^*}(t)\zeta(t)\big(c(t)-y(t)\big)\,dt+\pi_{\nu^*}(\tau_n)\zeta(\tau_n)W_{\nu^*}(\tau_n)\Big].\end{aligned}\tag{49}$$

Using (41), (42) and Itô's lemma shows that

$$\begin{aligned}\pi_{\nu^*}(\tau_n)\zeta(\tau_n)W_{\nu^*}(\tau_n) &= \int_0^{\tau_n}\pi_{\nu^*}(t)W_{\nu^*}(t)\,d\zeta(t)-\int_0^{\tau_n}\pi_{\nu^*}(t)\zeta(t)\big(c(t)-y(t)\big)\,dt \\ &\quad+\int_0^{\tau_n}\pi_{\nu^*}(t)\zeta(t)\big(\sigma(t)^\top\theta(t)+W_{\nu^*}(t)\kappa_{\nu^*}(t)\big)^\top dw(t) \\ &\quad+\int_0^{\tau_n}\pi_{\nu^*}(t)\big(\sigma(t)^\top\theta(t)+W_{\nu^*}(t)\kappa_{\nu^*}(t)\big)^\top\sigma(t)^{-1}\big(\nu(t)-\nu^*(t)\big)\,dt \\ &= \int_0^{\tau_n}\pi_{\nu^*}(t)W_{\nu^*}(t)\big(\sigma(t)^{-1}(\nu(t)-\nu^*(t))\big)^\top dw(t) \\ &\quad+\int_0^{\tau_n}\pi_{\nu^*}(t)\zeta(t)\big(\sigma(t)^\top\theta(t)+W_{\nu^*}(t)\kappa_{\nu^*}(t)\big)^\top dw(t) \\ &\quad+\int_0^{\tau_n}\pi_{\nu^*}(t)\Big[\theta(t)^\top\big(\nu(t)-\nu^*(t)\big)-\zeta(t)\big(c(t)-y(t)\big)\Big]\,dt,\end{aligned}\tag{50}$$

which implies, since the stochastic integrals in the above expression have zero expectation,

$$\mathrm{E}\Big[\int_0^{\tau_n}\pi_{\nu^*}(t)\zeta(t)(c(t)-y(t))\,dt+\pi_{\nu^*}(\tau_n)\zeta(\tau_n)W_{\nu^*}(\tau_n)\Big]=\mathrm{E}\Big[\int_0^{\tau_n}\pi_{\nu^*}(t)\theta(t)^\top(\nu(t)-\nu^*(t))\,dt\Big].$$

Substituting the above expression in (49) gives (46). □

Proof of Theorem 3. Assume that $(\psi^*,\nu^*)\in(0,\infty)\times\mathcal{N}^*$ solves (P^*), and that (32) holds. Define the consumption policy c_{ν^*} and the wealth process W_{ν^*} by (33) and (34), respectively (the latter is finite because of (32)). The argument used in the proof of Theorem 1 shows that $c_{\nu^*}\in\mathcal{C}_+^*$. In order to prove that c_{ν^*} is constrained-optimal we will proceed in two steps: first we will show that $U(c_{\nu^*})\geq U(c)$ holds for all $c\in\mathcal{B}(\mathcal{P},A)$, and then that $c\in\mathcal{B}(\mathcal{P},A)$.

Step 1: Since the process $B^{-1}y$ is bounded, (12) and (32) imply that

$$\mathrm{E}\Big[\int_0^T\pi_{\nu^*}(t)f(\psi\pi_{\nu^*}(t),t)\,dt\Big]<\infty\tag{51}$$

holds for all $\psi\in(0,\infty)$. By the optimality of ψ^*, we then have

$$\begin{aligned}0 &= \lim_{\varepsilon\to 0}\frac{J(\psi^*+\varepsilon,\nu^*)-J(\psi^*,\nu^*)}{\varepsilon} \\ &= \mathrm{E}\Big[\int_0^T\lim_{\varepsilon\to 0}\frac{\tilde{u}((\psi^*+\varepsilon)\pi_{\nu^*}(t),t)-\tilde{u}(\psi^*\pi_{\nu^*}(t),t)}{\varepsilon}\,dt+\int_0^T\pi_{\nu^*}(t)y(t)\,dt+W_0\Big] \\ &= \mathrm{E}\Big[\int_0^T\pi_{\nu^*}(t)(y(t)-c_{\nu^*}(t))\,dt\Big]+W_0,\end{aligned}$$

where the second equality follows from Lebesgue's dominated convergence theorem, using the fact that $\pi_{\nu^*}y$ is integrable and

$$\left|\frac{\tilde{u}\Big((\psi^*+\varepsilon)\pi_{\nu^*}(t),t\Big)-\tilde{u}\Big(\psi^*\pi_{\nu^*}(t),t\Big)}{\varepsilon}\right| \le \frac{\tilde{u}\Big((\psi^*-|\varepsilon|)\pi_{\nu^*}(t),t\Big)-\tilde{u}\Big(\psi^*\pi_{\nu^*}(t),t\Big)}{|\varepsilon|}$$
$$\le \pi_{\nu^*}(t)f\Big((\psi^*-|\varepsilon|)\pi_{\nu^*}(t),t\Big) \le \pi_{\nu^*}(t)f\Big((\psi^*/2)\pi_{\nu^*}(t),t\Big)$$

