HOW FAR APART CAN TWO RISKLESS INTEREST RATES BE? (ONE MOVES, THE OTHER ONE DOES NOT)

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How Far Apart Can Two Riskless Interest Rates Be?[***]

(One Moves, the Other One Does Not)

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Abstract

In the presence of transactions costs, no matter how small, arbitrage activity does not necessarily render equal all riskless rates of return. When two such rates follow stochastic processes, it is not optimal immediately to arbitrage out any discrepancy that arises between them. The reason is that immediate arbitrage would induce a definite expenditure of transactions costs whereas, without arbitrage intervention, there exists some, perhaps sufficient, probability that these two interest rates will come back together without any costs having been incurred. Hence, one can surmise that at equilibrium the financial market will permit the coexistence of two riskless rates which are not equal to each other. For analogous reasons, randomly fluctuating expected rates of return on risky assets will be allowed to differ even after correction for risk, leading to important violations of the Capital Asset Pricing Model. The combination of randomness in expected rates of return and proportional transactions costs is a serious blow to existing frictionless pricing models.

Investors, who have to pay transactions costs, optimally rebalance their portfolios at points in times that are random and are not easily observable. The financial econometrician, instead, measures rates of return on financial assets over regular, fixed intervals in time. Investors compare the rates of return on assets over the forthcoming holding periods while the econometrician testing the validity of an asset pricing model, arbitrarily attempts to compare them over successive weeks, months or years.

We would like to know whether it is possible meaningfully to compare the rates of return on two otherwise similar assets when the rates are measured at regular intervals, while investors trade at random times. The question cannot be adressed without a model of the way in which investors choose to rebalance or not to rebalance their portfolios. We first consider the case of two riskless assets in a portfolio. Then we extend the analysis to risky, long-lived assets such as equities.

If two interest rates on deposits were to remain unequal forever, it would pay to arbitrage out their difference immediately, even if transactions costs must be incurred in so doing. In the absence of discounting, and in the absence of any costs for rolling over the deposits, the interest differential earned by the arbitrage would eventually outweigh any finite transactions costs incurred at the outset of the arbitrage operation.

If, however, the spread between the two rates fluctuates randomly, it may no longer pay to start an arbitrage. The interest differential may not last long enough to cover profitably the transactions costs. This basic idea was put forth originally in Baldwin (1990) who also showed that the problem mathematically resembles Dixit's (1989) problem of stochastic entry and exit. The purpose of the present paper is to re-formulate it and exploit it in the context of an optimal portfolio choice problem with transactions costs.

We examine the portfolio choice of an investor with given relative risk aversion who has access to two riskless investments with instantaneous returns (infinitesimal maturity). One of these brings a rate of interest which is constant over time while the other yields a rate which varies according to a stochastic process. The process incorporates a reversion force which, in the long run, pulls the second rate towards the first one. We approach this problem of portfolio choice in the manner of Dumas and Luciano (1991). For a given portfolio imbalance, the investors allow some gap between the two rates to survive; this gap is called "the hysteresis band". We are interested in the size of this gap. We intend to show that the gap is of a different order of magnitude than the transactions costs.

Because deposits are not forcibly refunded and can be rolled over costlessly, the period over which a given investor holds the deposit -- the "holding period" -- is a decision variable.[1] As smaller and smaller transactions costs are considered, the allowable spread measured over the holding period is gradually compressed but the anticipated optimal holding period also shrinks because smaller transactions make it less costly to switch from one asset to other. Depending on the rates at which these two variables approach zero, the allowable annualized quoted spread may become small slowly or quickly. We show that it becomes small at a cubic-root rate.

Later on, we consider an arbitrage between a riskless asset with a

[1]The analysis is not limited to bank deposits. In fact, it applies to all long lasting assets. Shares of stock that pay no dividend are automatically "rolled over" until the investor explicitly sells them. Section 3 will be devoted to the analysis of rates of returns on equities. The analysis could, but will not, be generalized to shares that pay a partial dividend. Bonds would require a separate study because they are 100% refunded at the maturity date. That is one "transaction" that is forced on the bondholder.

constant rate and a risky asset with a stochastic, mean-reverting conditionally expected rate of return. We conclude that the CAPM must be badly violated because of the existence of transactions costs. This conclusion contrasts with the final observations of Constantinides (1986) who holds the view that small transactions costs only produce small deviations from the CAPM. The difference in the results is traceable to the difference in the assumed behavior of the conditionally expected return on the risky asset. Constantinides considers an expected return which is constant; we consider a stochastic, mean reverting one.

Mean reversion in expected returns on stocks has been studied empirically by Fama and French (1988) among others.[2]

The paper is organized as follows. In Section 1 we solve the basic portfolio problem considered by Baldwin (1990) in which investors are constrained to investing their entire wealth in one riskless asset or the other; we measure the resulting gap in interest rates. In Section 2, we allow continuous adjustment of the portfolio while still considering only two riskless assets. In Section 3, we optimize a portfolio made up of one riskless asset with a constant rate and one risky asset with a mean reverting expected return; we evaluate the deviation from the costless CAPM. Section 4 presents a calibrated numerical illustration.

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[2] Fama and French have shown that long-holding period returns display mean reversion. The behavior of long-period returns is the combined result of short-period mean behavior and volatility behavior. In our model, short-period volatility is assumed constant.

