LIMITING DIFFERENCES BETWEEN FORWARD AND FUTURES PRICES IN A LUCAS CONSUMPTION MODEL

by

Zvi Wiener
Simon Benninga
Aris Protopapadakis

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RODNEY L. WHITE CENTER FOR FINANCIAL RESEARCH
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6367

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Zvi Wiener
The Wharton School
University of Pennsylvania
WIENER@WHARTON.UPENN.EDU

Simon Benninga
School of Business
Hebrew University of Jerusalem
MSSIMON@PLUTO.MSCC.HUJI.AC.IL

Aris Protopapadakis
School of Business
University of Southern California
APROTOPAPADAKIS@SBA.USC.EDU

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ABSTRACT

A recent paper (Benninga-Protopapadakis 1994) considered a Lucas asset pricing model and showed that the pricing of forward and futures contracts was expressible as a simple matrix function. In this paper we derive limiting conditions for these differences and relate them to the eigenvectors of the state price matrix. We show that except for a zero-measure set of state price matrices, the differences are always small.

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1. Introduction

Since the publication of three papers by Cox, Ingersoll, Ross (1981), Jarrow and Oldfield (1981), and Richard and Sundaresan (1981), it has been understood that the difference between forward and futures prices is a function of the covariance between the futures prices and the term structure of interest rates. The empirical question of whether futures prices are indeed different from equivalent forward prices (where by "equivalent" we mean a forward price for the same commodity deliverable at the same delivery date as the futures contract) has been more vexatious. Empirical research seems to give contradictory answers, but this research has been hampered by limited sample size, its dependence on constructed forward prices, and the specificity of the time periods covered.

In a recent paper, Benninga and Protopapadakis (1994–henceforth BP) take a different approach to the determination of the difference between forward and futures prices. BP construct a simple Markovian model of the term structure of interest rates; the model is based on the well-known Lucas (1978) equilibrium model. The BP model has the advantage that the term structure of interest rates is a function of the the matrix of nominal state prices. Given this matrix, forward and futures prices are easily calculated. The model can also accomodate various degrees of risk aversion.

Within the framework of this model, BP conduct two kinds of "tests" to guage the difference between forward and futures prices: First, they construct a model of state prices based on historic Treasury-Bill data. Using this empirical state price matrix, they construct forward and futures prices for contracts on short-term interest rate instruments.[1] This test of the difference between forward and futures prices results in only minor differences between the two.

The second test constructed by Benninga and Protopapadakis involves the calculation of the difference between forward and futures prices using simulated state price matrices. For most simulated state price matrices, the differences between forward and futures prices are negligible, although BP do report that for highly diagonal state price matrices it is possible to simulate significant differences between forward and futures prices.

In this paper we extend the BP results by proving a result about the *limiting difference* between forward and futures prices. We show that this limiting difference is a function of the eigenvectors of the state price matrix. We further show that – except for a zero-measure set of state price matrices – these differences are always small. The importance of this result is that it shows that in the limit, the difference between

[1] BP report equivalent results for longer-term interest rate instruments.

forward and futures prices is almost surely bounded and small. We thus provide a theoretical basis for the "empirical" results of BP.

The structure of the paper is as follows: Section 2 reviews the BP model and notation. Section 3 proves our main result on the limiting differences between forward and futures prices. Section 4 discusses two examples which illustrate the result.

2. The model

In a Lucas model consumption over time (which may be stochastic) is given exogeneously and is consumed by a single, representative consumer. All assets are priced by this consumer's state prices, which are the probability and time-preference adjusted first-order consumption conditions. The usual version of the model focusses on real state prices; we employ a version of the model which has state-dependent inflation and which allows us (by means of the nominal state prices) to price nominal assets. Although the model includes neither production nor investment, these can easily be added by specifying appropriate linear production technologies. We follow the notation of Benninga and Protopapadakis:

- π_{ij} probability of going to state j, given that the system is currently in state i,
- $\tilde{c}_t$ is stochastic consumption at time t,
- α_j 1 + the consumption growth rate in state j,
- ω_j is the inverse of 1 + the inflation rate in state j,
- γ is the relative risk aversion of the representative consumer,
- δ is the representative consumer's pure-time preference factor,
- S is the number of states of the world at any date.

