DYNAMIC CONSUMPTION-PORTFOLIO CHOICE AND ASSET PRICING WITH NON-PRICE-TAKING AGENTS

by

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Abstract

This paper develops a continuous-time pure exchange model to theoretically study the dynamic consumption-portfolio problem of an agent who acts as a non-price-taker, and to analyze the implications of his behavior on the security prices and their dynamics. The non-price-taking behavior is modeled by allowing the non-price-taker's consumption stream to affect Arrow-Debreu prices. This allows us to employ martingale methods in a natural way, making the analysis highly tractable. We define non-price-taking equilibrium in an economy of N price-takers and one non-price-taker, and show the existence and uniqueness of this equilibrium under common assumptions about the agents' utility functions and dividend streams. Solving for the equilibrium consumption allocations reveals the existence of another driving factor apart from the aggregate consumption stream, the endowment stream of the non-price-taker, which leads to modified formulae for the interest rate and the consumption CAPM. We characterize the equilibrium consumption-portfolio allocations, and the Arrow-Debreu and security prices and their dynamics, i.e., the interest rate, market prices of risk, asset price volatility and risk premium. A variety of comparisons of equilibria between a price-taking and a non-price-taking economy are carried out, in some cases for general utility functions and in some cases for CARA utility of all agents. Intuition for the results is offered.

1 Introduction

Central to the equilibrium-based asset pricing models in finance is the competitive agents paradigm: each agent is atomistic relative to the market, and hence takes prices to be unaffected by his or her actions. For example, in the intertemporal asset pricing models, for given price processes each agent maximizes his expected lifetime utility through his choice of consumption and trading strategies. Aggregating all agents' demands for consumption and securities, the market clearing condition in all markets is imposed to arrive at characterizations of the equilibrium state and security price processes, and the agents' equilibrium consumption-portfolio policies.

An observation of today's security markets (and especially government bond markets) reveals the ever-increasing importance of large pension funds and financial institutions in the marketplace. "Large" investors are particularly prevalent in smaller security markets outside the U.S.A. (e.g., Belgium, France, Hong Kong, Singapore, Sweden). Such a "large" investor may have a significant effect on prices, and hence may prefer to choose a strategy taking the price impact of his own behavior into account. It is well-known that large trades do have a permanent price impact. This is attributable partially to the information a large trade reveals about future cash flows (Kraus and Stoll (1972), Holthausen, Leftwich, and Mayers (1987,1990), Seppi (1992)), but conceivably also to the effect of such a large position on security supply and demand. Internationally, there is certainly widespread anectodal evidence that large trades can affect prices independently of any information they may contain. It would be of interest to re-investigate the traditional equilibrium-based asset pricing models in the presence of "non-price-taking" investors, who take account of the price impact of the positions they take on (independent of any information revealed by their trading).

To our knowledge, the only work carried out towards this end has been by Lindenberg (1979). He works in a single-period mean-variance framework, with an exogeneous interest rate. Some of the agents in his model recognize that security prices depend on their demand and formulate their optimization problems accordingly. He finds that the non-price-taking investors hold an optimal portfolio that is unbalanced (i.e., contains differing percentages of the supply of shares of each security). A two-factor CAPM results, in which an asset's risk premium is driven by the covariance of its return with the market return and with the return on the aggregate portfolio of the price-affecting investors.

The objective of our paper is to study the effect of the presence of non-price-taking agents on dynamic market equilibrium. In contrast to Lindenberg's work, we focus on the dynamic aspects of consumption-portfolio choice and on the stochastic evolution of the asset and Arrow-Debreu prices, i.e., the interest rate, market price of risk and the security price volatilities and risk premia. Our emphasis is on carrying out comparisons between equilibria in price-taking economies (in which all agents are price-takers) and non-price-taking economies. We postulate as our starting point that one agent in the economy is a non-price-taker; we do not attempt to endogeneously justify his non-price-taking nature. In a frictionless economy such as ours, the size of agents' security holdings is not restricted by their wealth, so it should not be argued that an investor with large net wealth can affect prices more than a small investor; however, in practice given the presence of short sales constraints and transaction costs in the marketplace, it is plausible that some investors are able to make larger trades and hence affect prices more than others.

We develop a continuous-time model of a pure-exchange economy consisting of N price-taking agents and one non-price-taking agent. We model non-price-taking behavior in the securities markets by letting the non-price-taker take into account that the Arrow-Debreu prices are affected by his consumption choice. Since an agent finances his consumption choice through trading dynamically in the securities, he is effectively a non-price-taker in the security markets. This "consumption-based" formulation of the problem allows us to adapt the standard martingale optimization approach (Cox and Huang (1989, 1991), Karatzas, Lehoczky and Shreve (1987) and Pliska (1986)) in a natural way to incorporate non-price-taking behavior. This convenient choice of modeling method makes the analysis highly tractable as will be seen.

Our notion of equilibrium is in the spirit of the price leadership model in the oligopoly literature (e.g., Varian (1992, Chapter 16)). Accordingly, the non-price-taker chooses his consumption process, aware that prices will adjust so that the remaining (price-taking) agents' demands clear the consumption good market. The non-price-taking strategy we solve for can be thought of as a "self-commitment" strategy in which the non-price-taker chooses a plan at the beginning and then does not deviate from that plan. (We address this issue in Remark 1 of Section 3.1.) If the preferences of the price-takers are such that their representative agent's utility does not depend on their individual wealth allocations, the non-price-taker's consumption at any given time and state only affects the state price (the price of one unit of consumption) at that same time and state. This is the case we focus on for most of the paper. However, when the wealth allocation amongst the price-taking agents does affect their representative agent's preferences, the non-price-taker's consumption at any given time and state affects the whole state price process (at all times and states). We consider this case only briefly. We derive necessary and sufficient conditions for equilibrium and show the existence and uniqueness of equilibrium for HARA utility of the representative price-taker, under mild assumptions on the non-price-taker's preferences.

Solving for the equilibrium consumption allocations reveals that the non-price-taking agent deviates from his price-taking behavior by tending to move his consumption towards his endowment stream. We note that in the special case when his price-taking equilibrium behavior is to always consume exactly his endowment stream, and hence to not trade in the securities at all, the non-price-taker does not deviate from his price-taking strategy. In this case his non-price-taking nature has no effect on state or security prices, which is consistent with our earlier notion that the less an agent trades the less he may act as a non-price-taker. When the non-price-taker does trade, his endowment stream also appears as an extra factor, in addition to the aggregate consumption stream, in explaining the equilibrium interest rate, asset prices, and their volatilities and risk premia. We derive a two-factor consumption-based CAPM, stating that an asset's risk premium depends on the covariance of its return with changes in the non-price-taking agent's endowment stream as well as with changes in the aggregate consumption. (Detailed intuition follows within the body of the paper.)

To derive further implications of non-price-taking behavior, we specialize to the case of all agents' preferences exhibiting constant absolute risk aversion (CARA) and one risky asset.

When the non-price-taker is initially relatively wealthy compared to the rest of the market, he is found to react more to changes in the aggregate dividend than when he is a price-taker, and as a result his consumption drift and volatility increase. As a further consequence, if the economy is expanding, the non-price-taker on average postpones consumption compared with when he is a price-taker. The reverse is true when the non-price-taker has a relatively low endowment compared with the rest of the market; he reacts less to changes in the aggregate dividend meaning he reduces the volatility and drift of his consumption stream and, in an expanding economy, consumes on average more impatiently. This consumption behavior for the CARA utility leads to an increase in the volatility of the Arrow-Debreu price return (i.e., the market price of risk) when the non-price-taker is relatively wealthy, and a decrease when he is not so wealthy.

To analyze the effect of the non-price-taker on the agents' portfolio strategies and the market volatility, we derive representations of these quantities using tools from Malliavin calculus, in particular the Clark-Ocone formula. (For related applications of Malliavin calculus in finance, see Detemple and Zapatero (1991) - interest rate and risk premium formulae - and Ocone and Karatzas (1991) - optimal portfolio representations.) The optimal portfolio representations reveal that the CARA utility non-price-taker no longer holds half of the market in equilibrium. Additional terms in his optimal portfolio process suggest that (in an expanding economy) if he is relatively wealthy compared with the rest of the market he demands more of the risky asset and if he is less wealthy he demands less. An implication is that there will be less riskless lending and borrowing in the presence of the non-price-taker. A further consequence is that the market volatility appears to be increased by the presence of a relatively wealthy non-price-taking agent and decreased by the presence of a less wealthy non-price-taker.