for $|\varepsilon|<\psi^*/2$, because $\tilde{u}(\cdot,t)$ is decreasing and convex, $\frac{\partial}{\partial y}\tilde{u}(y,t)=-f(y,t)$, and $f(\cdot,t)$ is decreasing. Therefore

$$\mathrm{E}\Big[\int_0^T \pi_{\nu^*}(t)(c_{\nu^*}(t)-y(t))\,dt\Big]=W_0.$$

For any consumption process $c\in\mathcal{B}(\mathcal{P},A)$ we then have (from (24) and (26))

$$U(c_{\nu^*})-U(c)=\mathrm{E}\Big[\int_0^T (u(c_{\nu^*}(t),t)-u(c(t),t))\,dt\Big]\ge\psi^*\,\mathrm{E}\Big[\int_0^T \pi_{\nu^*}(t)(c_{\nu^*}(t)-c(t))\,dt\Big]\ge 0.$$

Hence, c_{ν^*} must be optimal provided it is A-feasible.

Step 2: By the martingale representation theorem, there exists an adapted process φ with $\int_0^T|\varphi(t)|^2\,dt<\infty$ a.s. such that

$$\pi_{\nu^*}(t)W_{\nu^*}(t)+\int_0^t \pi_{\nu^*}(\tau)(c_{\nu^*}(\tau)-y(\tau))\,d\tau=W_0+\int_0^t \varphi(\tau)^\top dw(\tau).$$

Defining the portfolio strategy θ by (42), it is immediately verified that (14) and (15) are satisfied and a comparison with (27) reveals that in order to prove that $c_{\nu^*}\in\mathcal{B}(\mathcal{P},A)$ we are only left to show that (43) and (44) are satisfied.

Fix an arbitrary $\nu\in\mathcal{N}$ and define the process ζ, as well as the stopping times τ_n, as in the proof of Theorem 2. For $\varepsilon\in(0,1)$, let $\nu_{\varepsilon,n}(t)=\nu^*(t)+\varepsilon[\nu(t)-\nu^*(t)]1_{\{t\le\tau_n\}}$. By (47), we then have

$$\left|\frac{\tilde{u}(\psi^*\pi_{\nu^*}(t),t)-\tilde{u}(\psi^*\pi_{\nu_{\varepsilon,n}}(t),t)}{\varepsilon}+\psi^*y(t)\Big(\frac{\pi_{\nu^*}(t)-\pi_{\nu_{\varepsilon,n}}(t)}{\varepsilon}\Big)\right|$$
$$\le \psi^*\pi_{\nu^*}(t)\frac{f(\psi^*e^{-\varepsilon n}\pi_{\nu^*}(t),t)+y(t)}{\varepsilon}\left|\frac{\pi_{\nu_{\varepsilon,n}}(t)}{\pi_{\nu^*}(t)}-1\right|$$
$$\le \psi^*K_n\pi_{\nu}^*(t)\Big(f(\psi^*e^{-n}\pi_{\nu^*}(t),t)+y(t)\Big)$$

where K_n is as in (48). Since the last expression is integrable (because of (51)), Lebesgue's dominated convergence theorem gives

$$\lim_{\varepsilon\downarrow 0}\frac{J(\psi^*,\nu^*)-J(\psi^*,\nu_{\varepsilon,n})}{\varepsilon}$$
$$=\lim_{\varepsilon\downarrow 0}\mathrm{E}\Big[\int_0^T\frac{\tilde{u}(\psi^*\pi_{\nu^*}(t),t)-\tilde{u}(\psi^*\pi_{\nu_{\varepsilon,n}}(t),t)}{\varepsilon}\,dt+\psi^*\int_0^T y(t)\Big(\frac{\pi_{\nu^*}(t)-\pi_{\nu_{\varepsilon,n}}(t)}{\varepsilon}\Big)\,dt\Big]$$
$$=\psi^*\,\mathrm{E}\Big[\int_0^T\pi_{\nu^*}(t)\zeta(t\wedge\tau_n)\Big(c_{\nu^*}(t)-y(t)\Big)\,dt\Big]=\psi^*\,\mathrm{E}\Big[\int_0^{\tau_n}\pi_{\nu^*}(t)\theta(t)^\top(\nu(t)-\nu^*(t))\,dt\Big],$$

where the last equality follows from Itô's lemma as in (50). Since $J(\psi^*,\nu_{\varepsilon,n})$ reaches a minimum at $\varepsilon=0$, the last term is nonpositive, and (43)–(44) then follow by taking $\nu=\nu^*+\rho$, $\rho\in\mathcal{N}$, and $\nu\equiv 0$ as in the proof of Theorem 2. This shows that $c_{\nu^*}\in\mathcal{B}(\mathcal{P},A)$.