1. THE CASE OF TWO RISKLESS ASSETS AND ALL OR NOTHING PORTFOLIO HOLDINGS

1.1 Problem formulation

Consider two assets. One of them has a constant riskless rate of return, which, without loss of generality in our context, we can set equal to zero. The other brings, over a small, fixed period of time, a rate of return, α, which is also riskless but which follows a mean-reverting stochastic process:

$$d\alpha = -\lambda\alpha dt + \sigma dz. \tag{1.1}$$

At any given time t, the dollar value of an investor's holding of the first asset is denoted x and the dollar value of his holding of the second asset is denoted y. Proportional transactions costs at the rate 1 - s are incurred when exchanging one asset into the other; these costs are proportional to the dollar value of the trade.

We seek an optimal portfolio policy in which the objective is to maximize the utility of terminal consumption at some later date T. The utility of terminal consumption is logarithmic so that the objective is stated as:

$$L(x, y, t; T) = \text{Max } E_t[\ell n(c_T)], \tag{1.2}$$

where: $c_T = x_T$.

In an attempt to discover a stationary optimal policy, we take T to infinity. Furthermore, we assume that the function L asymptotically exhibits linear growth, at some rate, β, to be determined:

$$L(x, y, \alpha, t; T) - \beta(T - t) \underset{T \to \infty}{\longrightarrow} J(x, y, \alpha). \tag{1.3}$$

In this section, we restrict the investor to holding all his wealth in the form of one asset or the other. Hence, the portfolio, apart from its size, can only be in one of two states. The only decision to be made at any given time is whether or not to switch the entire portfolio from one asset to the other. The investor will make that switch when α and the fixed rate are sufficiently far apart from each other. We seek the optimal choice of the trigger values $\underline{\alpha}$ and $\bar{\alpha}$ on each side of the constant value, 0, of the fixed rate of interest.

Exploiting the obvious homogeneity of the problem, define:

$$J(x, y, \alpha) = \ell n(x + y) + I(\theta, \alpha), \qquad \text{where: } \theta = y/(x + y). \tag{1.4}$$

In light of the restrictions imposed on the portfolio, θ is a binary variable which takes the value 0 or the value 1. For the remainder of this section we denote: $I_0(\alpha) = I(0, \alpha)$ and $I_1(\alpha) = I(1, \alpha)$. I_1 is the discounted utility function for a unit wealth that obtains when the investor is invested in the variable-rate asset; I_0 is the discounted utility for a unit of wealth that obtains when he is invested in the fixed interest-rate asset.

1.2 Probabilistic approach: backward induction

The relationship between the two functions I_1 and I_0 is given by equations (1.5) and (1.6) below. In equation (1.5), a backward, probabilistic reasoning gives the current value, $I_1(\alpha)$, of I_1. It is equal to:

. the value, $I_0(\underline{\alpha})$, of utility when the next switch out of the variable-rate asset will occur,

. plus the logarithm of the per unit loss in wealth produced by the transactions costs,

. plus the expected extra log-earnings, $E[\int_0^\tau \alpha_t dt \mid \alpha]$, produced by the variable-rate asset during the time until the switch,

. minus the effect of discounting over the expected time till the switch:

$$I_1(\alpha) = \ell ns + I_0(\underline{\alpha}) + E[\int_0^\tau \alpha_t dt \mid \alpha] - \beta E[\tau \mid \alpha]; \qquad \alpha > \underline{\alpha}. \tag{1.5}$$

Here, τ is the first-passage time of α to $\underline{\alpha}$. A similar backward reasoning, in (1.6), gives the current value, $I_0(\alpha)$, of the utility function I_0 when not invested:

$$I_0(\alpha) = \ell ns + I_1(\bar{\alpha}) - \beta E[\tau \mid \alpha]; \qquad \alpha < \bar{\alpha}. \tag{1.6}$$

In (1.6), τ is the first-passage time of α to $\bar{\alpha}$.

1.3 Equivalent analytical approach

Parenthetically, equations (1.5) and (1.6) can equivalently be obtained by imposing the condition that the value of the function L, defined in (1.2), executes a martingale process and subsequently introducing the changes of unknown function (1.3) and (1.4).

Hence, L has zero growth; J grows linearly at the unknown rate β; I_1 grows at the rate $\beta - \alpha$ because ℓny grows at the rate α when the portfolio is entirely made up of the variable-rate asset; similarly I_0 grows at the rate β.

These restrictions are written successively as follows:

$$- \lambda\alpha L_{\alpha} + (1/2)\sigma^2 L_{\alpha\alpha} = 0, \tag{1.7}$$

$$-\beta - \lambda\alpha J_{\alpha} + (1/2)\sigma^2 J_{\alpha\alpha} = 0, \tag{1.8}$$

$$-\beta + \alpha - \lambda\alpha I_1' + (1/2)\sigma^2 I_1'' = 0, \tag{1.9}$$

$$-\beta - \lambda\alpha I_0' + (1/2)\sigma^2 I_0'' = 0. \tag{1.10}$$

Equations (1.9) and (1.10), plus Value-matching boundary conditions, are equivalent to (1.5) and (1.6) by virtue of the Feynman-Kac formula, but they are more easily generalizable to the cases of sections 2 and 3 below than the probabilitic approach would be.