The remainder of the paper is organized as follows. Section 2 outlines the pure-exchange continuous-time framework of our model. In Section 3 we present the martingale formulation of the non-price-taking equilibrium and characterize the agents' equilibrium consumption allocations. Section 4 derives the modified consumption-based CAPM and interest rate formulae. In Section 5 we specialize to the CARA utility and one risky asset to derive further results for the equilibrium consumption-portfolio processes and the market and state prices and their dynamics. In Section 6 we summarize our conclusions and propose further work. The Appendices provide the proofs of all propositions and corollaries.

2 General Formulation

This section describes a continuous-time variation of the Lucas (1978) pure-exchange economy. The formulation follows along the lines of the continuous-time pure exchange general equilibrium models recently developed by Duffie and Huang (1985), Duffie (1986), Huang (1987), Duffie and Zame (1989) and Karatzas, Lehoczky and Shreve (1990, 1991).

We consider a finite horizon $[0, T]$ economy in which there is a single consumption good. All quantities (prices, endowments etc.) are expressed in units of this consumption good. We let $W = (W_1, ..., W_L)^\top$ be an L-dimensional Brownian motion on a complete probability space

$(\Omega, \mathcal{F}, \mathcal{P})$ and let $\{\mathcal{F}_t; t \in [0,T]\}$ be the augmentation by null sets of the filtration generated by W. All the uncertainty in the economy is represented by this L-dimensional Brownian motion. $\omega \in \Omega$ describes a state of nature in $[0,T]$.

2.1 Securities

We assume there are $L+1$ securities. One of them is an instantaneously riskless bond in zero net supply at all times; the remaining L securities are risky stocks, each in constant net supply of 1 and each paying out a dividend stream at rate $\delta_i(t)$ in $[0,T]$. We assume the following dynamics for the aggregate dividend process $\delta(t) \equiv \sum_{i=1}^{L} \delta_i(t)$:

$$d\delta(t) = \mu_\delta(t)dt + \sum_{j=1}^{L} \sigma_{\delta j}(t)dW_j(t), \quad t \in [0,T],$$

where $\mu_\delta(\cdot)$ and $\sigma_{\delta j}(\cdot)$ are $\mathcal{F}_t$-measurable processes.

The price of each security is modeled as a diffusion process relative to the Brownian filtration. The bond price dynamics are

$$dP_0(t) = P_0(t)r(t)dt, \quad t \in [0,T],$$

and the risky stock price dynamics are given by

$$dP_i(t) + \delta_i(t)dt = P_i(t)\left[\mu_i(t)dt + \sum_{j=1}^{L} \sigma_{ij}(t)dW_j(t)\right], \quad i = 1, \dots, L, \quad t \in [0,T].$$

Since these prices are ex-dividend we have $P_i(T) = 0$, $i = 1, \dots, L$. Here, the interest rate $r(\cdot)$ of the bond, the vector of drifts $\mu(\cdot) = (\mu_1(\cdot), \dots, \mu_L(\cdot))^\top$ and the volatility matrix $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}$ are $\mathcal{F}_t$-measurable processes, and in particular are allowed to be path dependent. The coefficients $\mu_i(t)$ and $\sigma_{ij}(t)$ are interpreted as the instantaneous expected return of the ith security and the instantaneous covariance of the ith security's return with the jth Brownian motion at time t.

Note that, the market in this set-up is dynamically complete (assuming $\sigma(\cdot)$ is invertible), since the number of risky securities is equal to the number of dimensions of uncertainty (L).

In our analysis, we use the martingale representation technology, which requires the construction of certain processes, related to the price dynamics. The details are given in Cox and Huang (1989, 1991), Karatzas, Lehoczky and Shreve (1987) and Pliska (1986). We will only present the required notions for our set-up and not concern ourselves with the details.

We define *the state price density process* $\xi(t)$ as a process with dynamics

$$d\xi(t) = -\xi(t)\left[r(t)dt + \theta(t)^\top dW(t)\right], \tag{1}$$

where $\theta(t)$ is the L-dimensional $\mathcal{F}_t$-measurable *market price of risk process* defined by

$$\theta(t) \equiv \sigma(t)^{-1}[\mu(t) - r(t)\mathbf{1}],$$

where $\mathbf{1}$ is an L-dimensional vector with every component equal to 1. $\xi(t,\omega)$ is interpreted as the Arrow-Debreu price (per unit of probability) of one unit of consumption good in state $\omega \in \Omega$ at time t.

The above construction of asset prices and the state price density process provides the following relationship between the asset prices and their future dividends, a result which follows from no arbitrage.

Lemma 1:

$$P_i(t) = \frac{1}{\xi(t)} E\left[\int_t^T \xi(s)\delta_i(s)ds \mid \mathcal{F}_t\right], \quad i = 1, ..., L, \quad t \in [0,T]. \tag{2}$$

2.2 Agents' Preferences and Endowments

We assume there are $N+1$ agents, m and $n = 1, ..., N$, in the economy. Agent m is a non-price-taker in the sense that he takes into account the fact that his consumption process $c_m(t)$, $t \in [0,T]$, affects the state price density process $\xi(t)$, $t \in [0,T]$. We will discuss the way agent m affects prices in Section 3. The other agents n are price-takers. Each agent, k, is endowed at time zero with e_{ki} shares of risky security i, such that

$$\sum_{k=m,n} e_{ki} = 1, \quad i = 1, ..., L,$$

i.e., the initial supply of each security is one share. Then, define the initial wealth of agent k as

$$\begin{aligned} x_{ko} &\equiv \sum_{i=1}^{L} P_i(0)e_{ki} \\ &= \frac{1}{\xi(0)} E\left[\int_0^T \xi(t)\delta_k(t)dt\right], \quad k = m, n, \end{aligned}$$

where $\delta_k(t) \equiv \sum_{i=1}^{L} e_{ki}\delta_i(t)$ and the second equality follows from Lemma 1. $\delta_k(t)$ can be interpreted as the endowment stream of agent k since it is the sum of dividend streams from the initial endowments of the agent. We assume that no further net shares of securities are issued after time zero, so that the total supply of each risky security is one share at all times.

For each agent k, we define a consumption process $c_k(t)$, and an L-dimensional portfolio process $\pi_k(t) = (\pi_{k1}(t), ..., \pi_{kL}(t))^\top$, where $\pi_{ki}(t)$ denotes the amount (in units of the consumption good) invested in the ith risky security by the kth agent at time t. We can express each component of an agent's portfolio process as $\pi_{ki}(t) = \alpha_{ki}(t)P_i(t)$, where $\alpha_{ki}(t)$ is the number of shares of asset i held by agent k at time t. Denoting $X_k(t)$ to be the wealth of the kth agent, $X_k(t) - \pi_k(t)^\top\mathbf{1}$ is the amount invested in the bond. We can write the dynamics of the kth agent's wealth process as

$$dX_k(t) = r(t)X_k(t)dt - c_k(t)dt + \pi_k(t)^\top[\mu(t) - r(t)\mathbf{1}]dt + \pi_k(t)^\top\sigma(t)dW(t). \tag{3}$$

In this economy, each agent is assumed to derive time-additive, state-independent utility $u_k(c_k(t))$ from consumption at all times in $[0,T]$. We will assume the utility functions are strictly increasing and strictly concave. The agents trade in the securities in order to hedge the risk associated with the dividend streams of their initial endowments and to finance their optimal (expected utility maximizing) consumption $\hat{c}_k(t)$. Throughout the paper, a symbol with a $\hat{}$ will denote the optimal quantity corresponding to $\hat{c}_k(t)$ and its associated portfolio process $\hat{\pi}_k(t)$. A symbol with a * will denote equilibrium in a non-price-taking economy, a symbol with a $\bar{}$ equilibrium in a price-taking economy (where all agents are price-takers).