□

Appendix B

This Appendix is devoted to the proof of Theorem 4. Since the proof is rather long, we first provide a brief outline. We will start by showing (Lemma B1) that, under the assumptions of Theorem 4, a sufficient condition for the minimum in (P^*) to be attained is that for all $\psi \in (0, \infty)$ there exists a solution to the problem

$$\min_{\nu \in \mathcal{N}^*} J(\psi, \nu). \tag{52}$$

However, the above problem is not easy to attack directly, since the set $\mathcal{N}^*$ does not have any obvious topological structure. We will therefore reexpress (52) as a problem formulated directly over a space of martingales, as follows. Recall from Propositions 1 and 2 that for any $\nu \in \mathcal{N}^*$ we can extend the corresponding process ξ_ν to a martingale on $[0, T]$ by setting $\xi_\nu(T) = \lim_{t \uparrow T} \xi_\nu(t)$. Let $\mathcal{M}(Q_0)$ denote the space of Q_0-martingales on $[0, T]$, where Q_0 denotes the probability measure with $dQ_0/dP = \xi_0(T)$. Since any martingale $\eta \in \mathcal{M}(Q_0)$ can be uniquely identified with its terminal random variable $\eta(T) \in L^1(Q_0)$, the space $\mathcal{M}(Q_0)$ comes equipped with a natural topology, which is generated by the norm $\|\eta\|_{\mathcal{M}(Q_0)} = \|\eta(T)\|_{L^1(Q_0)}$.[11] Moreover, we will show in Lemma B3 that

$$\left\{\xi_\nu : \nu \in \mathcal{N}^*\right\} = \left\{\xi_0 \eta : \eta \in H,\ \eta(t) > 0\ \forall t \in [0, T)\right\}$$

where H is a closed, convex subset of $\mathcal{M}(Q_0)$ to be defined below. This suggests that we can rewrite (52) as an equivalent problem

$$\min_{\eta \in H} \hat{J}(\psi, \eta),$$

since we will prove in Lemma B4 that the constraint that $\eta(t) > 0$ for all $t \in [0, T)$ is not binding. Existence of the above minimum can then be shown using conjugate duality techniques from Rockafellar (1974, 1975).

We will now proceed to substantiate the previous sketch through a series of lemmas.

Lemma B1. *Under assumptions (a)–(c) of Theorem 4, if for all $\psi \in (0, \infty)$ there exists a solution to (52), then there exists a solution to (P^*).*

PROOF. Let $V(\psi)$ denote the value function in (52). We will prove below that V is strictly convex and continuous on $(0, \infty)$ and that it satisfies the coercitivity conditions $V(0+) = V(\infty) = \infty$. Therefore, V must attain a (unique) minimum on $(0, \infty)$, and hence (P^*) has a solution.

Let $\psi_1, \psi_2 > 0$ be arbitrary and let ν_i $(i = 1, 2)$ denote the solution to (52) with $\psi = \psi_i$. It is easily verified by Itô's lemma and the convexity of $\tilde{A}$ that the set $\{\xi_\nu : \nu \in \mathcal{N}^*\}$ is convex. Therefore, for all $\lambda \in (0, 1)$ there exists a $\nu_\lambda \in \mathcal{N}^*$ such that

$$\xi_{\nu_\lambda} = \frac{\lambda \psi_1 \xi_{\nu_1} + (1 - \lambda)\psi_2 \xi_{\nu_2}}{\lambda \psi_1 + (1 - \lambda)\psi_2}.$$

[11]See section IV.1(d) in Jacod (1979) for a description of this topology.

The strict convexity of V then follows immediately from the strict convexity of $\tilde{u}(\cdot, t)$, as

$$\begin{aligned} V(\lambda\psi_1 + (1-\lambda)\psi_2) &\leq J(\lambda\psi_1 + (1-\lambda)\psi_2, \nu_\lambda) \\ &\leq \lambda J(\psi_1, \nu_1) + (1-\lambda) J(\psi_2, \nu_2) = \lambda V(\psi_1) + (1-\lambda) V(\psi_2). \end{aligned}$$

Since V is convex and finite on $(0, \infty)$ (by condition (c)), it must be continuous there (Rockafellar (1974), Corollary 8A).

Let $\varepsilon > 0$ be such that $y/B + W_0/T > \varepsilon$ (such a ε exists by condition (b) of Theorem 4). By (2), Fubini's theorem, the properties of $\tilde{u}$ and Jensen's inequality we then have

$$V(\psi) \geq \min_{\nu \in \mathcal{N}^*} \int_0^T \tilde{u}(\psi e^{K_r} \operatorname{E} \xi_\nu(t), t)\, dt + \psi T \varepsilon \geq \int_0^T \tilde{u}(\psi e^{K_r}, t)\, dt + \psi T \varepsilon \tag{53}$$

and therefore it follows from condition (a) and Lemma 3 that $V(\psi) \to \infty$ as $\psi \downarrow 0$. Also, it is easily verified that condition (a) of Theorem 4 implies $\tilde{u}(y, t) \geq -\tilde{k}(1 + y^{(b-1)/b})$ for some $\tilde{k} > 0$, so that (53) gives

$$V(\psi) \geq -\tilde{k} T \Big(1 + (\psi e^{K_r})^{\frac{b-1}{b}}\Big) + \psi T \varepsilon,$$

which shows that $V(\psi) \to \infty$ as $\psi \uparrow \infty$. □

Next, we recall that we have from (6) that $0 \in \mathcal{N}^*$ and that the probability measure Q_0 with $dQ_0/dP = \xi_0(T)$ is equivalent to P on $(\Omega, \mathcal{F})$, so that we can use the term "almost surely" unambiguously. Also, it follows from Girsanov's and Lévy's theorems (Jacod and Shiryaev (1987), Theorems III.3.11 and II.4.4) that the process

$$w_0(t) = w(t) - \int_0^t \kappa_0(\tau)\, d\tau$$

is a standard Brownian motion under Q_0.