1.4 Solution

Returning to the backward, probabilistic approach, we first calculate the expected-earnings integral, $E[\int_0^{\tau}\alpha_t dt \mid \alpha]$, which appears in equation (1.5). An analogous calculation is performed in Karlin and Taylor (1981).[3] The answer in our case is:

$$E[\int_0^{\tau}\alpha_t dt \mid \alpha] = (\alpha - \underline{\alpha})/\lambda; \qquad \alpha > \underline{\alpha}. \tag{1.11}$$

For the purpose of interpretation, recall that the value of this integral is the expected cumulative earnings on the variable-rate asset until the next switch to the fixed-rate asset, which will occur at τ, the first time that α

[3] pages 196-197.

reaches $\underline{\alpha}$ from above.[4]

These expected earnings are always non negative, which may be surprising. In order to understand this result, it is important to keep in mind that the event $\alpha = \underline{\alpha}$ stops the sample paths over the which the integral is calculated. Hence, earnings that are below $\underline{\alpha}$ are censored out, whereas excursions of large positive earnings are included in the sum. It may also be surprising to the reader that these expected earnings increase as $\underline{\alpha}$ is set to a lower, presumably negative value. The answer to this puzzle is again that setting $\underline{\alpha}$ lower takes the earnings into a somewhat lower negative zone but also allows some additional, possibly long excursions into positive values that would otherwise be censored out.[5]

The calculation of the expected first-passage time of an Ornstein-Uhlenbeck process, $E[\tau \mid \alpha]$, is performed in Ricciardi and Sato (1988). In contrast to a standard Brownian motion, an Ornstein-Uhlenback process always has a finite expected hitting time. Ricciardi and Sato define a function ϕ_1 as follows:

$$\phi_1(\alpha) = (1/(2\lambda)) \sum_{n=1}^{\infty} [(2\sqrt{\lambda}/\sigma)\ \alpha]^n\ \Gamma(n/2)/n!, \tag{1.12}$$

which is easily programmed on a computer. Depending on the situation, $\phi_1(\alpha)$ or

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[4]We expect that $\underline{\alpha} < 0$.

[5]The reader might also wonder why the investor would ever want to switch to a zero-rate of return asset when the value of his earnings on the variable-rate asset till the next switch is currently expected to be negative. He will do it (see optimization below) when α is negative enough because that will further enhance his expected earnings. Earnings of the near future are negative; by switching he avoids those. The later switch back to the variable-rate asset will occur only when α is positive and large enough again ($\alpha = \tilde{\alpha}$).

$-\phi_1(-\alpha)$ serves to compute expected hitting time.

In equation (1.11), the expected earnings and the expected hitting time are inserted as follows:

$$I_1(\alpha) = \ell ns + I_0(\underline{\alpha}) + (\alpha - \underline{\alpha})/\lambda - \beta[-\phi_1(-\alpha) + \phi_1(-\underline{\alpha})], \tag{1.13}$$

while, in equation (1.6), the correct expression is:

$$I_0(\alpha) = \ell ns + I_1(\bar{\alpha}) - \beta[\phi_1(\bar{\alpha}) - \phi_1(\alpha)]. \tag{1.14}$$

The functions I_0 and I_1 given by (1.13) and (1.14) are solutions of the differential equations (1.9) and (1.10).

The values of I_1 and I_0 are easily eliminated between equations (1.13) and (1.14) to get a single equation:

$$0 = 2\ell ns + (\bar{\alpha} - \underline{\alpha})/\lambda - \beta[\phi_1(-\underline{\alpha}) - \phi_1(-\bar{\alpha}) + \phi_1(\bar{\alpha}) - \phi_1(\underline{\alpha})]. \tag{1.15}$$

This equation lends itself to a satisfactory interpretation. The sum of the first two terms of the right-hand side, $2\ell ns + (\bar{\alpha} - \underline{\alpha})/\lambda$, equals the expected net log-earnings (per unit of wealth) from a round-trip between the two assets: $2\ell ns$ is the per unit log-transactions costs and $(\bar{\alpha} - \underline{\alpha})/\lambda$ is the expected log-earnings during the part of the round trip where the varibale-rate asset is held. β is, of course, the expected rate of growth of wealth (or the expected increment of log utility per unit of time). β is multiplied by the term between square brackets which is simply equal to the expected duration of a round trip.

Hence, Equation (1.15) serves to calculate the expected rate of growth of wealth produced by a given ($\underline{\alpha}$, $\bar{\alpha}$) switching policy; it is equal to the expected net earnings during a round trip divided by the expected time that the round trip takes:

$$\beta = \frac{2\ell ns + (\bar{\alpha} - \underline{\alpha})/\lambda}{\phi_1(-\underline{\alpha}) - \phi_1(-\bar{\alpha}) + \phi_1(\bar{\alpha}) - \phi_1(\underline{\alpha})}. \tag{1.16}$$

1.5 Optimization

We need to write that the choice of $\underline{\alpha}$ and $\bar{\alpha}$ is optimal. Two Smooth-pasting conditions will accomplish that task. They are:

$$I_1'(\underline{\alpha}) = I_0'(\underline{\alpha}),$$

and:

$$I_1'(\bar{\alpha}) = I_0'(\bar{\alpha}),$$

otherwise written (based on (1.13) and (1.14)):

$$1/\lambda - \beta\ \phi_1'(-\underline{\alpha}) = \beta\ \phi_1'(\underline{\alpha}) \tag{1.17a}$$

and:

$$1/\lambda - \beta\ \phi_1'(-\bar{\alpha}) = \beta\ \phi_1'(\bar{\alpha}). \tag{1.17b}$$

It is easy to check that equations (1.17) are the straightforward first-order conditions of the maximization of the rate of growth, β, calculated as in (1.16), with respect to the choice of $\underline{\alpha}$ and $\bar{\alpha}$.