3 Agents' Optimization and Equilibrium

This section presents the agents' optimization problems and defines equilibrium in a multi-agent economy consisting of one non-price-taking agent and N price-taking agents. Conditions under which the equilibrium allocations and Arrow-Debreu prices deviate from those in a price-taking economy are summarized and general statements made about how they deviate.

We use the martingale representation approach to solve the agents' optimization problems. Under this approach each agent k's $(k = m, n)$ dynamic optimization problem is converted into the following static variational problem:

$$\max_{c_k(\cdot)} E\left[\int_0^T u_k(c_k(t))dt\right]$$
$$\text{subject to} \quad E\left[\int_0^T \xi(t)c_k(t)dt\right] \leq E\left[\int_0^T \xi(t)\delta_k(t)dt\right].$$

The price-takers solve their optimization problem taking the state price density process $\xi(\cdot)$ as given. Using the Lagrangian method, the first order conditions of the price-takers n are

$$u_n'(\hat{c}_n(t)) = y_n\xi(t), \quad t \in [0,T], \quad n = 1, \ldots, N,$$

where each y_n is the Lagrange multiplier associated with agent n's static budget constraint. For a price-taker, the incremental satisfaction obtained from an extra unit of consumption at time t and state ω is proportional to the cost $\xi(t,\omega)$ of that extra unit of consumption; otherwise the price-taker could make himself better off by shifting consumption to or from this state and time. Hence their optimal consumptions are given by

$$\hat{c}_n(t) = I_n(y_n\xi(t)), \quad t \in [0,T], \quad n = 1, \ldots, N, \tag{4}$$

where $I_n(\cdot)$ is the inverse of n's marginal utility, and the constants y_n are such that the budget constraints hold with equality, i.e., the y_n satisfy

$$E\left[\int_0^T \xi(t)I(y_n\xi(t))dt\right] = E\left[\int_0^T \xi(t)\delta_n(t)dt\right], \quad n = 1, \ldots, N. \tag{5}$$

For notational and analytical convenience we introduce a representative agent formulation for the price-takers (following, for example, Huang (1987)). We define the price-taker representative agent's utility function by

$$U(c;\Lambda) \equiv \max_{c_1,\dots,c_N} \sum_{n=1}^{N} \lambda_n u_n(c_n)$$

subject to $\sum_{n=1}^{N} c_n = c$, where $\Lambda \equiv (\lambda_1, \dots, \lambda_N) \in (0,\infty)^N$. It can be shown that the inverse of $U'(c;\Lambda)$ is given by $J(h;\Lambda) \equiv \sum_{n=1}^{N} I_n(h/\lambda_n)$. Hence, from (4), the aggregate optimal consumption over all price-takers is given by

$$\sum_{n=1}^{N} \hat{c}_n(t) = J(\xi(t); 1/y_1, \dots, 1/y_N). \tag{6}$$

In contrast to the price-takers, the non-price-taker solves his optimization problem taking into account the effect of his consumption process $c_m(t)$ on the state price process $\xi(t)$. In subsection 3.1 we consider cases for which the non-price-taker's time t, state ω consumption $c_m(t,\omega)$ affects only the state price density at that time and state, i.e., there is a mapping $\xi(t,\omega) = \Xi(c_m(t,\omega); t, \omega)$. In Subsection 3.2 we consider the more general case of his consumption $c_m(t,\omega)$ affecting the whole process $\xi(t)$.

3.1 The Case of Price-Taker Representative Agent Independent of Individual Weights

In this subsection and throughout most of the paper we take the case where the representative agent's utility function can be written as

$$U(c;\Lambda) = h(\Lambda)U(c).$$

Examples of this include the case of only one price-taker, or all price-takers having the same log or power or negative exponential utility function. In general, it can be shown that in equilibrium the vector $(y_1, \dots, y_N)$ is only determined up to a multiplicative constant, so without loss of generality we can let $h(\Lambda) = 1$, i.e., $U(c;\Lambda) = U(c)$, and $J(h;\Lambda) = J(h)$. We will see that in this case the non-price-taker's time t, state ω consumption $c_m(t,\omega)$ only affects the state price at that time and state $\xi(t,\omega)$, and hence the analysis and interpretation are much simplified. We may define a mapping

$$\xi(t,\omega) = \Xi(c_m(t,\omega); t, \omega),$$

to represent the effect of agent m's consumption on the state price at a time and state. Then the first order conditions of the non-price-taker, m, are

$$u'_m(\hat{c}_m(t)) = y_m \left[\xi(t) + \Xi'(\hat{c}_m(t); t, \omega)(\hat{c}_m(t) - \delta_m(t))\right], \quad t \in [0,T], \tag{7}$$

where the constant y_m is such that the solution $\hat{c}_m(t)$ satisfies

$$E\left[\int_0^T \xi(t)\hat{c}_m(t)dt\right] = E\left[\int_0^T \xi(t)\delta_m(t)dt\right]. \tag{8}$$

For the non- price-taker, the incremental satisfaction from an extra unit of consumption at time t and state ω is proportional to the cost $\xi(t,\omega)$ of that extra unit of consumption, plus an additional "cost" term due to the direct effect of this incremental change in $c(t,\omega)$ on the price of consumption $\xi(t,\omega)$.

Due to the fact that he affects prices the non-price-taker has an extra term on the right-hand side of his first order condition. Inspection of equation (7) provides some clues about the non-price-taking agent's behavior. It seems reasonable to assume that the state price density at time t, state ω is increasing in his consumption in that state (i.e., $\Xi'(\hat{c}_m(t,\omega);t,\omega) > 0$), since the state price represents the price of a unit of consumption at that time in that state. (We will see later that this is consistent with prices adjusting to clear the market, for concave utility functions.) So when $(\hat{c}_m(t) - \delta_m(t))$ is positive the additional term in m's first order condition is positive implying $u'_m(\hat{c}_m(t))$ is higher, or $\hat{c}_m(t)$ is lower, than if the additional term were not present; and vice versa when $(\hat{c}_m(t) - \delta_m(t))$ is negative. So the presence of the additional term always causes the non-price-taker to deviate towards his "own" dividend, $\delta_m(t)$. The intuition for this behavior is that when $\hat{c}_m(t)$ is greater than $\delta_m(t)$ the non-price-taker is a net "buyer" of consumption in that state; he is consuming more than he is entitled to from his initial endowment. So it is in his interest to reduce $\xi(t)$, the price of consumption in that state, and he believes he can do this by decreasing his consumption. Similarly, when $\hat{c}_m(t)$ is less than $\delta_m(t)$ he is a net seller of consumption. Note also that the extent of his deviation towards his dividend depends on the magnitude of $\Xi'(\hat{c}_m(t,\omega);t,\omega)$. This term is a measure of how much the non-price-taker is able to affect prices and so how much it is worthwhile for him to deviate from his price-taking behavior.

Equilibrium in an economy with one non-price-taker and N price-takers is defined as a set of prices and consumption-portfolio processes $(c^*_m(t), \pi^*_m(t), c^*_n(t), \pi^*_n(t))$ such that the price-takers choose their optimal consumption and portfolio strategy at the given prices, the non-price-taker chooses his optimal consumption and portfolio strategy taking account of the fact that prices respond to clear the market, and the prices are such that the good and the security markets do clear, i.e.,

$$c^*_m(t) + \sum_{n=1}^{N} c^*_n(t) = \delta(t), \quad t \in [0,T], \tag{9}$$

$$\pi^*_{mi}(t) + \sum_{n=1}^{N} \pi^*_{ni}(t) = P_i(t), \quad t \in [0,T], \quad i = 1, ..., L, \tag{10}$$

$$X^*_m(t) + \sum_{n=1}^{N} X^*_n(t) = \sum_{i=1}^{L} P_i(t), \quad t \in [0,T]. \tag{11}$$

The non-price-taker determines his equilibrium allocations recognizing the fact that his dynamic consumption-portfolio choice affects the Arrow-Debreu prices. Let us now determine the dependence of the state prices on his choice of consumption. Recalling equation (6), clearing in the consumption good (9) implies

$$c_m(t) = \delta(t) - J(\xi(t)). \tag{12}$$

This expression can be interpreted as the "residual supply curve", analogous to the notion of residual demand in the price leadership model of oligopoly theory (e.g., Varian (1992, Chapter 16)). So we have the following behavior for $\xi(t)$ as a function of the non-price-taker's consumption:

$$\xi(t) \equiv \Xi(\hat{c}_m(t); t, \omega) = U'(\delta(t) - c_m(t)). \tag{13}$$

Hence

$$\Xi'(\hat{c}_m(t); t, \omega) = U''(\delta(t) - c_m(t)),$$

which is positive for concave utility functions, as posited earlier. We can substitute this into the non-price-taker's first order condition to obtain equation (14).

