NUMERICAL EVALUATION OF THE CRITICAL PRICE AND AMERICAN OPTIONS

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Abstract

An approximate solution to the American put value is proposed and implemented numerically. Relaxation techniques enable the critical price to be determined with high accuracy. The method uses a modification of the quadratic approximation of MacMillan and Barone-Adesi and Whaley which gives an expression for the critical price. Numerical experimentation and iterative methods quickly provide highly accurate solutions.

Key Words: American options, critical price, numerical approximations.

1. Introduction

The valuation of American options has challenged financial economists for many years. Mathematically the problem is related to optional stopping and free boundary problems, situations which rarely have explicit, closed form solutions. Earlier theoretical work on American options includes the articles by McKean (1965), van Moerbeke (1976), Bensoussan (1984), and Karatzas (1988). Papers discussing numerical solutions include Brennan and Schwartz (1977) and Jaillet, Lamberton and Lapeyre (1990). In this paper we initially propose a solution to the American put which is a modification of the approximations of MacMillan (1986) and Barone-Adesi and Whaley (1987). The proposed solution is itself only an approximation but relaxation techniques and numerical methods enable the critical price, or free boundary, to be determined relatively quickly and to a high degree of accuracy. This is the paper's contribution. Knowledge of the critical price enables the option values to be determined, as well as related hedging strategies. Related results for bond options will be presented in a later article.

2. American Put Options

For discussions of the American put see the articles by Carr, Jarrow and Myneni (1992), Jacka (1991) and Kim (1990). The usual assumptions and analysis lead to the situation where, under an equivalent probability measure, the asset price is a process described by the equation

$$dS = S(t)\big(bdt + \sigma dw(t)\big). \tag{2.1}$$

Here b and σ are constants and $w(t)$ is a standard Brownian motion on $(\Omega, \mathcal{F}, P)$, $0 \le t \le t^*$, where t^* is the expiration time. The value at time

$t \leq t^*$ of the American put option is the Snell envelope

$$V_A(S,t) = \sup_{\tau \in [t,t^*]} E[e^{-r(\tau - t)}(X - S(\tau))^+|S]. \qquad (2.2)$$

Here we use the notation

$$x^+ = x \quad \text{if} \quad x \geq 0$$
$$= 0 \quad \text{otherwise.}$$

Here $S(t)$ is the asset price at time t, X is the strike price and the supremum is taken over all stopping times τ which take values in $[t, t^*]$.

Associated with V_A is the continuation region

$$C := \{(S,t) : V_A(S,t) > (X - S)^+\}. \qquad (2.3)$$

Within the continuation region it is better not to exercise the put option immediately because it is possible, on average, to obtain a greater amount by exercising the option at a later time. For a fixed t write

$$S^*(t) = \sup \{S : S \notin C\}. \qquad (2.4)$$

Definition 2.1. $S^*(t)$ is the critical price.

It is shown in Jacka that if $S > S^*(t)$ then $(S,t) \in C$, while if $S \leq S^*(t)$ then $(S,t) \notin C$.

Define the stopping time τ^* for $(S,t) \in C$ by

$$\tau^* = \inf \{s : S(s) \leq S^*(s)\} \wedge t^*. \qquad (2.5)$$

Then τ^* is the optimal stopping time, that is, it achieves the supremum in (2.2). V_A and S^* are the unique solutions of the free boundary problem:

$$LV := \frac{\partial V}{\partial t} - rV + bS\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} = 0 \tag{2.6}$$

$$\begin{aligned} V(S,t^*) &= (X-S)^+, \qquad V(\infty,t) = 0, \\ V(S^*,\tau^*) &= (X-S^*)^+, \quad \frac{\partial V}{\partial S}\,\Big|_{S=S^*} = -1. \end{aligned} \tag{2.7}$$

Now, the European put can be exercised only at the expiration time t^* and has a value

$$V_E(S,t) = E[e^{-r(t^*-t)}(X - S(t^*))^+|S]. \tag{2.8}$$

V_E is a solution of the parabolic equation $LV = 0$ and has a final boundary condition $V_E(S,t^*) = (X - S(t^*))^+$.