Because we have been able to express the functions I_1 and I_0 explicitly in (1.13) and (1.14), the difficult variational problem that we were facing has been reduced to the solution of a system of three algebraic equations in three real numbers (1.15, 1.17). Furthermore, in that system the unknown number β appears linearly so that it can be easily eliminated, leaving two equations in two unknowns. A further simplification is reached since one can easily show symmetry: $\underline{\alpha} = -\bar{\alpha}$. Hence, we are left with just one equation in one unknown number. That number must be found numerically.

1.6 <u>The hysteresis band</u>

FIGURE 1 GOES HERE

We have solved the system (1.15, 1.17) repeatedly for various values of s, from the value 1 downward, corresponding to increasing rates of transactions costs. Figure 1 shows the values of $\underline{\alpha}$ and $\bar{\alpha}$ against the value of s (outer curves).[6] The interesting result is that, as s --> 1, the slope of these curves approaches infinity. As the rate of transactions costs goes to zero, the spread that the investor allows to survive between the two riskless rates, goes to zero at a slower pace.

In the absence of transactions costs, arbitrage would force α to be pegged at the value 0. Transactions costs allow wide deviations from the arbitrage result. We can quantify the rate at which the range of deviations

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[6] We discuss the choice of parameter values in Section 4 below. For the time being, we focus on the qualitative features of the solution.

approaches zero:

Statement 1: As ℓns approaches zero, the range of fluctuations of α, over which no transaction takes place, approaches zero like $(\ell ns)^{1/3}$.

Proof: Call $z = \bar{\alpha} = -\underline{\alpha}$ the common unknown value of the interest rate bounds. Eliminate β between (1.15) and (1.16) or (1.17), to get:

$$-\ell ns = z/\lambda - \frac{1}{\lambda}\,\frac{\phi_1(z) - \phi_1(-z)}{\phi_1'(z) + \phi_1'(-z)}\,. \tag{1.18}$$

The expansion of $\phi_1(z)$ was provided in (1.12). The expansion of $\phi_1'(z)$ is:

$$\phi_1'(z) = (1/(2\lambda)) \sum_{n=1}^{\infty} (2\sqrt{\lambda}/\sigma)^n z^{n-1} \Gamma(n/2)/(n-1)!\,. \tag{1.19}$$

From these we can get the expansion of the right-hand side of (1.18). The result is:[7]

$$-\ell ns = \frac{1}{12\lambda}\left[\frac{2\sqrt{\lambda}}{\sigma}\right]^2 z^3, \tag{1.20}$$

or:

$$z = (-3\ \ell ns\ \sigma^2)^{1/3}. \tag{1.21}$$

Q.E.D.

Cubic rates of convergence for similar limit problems have been found in different contexts by Dixit (1990), Fleming *et al* (1990) and Svensson (1991).

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[7] $\Gamma(1/2)/\Gamma(3/2) = 2$.

Equation (1.21) shows that, for small transactions costs, two parameters only play a role in the determination of the hysteresis band, viz. σ and s. Mean reversion, λ, is not present. For finite transactions costs, the band remains very insensitive to the value of λ. Figure 1 displays the approximate values of $\underline{\alpha}$ and $\bar{\alpha}$ as given by (1.21) (outer curves); they virtually identical to the exact values over the range of transactions costs shown.

1.7 <u>The expected rate of growth and the expected frequency of transactions</u>

Equation (1.12) makes it plain that the expected time between two transactions is of the same order of magnitude, i.e. 1/3, as the barrier position itself. From the identity (1.15) and the leading term in the expansion (1.12) of the function ϕ_1, one can deduce that the limit of the expected rate of growth as transactions costs are taken to zero is equal to the following number β^*:

$$\beta^* = [2(\surd\lambda/\sigma)\Gamma(1/2)]^{-1}. \tag{1.22}$$

Substituting (1.21) into the first terms of (1.12), the expected duration of a round trip is approximately equal to:

$$\frac{(-6\ \ell ns\ \sigma^2)^{1/3}}{\lambda\ \beta^*} + \frac{1}{\lambda}\left[\frac{2\surd\lambda}{\sigma}\right]^3 \frac{\Gamma(3/2)}{6}(-3\ \ell ns\ \sigma^2). \tag{1.23}$$

Finally, the value of the expected growth rate in a neigborhood of s = 1 is given by:

$$\beta = \beta^* - 2\ \beta^*\ [\tfrac{1}{\lambda}\ (3\sigma^2)^{1/3}\ ]^{-1}(-\ \ell ns)^{2/3}. \tag{1.24}$$

As in the case of the boundary positions, these approximate expressions are extremely accurate over a range of transactions costs from zero to several percentage points. The assumption of "small" transactions costs allows the derivation of accurate analytical expressions.

2. THE CASE OF TWO RISKLESS ASSETS AND CONTINUOUS PORTFOLIO HOLDINGS

When the two asset holdings x and y are allowed to vary continuously, the state transition equations are:

$$dx = s\,d\ell - du; \tag{2.1a}$$

$$dy = \alpha y\,dt - d\ell + s\,du; \tag{2.1b}$$

$$d\alpha = -\lambda\alpha\,dt + \sigma\,dz. \tag{2.1c}$$

Here u, and ℓ, are two nondecreasing stochastic processes which increase only when (respectively) some amount of fixed-rate, or variable-rate asset is sold. We call $\underline{\alpha}(\theta)$ and $\bar{\alpha}(\theta)$ the upper and lower trigger values of α which depend on the current composition, $\theta = y/(x + y)$, of the portfolio.