Proposition 1: *The necessary and sufficient condition for equilibrium is that there exists a process $c_m^*(t)$ and a nonnegative number y_m satisfying*

$$u_m'(c_m^*(t)) = y_m \left[U'(\delta(t) - c_m^*(t)) U''(\delta(t) - c_m^*(t))(c_m^*(t) - \delta_m(t)) \right], \quad t \in [0, T], \tag{14}$$

and

$$E\left[\int_0^T U'(\delta(t) - c_m^*(t)) c_m^*(t) dt\right] = E\left[\int_0^T U'(\delta(t) - c_m^*(t)) \delta_m(t) dt\right], \tag{15}$$

given that agents n follow their optimal consumption determined from equation (4) and that $\xi^*(t)$ clears the consumption good market, i.e., satisfies (12).

Assume the non-price-taker's utility function is defined over some domain (c_∞, ∞), where $c_\infty \geq -\infty$, and satisfies $\lim_{c\to\infty} u_m'(c) = 0$ and $\lim_{c\to c_\infty} u_m'(c) = \infty$. Assume the representative price-taking agent's utility function is of the HARA form, i.e., $U(c) = \frac{1-\gamma}{\gamma}\left(\frac{\beta c}{1-\gamma} + \eta\right)^\gamma$, where $\beta > 0$, $\eta \geq 0$, $\gamma \neq 1$, $\eta = 1$ if $\gamma = -\infty$, defined over the domain where $\left(\frac{\beta c}{1-\gamma} + \eta\right) > 0$. Then, for agents' "own" dividend processes, $\delta_m(t)$ and $\sum_{n=1}^N \delta_n(t)$ within the domains of their respective utility functions, i.e., $\delta_m(t) > c_\infty$ and $\left(\frac{\beta \sum_{n=1}^N \delta_n(t)}{1-\gamma} + \eta\right) > 0$, $t \in [0, T]$, a.s., the equilibrium in the non-price-taking economy exists and is unique.

We will assume from now on that equilibrium exists. As seen in Proposition 1 this is indeed the case for the HARA family of utility functions. The HARA family includes power utility ($\beta = 1$, $\eta = 0$, $\gamma < 1$), log utility ($\beta = 1$, $\eta = 0$, $\gamma = 0$) and negative exponential utility ($\gamma = -\infty$, $\eta = 1$), some of the most commonly used utility functions in finance. Note also that for $u_m(c)$ and $U(c)$ both negative exponential (as in Section 5), since $c_\infty = -\infty$ and negative exponential is defined over all $c \in (-\infty, \infty)$, there are no restrictions on $\delta_m(t)$ and $\sum_{n=1}^N \delta_n(t)$. For power or log utility, the stated restrictions on $\delta_m(t)$ and $\sum_{n=1}^N \delta_n(t)$ correspond to requiring both to be positive.

If it exists, the solution to equation (14) will be of the form $c_m^*(t, \omega) = c_m^*(\delta(t, \omega), \delta_m(t, \omega); y_m)$. In a price-taking economy with heterogeneous agents, the equilibrium consumption allocations, $c_m(t)$ and $c_n(t)$, are only a function of the aggregate dividend $\delta(t)$. In the non-price-taking economy we have an additional state variable: $\delta_m(t)$, the dividend stream from m's initial asset endowments. An immediate consequence is that, unlike in a price-taking economy (Breeden (1979)), the equilibrium consumption streams of the $N + 1$ agents, $c_m^*(t), c_n^*(t)$, are no longer (instantaneously) perfectly correlated with each other and with the aggregate dividend, $\delta(t)$.

This in turn leads to a modified two-factor consumption CAPM, as will be demonstrated in Section 4.

As was posited earlier from equation (7), equation (14) reveals that the non-price-taker's consumption tends to move towards $\delta_m(t)$ compared with his price-taking equilibrium consumption. (Since y_m and $\xi(t)$ differ across economies we cannot say that $c_m^*(t)$ is always closer to $\delta_m(t)$ than is m's consumption in the price-taking economy, but it is reasonable to assume that this will be the case when agent m is enough of a net buyer or seller of consumption good.) This behavior leads us to the intuition that the instantaneous volatility of the difference between $c_m(t)$ and $\delta_m(t)$ is lower in the non-price-taking economy. We formalize this idea in Section 5 when we put more structure into the model. Rewriting equation (14) as

$$u_m'(c_m^*(t)) = y_m U'(\delta(t) - c_m^*(t)) \left[1 - \frac{U''(\sum_{n=1}^N c_n^*(t))}{U'(\sum_{n=1}^N c_n^*(t))}(c_m^*(t) - \delta_m(t))\right], \quad t \in [0,T], \qquad (16)$$

we see that the extent of agent m's deviation towards $\delta_m(t)$ depends on the absolute risk aversion, $-U''/U'$, of the representative price-taker. The more risk averse the price-takers, the less their consumption reacts to changes in the state price or, conversely, the more the state price reacts to their (and hence also the non-price-taker's) consumption and so the more it is in the non-price-taker's interest to deviate. In the limit of risk neutral price-takers, the non-price-taker cannot affect the state price at all and so he will not deviate from his price-taking behavior.

By substituting (13) into (14), we see that equilibrium implies the following condition on $\xi^*(t)$:

$$u'(\delta(t) - J(\xi^*(t))) = y_m \left[\xi^*(t) U''(J(\xi^*(t)))(\delta(t) - \delta_m(t) - J(\xi^*(t)))\right], \quad t \in [0,T].$$

If this equation has a solution it will be of the form $\xi^*(t,\omega) = \xi^*(\delta(t,\omega), \delta_m(t,\omega); y_m)$. Unlike in the price-taking case, $\xi^*(t)$ is also driven by the extra factor $\delta_m(t)$. Compared with the analogous equation in the price-taking economy, equation (15) has an additional term on its right hand side. Whenever $J(\xi^*(t))$ deviates from $\delta(t) - \delta_m(t)$, i.e., $\xi^*(t)$ deviates from $U'(\delta(t) - \delta_m(t))$, the additional term is non-zero, and so the equilibrium state price is affected by the presence of the non-price-taking agent. Since $U''(\cdot) < 0$, when $\xi^*(t) > U'(\delta(t) - \delta_m(t))$, the extra term is positive and so tends to decrease $\xi^*(t)$, since the left hand side is decreasing in $\xi^*(t)$. When $\xi^*(t) < U'(\delta(t) - \delta_m(t))$, the extra term increases $\xi^*(t)$. Hence the extra term always causes $\xi(t)$ to move closer to $U'(\delta(t) - \delta_m(t))$. This arises from the consumption behavior of the non-price-taking agent discussed earlier.

Equation (16) suggests that, in addition to the risk aversion of the representative price-taker, the extent of the non-price-taker's effect on the economy is also driven by $(\delta(t) - c_m^*(t))$, or how much of a trader he is. The following proposition establishes formally that in the limit when agent m does not trade then there is no effect on the equilibrium of making him a non-price-taker.

Proposition 2:
(a) Suppose the agents' initial endowments are such that agent m does not trade in equilibrium in the economy where all are price-takers. Then in equilibrium in the economy where agent m is a non-price-taker, he will not deviate from his price-taking consumption-portfolio strategy.

(b) Similarly, if equilibrium in the non-price-taking economy is such that agent m does not trade, then he must not be deviating from his equilibrium price-taking strategy.