Write

$$d_1(S,t) = \{\ell n\,(\frac{S}{X}) + (b + \frac{\sigma^2}{2})(t^* - t)\}/\sigma\,\sqrt{t^* - t} \tag{2.9}$$

and $d_2(S,t) = d_1(S,t) - \sigma\,\sqrt{t^* - t}$. $N(\cdot)$ will denote the standard normal distribution. The Black-Scholes formula gives

$$V_E(S,t) = Xe^{-r(t^*-t)}N\big(-d_2(S,t)\big) - Se^{(t-r)(t^*-t)}N\big(-d_1(S,t)\big). \tag{2.10}$$

Clearly the American put value is greater than the corresponding European put value. The early exercise premium is

$$\varepsilon(S,t) := V_A(S,t) - V_E(S,t). \tag{2.11}$$

It is shown in Jacka that, with $S^*(s)$ the critical price defined by (2.4),

$$\varepsilon(S,t) = \int_t^{t^*} \left(rXe^{-r(s-t)}N(-h_2) - (r-b)Se^{-(r-b)(s-t)}N(-h_1)\right)ds \qquad (2.12)$$

where $h_1 = \{\ell\text{n}\,(\frac{S}{S^*(s)}) + (b+\frac{\sigma^2}{2})(s-t)\}/\sigma\,\sqrt{s-t}$ and $h_2 = h_1 - \sigma\,\sqrt{s-t}$.

The boundary conditions (2.7) imply

$$V_E(S^*,t) + \varepsilon(S^*,t) = X - S^* \qquad (2.13)$$

$$\left(\frac{\partial V_E}{\partial S}\right)_{S=S^*} + \left(\frac{\partial \varepsilon}{\partial S}\right)_{S=S^*} = -1. \qquad (2.14)$$

3. An approximation for the Early Exercise Premium

Equations (2.5) and (2.6) provide an integral equation for $S^*(t)$ which could be solved backwards in time from t^*. This procedure is, however, exceptionally involved with the value $S^*(t)$ being given in terms of an integral involving all later values $S^*(s)$, $t \leq s \leq t^*$. We initially suggest instead an approximate form for $\varepsilon(S,t)$, namely:

$$\widetilde{\varepsilon}(S,t) = A(t)\left(\frac{S}{S^*(t)}\right)^{q(t)}. \qquad (3.1)$$

Here $A(t)$ and $q(t)$ are functions of t and are to be determined. Substituting $\widetilde{\varepsilon}$ for ε in (2.6) and (2.7) we require

$$V_E\big(S^*(t),t\big) + A(t) = X - S^*(t) \qquad (3.2)$$

$$-1 = -e^{(b-r)(t^*-t)}N\big(-d_1(S^*(t),t)\big) + \frac{A(t)q(t)}{S^*(t)}\,. \qquad (3.3)$$

However, we would also like $L\widetilde{\varepsilon} = 0$, that is

$$\frac{1}{2}\sigma^2 q(t)(q(t)-1)A(t)(\frac{S}{S^*(t)})^{q(t)} - rA(t)(\frac{S}{S^*(t)})^{q(t)} + A(t)bq(t)(\frac{S}{S^*(t)})^{q(t)} + \frac{\partial}{\partial t}\,(A(t)(\frac{S}{S^*(t)})^{q(t)}) = 0. \qquad (3.4)$$