Between transactions, $dx = 0$ and $dy = \alpha y\,dt$ so that the portfolio composition, θ, satisfies the following time-differential equation:

$$d\theta = \alpha\theta(1 - \theta)dt. \tag{2.2}$$

Over the domain of no transactions, therefore, the value function, I(α,

θ), satisfies the following partial differential equation:[8]

$$-\beta + \alpha\theta - \lambda\alpha I_{\alpha} + (1/2)\sigma^2 I_{\alpha\alpha} + \alpha\theta(1 - \theta)I_{\theta} = 0. \tag{2.3}$$

We solve this partial differential equation by first discretizing it over the values of θ. We pick $\theta \in \{\theta_i;\ i = 0,\ldots n\}$. Then we need to find n+1 functions $I(\alpha, \theta_i)$, analogous to the functions $I_0(\alpha)$ and $I_1(\alpha)$ in the previous section. When α reaches $\underline{\alpha}_{i-1} = \underline{\alpha}(\theta_i)$, the portfolio proportion is dropped to θ_{i-1}; when α reaches $\bar{\alpha}_i = \bar{\alpha}(\theta_i)$, the portfolio proportion is increased to θ_{i+1}.

Given the existence of proportional transactions costs, the utility impact of switching may be computed as follows. First, on the way down from θ_i to θ_{i-1}:

$$\theta_i = \frac{y}{x + y}\ ; \qquad \theta_{i-1} = \frac{y - \Delta y}{x + s\Delta y + y - \Delta y}\ , \tag{2.4}$$

which implies:

$$\Delta y = (x + y)\ \frac{\theta_i - \theta_{i-1}}{\theta_{i-1}(s-1) + 1}\ . \tag{2.5}$$

Matching the values of the indirect utility before and after the change in portfolio composition, we have:

$$\ell n(x + y) + I(\underline{\alpha}, \theta_i) = \ell n(x + s\Delta y + y - \Delta y) + I(\underline{\alpha}, \theta_{i-1}). \tag{2.6}$$

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[8]This P.D.E. is analogous to the pair of equations (1.9) and (1.10) above.

From (2.5):

$$\frac{x + s\Delta y + y - \Delta y}{x + y} = 1 + (s - 1)\,\frac{\theta_i - \theta_{i-1}}{\theta_{i-1}(s-1)+1}. \tag{2.7}$$

Call $\underline{\pi}_{i-1}$ the right-hand side of (2.7). Since (2.6) may be rewritten as:[9]

$$I(\underline{\alpha}_{i-1}, \theta_i) = \ell n(\underline{\pi}_{i-1}) + I(\underline{\alpha}_{i-1}, \theta_{i-1}), \tag{2.8}$$

we conclude that the transaction-cost related utility loss on the way down is $\ell n(\underline{\pi}_{i-1})$. Equation (2.8) is a Value-matching condition.

The transition on the way up from θ_i to θ_{i+1} is handled in a similar way. Let:[10]

$$\bar{\pi}_i = 1 + (s - 1)\,\frac{\theta_i - \theta_{i+1}}{\theta_{i+1}(s-1)-s}, \qquad i = 0,\dots n-1. \tag{2.9}$$

The transactions-cost related utility loss on the way up is $\ell n(\bar{\pi}_i)$, resulting in a second set of Value-matching conditions.

Finally, we need to write that the choice of $\underline{\alpha}_{i-1}$ and $\bar{\alpha}_i$ is optimal for each i. Smooth-pasting necessary conditions accomplish that task:

$$I_\alpha(\underline{\alpha}_{i-1}, \theta_i) = I_\alpha(\underline{\alpha}_{i-1}, \theta_{i-1}), \quad i = 1,\dots n; \tag{2.10}$$

$$I_\alpha(\bar{\alpha}_i, \theta_i) = I_\alpha(\bar{\alpha}_i, \theta_{i+1}), \qquad i = 0,\dots n-1. \tag{2.11}$$

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[9]Assume $\theta_{i-1} < 1/(1-s)$, so that $\underline{\pi}_{i-1} > 1$.

[10]Assume $\theta_1 > -s/(1-s)$ so that $\bar{\pi} > 1$.

FIGURE 2 GOES HERE

The solution of the system (2.3, 2.8-2.11) is obtained numerically.[11] Under the assumption of no short-selling, $\theta_0 = 0$ and $\theta_n = 1$. Furthermore, at $\theta = 0$ and $\theta = 1$, the P.D.E. (2.3) is locally and ordinary differential equation. Hence, the functions I_0 and I_1, already obtained in section 1, namely:

$$I(\alpha, 1) = I(\underline{\alpha}_{n-1}, 1) + (\alpha - \underline{\alpha}_{n-1})/\lambda - \beta[-\phi_1(-\alpha) + \phi_1(-\underline{\alpha}_{n-1})], \quad (2.12)$$

$$I(\alpha, 0) = I(\bar{\alpha}_0, 0) - \beta[\phi_1(\bar{\alpha}_0) - \phi_1(\alpha)]. \quad (2.13)$$

serve as boundary conditions at the extreme values of θ. The resulting portfolio-adjustment boundary is shown in Figure 2. Numerical experiments indicate that the barrier has the following property:

Statement 2: The optimal barrier is a (steep) straight line whose middle point is located at the optimal switching point of the binary policy.