(c) If the initial endowments are such that agent m does trade in equilibrium in the economy where all are price-takers, in the non-price-taking equilibrium the non-price-taker m will deviate from his price-taking strategy.

Earlier we discussed the intuition for the non-price-taking agent's consumption behavior compared with the price-taking case. So why does he not deviate to exercise his price-affecting power in the case where his price-taking optimal behavior was to not trade and consume his "own" dividend? We will argue that he always makes himself worse off by deviating. Suppose in some state he increases his consumption a little so that $c_m(t) > \delta_m(t)$. Now he is a net buyer of consumption good in that state. Unfortunately, by increasing $c_m(t)$ he simultaneously increases $\xi(t)$, i.e., raises the price of consumption in that state, which will affect him adversely. If he decreases $c_m(t)$, he will be a net seller of consumption but at the same time reduce its price; deviation in either direction has an adverse effect. Part (c) of Proposition 2 states that generically the non-price-taking equilibrium will differ from the price-taking equilibrium.

Note that, although seemingly intuitive, the condition $e_m = 0$ does not always imply that the non-price-taker does not deviate from his price-taking behavior. So, even if he has no initial wealth it may still be the case that he changes the equilibrium by being a non-price-taker. For example, for the CARA utility and one risky asset case considered in Section 5, it is not $e_m = 0$ but $e_m = 1/2$ for which there is no deviation from the price-taking equilibrium. Both Proposition 2 and this example illustrate that the non-price-taker's influence on the equilibrium does not depend on how wealthy he is, but on how much of a trader he is. For the case of two CARA utility agents where one has no initial wealth, the agents do still trade in the price-taking equilibrium and so the condition for Proposition 2 part (a) does not hold. In the case of CRRA (constant relative risk aversion) utility agents, however, if agent m has no initial wealth ($e_m = 0$) he will never consume and never trade in the price-taking economy and so, as stated by Proposition 2 part (a), his non-price-taking strategy will not deviate from his price-taking one. The same is true for any other utility function satisfying $\lim_{c \to 0} u'(c) = \infty$ because then the consumption can never go negative and so an agent with no wealth can never afford to consume or trade. The reader should keep in mind, though, that the reason he has no effect is not that he has no wealth, but that he does not trade at all.

An appropriate question is whether the non-price-taker gains any advantage through taking into account his effect on prices. Can he do better than if he were a price-taker? Proposition 3 states that indeed the non-price-taker will be at least as well off as if he were a price-taker. This is because when solving his optimization problem the non-price-taking agent always has the option of choosing his price-taking equilibrium consumption.

Proposition 3: *In equilibrium, in the non-price-taking economy, agent m's expected lifetime utility from consumption is greater than or equal to that in equilibrium in the price-taking economy, i.e.,*

$$E\left[\int_0^T u_m(c_m^*(t))dt\right] \geq E\left[\int_0^T u_m(\bar{c}_m(t))dt\right].$$

We have seen in this section that, unlike for the price-takers, the non-price-taker's equilibrium consumption is not simply proportional to the state price. Hence his marginal rate of substitution between consumption in different states and times is not necessarily equal to the ratio of the state prices, whereas each agent n's is. As stated below in Proposition 4, this discrepancy between agents' marginal rates of substitution, implies that the non-price-taking equilibrium consumption allocations are not pareto optimal.

Proposition 4: *If the non-price-taking equilibrium differs (with probability greater than zero) from the price-taking equilibrium, then the non-price-taking equilibrium allocations are not pareto optimal.*

Remark 1: In this paper we solve the problem where the agents choose their plans at time 0 and do not subsequently deviate from these plans, i.e., we have characterized a self-commitment solution. A natural question to ask in this setting is whether the agents have any incentive to deviate at a later date, in other words, whether their strategies are time-inconsistent (Sargent (1987, p.11), Merton(1990, p.177)).

To address this issue, we look at each agent's optimization problem from some intermediate time $s \in (0, T)$ onwards, for given asset holdings at s, i.e.,

$$\max_{c_k(\cdot)} E\left[\int_s^T u_k(c_k(t))dt \mid \mathcal{F}_s\right]$$

$$\text{subject to} \quad E\left[\int_s^T \xi(t)c_k(t)dt \mid \mathcal{F}_s\right] \leq \xi(s)\alpha_{k0}(s)P_0(s) + E\left[\int_s^T \xi(t)\delta_k^s(t)dt \mid \mathcal{F}_s\right].$$

where $\delta_k^s(t) \equiv \sum_{i=1}^L \alpha_{ki}(s)\delta_i(t)$. It is well-known that the standard price-taking (i.e. agent n's) strategy is time-consistent. This can be seen by noting that his first order conditions of the above problem again lead to equation (4), where y_n must be the same number as before since his commitment solution must solve his budget constraint from time s onwards.

On the other hand the non-price-taker's strategy is not time-consistent. For simplicity, take a time and state where $\alpha_{m0}(s) = 0$. This assumption allows us not to concern ourselves with the bond price but does not affect any of our conclusions. The first order conditions of the above problem become

$$u_m'(c_m(t)) = y_m^s\left[U'(\delta(t) - c_m(t)) - U''(\delta(t) - c_m(t))(c_m(t) - \delta_m^s(t))\right], \quad t \in [s, T],$$

where y_m^s is such that the non-price-taker's budget constraint at time s holds with equality. Generically the solution will differ from the commitment solution, since $\delta_m^s(t)$ differs from $\delta_m(t)$.

Note that this solution of the problem from time s onwards has some of the familiar features associated with our commitment solution. Now the non-price-taker deviates towards $\delta_m^s(t)$, the dividend stream from his time s asset holdings, and we may think of $c_m(t)-\delta_m^s(t)$ as representing how much of a net buyer or seller of consumption good he is. Much of our intuition can be reapplied to this solution.

Since the non-price-taker has an incentive to deviate from his commitment strategy as time unfolds, it would be valuable for comparison to also solve for his subgame perfect strategy, by backward induction. Our preliminary analysis suggests that this problem is intractable in our framework for the type of non-price-taking agents we are considering. So the commitment solution may be considered as a tractable benchmark solution to a problem that is more generally intractable. One comparison that can be made is that the non-price-taker is better off to follow his commitment strategy than his subgame perfect strategy. This is clear because the subgame perfect strategy must satisfy his $[0,T]$ budget constraint with $\xi(t)$ always satisfying equation (13), and by definition, the commitment strategy maximizes his expected lifetime utility over all strategies which satisfy these two conditions. When looking for the subgame perfect strategy we are restricting the non-price-taker to only follow strategies which are optimal in all subgames.

Given this discussion, one interpretation of our solution in this paper is that we are expanding the strategy space of the agents, from the space of only subgame perfect strategies, to include commitment strategies. Given the non-price-taker is better off by following a self-commitment strategy, it is reasonable to assume that he will create some mechanism to force himself to commit, and so it is not unreasonable to include these types of strategies. In this paper we do not attempt to discuss the means by which the non-price-taker might force himself to commit, instead we focus on the dynamic consumption and price behavior.

Remark 2: Instead of the agents being initially endowed with units of positive net supply assets paying dividend streams $\delta_m(t)$ and $\delta_n(t)$, we could have modeled them as having continuous endowment streams $\epsilon_k(t)$ in the consumption good. The agents would then create zero net supply securities to complete the market and the economy would be effectively the same as we have here. The extra state variable would then be $\epsilon_m(t)$ instead of $\delta_m(t)$.