Now

$$\frac{\partial}{\partial t}\,(A(t)(\frac{S}{S^*(t)})^{q(t)}) = \frac{dA(t)}{dt}\,(\frac{S}{S^*(t)})^{q(t)} - \frac{dS^*(t)}{dt}(\frac{A(t)q(t)}{S}(\frac{S}{S^*(t)})^{q(t)+1}) + \frac{dq(t)}{dt}\,A(t)(\frac{S}{S^*(t)})^{q(t)}\ell\text{n}\,(\frac{S}{S^*(t)}). \qquad (3.5)$$

Substituting (3.5) into (3.4) and dividing by $A(t)(\frac{S}{S^*(t)})^{q(t)}$ this would imply

$$\frac{1}{2}\,\sigma^2 q(t)(q(t)-1) - r + bq(t) + [\frac{1}{A(t)}\,\frac{dA(t)}{dt} - \frac{q(t)}{S^*(t)}\,\frac{dS^*(t)}{dt}] + \ell\text{n}\,(\frac{S}{S^*(t)})\,\frac{dq(t)}{dt} = 0. \qquad (3.6)$$

This equation implies that q is not independent of S, so $\varepsilon(S,t)$ is not of the form (3.1). However, a useful approximation is obtained by neglecting the last term of (3.6). That is, we consider $q(t)$ to be a solution of

$$\frac{1}{2}\sigma^2 q(t)(q(t)-1) - r + bq + [\frac{1}{A(t)}\,\frac{dA(t)}{dt} - \frac{q(t)}{S^*(t)}\,\frac{dS^*(t)}{dt}] = 0. \qquad (3.7)$$

This approximation is reasonable when $\ell\text{n}\,(\frac{S}{S^*(t)})\,\frac{dq(t)}{dt}$ is small. This is the case when $\frac{dq}{dt}$ is small, (at long maturities) or in a neighborhood of $S^*(t)$.

From (3.2) we see

$$\frac{dA(t)}{dt} = [e^{(b-r)(t^*-t)}N(-d_1(S^*(t),t)) - 1]\,\frac{dS^*(t)}{dt} - \frac{\partial V_E(S^*(t),t)}{\partial t}\,.$$

Using (3.3)

$$\frac{1}{A(t)}\,\frac{dA(t)}{dt} - \frac{q(t)}{S^*(t)}\,\frac{dS^*(t)}{dt} = -\,\frac{1}{A(t)}\,\frac{\partial V_E\big(S^*(t),t\big)}{\partial t}\,. \tag{3.8}$$

Write $g(t) = \frac{1}{A(t)}\,\frac{\partial V_E\left(S^*(t),t\right)}{\partial t}$,

$$M = \frac{2r}{\sigma^2}\,,\quad N = \frac{2b}{\sigma^2}\,,\quad G(t) = \frac{2g(t)}{\sigma^2} \tag{3.9}$$

so that (3.7) becomes

$$q(t)^2 + (N-1)q(t) - \big(M + G(t)\big) = 0. \tag{3.10}$$

To satisfy the boundary conditions at infinity for puts we consider the quadratic root

$$q(t) = \frac{1}{2}\,\big(1 - N - \sqrt{(1-N)^2 + 4(M + G(t))}\,. \tag{3.11}$$

4. Numerical Experimentation

By attempting to find an expression for the early exercise premium of the form $\widetilde{\varepsilon}(S,t) = A(t)\,\left(\frac{S}{S^*(t)}\right)^{q(t)}$, and by dropping the last term of (3.6), we have obtained three equations, (3.2), (3.3) and (3.10), in the three unknowns: $A(t), q(t), S^*(t)$. Write these in the form

$$S^*(t) = \frac{\big(X - V_E(S^*(t))\big)q(t)}{-1 + q(t) + e^{(b-r)(t^*-t)}N\big(-d_1(S^*(t),t)\big)} \tag{4.1}$$

$$A(t) = -V_E\big(S^*(t)\big) - S^*(t) + X \tag{4.2}$$

$$q(t)^2 + (N-1)q(t) - \big(M + G(t)\big) = 0. \tag{4.3}$$

For a fixed value of t these equations can be solved using the following procedure:

1) give a trial value of $S^*(t)$
2) calculate the new $A(t)$ from (4.2)
3) calculate the new $q(t)$ from (4.3)
4) calculate the new $S^*(t)$ from (4.1).