The location of the boundary implies that the "cubic" property (Statement 1) applies equally to the width of the hysteresis band in this case.

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[11] The desired degree of numerical accuracy was achieved by using the fact that the differential equation (2.3) without the term in I_θ has an analytical solution. Numerical errors in the solution of (2.3) were all but eliminated by comparing the numerical solution of the truncated equation with its analytical solution. This is called the "control variate" technique; (See e.g. Hull and White (1988)).

3. THE CASE OF ONE RISKLESS AND ONE RISKY, MEAN-REVERTING ASSET

3.1 Optimal portfolio policy

When not only the expected rate of return on one of the two assets follows a stochastic process but its rate of return is also risky, the state transition equations are:

$$dx = sd\ell - du; \tag{3.1a}$$

$$dy = \mu y dt + \sigma_1 dz_1 - d\ell + sdu; \tag{3.1b}$$

$$d\mu = \lambda(\gamma - \mu)dt + \sigma dz. \tag{3.1c}$$

Here, μ is the conditionally expected rate of return on the risky asset, σ_1 is the conditional standard deviation of the rate of return on that asset. The expected rate of return, μ, is assumed to be mean reverting. We call γ the center of reversion. The white noise shock, dz_1, affecting the rate of return on the asset is assumed independent of the white noise shock, dz, affecting the expected rate of return.

Introduce a change of state variable:

$$\alpha_t = \mu_t - \gamma, \tag{3.2}$$

and observe that the investor's frictionless demand for the risky asset at any given time would be given by:

$$\theta_t = \mu_t/\sigma_1^2 = (\alpha_t + \gamma)/\sigma_1^2,$$

or: $$\theta_t - \gamma/\sigma_1^2 = \alpha_t/\sigma_1^2, \tag{3.3}$$

which means that the frictionless demand schedule is symmetric around the point ($\alpha = 0$, $\theta = \gamma/\sigma_1^2$). The portfolio demand with transactions costs will inherit the same symmetry property.

Between transactions, the stochastic differential equation governing the evolution of the portfolio composition, θ, is:

$$d\theta = \theta(1 - \theta)(\alpha + \gamma - \theta\sigma_1^2)dt + \sigma_1\theta(1 - \theta)dz_1. \tag{3.4}$$

Over the domain of no transactions, the value function, $I(\alpha, \theta)$, satisfies the following partial differential equation:

$$-\beta + (\alpha + \gamma)\theta - (1/2)\theta^2\sigma_1^2 - \lambda\alpha I_\alpha + (1/2)\sigma^2 I_{\alpha\alpha}$$

$$+ \theta(1 - \theta)(\alpha + \gamma - \theta\sigma_1^2)I_\theta + (1/2)\sigma_1^2\theta^2(1 - \theta)^2 I_{\theta\theta} = 0. \tag{3.5}$$

The Value-matching and Smooth-pasting boundary conditions remain as in (2.8-2.11). Under the assumption of no shortselling ($\theta_0 = 0$, $\theta_n = 1$), the boundary conditions corresponding to the extreme values of θ are now:

$$I(\alpha,1) = I(\underline{\alpha}_{n-1},1) + (\alpha - \underline{\alpha}_{n-1})/\lambda + (-\beta + \gamma - \sigma_1^2/2)[-\phi_1(-\alpha) + \phi_1(-\underline{\alpha}_{n-1})], \tag{3.6}$$

$$I(\alpha,0) = I(\bar{\alpha}_0,0) - \beta[\phi_1(\bar{\alpha}_0) - \phi_1(\alpha)]. \tag{3.7}$$

FIGURE 3 GOES HERE

The optimal policy that solves this sytem has been obtained numerically by the same method that has been outlined in Section 2. It is displayed in Figure 3 and described by Statement 3.[12]

Statement #3: The optimal barrier is positioned slightly outside the hysteresis band of the riskless case constructed around the frictionless demand.

From Statement 3, it follows that the cubic property (Statement 1) is valid again.

3.2 Equilibrium and deviations from the C.A.P.M.

We now briefly discuss the equilibrium of an economy with two production techniques available in infinitely elastic supply. One is riskless and brings a zero return; the other is risky and brings a mean-reverting expected return. The economy is populated with identical logarithmic investors, each one of whom chooses a portfolio of techniques in the manner that we have just described. In this economy, let the variable θ -- provided its value[13] is between 0 and 1 -- describe the composition of the aggregate, "market" portfolio. This variable changes over time as the expected return, μ, on the

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[12]The parameter values underlying Figure 3 are discussed in Section 4.

[13]If $\theta = 0$ or 1, a switch in regime occurs: only one asset is available to all investors and no portfolio decision has to be made by any of them.

risky technology fluctuates. The classic Capital Asset Pricing Model would require that:[14]

$$\mu_t = \sigma_1^2 \theta_t, \tag{3.8}$$

which is simply the inverse of the frictionless demand (3.2), and which is shown as the solid line on Figure 3.

In an economy with transactions costs, the expected return, $\mu_t = \alpha_t + \gamma$, is allowed to fluctuate within a wide interval given horizontally by the hysteresis band of Figure 3, without any adjustment in the aggregate portfolio. Any fluctuation within that band is interpretable as a deviation from the CAPM. This shows that deviations from the CAPM can be large, even with small transactions costs, provided that expected returns fluctuate randomly.