We suppose that the representative consumer maximizes a time-separable expected utility function, and we let $\tilde{c}$ denote the lifetime, state-dependent, consumption stream. Then we may write the consumer's expected lifetime utility as:

$$EU(\tilde{c}) = \sum_{t=0}^{\infty} \delta^t Eu(\tilde{c}_t), \text{ where } u(x) = \frac{x^{1-\gamma}-1}{1-\gamma}.$$

Uncertainty in the model is generated by the random consumption endowments and inflation. If time t, state i consumption is c_t, then time $t+1$, state j $(j = 1, ..., S)$ consumption will given by $c_{t+1} = \alpha_j c_t$; furthermore, the inflation rate in state j at time t is denoted by $1/\omega_j - 1$. The probability of the transition from state i at time t to state j at time $t+1$ is time-independent and is denoted by π_{ij}. We assume no transactions costs or trading restrictions. This means that asset markets are complete, and that the representative consumer's probability-adjusted marginal rates of substitution are the real state prices which determine the prices of all real assets in the economy. If we assume that in addition to consumption growth, inflation is also Markovian, the state prices are time-independent, and can be denoted by an $S \times S$ matrix

$B \equiv [b_{ij}]$. Benninga and Protopapadakis (1983) show that in an economy of this type, nominal state prices can be defined by,

$$b_{ij} = \frac{\delta \pi_{ij} u'(\alpha_j c)}{u'(c)} = \delta \pi_{ij} \left[\frac{1}{\alpha_j}\right]^{\gamma} \omega_j \, .$$

Here c denotes the consumption at any state i at date t.

Let B be denote the matrix of nominal state prices. The vector of (state-dependent) period-n nominal discount factors is given by,

$$I(n) = B^n I(0) \, .$$

where $I(0)$ is an S-dimensional unit column vector.

For future reference we note the following properties of B:

1. Each entry of B is non-negative.
2. The row-sum of each line is the inverse of one-plus the one period interest rate. It follows that when one-period interest rates are finite and positive, no line of B is zero.
3. Although not strictly necessary for the subsequent results, it is convenient to assume that B is *sub-stochastic*: Each entry is non-negative, and each row sum is less than or equal to 1.[2]

3. The prices of forward and futures contracts

In this section we introduce a normalization procedure on the state price matrix which allows us to calculate both forward and futures prices. We shall restate the Benninga-Protopapadakis results in terms of this normalization procedure and then go on to prove our main result.

We define the following normalization procedure for any matrix A with non-negative elements in which there is no zero line:

$$n(A) = N_A \cdot A,$$

where N_A is a diagonal matrix, each entry of which is the inverse of the row-sum of the corresponding line of A. The function $n(A)$ transforms any non-negative matrix into a *stochastic* matrix: A *stochastic* matrix is a matrix with non-negative entries each of whose rows sum to 1. Another way of viewing $n(A)$ is that the procedure $n(A)$ transforms any positive state-price matrix into its equivalent Harrison-Kreps (1978) *risk-neutral valuation matrix.* The economic interpretation of the procedure $n(A)$ is that $n(A) \cdot V$ first *discounts* the vector V by multiplying it times the state-price matrix A and then *grosses up* this discounted valuation by the *accumulation factors* appropriate to the matrix A.

Let $V_t(s)$ be the time-t price in state s of the world of a specific fixed income security, and let $V_t = V_t(1), ..., V_t(S)$. When the term structure is determined by a Lucas asset pricing model of the type described in the previous section, it is easily shown that any vector of prices for interest-bearing securities (for example,

[2] If the state price matrix is not sub-stochastic, then there will necessarily be some state of the world for which the one-period real interest rate is negative.

a bond with coupon payments, or a certificate of deposit with add-on interest is *time-independent.* In what follows we shall denote such a vector by V. Benninga-Protopapadakis (1994) prove the three following propositions.