Using the new value for $S^*(t)$ the procedure is repeated, generating at the n^{th} iteration values $(S^*(t)_n, A(t)_n, q(t)_n)$. If for some n

$$|S^*(t)_n - S^*(t)_{n-1}| + |A(t)_n - A(t)_{n-1}| + |q(t)_n - q(t)_{n-1}| \leq 10^{-4}$$

the procedure stops and $S^*(t)_n, A(t)_n, q(t)_n$ are taken to be the values of $S^*(t), A(t), q(t)$. (Originally the procedure was stopped when the difference was less than 10^{-8}, but the increased accuracy of the final result was negligible. An error message was to be printed if the cut-off criterion was not met after 10^4 iterations.) Once the iteration is stopped for some value of t the procedure begins at the next time value. The investigation was empirical. The variation between iterations quickly became small.

This procedure, therefore, provides us with values of $S^*(t)$, $0 \leq t \leq t^*$. These, however, can only be approximate values because they are calculated using the approximation $\tilde{\varepsilon}(S,t)$ for the early exercise premium.

As a measure of accuracy of the approximation the integral expression (2.12) for $\varepsilon(S^*(t),t)$ can be calculated and compared with the value of $\tilde{\varepsilon}(S^*(t),t) = A(t)$ given by the iterative scheme. (All integrals were evaluated using Bode's rule, cf. Dahlquist and Bjorck (1974).) For the scheme described above $\varepsilon(S^*(t),t)$ was not equal to $A(t)$, though the percentage error $(\varepsilon(S^*(t),t) - A(t))/A(t)$ was small, of the order of 2%. It was clear that, with values of $S^*(t), A(t), q(t)$ determined by the iterative scheme, $\tilde{\varepsilon}(S,t)$ is only an approximate solution; the question was, how might this approximate solution be improved? In the above development the final term of (3.6) was dropped; consequently the function $G(t)$ in the quadratic equation (3.9) is only an approximation. Rather than adding a correction term a factor λ was introduced, a 'relaxation constant,' so that (3.9) was replaced by

$$q(t)^2 + (N-1)q(t) - \big(M + \lambda G(t)\big) = 0. \tag{4.4}$$

The iterative scheme, using steps 1) to 4), was then followed for equations (4.1), (4.2) and (4.4) with different values of $\lambda \in [1,2]$. The value of λ was chosen so that $\varepsilon\big(S^*(t),t\big) = A(t)$.

Three cases were considered: $r = b$, $r = \frac{b}{2}$ and $b = 0$. In each case σ varied in the range 0.2 to 0.4 and r from 0.04 to 0.20.

In the first case, when $r = b$, λ was determined empirically by

$$\lambda = 1.2952 + 4.3338 \times 10^{-2} M - 4.6591 \times 10^{-3} M^2 + 2.1452 \times 10^{-4} M^3.$$

Here again $M = 2r/\sigma^2$. It appeared that $\big(\varepsilon(S^*(t),t) - A(t)\big)/A(t)$ was extremely sensitive to changes in λ. The error dropped to well below 1% except for very short times to expiration, for example when only 3 months remained of a 5 year option. This was probably due to computational problems near $t = 0$ rather than problems in the method, because for short times, using the same λ, finer grids reduced the error.

In the second case, when $b = 0$, the following empirical formula for λ gave good results:

$$\lambda = 1.2495 - (4.15)10^{-2}\sigma. \tag{4.5}$$

In the third case the formula chosen was:

$$\lambda = 1.227 + 0.12066M - 4.2737 \cdot 10^{-2}M^2 + 5.453 \cdot 10^{-3}M^3. \tag{4.6}$$

Taking $\lambda = 1$ we found that $S^*(t)$ crossed its asymptotic value. A conjecture is that the correct value of λ is the smallest value such that $S^*(t)$ is indeed monotone.