3.2 Extension to Representative Price-Taking Agent Not Independent of Individual Weights

Here we extend the analysis of the previous subsection to the case of one non-price-taker and multiple price-takers whose representative agent utility function is not independent of the individual price-takers' weights. Recalling equation (6) clearing in the consumption good (9) implies $\xi(t)$ and $c_m(t)$ are related by

$$c_m(t) = \delta(t) - J(\xi(t);\Lambda),$$

or

$$\xi(t) = U'(\delta(t) - c_m(t);\Lambda). \tag{17}$$

The significant difference between these expressions and equations (12)-(13) is that now agent m's consumption at time t in state ω, depends not only on $\xi(t,\omega)$ but also on the weights Λ. Since these weights are determined from the budget constraints of the price-taking agents (equation (5)) and since the budget constraints are in terms of present values, driven by the whole process of $\xi(\cdot)$, $c_m(t,\omega)$ depends on the whole process $\xi(\cdot)$. Alternatively, we may state that $c_m(t,\omega)$ affects the whole state price density process $\xi(\cdot)$; there is no longer a mapping from $c_m(t,\omega)$ to $\xi(t,\omega)$. Now, when the non-price-taker chooses his optimal consumption he has to worry about the wealth effects of this choice, since the other agents' weights now appear in equation (17). He has to be concerned about the externalities he is imposing on the other agents by his choice of consumption (and hence state price process), which will determine the distribution of wealth across these agents. As a result the analysis of his optimization problem becomes much more complicated, as can be seen from his first order condition, presented in Proposition 5.

Proposition 5: *The necessary and sufficient condition for equilibrium is that there exists a process $c_m^*(t)$ and $N+1$ nonnegative numbers $(y_m, y_1, ..., y_N)$ satisfying*

$$u_m'(c_m^*(t)) = y_m\left[U'(\delta(t)-c_m^*(t);\Lambda) - U''(\delta(t)-c_m^*(t);\Lambda)(c_m^*(t)-\delta_m(t))\right]$$
$$-\sum_{n=1}^{N} K_n \frac{U''(\delta(t)-c_m^*(t);\Lambda)}{u_n''(I_n(y_nU'(\delta(t)-c_m^*(t);\Lambda)))} *$$
$$\left[y_nU'(\delta(t)-c_m^*(t);\Lambda) + u_n''(I_n(y_nU'(\delta(t)-c_m^*(t);\Lambda)))(I_n(y_nU'(\delta(t)-c_m^*(t);\Lambda)) - \delta_m(t))\right], \tag{18}$$

$$E\left[\int_0^T U'(\delta(t)-c_m^*(t);\Lambda)c_m^*(t)dt\right] = E\left[\int_0^T U'(\delta(t)-c_m^*(t);\Lambda)\delta_m(t)dt\right], \tag{19}$$

and

$$E\left[\int_0^T U'(\delta(t)-c_m^*(t);\Lambda)I_n(y_nU'(\delta(t)-c_m^*(t);\Lambda))dt\right] = E\left[\int_0^T U'(\delta(t)-c_m^*(t);\Lambda)\delta_n(t)dt\right], \quad n=1,...,N, \tag{20}$$

given that agents n follow their optimal consumption determined from equation (4) and that $\xi^*(t)$ clears the consumption good market, i.e., satisfies (17). The constants K_n are given in the proof in Appendix B.

If $U(c;\Lambda) = h(\Lambda)U(c)$, then $K_n = 0$ for all n.

Equation (18) is similar to equation (14) but with N extra terms on the right hand side. Again, the incremental satisfaction the non-price-taker gets from an extra unit of consumption at time t and state ω must be equal to the total "costliness" to him of that extra unit of consumption. As in Section 3.1, the first and second terms on the right hand side of (18) are the cost $\xi(t,\omega)$ of that extra unit of consumption and the costliness to him due to the direct effect of $c_m(t;\omega)$ on $\xi(t,\omega)$. Note that the second term will again have the effect of making $c_m^*(t)$ tend towards $\delta_m(t)$ as compared with a price-taking economy.

In this case, however, an extra unit of consumption is costly to agent m in a third way, represented by the extra N terms in (18). The non-price-taker also realizes now that an extra unit of $c_m(t;\omega)$ can affect the effective wealths of the other agents which in turn affect the whole state price process $\xi(\cdot)$, and hence his satisfaction at all other times and states. In determining his optimal consumption process $c_m^*(t)$ he must also take this into account. For example, it may turn out that it is in his interests to make price-taker 3 more wealthy relative to price-taker 1, and these extra terms capture this awareness. We argue in the proof of Proposition 5 that the extra terms in (18) are indeed the indirect incremental change in agent m's expected lifetime utility $E\left[\int_0^T u_m'(c_m(t)dt\right]$ via the effect of an extra unit of $c_m(t;\omega)$ on each of the other agent's budget constraints.

As a final note, the last statement of Proposition 5 shows that expression (18) indeed collapses to our previous expression (14) in the case when the price-taker representative agent utility function is independent of the individual weights.

4 The Equilibrium Interest Rate and the Consumption-Based CAPM

Here we look more closely at the effect of the presence of a non-price-taking agent on equilibrium asset and state prices, and on the consumption-based CAPM. We will discuss only the simpler case of representative price-taking agent's utility independent of individual weights, as in Subsection 3.1.

It is well-known that in an economy with $N+1$ price-taking agents with (possibly heterogeneous) time-additive, state-separable preferences, the equilibrium interest rate $\bar{r}(t)$ is given by

$$\bar{r}(t) = \bar{\lambda}_\delta(t)\mu_\delta(t) + \bar{\nu}_\delta(t)\|\sigma_\delta(t)\|^2; \quad \bar{\lambda}_\delta(t) > 0$$

where $\bar{\lambda}_\delta(t) \equiv -V''(\delta(t);\bar{Y})/V'(\delta(t);\bar{Y})$, $\bar{\nu}_\delta(t) \equiv -V'''(\delta(t);\bar{Y})/(2V'(\delta(t);\bar{Y}))$ and $\bar{Y} \equiv (\bar{y}_m, \bar{y}_1, ..., \bar{y}_N)$. Here $V(\cdot;\bar{Y})$ is the utility function of the representative agent of all agents (including m). (This representation is just the pure-exchange, multi-agent version of Cox, Ingersoll and Ross (1985).) The interest rate depends only on the aggregate dividend and its moments. Since $\bar{\lambda}_\delta(t) > 0$ for concave utility functions, the interest rate is positively related to $\mu_\delta(t)$. Furthermore, if $V'''(\delta(t);\bar{Y}) > 0$ then the interest rate process is negatively related to the variance of the aggregate dividend $\|\sigma_\delta(t)\|^2$.

The pure exchange version of Breeden's CCAPM (1979, 1986) with two price-taking agents is (see Duffie and Zame (1989) or Karatzas, Lehoczky and Shreve (1990))

$$\bar{\mu}(t) - \bar{r}(t)\mathbf{1} = \bar{\lambda}_\delta(t)\text{cov}\left(\frac{d\bar{P}(t)}{\bar{P}(t)}, d\delta(t)\right); \quad \bar{\lambda}_\delta(t) > 0.$$

The stocks' risk premia are driven by a single factor $\delta(t)$. The risk premium of an asset is positively related to the (instantaneous, conditional) covariance of its return with $d\delta(t)$, the change in the aggregate consumption.

In our economy with N price-taking agents and one non-price-taking agent, the interest rate and the risk premia are now driven by two factors, $\delta(t)$ and $\delta_m(t)$, as summarized in Proposition 6.

Proposition 6: *In an economy with N price-taking and one non-price-taking agent the equilibrium interest rate is given by:*

$$r^*(t) = \eta_\delta^*(t)\mu_\delta(t) + \eta_{\delta_m}^*(t)\mu_{\delta_m}(t) + \nu_\delta^*(t)\|\sigma_\delta(t)\|^2 + \nu_{\delta_m}^*(t)\|\sigma_{\delta_m}(t)\|^2 + \kappa^*(t)\sigma_\delta(t)^\top\sigma_{\delta_m}(t), \quad (21)$$

and the risk premia of risky securities are given by:

$$\mu^*(t) - r^*(t)\mathbf{1} = \lambda_\delta^*(t)cov\left(\frac{dP(t)}{P(t)}, d\delta(t)\right) + \lambda_{\delta_m}^*(t)cov\left(\frac{dP(t)}{P(t)}, d\delta_m(t)\right), \quad (22)$$

where $\mu_{\delta_m}(t)$ and $\|\sigma_{\delta_m}(t)\|$ are the drift and volatility of $\delta_m(t)$, and

$$\lambda_\delta^*(t) \equiv -\frac{U''(\delta(t) - c_m^*(t))}{U'(\delta(t) - c_m^*(t))}\left\{\frac{u_m''(c_m^*(t)) + y_m U''(\delta(t) - c_m^*(t))}{u_m''(c_m^*(t)) - y_m U'''(\delta(t) - c_m^*(t))(c_m^*(t) - \delta_m(t)) + 2y_m U''(\delta(t) - c_m^*(t))}\right\}$$

$$\lambda_{\delta_m}^*(t) \equiv \frac{y_m U''(\delta(t) - c_m^*(t))}{u_m''(c_m^*(t)) + y_m U''(\delta(t) - c_m^*(t))}\lambda_\delta^*(t).$$

The interest rate now depends on both $\delta(t)$ and $\delta_m(t)$. The coefficients (η, ν, κ) in the interest rate formula for general utility functions are rather complicated, with ambiguous signs, and so not especially illuminating. Hence we do not present them here. For general utility functions the signs of the dependence of $r^*(t)$ on $\mu_\delta(t)$, $\mu_{\delta_m}(t)$, $\|\sigma_\delta(t)\|^2$, $\|\sigma_{\delta_m}(t)\|^2$ and $\sigma_\delta(t)^\top\sigma_{\delta_m}(t)$ are not clear.