Proposition 1 (Determination of forward prices): *Let V be a vector of time-independent commodity prices. Then the vector of forward prices at date t for assets deliverable at date $t+m$ is given by:*

$$G(t,t+m) = n(B^m) \cdot V \, .$$

The intuition of Proposition 1 is that a forward price for delivery m periods hence is simply the discounted asset price grossed up by the appropriate accumulation factors. V is the vector of asset prices at date $t+m$; $n(B^m) \cdot V$ first discounts the vector V to the present at the appropriate m-period discount factors and then applies the m-period accumulation factors to these discounted prices to find the appropriate forward prices.

Proposition 2 (Determination of futures prices): *Let V be a vector of time-independent commodity prices. Then the vector of futures prices at date t for assets deliverable at date $t+m$ is given by:*

$$H(t,t+m) = n(B)^m \cdot V \, .$$

The interpretation of Proposition 2 has to do with marking-to-market in futures markets. Because of marking-to-market, a futures contract is priced as if it were a sequence of rolled-over one-period forward contracts. Thus a one-period futures contract is priced as $H(t,t+1) = n(B) \cdot V$ (the same as a one-period forward contract), and a two-period futures contract is priced as $H(t,t+2) = n(B) \cdot H(t,t+1) = n(B)^2 \cdot V$, etc. It follows from these two propositions that the *difference* between forward and futures prices is given by

$$G(t,t+m) - H(t,t+m) = [n(B^m) - n(B)^m] \cdot V \, .$$

Furthermore, it is readily proven:

Proposition 3 (Sufficient conditions for equality of futures and forward prices): *The following conditions are sufficient for the equality of forward and futures prices:*

3.1 The matrix B of state prices is diagonal.

3.2 The row-sums of B are equal.

3.3 Asset prices $V(s)$, $s = 1, ..., S$ are equal across states.

Proof:

We prove this proposition to show the simplification achieved by our new notation.

3.1. If the matrix B is diagonal, then both $n(B^m)$ and $n(B)^m$ are the unit matrix, and hence the difference between forward and futures prices is zero.

3.2. To prove this property we first note that the product of two matrices with constant row sums is a matrix with constant row sum equal to the product of the row sums of the multipliers. Using this fact we obtain immediately that if B has a constant row sum ρ then $N(B) = \frac{B}{\rho}$ and $N(B)^m = \frac{B^m}{\rho^m}$. On the other hand, B^m has constant row sum equal to ρ^m. Thus $N(B^m) = N(B)^m$.

3.3. If asset prices V are equal across states, then since the normalizing procedure leads to a stochastic matrix, it follows that both $n(B)^m \cdot V = V$ and $n(B^m) \cdot V = V$.

This proves Proposition 3.

It follows from Proposition 3 that for a flat term stucture there is no difference between futures and forward prices. By continuity, as the variation in one-period interest rates becomes small (and consequently the term structure becomes flatter), the difference between futures and forward prices becomes smaller.

We shall establish a formula for the limiting value of the difference between the forward and futures pricing matrices $n(B^m) - n(B)^m$. Our results depend on the following Lemma:

Lemma. Let C be an $n \times n$ matrix with nonnegative elements, such that there exists a real positive eigenvalue λ which is strictly bigger in modulus than the rest of spectrum. We assume that this eigenvalue is simple; i.e., it has a unique eigenvector and there is no Jordan block which corresponds to it. Then $\lim_{m\to\infty} \frac{C^m}{\lambda^m}$ is equal to the tensor product of the right and left (normalized) eigenvectors corresponding to λ.