Let $\mathcal{M}(Q_0)$ denote the space of Q_0-martingales on $[0, T]$ and define the set

$$\mathcal{M}(Q_0, \tilde{A}) = \Big\{\eta \in \mathcal{M}(Q_0) : \exists \psi \in \Psi(\tilde{A}) \text{ s.t. } \eta(\cdot) = \eta(0) + \int_0^{\cdot} (\sigma(t)^{-1}\psi(t))^\top dw_0(t)\Big\},$$

where $\Psi(\tilde{A})$ denote the set of adapted n-dimensional processes ψ such that

$$\int_0^t |\sigma(\tau)^{-1}\psi(\tau)|^2\, d\tau < \infty \quad \text{a.s.}$$

for all $t \in [0, T)$ and

$$\psi(t, \omega) \in \tilde{A} \qquad (\lambda \times P)\text{-a.e..}$$

Letting $\mathcal{M}_+(Q_0, \tilde{A})$ denote the set of nonnegative martingales in $\mathcal{M}(Q_0, \tilde{A})$, we have

Lemma B2. *The set*

$$H = \Big\{\eta \in \mathcal{M}_+(Q_0, \tilde{A}) : \|\eta\|_{\mathcal{M}(Q_0)} = 1\Big\}$$

is convex and closed in $\mathcal{M}(Q_0)$.

PROOF. Convexity follows immediately from the convexity of $\tilde{A}$. Suppose that $\{\eta_n\} \subset H$ and $\|\eta_n - \eta\|_{\mathcal{M}(Q_0)} \to 0$ as $n \uparrow \infty$ for some $\eta \in \mathcal{M}(Q_0)$. Then clearly $\|\eta\|_{\mathcal{M}(Q_0)} = 1$ and $\eta(t) \geq 0$ for all t (since $\eta(T)$ is also the a.s. limit of a subsequence of $\{\eta_n(T)\}$). By the martingale representation theorem (Jacod and Shiryaev (1987), Theorem III.4.33),

$$\eta(t) = 1 + \int_0^t (\sigma(\tau)^{-1}\psi(\tau))^\top \, dw_0(\tau)$$

for some $\psi \in \Psi(\mathbb{R}^n)$, and we are only left to show that $\psi(t, \omega) \in \tilde{A}$ $(\lambda \times P)$-a.e..

We will show below that the set of martingales $\hat{\eta} \in H$ with $\|\hat{\eta}(T)\|_{L^\infty(Q_0)} < \infty$ is dense in H. We can therefore assume without loss of generality that each η_n is (essentially) bounded. Fix an arbitrary process $\rho \in \Psi(A)$, and for each integer k define the stopping time

$$\tau_k = T \wedge \inf\Big\{t \in [0,T] : \eta(t) > k \text{ or } \Big|\int_0^t \rho(\tau)^\top \sigma(\tau)\, dw_0(\tau)\Big| > k\Big\}.$$

By Itô's lemma

$$\begin{aligned}\eta(t \wedge \tau_k) \int_0^{t\wedge\tau_k} \rho(\tau)^\top \sigma(\tau)\, dw_0(\tau) &= \int_0^{t\wedge\tau_k} \eta(\tau)\rho(\tau)^\top\sigma(\tau)\, dw_0(\tau) \\ &+ \int_0^{t\wedge\tau_k} \Big(\int_0^\tau \rho(s)^\top \sigma(s)\, dw_0(s)\Big)(\sigma(\tau)^{-1}\psi(\tau))^\top \, dw_0(\tau) + \int_0^{t\wedge\tau_k} \rho(\tau)^\top\psi(\tau)\, d\tau,\end{aligned}$$

Since the stochastic integrals in the previous expression are Q_0-martingales, we have

$$\begin{aligned}\mathrm{E}^{Q_0}\Big[\int_0^{t\wedge\tau_k} \rho(\tau)^\top\psi(\tau)\, d\tau\Big] &= \mathrm{E}^{Q_0}\Big[\eta(t\wedge\tau_k)\int_0^{t\wedge\tau_k} \rho(\tau)^\top\sigma(\tau)\, dw_0(\tau)\Big] \\ &= \lim_{n\uparrow\infty} \mathrm{E}^{Q_0}\Big[\eta_n(t\wedge\tau_k)\int_0^{t\wedge\tau_k}\rho(\tau)^\top\sigma(\tau)\, dw_0(\tau)\Big] \;\leq\; 0,\end{aligned}$$

where the second equality follows from the fact that $\eta_n(t) \to \eta(t)$ in $L^1(Q_0)$ for all $t \in [0,T]$ and the dominated convergence theorem, while the inequality follows from the fact that $\eta_n \in \mathcal{M}(Q_0, \tilde{A})$, $\rho \in \Psi(A)$, and another application of Itô's lemma (using the fact that each η_n is bounded). Since $\rho \in \Psi(A)$ was arbitrary, the above implies the existence of a set $E \subset [0,T) \times \Omega$ having full $(\lambda \times P)$ measure such that

$$x^\top \psi(t,\omega) \leq 0, \qquad \forall (t,\omega) \in E,\ x \in A.$$

By Theorem 14.1 in Rockafellar (1970), this implies $\psi \in \Psi(\tilde{A})$. Therefore, $\eta \in H$ and H is closed.