We illustrate explictly the above procedures by considering two typical cases. We first consider the case with $\sigma = .3,\ b = .08$. Our approximating cubic polynomial expression then yields $\lambda = 1.3587$. Figure 1 shows the plot of S^* against time for a period of 20 years. Observe that S^* is monotonically decreasing to its asymptotic value at $t = \infty$. More detailed calculations are shown for the period up to $t = 5$ years in Table 1. The second column clearly shows once again the monotonicity of S^*. The third and fourth columns show the values of ε (denoted by E here) and A respectively. The last column shows the percentage error of $(E - A)/A$, and we note that for periods longer than 6 months, this error is less than 1%. Indeed for periods longer than 2 years, the percentage error is less than one per thousand. Table 2 is similar to Table 1, except that now $b = 0.0$, and λ is calculated by the analytical formula suggested above. In this case the percentage error of $(\varepsilon - A)/A$ are again smaller than 1%. The case $b = r/2$, in Table 3, shows again very small errors.

We remark that the above calculations were performed on an IBM 320H RISC workstation and typically required just a few minutes to complete. This shows that our procedure may be carried out if access to relatively modest computing facilities is ensured.

5. Conclusions

We proposed a new approximate solution to the American put value, and implemented this procedure numerically. Our approach involved the introduction of a new (relaxation) parameter λ in a modification of the approximations of MacMillan and Barone-Adesi and Whaley. This parameter is chosen by requiring the additional condition $\varepsilon(S^*(t), t) = A(t)$ hold. We illustrated these ideas by explicitly considering two typical situations, and found that the error of $(\varepsilon - A)/A$ was considerably smaller than was the case for the case $\lambda = 1$. We also found that S^* was montone in t, unlike the situation when $\lambda \equiv 1$. Finally, rapid implementation of the calculations only required access to a work station.