Proof: Denote the eigenvectors by $x, y \in \mathbb{R}^n$, so that $Cx = \lambda x$, and $y^T C = \lambda y^T$. The Jordan form of C is $C = PJP^{-1}$, where columns of P form a system of eigenvectors. Without loss of generality we assume that λ corresponds to the first element of J. Then

$$\frac{C^m}{\lambda^m} = P \begin{pmatrix} 1 & 0_{1,n-1} \\ 0_{n-1,1} & \tilde{C}_{n-1,n-1} \end{pmatrix}^m P^{-1}.$$

Here $\tilde{C}$ stands for some matrix whose spectrum is strictly less than 1 in absolute value. This yields that

$$\lim_{m\to\infty} \frac{C^m}{\lambda^m} = a\, b^T,$$

where a is the first column of P and b^T is the first row of P^{-1} (cf. Gantmacher 1964, p. 53). This means that a and b are the right and left principal eigenvectors of C corresponding to λ and normalized by the condition $b^T a = 1$, since $P \cdot P^{-1} = id$, where id is the identity matrix. This finishes the proof.

The conditions of the Lemma are very general and not restrictive.[3] In the remainder of this section we show how the Lemma can be applied to find the limiting difference between futures and forward prices.

Let λ be the principal – maximal in modulus – eigenvalue of B and let x and y be the corresponding right and left eigenvectors, respectively. By the Lemma we see that the difference $B^m - \lambda^m x\, y^T$, tends to

[3] Matrices with a multiple principle eigenvalue constitute a set of measure zero. In the neighborhood of this set, our Lemma will still hold, but convergence to the limit will be slow. The rate of convergence will depend on the distance between the principle (Perron) eigenvalue and the next-closest eigenvalue.

zero as m increases. The block-form of this limit is:

$$B^m \approx \lambda^m \begin{pmatrix} x_1 \, y^T \\ x_2 \, y^T \\ \vdots \\ x_n \, y^T \end{pmatrix} .$$

This difference is very small for big m. The normalization procedure for forward prices, $n(B^m)$ may be written as

$$n(B^m) = N_m \cdot B^m \quad ,$$

where N_m is a diagonal matrix which can be approximated by

$$N_m \approx \frac{1}{\lambda^m \sum_{i=1}^{n} y_i} \, diag\left\{ \frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n} \right\} .$$

Thus the forward pricing matrix $n(B^m)$ will be arbitrarily (for big m) close to the following matrix of rank 1 written in block-form:

$$N_m \, B^m \approx \frac{1}{\sum_{i=1}^{n} y_i} \begin{pmatrix} y^T \\ y^T \\ \vdots \\ y^T \end{pmatrix} .$$

Since an eigenvector is defined up to multiplication by a constant we can normalize it by $\sum_{i=1}^{n} y_i = 1$.

We now consider the futures pricing matrix $n(B)^m$. Denote by u, v^T the eigenvectors of $n(B)$ corresponding to its principal eigenvalue μ. In analogous way one can show that

$$\frac{n(B)^m}{\mu^m} \rightarrow \begin{pmatrix} u_1 \, v^T \\ u_2 \, v^T \\ \vdots \\ u_n \, v^T \end{pmatrix} .$$

Since the matrix $n(B)$ is stochastic its principal eigenvalue is equal to 1 and its right eigenvector is $u = (1, \; 1, \cdots, 1)^T$ and $\sum_{i=1}^{n} v_i = 1$. Thus $n(B)^m \rightarrow (v, v, \cdots, v)^T$.

Combining the results for the forward and futures pricing matrices, we obtain an explicit formula for the limiting difference:

$$\lim_{m \to \infty} [n(B)^m - n(B^m)] = \begin{pmatrix} v^T - y^T \\ v^T - y^T \\ \vdots \\ v^T - y^T \end{pmatrix} . \qquad (1)$$

Both vectors are normalized, so the difference between forward and future contracts can be easily estimated (in the general case) by the angle between the principal left eigenvectors of B before and after normalization. Thus the angle between y and v determines the limiting difference between the forward and futures prices.

4. Examples

In this section we discuss two examples. The first example implements the Lemma and shows how the limiting difference between forward and futures prices can be approximated by our procedure.