Finally, we prove our previous claim that the set $\{\hat{\eta} \in H : \|\hat{\eta}(T)\|_{L^\infty(Q_0)} < \infty\}$ is dense in H. Let $\eta \in H$ be arbitrary and define the stopping times $\tau_n = T \wedge \inf\{t \in [0,T] : \eta(t) > n\}$ and the sequence $\{\eta_n\}$ with $\eta_n(t) = \eta(t \wedge \tau_n)$. It is immediate to see that $\eta_n \in H$ for all n and $\|\eta_n(T)\|_{L^\infty(Q_0)} \leq n$. Moreover $\eta_n(T) \to \eta(T)$ a.s. as $n \uparrow \infty$ because $\tau_n \uparrow T$ a.s.. Since $\{\eta_n(T)\} = \{\eta(\tau_n)\}$ is uniformly integrable, it follows from the mean convergence theorem that $\|\eta_n(T) - \eta(T)\|_{L^1(Q_0)} \to 0$, and hence $\eta_n \to \eta$ in $\mathcal{M}(Q_0)$. □

Our interest in the set H is motivated by the following result.

Lemma B3. *We have*

$$\Big\{\xi_\nu : \nu \in \mathcal{N}^*\Big\} = \Big\{\xi_0\eta : \eta \in H,\ \ \eta(t) > 0\ \forall t \in [0,T)\Big\}.$$

Proof. First, note that for $\nu \in \mathcal{N}^*$ we have $\xi_\nu(t) = \xi_0(t)\eta_\nu(t)$, where

$$\eta_\nu(t) = \exp\Big(\int_0^t (\sigma(\tau)^{-1}\nu(\tau))^\top\, dw(\tau) - \frac{1}{2}\int_0^t (|\sigma(\tau)^{-1}\nu(\tau)|^2 + 2\kappa_0^\top \sigma(\tau)^{-1}\nu(\tau))\, d\tau\Big).$$

Clearly, $\eta_\nu(t) > 0$ for all $t \in [0,T)$. Also, since ξ_ν is a martingale under P, η_ν is a martingale under Q_0 (Jacod and Shiryaev (1987), Proposition III.3.8), and hence $\|\eta_\nu(T)\|_{L^1(Q_0)} = \eta_\nu(0) = 1$. By Itô's lemma

$$\begin{aligned}\eta_\nu(t) &= 1 + \int_0^t \eta_\nu(\tau)(\sigma(\tau)^{-1}\nu(\tau))^\top\, dw(\tau) - \int_0^t \eta_\nu(\tau)(\sigma(\tau)^{-1}\nu(\tau))^\top \kappa_0(\tau)\, d\tau \\ &= 1 + \int_0^t \eta_\nu(\tau)(\sigma(\tau)^{-1}\nu(\tau))^\top\, dw_0(\tau)\end{aligned}$$

for all $t \in [0,T]$. Taking $\psi = \eta_\nu \nu$ and using the fact that $\nu \in \tilde{A}$, $\eta_\nu \geq 0$ and $\tilde{A}$ is a cone, it is then immediate to see that $\psi \in \Psi(\tilde{A})$, and hence $\eta_\nu \in H$.

Conversely, suppose that $\eta \in H$ and $\eta(t) > 0$ for $t \in [0,T)$. Then there exists a $\psi \in \Psi(\tilde{A})$ such that

$$\eta(t) = 1 + \int_0^t (\sigma(\tau)^{-1}\psi(\tau))^\top dw_0(\tau) = 1 + \int_0^t (\sigma(\tau)^{-1}\psi(\tau))^\top dw(\tau) - \int_0^t (\sigma(\tau)^{-1}\psi(\tau))^\top \kappa_0(\tau)\, d\tau.$$

Setting $\nu = \psi/\eta \in \mathcal{N}$, it then follows by Itô's lemma that

$$\xi_0(t)\eta(t) = \exp\Big(\int_0^t \kappa_\nu(\tau)^\top\, dw(\tau) - \frac{1}{2}\int_0^t |\kappa_\nu(\tau)|^2\, d\tau\Big).$$

Also, since η is a Q_0-martingale on $[0,T]$, $\xi_0\eta$ is a P-martingale on $[0,T]$. Therefore, $\nu \in \mathcal{N}^*$ (by Proposition 2). □