4. CALIBRATION

We now wish to quantify the gap in expected returns that can survive in an economy with realistic parameter values. A calibration exercise was conducted both for the problem with two riskless assets (as in Sections 1 and 2) and for the problem with one risky and one riskless asset (as in Section 3). Parameter values were obtained from the empirical literature on mean reversion in interest rates and stock returns. Our principal sources were Jegadeesh (1991) and Chan *et al.* (1992). The crucial parameters are the degree

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[14] The portfolio composition, θ_t, and the risk measurement, beta, traditionally used in writing the CAPM, would be equal to each other in our case example.

of mean reversion, λ and the volatility, σ, in expected returns.

For the case of two riskless assets, we have chosen the values: λ = 1.25%/year and σ = 2.6%/year. The value of λ implies that it takes eighty years on average for the interest rate to revert to its long-run value. This is a very small value of the reversion parameter, which is equal to half of the value of λ estimated by Chan *et al.*.

Figures 1 and 2 were drawn for these parameter values. Consider the situation where transactions costs are collected at the rate of 0.1%, s = 0.999. The figures indicate that the combined effect of such small transactions costs and fluctuating expected returns is enough to produce a dramatic hysteresis effect in the rebalancing of the portfolio. Specifically, a gap of 1.005%/year in interest rates must exist before a decision is made to switch from one asset to the other.

The case of one risky and one riskless asset is calibrated in a like manner. The evidence concerning mean reversion in stock returns is not as conclusive as that concerning interest rates; nonetheless, we used the same value of the mean reversion parameter, which was in any case a low one. The parameter values chosen are: λ = 1.25%/year, σ_1 = 15%/year, σ = 2.6%/year, $\gamma = (0.15)^2/2$ = 1.12%/year. Again, we use transactions costs of 0.1%.

With these values, at $\theta = 1$, some wealth is transfered from the risky asset to the riskless as soon as the expected return on the risk asset falls below 0.06%/year. It must be 2.19% before the investor wishes to transfer some wealth from riskless to risky asset (but the constraint $\theta \leq 1$ prevents him from doing that).

Conclusion

Hysteresis bands have not been discovered in this paper. Neither has been the fact that hysteresis bands tend to remain comparatively large when the costs that created them become small. The new result of this paper is that these ideas apply to pricing models so that classic CAPMs are subject to wide hysteresis-band violations when conditionally expected returns follow a stochastic, mean reverting process. Our results also imply that arbitrage models must be drastically revised to take into account the combined effect of stochastic expected returns and transactions costs.

The qualitative point made by this paper regarding violations of frictionless pricing models does not depend on our assumption that investors have unit risk aversion. Regardless of his degree of risk aversion, a risk averse investor would always chose hysteretic rebalancing decisions. The unit risk aversion assumption has only simplified the calculations and allowed us in the simplest cases to obtain closed-form solutions.

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Figure 1: the effect of transactions costs