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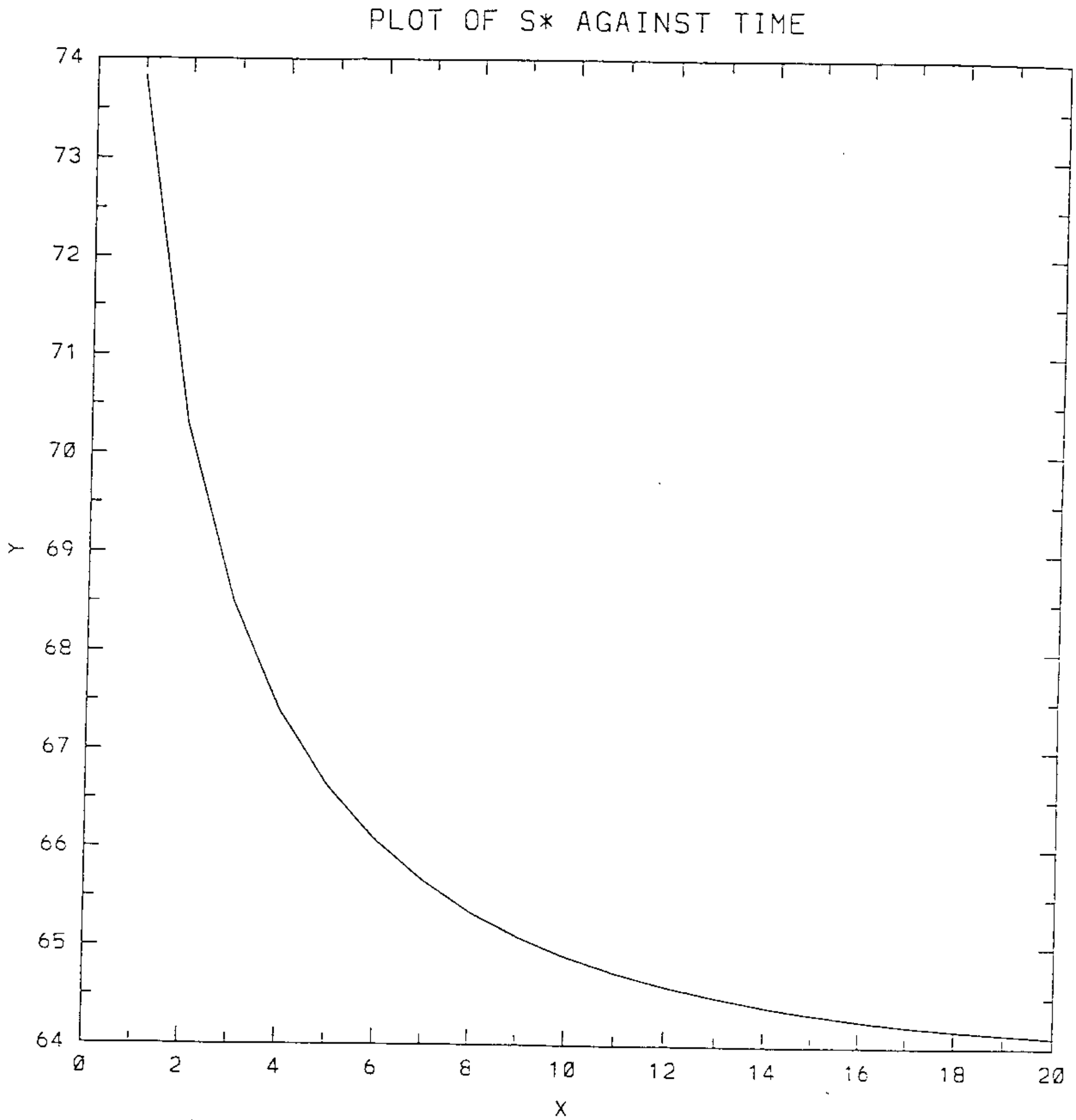

PLOT OF S* AGAINST TIME
74
73
72
71
70
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68
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66
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64
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10
12
14
16
18
20
X

Fig. 1

TABLE 1

SIGMA = .3000 B = .0800 R = .0800

LAMBDA = 1.3587

TIME	S*	E(T*)	A(T*)	VALUE	%(E-A)/A
.2500	81.3583	1.2791	1.2925	18.6283	-1.0355
.5000	77.6120	2.3972	2.4182	22.3670	-.8693
.7500	75.3826	3.4310	3.4574	24.5909	-.7643
1.0000	73.8237	4.4257	4.4328	26.1692	-.1608
1.2500	72.6448	5.3449	5.3564	27.3438	-.2139
1.5000	71.7099	6.2209	6.2359	28.2751	-.2406
1.7500	70.9445	7.0586	7.0766	29.0374	-.2546
2.0000	70.3032	7.8818	7.8827	29.6960	-.0108
2.2500	69.7561	8.6531	8.6572	30.2398	-.0476
2.5000	69.2829	9.3956	9.4027	30.7100	-.0756
2.7500	68.8689	10.1116	10.1215	31.1212	-.0980
3.0000	68.5033	10.8198	10.8152	31.5013	.0424
3.2500	68.1779	11.4867	11.4854	31.8234	.0112
3.5000	67.8862	12.1317	12.1336	32.1119	-.0154
3.7500	67.6234	12.7559	12.7608	32.3717	-.0388
4.0000	67.3852	13.3754	13.3683	32.6219	.0531
4.2500	67.1685	13.9604	13.9568	32.8350	.0254
4.5000	66.9705	14.5275	14.5275	33.0296	.0006
4.7500	66.7889	15.0776	15.0809	33.2078	-.0220
5.0000	66.6219	15.6246	15.6180	33.3848	.0425