The CCAPM is now a two-beta CCAPM, driven by the covariance of the price return with both $d\delta(t)$ and $d\delta_m(t)$. When $U(\cdot)$ is strictly concave, both λ's are nonzero and have the same sign, so the dependences of an asset's risk premium on the covariance of its return with the non-price-taker's dividend and with the aggregate consumption are of the same sign. At the equilibrium, λ_δ and λ_{δ_m} are positive for price-taker representative agent with HARA utility under the conditions required for the existence in Proposition 1.[1] It is an open question whether for other utility functions the λ's can go negative, depending on the signs (and sizes) of $U'''(\cdot)$ and $(c_m^*(t) - \delta_m(t))$. If so, the usual implication of the consumption CAPM would fail; the risk premium would be negatively related to the covariance of the asset return with changes in the aggregate consumption. A traditional one-factor consumption CAPM arises when $\delta(t)$ and $\delta_m(t)$ are (instantaneously) perfectly correlated, collapsing the two terms together. For the case of one risky asset, for example, $\delta(t)$ and $\delta_m(t)$ are perfectly correlated.

[1] A sufficient condition for the λ's to be positive is that

$$U'''(\delta_m(t) - c_m^*(t))(c_m^*(t) - \delta_m(t)) - 2U''(\delta_m(t) - c_m^*(t)) \geq 0.$$

This term is $R'(c_m^*(t); y_m)/y_m$ in the proof of Proposition 1 (Appendix B) where it is shown to always be nonnegative for HARA representative price-taker at the equilibrium.

5 The CARA Utility and One Risky Asset Case

In order to derive further results, we now put more structure into our general set-up by specializing to the case of one risky asset (and one dimension of uncertainty) and to a specific utility function for both agent m and the price-taker representative agent. From now on, we will talk of the price-takers as one agent, although this could be a representative agent over multiple price-takers. There is still a risk-free bond in addition to the one risky asset. With power or log utility, in the price-taking equilibrium (when m is also a price-taker) with one risky asset each agent holds on to his initial endowment of the risky asset and no trading occurs. Hence, according to Proposition 2, a power or log utility non-price-taking agent does not deviate from his price-taking behavior and there is no effect on the optimal strategies or equilibrium prices. So we choose the case of exponential utility in which trading does take place in the price-taking equilibrium even in one risky asset. The utility function of both agents is of the form $u(c) = -\exp\{-ac\}/a; \quad a > 0$. The agents are initially endowed with e_m and e_n units of the risky asset; then $\delta_k(t) \equiv e_k\delta(t)$.

For some of the results in this section we will make the following assumption on the dividend process.[2]

Assumption A1: $\mu_\delta(t) \equiv \mu_\delta$, $\sigma_\delta(t) \equiv \sigma_\delta$ are constants, i.e., the dividend process is driven by an arithmetic Brownian motion.

We will generally assume that $\mu_\delta(t)$ is positive, i.e., that the economy is expanding.

We specified an economy with one representative price-taker and one non-price-taker in Section 3. Let us here define a benchmark "price-taking" economy as one with two exponential utility price-taking agents. Each agent m, n solves his optimization problem at given prices. The equilibrium price processes are such that all markets clear. It can be shown that the equilibrium state price density process in this price-taking economy is

$$\bar{\xi}(t) = \frac{1}{\bar{y}_m^{1/2}} \exp\left\{-\frac{1}{2}a\delta(t)\right\}, \tag{23}$$

and the equilibrium consumption processes are

$$\bar{c}_m(t) = \frac{1}{2}\delta(t) - \frac{1}{2a}\ln(\bar{y}_m), \tag{24}$$

$$\bar{c}_n(t) = \frac{1}{2}\delta(t) + \frac{1}{2a}\ln(\bar{y}_m), \tag{25}$$

where $\bar{y}_m$ is given by

$$\frac{1}{2a}\ln(\bar{y}_m) = \left(\frac{1}{2} - e_m\right) \frac{E\left[\int_0^T \delta(t)\exp\left\{-\frac{1}{2}a\delta(t)\right\} dt\right]}{E\left[\int_0^T \exp\left\{-\frac{1}{2}a\delta(t)\right\} dt\right]}. \tag{26}$$

[2]This process has the undesirable feature that the dividend can go negative since it is normally distributed at each point in time. However, the probability of this happening can be made arbitrarily small, and hence we will not concern ourselves with this inconvenience.

When $e_m > 1/2$, agent m consumes more than half of the dividend $\delta(t)$, and when $e_m < 1/2$ he consumes less than half of $\delta(t)$. We will use the notation $\bar{}$ to denote equilibrium quantities in the benchmark economy and the notation * to denote equilibrium quantities in the non-price-taking economy. In this section we compare the equilibrium consumption allocations and their dynamics in the two economies. We then compare state and market price dynamics, as well as the equilibrium portfolio strategies and wealths of the agents. The results for the benchmark economy are straightforward to derive and will often be quoted without proof.

5.1 The Equilibrium Consumption Allocations

In the non-price-taking economy the equilibrium consumptions of the two agents are given by (14) and by (6) where ξ satisfies (13). Hence for exponential utility $c_m^*(t)$ and $c_n^*(t)$ solve

$$c_m^*(t) = \frac{1}{2}\delta(t) - \frac{1}{2a}\ln\left[1 + a(c_m^*(t) - \delta_m(t))\right] - \frac{1}{2a}\ln(y_m), \tag{27}$$

$$c_n^*(t) = \frac{1}{2}\delta(t) + \frac{1}{2a}\ln\left[1 + a(c_m^*(t) - \delta_m(t))\right] + \frac{1}{2a}\ln(y_m), \tag{28}$$

where y_m is such that the process $c_m^*(t)$ satisfies

$$E\left[\int_0^T c_m^*(t)\exp\left\{-a(\delta(t) - c_m^*(t))\right\}dt\right] = e_m E\left[\int_0^T \delta(t)\exp\left\{-a(\delta(t) - c_m^*(t))\right\}dt\right].$$

In Proposition 1 we established the existence of equilibrium for this case.

In the benchmark economy the agents consumed half each of the dividend plus or minus a constant depending on their initial endowments. In the non-price-taking economy there are additional terms which are not constants. As discussed in Section 3, the agents' consumption streams depend on $\delta_m(t)$ as well as $\delta(t)$, although in this case of one asset $\delta_m(t)$ is perfectly correlated with $\delta(t)$.

The following proposition compares the levels of consumption of agent m across economies, and the levels of consumption of agent m and n within economies.