Example 1: The first examples uses the empirical nominal state price matrix derived by Benninga and Protopapadakis (1994) from Treasury bill data:

$$B = \begin{pmatrix} 0.895 & 0.099 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.11 & 0.441 & 0.22 & 0.109 & 0.111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.099 & 0.595 & 0.197 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.11 & 0.22 & 0.328 & 0.221 & 0.109 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.09 & 0 & 0.269 & 0.271 & 0.179 & 0.179 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.271 & 0.447 & 0.179 & 0.089 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.111 & 0 & 0.328 & 0.437 & 0.109 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.179 & 0.179 & 0.268 & 0.358 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.098 & 0 & 0.197 & 0.197 & 0.293 & 0.197 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.089 & 0.269 & 0.532 & 0.09 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.108 & 0.549 & 0.32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.123 & 0.122 & 0.123 & 0.6 \end{pmatrix}$$

By our Lemma, the limiting difference between forward and futures prices for this matrix should equal the difference between the principal left eigenvector y^T of the matrix B (for the forward prices) and the principal left eigenvector v^T of the matrix $n(B)$ (for the futures prices). These vectors are given by:[4]

$$y^T = \{0.0602, 0.0485, 0.0619, 0.0609, 0.0834, 0.0953, 0.0773, 0.104, 0.111, 0.118, 0.0979, 0.0816\},$$

$$v^T = \{0.0467, 0.0419, 0.0538, 0.0548, 0.0782, 0.094, 0.0759, 0.107, 0.118, 0.129, 0.108, 0.0931\}.$$

The eigenvalues of B are given by:

$$\{0.984, 0.936, 0.871, 0.745, 0.5, 0.412, 0.359{+}0.023\,i, 0.359{-}0.023\,i, 0.224{+}0.112\,i, 0.224{-}0.112\,i, -0.143, -0.019\},$$

and the eigenvalues of $n(B)$ are given by:

$$\{1., 0.948, 0.881, 0.757, 0.510, 0.424, 0.363{+}0.023\,i, 0.363{-}0.023\,i, 0.227{+}0.114\,i, 0.227{-}0.114\,i, -0.146, -0.019\}$$

The vectors y^T and v^T correspond to the largest eigenvalues of these systems (respectively). The speed of convergence is a function of the distance between the principal eigenvalue and the eigenvalue closest to the principal for each of the systems. After 100 iterations, the difference between the forward and futures prices is given by the matrix:

4 All calculations reported were done in *Mathematica*.

$$\text{forward} - \text{futures} = n(B^{100}) - n(B)^{100} =$$

$$\begin{pmatrix}
0.0143 & 0.0068 & 0.00833 & 0.00622 & 0.00523 & 0.0012 & 0.00129 & -0.00248 & -0.00715 & -0.0116 & -0.0105 & -0.0117 \\
0.0139 & 0.00667 & 0.00821 & 0.00616 & 0.0052 & 0.00125 & 0.00133 & -0.00238 & -0.00701 & -0.0114 & -0.0103 & -0.0116 \\
0.0137 & 0.00662 & 0.00816 & 0.00612 & 0.00518 & 0.00126 & 0.00135 & -0.00234 & -0.00696 & -0.0113 & -0.0103 & -0.0115 \\
0.0137 & 0.00661 & 0.00814 & 0.00611 & 0.00518 & 0.00127 & 0.00135 & -0.00233 & -0.00694 & -0.0113 & -0.0103 & -0.0115 \\
0.0136 & 0.00657 & 0.00811 & 0.00609 & 0.00517 & 0.00128 & 0.00136 & -0.0023 & -0.0069 & -0.0113 & -0.0102 & -0.0115 \\
0.0134 & 0.00653 & 0.00806 & 0.00606 & 0.00515 & 0.00129 & 0.00137 & -0.00227 & -0.00685 & -0.0112 & -0.0102 & -0.0114 \\
0.0133 & 0.00647 & 0.00801 & 0.00603 & 0.00514 & 0.0013 & 0.00138 & -0.00223 & -0.00679 & -0.0111 & -0.0101 & -0.0113 \\
0.0132 & 0.00646 & 0.00799 & 0.00602 & 0.00513 & 0.00131 & 0.00138 & -0.00222 & -0.00678 & -0.0111 & -0.0101 & -0.0113 \\
0.0131 & 0.00643 & 0.00796 & 0.006 & 0.00512 & 0.00131 & 0.00139 & -0.0022 & -0.00674 & -0.011 & -0.01 & -0.0113 \\
0.0131 & 0.00641 & 0.00794 & 0.00598 & 0.00511 & 0.00132 & 0.00139 & -0.00219 & -0.00672 & -0.011 & -0.01 & -0.0113 \\
0.013 & 0.00637 & 0.0079 & 0.00596 & 0.00509 & 0.00133 & 0.0014 & -0.00216 & -0.00668 & -0.011 & -0.00996 & -0.0112 \\
0.013 & 0.00638 & 0.00791 & 0.00597 & 0.0051 & 0.00132 & 0.0014 & -0.00217 & -0.00669 & -0.011 & -0.00998 & -0.0112
\end{pmatrix}$$