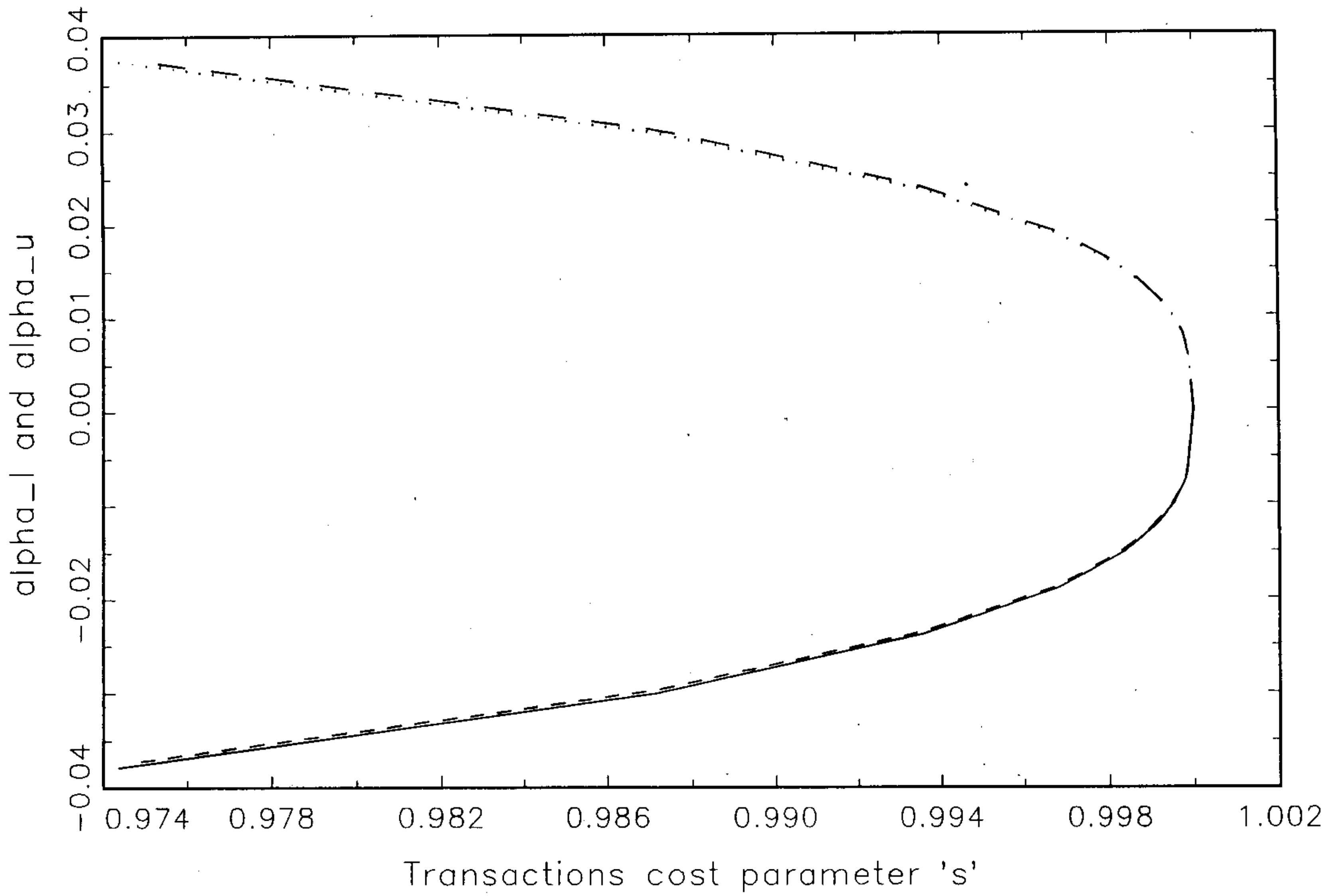

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Figure 2: the barrier in the riskless case

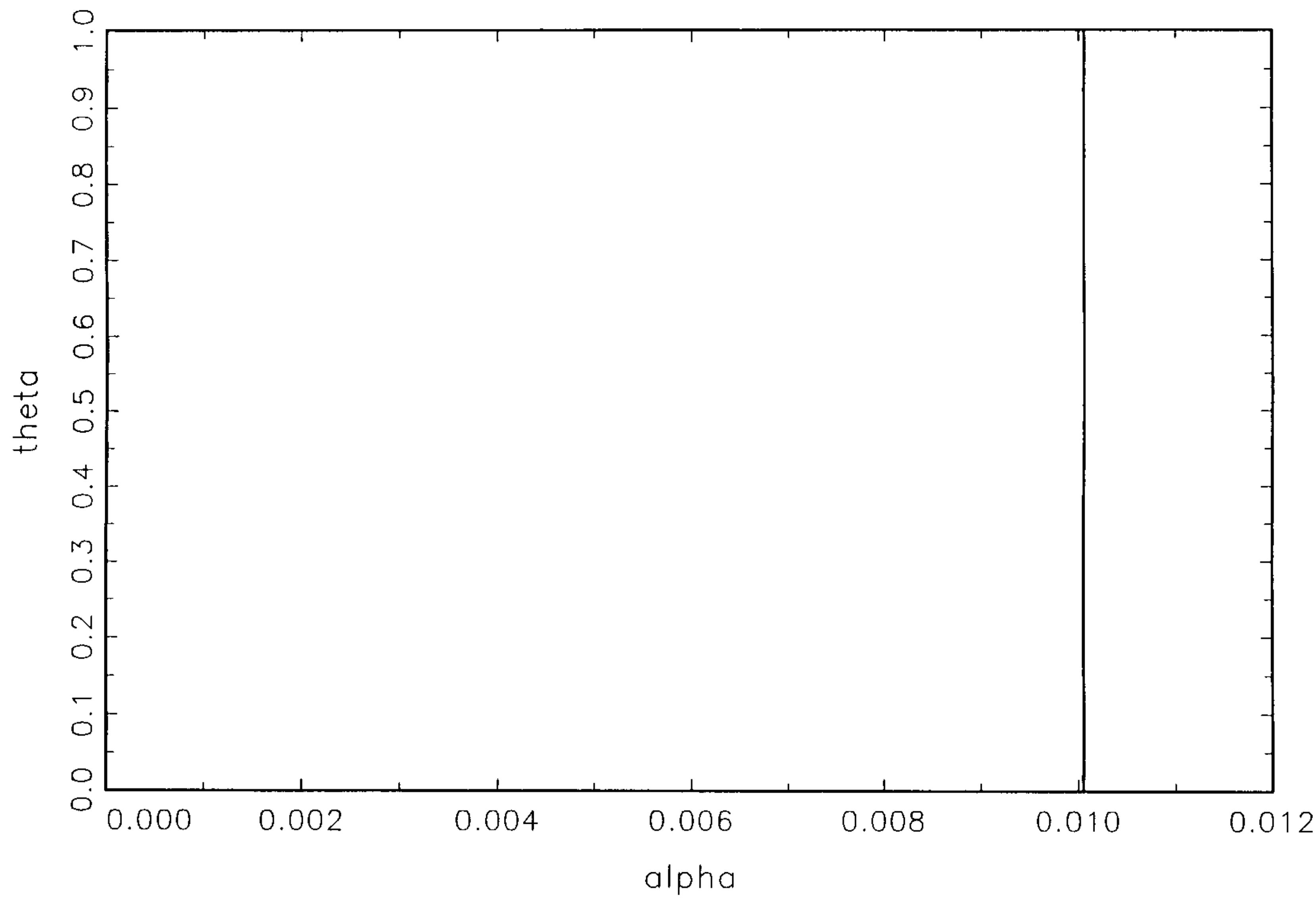

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Figure 3: comparing barriers