TABLE 2

SIGMA = .4000 B = .0000 R = .1200
LAMBDA = 1.2329

TIME	S*	E(T*))	A(T*)	VALUE	%(E-A)/A
.2500	68.0897	.7736	.7764	31.9075	-.3710
.5000	62.3248	1.7808	1.7851	37.6709	-.2415
.7500	59.0236	2.8515	2.8563	40.9716	-.1690
1.0000	56.7810	3.9474	3.9521	43.2144	-.1180
1.2500	55.1256	5.0517	5.0558	44.8704	-.0797
1.5000	53.8408	6.1551	6.1582	46.1561	-.0500
1.7500	52.8091	7.2519	7.2539	47.1890	-.0266
2.0000	51.9601	8.3385	8.3391	48.0392	-.0082
2.2500	51.2482	9.4122	9.4116	48.7524	.0062
2.5000	50.6423	10.4714	10.4696	49.3595	.0174
2.7500	50.1205	11.5147	11.5117	49.8825	.0259
3.0000	49.6666	12.5413	12.5373	50.3375	.0322
3.2500	49.2684	13.5506	13.5456	50.7366	.0366
3.5000	48.9165	14.5420	14.5362	51.0892	.0394
3.7500	48.6037	15.5152	15.5089	51.4026	.0409
4.0000	48.3241	16.4701	16.4633	51.6827	.0413
4.2500	48.0729	17.4065	17.3994	51.9342	.0406
4.5000	47.8464	18.3244	18.3172	52.1608	.0391
4.7500	47.6413	19.2238	19.2167	52.3658	.0370
5.0000	47.4549	20.1048	20.0979	52.5520	.0342

TABLE 3

SIGMA = .4000 B = .0600 R = .1200
LAMBDA = 1.2915

TIME	S*	E(T*))	A(T*)	VALUE	%(E-A)/A
.2500	73.4557	1.3277	1.3349	26.5371	-.5402
.5000	68.3948	2.6388	2.6477	31.5963	-.3386
.7500	65.4637	3.9131	3.9218	34.5275	-.2240
1.0000	63.4568	5.1494	5.1569	36.5357	-.1453
1.2500	61.9659	6.3488	6.3544	38.0285	-.0876
1.5000	60.8024	7.5123	7.5156	39.1943	-.0441
1.7500	59.8632	8.6414	8.6423	40.1358	-.0108
2.0000	59.0867	9.7373	9.7359	40.9147	.0147
2.2500	58.4326	10.8012	10.7975	41.5711	.0342
2.5000	57.8734	11.8343	11.8286	42.1324	.0488
2.7500	57.3897	12.8377	12.8301	42.6180	.0594
3.0000	56.9671	13.8123	13.8031	43.0422	.0668
3.2500	56.5948	14.7591	14.7486	43.4158	.0716
3.5000	56.2644	15.6790	15.6674	43.7472	.0741
3.7500	55.9696	16.5729	16.5605	44.0428	.0748
4.0000	55.7049	17.4415	17.4286	44.3080	.0740
4.2500	55.4663	18.2856	18.2725	44.5468	.0718
4.5000	55.2502	19.1059	19.0928	44.7629	.0686
4.7500	55.0538	19.9032	19.8904	44.9590	.0645
5.0000	54.8746	20.6782	20.6659	45.1377	.0596

Captions

FIGURE 1. S^* as a function of t for $\sigma = .3$, $b = .08$, $r = .08$, and $\lambda = 1.3587$.

TABLE 1. Detailed analyses of results for $\sigma = .3$, $b = .08$, $r = .08$ and $\lambda = 1.3587$ for a five year period.

TABLE 2. Detailed analyses of results for $\sigma = .4$, $b = 0.0$, $r = .12$ for a five year period.

TABLE 3. Detailed analyses of results for $\sigma = .4$, $b = 0.06$, $r = .12$ for a five year period.