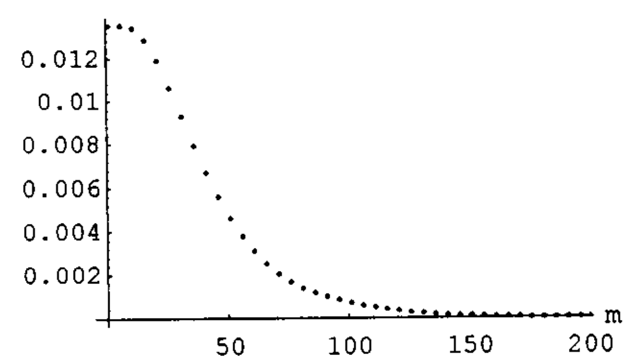
Each line of this difference should be compared to the predicted limiting difference (1):

$$v^T - y^T = \{0.0135, 0.00658, 0.00812, 0.0061, 0.00519, 0.00129, 0.00137, -0.00229, -0.0069, -0.011, -0.0102, -0.0115\}.$$

The speed of convergence is shown in the following graph. The y-axis shows the maximal absolute entry in the matrix of the difference between the right and left-hand sides of equation (1) for m iterations.

ACTUAL VERSUS PREDICTED LIMITING DIFFERENCES

OF FORWARD MINUS FUTURES PRICES



Example 2: In the second example we show a case where the conditions of the Lemma do not hold and where, consequently, the limiting difference of the forward and futures prices is *not* given by the procedure we describe in the Lemma. Consider the case where the matrix B is given by:

$$B = \begin{pmatrix} 0.8 & 0.1 \\ 0 & 0.8 \end{pmatrix}.$$

Since this matrix has a Jordan block, it violates the conditions of the Lemma.[5] The eigenvectors v and y are equal. However, the limiting difference between the forward and futures prices for this case is **not** zero.

[5] As noted in footnote 3, such matrices are a set of zero measure.

For example after 100 iterations, the difference is given by the matrix:

$$\text{forward} - \text{futures} = \mathrm{n}(\mathrm{B}^{100}) - \mathrm{n}(\mathrm{B})^{100} = \begin{pmatrix} 0.0741 & -0.0741 \\ 0 & 0 \end{pmatrix}.$$

5. Conclusions

A recent paper by Benninga and Protopapadakis (1994) uses the term structure derived from a standard Lucas (1978) model of capital market equilibrium under uncertainty to price forward and futures contracts on interest-rate dependent securities. In this paper we extend the BP results. We derive the limiting differences between forward and futures prices as the contract maturity date $m \to \infty$.

The main application of our result is for the case of interest-rate futures contracts. The spot prices of the assets underlying these contracts are determined by the term structure; when the term structure is time-independent (as it is in the Lucas model), the distribution of these spot prices will, as a result, also be time- independent. For this case of time-independent spot prices, our result shows that the limiting difference between the forward and futures prices is a function of the eigenvectors of the state price matrix, and that – except for a zero-measure set of state price matrices – these differences are always small. We thus provide a theoretical basis for the results reported by Benninga and Protopapadakis.

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