For a fixed but arbitrary $\psi \in (0,\infty)$, define the functional $\hat{J}_\psi(\eta) : H \mapsto \mathbb{R} \cup \{\infty\}$ by

$$\begin{aligned}\hat{J}_\psi(\eta) &= \mathrm{E}\Big[\int_0^T \tilde{u}\Big(\psi B(t)^{-1}\xi_0(t)\eta(t), t\Big)\, dt + \psi \int_0^T B(t)^{-1}\xi_0(t)\eta(t)y(t)\, dt + \psi W_0\Big] \\ &= \mathrm{E}^{Q_0}\Big[\int_0^T \xi_0(t)^{-1}\tilde{u}\Big(\psi B(t)^{-1}\xi_0(t)\eta(t), t\Big)\, dt + \psi \int_0^T B(t)^{-1}\eta(t)\tilde{y}(t)\, dt\Big],\end{aligned}$$

where $\tilde{y}(t) = y(t) + \frac{W_0}{T}B(t)$, and consider the problem

$$\inf_{\eta \in H} \hat{J}_\psi(\eta). \qquad (P^{**})$$

It should be clear from the definition of $\hat{J}_\psi$ and Lemma B3 that if the above infimum is attained by some $\eta^* \in H$ and $\eta^*(t) > 0$ for all $t \in [0,T)$, then there exists a $\nu^* \in \mathcal{N}^*$ such that $J(\psi, \nu^*) = \inf_{\nu \in \mathcal{N}^*} J(\psi, \nu)$. In fact, the following lemma shows that the requirement that $\eta^*(t) > 0$ for all $t \in [0,T)$ is always satisfied under our assumptions.

Lemma B4. *Under conditions (a) and (c) of Theorem 4, if η^* solves (P^{**}), then $\eta^*(t) > 0$ for all $t \in [0,T)$.*

PROOF. Suppose $\eta^* \in H$ solves (P^{**}). Then $\hat{J}_\psi(\eta^*) < \infty$ by condition (c) in Theorem 4. Also, letting $\tau = T \wedge \inf\{t \in [0,T] : \eta^*(t) = 0\}$, it follows by Lemma III.3.6 in Jacod and Shiryaev (1987) that $\eta^* = 0$ a.s. on $[\tau, T]$. Since $\tilde{u}(0+,t) = u(\infty,t) = \infty$ by condition (a), this implies $\tau \geq T$. □

We next remark some properties of the functional $\hat{J}_\psi$ that will be useful in the sequel.

Lemma B5. *Under assumption (a) of Theorem 4, $\hat{J}_\psi$ is (i) strictly convex and (ii) lower semicontinuous: for every $\eta \in H$ and $\{\eta_n\} \subset H$ with $\|\eta_n - \eta\|_{\mathcal{M}(Q_0)} \to 0$, we have*

$$\liminf_{n\uparrow\infty} \hat{J}_\psi(\eta_n) \geq \hat{J}_\psi(\eta).$$

PROOF. Strict convexity is immediate by the strict convexity of $\tilde{u}$. Suppose that $\hat{J}_\psi$ is not lower semicontinuous. Then there is a $\eta \in H$ and $\{\eta_n\} \subset H$ with $\|\eta_n - \eta\|_{\mathcal{M}(Q_0)} \to 0$ such that

$$\hat{J}_\psi(\eta_n) < \hat{J}_\psi(\eta) \quad \text{for all } n. \tag{54}$$

On the other hand, since $\eta_n \to \eta$ in $\mathcal{M}(Q_0)$, $\eta_n(t) \to \eta(t)$ in $L^1(Q_0)$ for all $t \in [0,T]$. Therefore, $\eta_n \to \eta$ in $L^1(\lambda \times Q_0)$ (where λ denotes the Lebesgue measure on $[0,T)$), and hence there exists a subsequence $\{\eta_{n_k}\}$ such that $\eta_{n_k} \to \eta$ $(\lambda \times P)$-a.e.. Also, by (2) and condition (a) of Theorem 4, there exists a constant $\tilde{k} > 0$ such that

$$\begin{aligned}
&\left(\tilde{u}(\psi B(t)^{-1}\xi_0(t)\eta_{n_k}(t), t) + \psi B(t)^{-1}\xi_0(t)\eta_{n_k}(t)\tilde{y}(t)\right)^- \\
&\quad \leq \; X_{n_k}(t) \; \equiv \; \tilde{k}\left(1 + (\psi e^{K_r}\xi_0(t)\eta_{n_k}(t))^{\frac{b-1}{b}}\right).
\end{aligned}$$

We will show below that the family $\{X_{n_k}\}$ is $(\lambda \times P)$-uniformly integrable. Therefore, it follows from Fatou's lemma for random variables uniformly integrable from below that

$$\liminf_{n_k\uparrow\infty} \hat{J}_\psi(\eta_{n_k}) \geq \hat{J}_\psi(\eta).$$

This contradicts (54) and thus establishes the lower semicontinuity of $\hat{J}_\psi$.

Finally, to prove our claim that the family $\{X_{n_k}\}$ is $(\lambda \times P)$-uniformly integrable, we need to show that

$$\lim_{K\uparrow\infty} \sup_k \int_{|X_{n_k}|>K} |X_{n_k}(t,\omega)| \, d(\lambda \times P) \to 0.$$

To this extent, it is clearly enough to show that for all $\varepsilon > 0$ we can find a $K > 0$ such that

$$\int_{|X_{n_k}|>K} |X_{n_k}(t,\omega)| \, d(\lambda \times P) \leq \varepsilon \qquad \text{for all } k.$$

Fix $\varepsilon > 0$. Since the function $k(1 + y^{(b-1)/b})/y \to 0$ as $y \to 0$, we can find a $y_\varepsilon > 0$ such that $\tilde{k}(1 + y^{(b-1)/b}) \leq \frac{\varepsilon}{\psi e^{K_r T}} y$ for all $y \geq y_\varepsilon$. Letting $K = \tilde{k}(1 + y_\varepsilon^{(b-1)/b})$, we then have

$$\begin{aligned}
\int_{|X_{n_k}|>K} |X_{n_k}(t,\omega)| \, d(\lambda \times P) \; &= \; \int_{\psi e^{K_r}\xi_0\eta_{n_k} > y_\varepsilon} \tilde{k}\left(1 + (\psi e^{K_r}\xi_0(t)\eta_{n_k}(t))^{\frac{b-1}{b}}\right) d(\lambda \times P) \\
&\leq \; \frac{\varepsilon}{\psi e^{K_r}T} \int_{[0,T)\times\Omega} \psi e^{K_r}\xi_0(t,\omega)\eta_{n_k}(t,\omega) \, d(\lambda \times P) \; = \; \varepsilon.
\end{aligned}$$

□

The following lemma, which uses the theory of conjugate duality in Rockafellar (1974, 1975), is the critical step in establishing that the infimum in (P^{**}) is attained.

Lemma B6. *Under conditions (a)–(c) of Theorem 4, the sets*

$$\left\{\eta \in H : \hat{J}_\psi(\eta) \leq a\right\}$$

are weakly compact in $\mathcal{M}(Q_0)$ for all $a \in \mathbb{R}$.[12]

PROOF. Let λ^* denote the measure on $[0,T]$ equal to the Lebesgue measure on $[0,T)$ plus a point mass at T, let $v = \lambda^* \times Q_0$, and consider the spaces $L^1(\Omega \times [0,T], \mathcal{O}, v)$ and $L^\infty(\Omega \times [0,T], \mathcal{O}, v)$ of integrable and essentially bounded optional processes, respectively.[13] Then $H \subset L^1(v)$. We will take the norm topology on $L^1(v)$ and the L^1-Mackey topology on $L^\infty(v)$, so that they become paired spaces in the sense of Rockafellar (1974, p. 13).[14] Let $f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be defined by

$$f(t,\omega,x) = \begin{cases} -\xi_0(t,\omega)^{-1} u\Big(\tilde{y}(t,\omega) - \psi^{-1} B(t,\omega) x, t\Big), & \text{if } t < T,\ x \leq \psi B(t,\omega)^{-1} \tilde{y}(t,\omega) \\ 0, & \text{if } t = T,\ |x| \leq 1 \\ \infty, & \text{otherwise} \end{cases}$$

and let $f^*(t,\omega,z) = \sup_{x \in \mathbb{R}}[xz - f(t,\omega,x)]$ denote the convex conjugate of f. It is then immediately verified that

$$f^*(t,\omega,z) = \begin{cases} \xi_0(t,\omega)^{-1} \tilde{u}\Big(\psi B(t,\omega)^{-1} \xi_0(t,\omega) z, t\Big) + \psi B(t,\omega)^{-1} z \tilde{y}(t,\omega), & \text{if } t < T,\ z \geq 0 \\ |z|, & \text{if } t = T \\ \infty, & \text{otherwise,} \end{cases}$$

so that the integral functional $I_{f^*} : L^1(v) \to \mathbb{R} \cup \{\infty\}$ defined by

$$I_{f^*}(\eta) = \int_{\Omega \times [0,T]} f^*(t,\omega,\eta(t,\omega))\, dv$$

equals $\hat{J}_\psi + 1$ on H. Define the integral functional $I_f : L^\infty(v) \to \mathbb{R} \cup \{\infty\}$ similarly.

Since $I_f(0) < \infty$ (because $\tilde{y}$ is bounded below away from zero) and there exists by assumption an $\eta \in H$ such that $I_{f^*}(\eta) = \hat{J}_\psi(\eta) + 1 < \infty$, it follows from Theorem 3C and Corollary 3D in Rockafellar (1975) that I_f is lower semicontinuous in the L^1-weak topology on $L^\infty(v)$, and that $I_f^* = I_{f^*}$ on $L^1(v)$, where I_f^* denotes the convex conjugate of I_f.

Next, we intend to show that I_f is bounded above on a neighborhood of 0. By assumption (b) of Theorem 4, there exists an $\varepsilon > 0$ such that $\tilde{y}/B = y/B + W_0/T > \varepsilon$, $(\lambda \times P)$-a.e. (and hence $(\lambda \times Q_0)$-a.e.), and without loss of generality we can assume that $\psi\varepsilon \leq 1$. Let

$$\alpha = -\int_0^T u\Big(\frac{\varepsilon}{2} e^{-K_r}, t\Big)\, dt < \infty,$$

[12]A set $H_0 \subset \mathcal{M}(Q_0)$ is *weakly compact* if the set $\{\eta(T) : \eta \in H_0\}$ is compact in the weak topology induced by $L^\infty(Q_0)$ on $L^1(Q_0)$.

[13]The *optional sigma-field* $\mathcal{O}$ is the sigma-field on $[0,T]$ generated by the adapted right-continuous processes. An *optional process* is a process measurable with respect to $\mathcal{O}$.

[14]The L^1*-Mackey topology* on $L^\infty(v)$ is the finest locally convex topology such that the dual space of $L^\infty(v)$ is $L^1(v)$.

where K_r is the constant of (2), and consider the level set $C = \{x \in L^\infty(v) : I_f(x) \leq \alpha\}$. Clearly, it is enough to show that $0 \in \text{int}(C)$. First, note that C is convex and L^1-weakly closed by the convexity and weak lower semicontinuity of I_f, and hence closed in the L^1-Mackey topology (since this topology is finer than the L^1-weak topology). Moreover, it is immediately verified that $\{x \in L^\infty(v) : \|x\|_{L^\infty(v)} < \psi\varepsilon/2\} \subset C$, and hence $0 \in \text{core}(C)$.[15] Since $L^1(v)$ is weakly complete (Dunford and Schwartz (1988), Theorem IV.8.6), $L^\infty(v)$ with the L^1-Mackey topology is a "barrelled space" (Schaefer (1986), Theorem IV.5.5), and hence $\text{int}(C) = \text{core}(C)$ (Rockafellar (1974), p. 31). Therefore, $0 \in \text{int}(C)$.

Since I_f is bounded above on $\text{int}(C)$, it follows by Theorem 10(b) in Rockafellar (1974) that the level sets

$$\left\{\eta \in L^1(v) : I_{f^*}(\eta) \leq a\right\}$$

are weakly compact for all $a \in \mathbb{R}$, and hence weakly sequentially compact (Dunford and Schwartz (1988)–henceforth DS—Theorem V.6.1).[16] Clearly, this implies that the sets

$$\left\{\eta \in H : I_{f^*}(\eta) \leq a\right\} = \left\{\eta \in H : \hat{J}_\psi(\eta) \leq a - 1\right\}$$

are weakly sequentially compact in $\mathcal{M}(Q_0)$. Finally, since the latter sets are convex and closed in $\mathcal{M}(Q_0)$ by Lemmas B2 and B5, they are weakly closed (DS, Theorem V.3.13), and hence weakly compact (DS, Theorem V.6.1).

□

We now easily obtain the following result.

Lemma B7. *Under conditions (a)–(c) of Theorem 4, the infimum in (P^{**}) is attained by some $\eta^* \in H$.*

PROOF. Let $\{\eta_n\} \subset H$ be a sequence such that

$$\hat{J}_\psi(\eta_n) \downarrow \alpha = \inf_{\eta \in H} \hat{J}_\psi(\eta)$$

and consider the level sets $\{\eta \in H : \hat{J}_\psi(\eta) \leq \hat{J}_\psi(\eta_n)\}$. By the previous lemma, these sets are weakly compact. Since a nest of nonempty compact sets has nonempty intersection, it follows that the set $\{\eta \in H : \hat{J}_\psi(\eta) \leq \alpha\}$ is nonempty, and hence that the infimum is attained.

□

PROOF OF THEOREM 4. Let $\eta^* \in H$ attain the infimum in (P^{**}) (such a η^* exists by Lemma B7). Then $\eta^*(t) > 0$ for all $t \in [0, T)$ by Lemma B4 and hence $\xi_0\eta^* = \xi_{\nu^*}$ for some $\nu^* \in \mathcal{N}^*$ by Lemma B3. By the definition of $\hat{J}_\psi$ and Lemma B3 again, we conclude that $J(\psi, \nu^*) = \inf_{\nu \in \mathcal{N}^*} J(\psi, \nu)$. Since $\psi \in (0, \infty)$ was arbitrary, it follows from Lemma B1 that (P^*) has a solution.

[15] If C is a subset of a vector space X, then

$$\text{core}(C) = \left\{y \in C : \forall x \in X, \exists \varepsilon > 0 \text{ s.t. } y + \delta x \in C \text{ for } |\delta| \leq \varepsilon\right\}.$$

[16] A subset A of a topological vector space X is *weakly sequentially compact* if every sequence $\{x_n\} \subset A$ contains a subsequence which converges weakly to a point in X.

Finally, we show that (32) is satisfied. Taking $c = f(y,t)$, condition (d) of Theorem 4 gives

$$yf(y,t) \leq a + (1-b)u(f(y,t),t).$$

Subtracting $(1-b)yf(y,t)$ from both members and recalling the definition of $\tilde{u}$ shows that

$$yf(y,t) \leq \frac{a}{b} + \frac{1-b}{b}\tilde{u}(y,t).$$

Therefore,

$$\mathrm{E}\left[\int_0^T \pi_{\nu^*}(t) f(\psi^* \pi_{\nu^*}(t), t)\, dt\right] \leq \frac{a}{b\psi^*} + \frac{1-b}{b\psi^*}\,\mathrm{E}\left[\int_0^T \tilde{u}(\psi^* \pi_{\nu^*}(t), t)\, dt\right] < \infty,$$

where the last inequality follows from the fact that $|J(\psi^*, \nu^*)| < \infty$ and y/B is bounded. □

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