Proposition 7:
(a) For $e_m > 1/2$:

1. *When $\delta(t) > \delta_{crit}$, $c_m^*(t) > \bar{c}_m(t)$; when $\delta(t) < \delta_{crit}$, $c_m^*(t) < \bar{c}_m(t)$; and when $\delta(t) = \delta_{crit}$, $c_m^*(t) = \bar{c}_m(t)$, where δ_{crit} is the constant given by*

$$\delta_{crit} = \frac{1}{a(e_m - 1/2)}\left[1 - \frac{\bar{y}_m}{y_m} - \frac{1}{2}\ln(\bar{y}_m)\right].$$

2. *Although $\bar{c}_m(t) > \bar{c}_n(t)$ almost surely in the benchmark economy, in the non-price-taking economy we may have $c_m^*(t) < c_n^*(t)$.*

(b) *For* $e_m < 1/2$:

1. *When* $\delta(t) > \delta_{crit}$, $c_m^*(t) < \bar{c}_m(t)$; *when* $\delta(t) < \delta_{crit}$, $c_m^*(t) > \bar{c}_m(t)$; *and when* $\delta(t) = \delta_{crit}$, $c_m^*(t) = \bar{c}_m(t)$.

2. *Although* $\bar{c}_m(t) < \bar{c}_n(t)$ *a.s. in the benchmark economy, in the non-price-taking economy we may have* $c_m^*(t) > c_n^*(t)$.

(c) *For* $e_m = 1/2$: $c_m^*(t) = \bar{c}_m(t)$, $\forall\ \delta(t)$.

From part (a)1. of Proposition 7 we see that when the non-price-taking agent initially owns more than half of the market, as the aggregate dividend increases above a critical level he tends to consume more than when he is a price-taker and as it decreases below the critical level he tends to consume less; in a sense he is reacting more to changes in aggregate dividend. The intuition for this can be seen by considering m's consumption behavior in the price-taking economy. It is straightforward to show that in the price-taking economy, for $e_m > 1/2$, when the aggregate dividend is relatively high, agent m consumes less than his endowment, i.e., in that state, he is a net seller of consumption, and so would like the price of consumption to be high. Hence when we move to the economy in which agent m is a non-price-taker, we see agent m increasing his consumption, and thereby increasing the price of consumption in that state. He does the opposite when the aggregate dividend is low. For the case when the non-price-taker is initially less wealthy, the opposite happens, as stated in part (b)1. Part (c) of the proposition illustrates a special case of Proposition 2. When $e_m = 1/2$, in the benchmark economy $\ln(\bar{y}_m) = 0$ and so $\bar{c}_m(t) = \delta(t)/2 = \delta_m(t)$ and no trade takes place. Then there is no effect of the presence of one non-price-taking agent.

In the case of δ growing over time on average, Proposition 7 implies that, when initially endowed with more than half of the market, the non-price-taker on average postpones consumption to later in his lifetime relative to if he were a price-taker. When initially endowed with less than half of the market, he on average consumes more towards the beginning of his lifetime, i.e., is more impatient, than if he were a price-taker. It should be emphasized that δ_{crit} depends on the y_m's, which in turn depend on the distribution of the whole $\delta(t)$ process. So if we exogeneously increase the overall level of the $\delta(t)$ process, δ_{crit} increases too. It seems intuitive that no matter how we choose our exogeneous δ process there will always be both states in which $\delta(t) > \delta_{crit}$ and states in which $\delta(t) < \delta_{crit}$.

Parts 2. of Proposition 7 compare agents m and n within economies; note that in case (a), even though the non-price-taker initially owns more than half of the market, there will be states in which he chooses to consume less than the less-endowed price-taker.

Next we compare the dynamics of the agents' consumption streams across economies. We define the drift $\mu_{c_k}(t)$ and volatility $\sigma_{c_k}(t)$ of agent k's consumption by

$$dc_k(t) = \mu_{c_k}(t)dt + \sigma_{c_k}(t)dW(t), \quad k = m, n.$$

In the benchmark economy the equilibrium drift and volatility of agent m's (and agent n's) consumption are given by

$$\bar{\mu}_{c_m}(t) = \frac{1}{2}\mu_\delta(t), \quad \bar{\sigma}_{c_m}(t) = \frac{1}{2}\sigma_\delta(t).$$

Proposition 8: *In the non-price-taking economy, the equilibrium drift and volatility of the non-price-taking agent's consumption are given by*

$$\mu^*_{c_m}(t) = g(t)\mu_\delta(t) + \frac{2f(t)}{1+f(t)}a\left[g(t)^2 + g(t) - \frac{3}{4}\right]\sigma_\delta(t)^2, \tag{29}$$

and

$$\sigma^*_{c_m}(t) = g(t)\sigma_\delta(t), \tag{30}$$

where

$$f(t) \equiv \frac{1}{y_m}\exp\left\{-2a(c^*_m(t) - \frac{1}{2}\delta(t))\right\}; \quad f(t) > 0 \tag{31}$$

and

$$g(t,\omega) \equiv \frac{e_m + f(t,\omega)}{1+2f(t,\omega)} = \frac{\partial c^*_m(t,\omega)}{\partial\delta(t,\omega)}; \quad g(t,\omega) \in (0,1). \tag{32}$$

As a consequence, for $e_m > 1/2$, *we have* $|\sigma^*_{c_m}(t)| > |\bar{\sigma}_{c_m}(t)|$ *and* $\mu^*_{c_m}(t) > \bar{\mu}_{c_m}(t)$; *and for* $e_m < 1/2$, *we have* $|\sigma^*_{c_m}(t)| < |\bar{\sigma}_{c_m}(t)|$ *and* $\mu^*_{c_m}(t) < \bar{\mu}_{c_m}(t)$.

The comparative statics results of Proposition 8 are derived from the properties of the process $g(t)$, which captures how the non-price-taker reacts to changes in the aggregate dividend. For example, for $e_m > 1/2$, $g(t) > 1/2$ since the non-price-taker reacts more to changes in $\delta(t)$. We see that if the non-price-taking agent initially owns more than half of the market, he increases the drift and volatility of his consumption compared with the price-taking case. In other words, his consumption stream is riskier than when he is a price-taker. On the other hand, if he initially owns less than half of the market his consumption stream is less risky when he is a non-price-taker than when he is a price-taker. To see why these results hold, suppose that the non-price-taker initially owns 3/4 of the market and consider two consumption strategies: (1) he holds on to his initial endowment at all times (absorbing 3/4 of the dividend risk and growth), or (2) he follows his price-taking strategy (sharing the dividend risk and growth equally with the other agent). Recall from Section 3 that the non-price-taker deviates from his price-taking consumption (strategy 2) towards his endowment (strategy 1), and hence in this case he increases his consumption drift and volatility.

The following corollary captures the anticipated result that the non-price-taker tends to move his consumption closer to $\delta_m(t)$.

Corollary 1: *The instantaneous volatility of the difference between* $c_m(t)$ *and* $\delta_m(t)$, *i.e.,* $|\ \sigma_{c_m}(t) - \sigma_{\delta_m}(t)\ |$, *is lower in the economy where* m *is a non-price-taker than in the benchmark economy.*

Since agents derive utility from consumption, an interesting question is whether the non-price-taker is able to consume more on average than when he is a price-taker. Proposition 7 revealed that his consumption is sometimes lower and sometimes higher depending on the level of $\delta(t)$. Nevertheless, as a corollary to Proposition 8, we can compare expectations of future consumption by appealing to results for mean comparison of solutions to stochastic differential equations (Ikeda and Watanabe (1989), Hajek (1985)).[3] Existing mean comparison theorems do not accommodate comparisons between two processes both of whose drifts and volatilities are stochastic (unless one process is Markov), so we make assumption A1 in order to appeal to existing mean comparison theorems. The mean comparison theorem we use is valid for comparison between processes of which one has a dominating constant drift and volatility. This theorem proves Corollaries 2 and 3 which is for the case of $e_m < 1/2$; I anticipate that the analogous theorem is true to prove the corresponding cases for $e_m > 1/2$, but to my knowledge no such theorem exists.

Corollary 2: *Assume A1. Then for $e_m < 1/2$, when $\delta(t) > \delta_{crit}$*

$$E\left[c_m^*(s) \mid \mathcal{F}_t\right] \leq E\left[\bar{c}_m(s) \mid \mathcal{F}_t\right], \quad s > t,$$

and so

$$E\left[\int_t^T c_m^*(s)ds \mid \mathcal{F}_t\right] \leq E\left[\int_t^T \bar{c}_m(s)ds \mid \mathcal{F}_t\right].$$

If $\delta(t)$ drifts upwards on average, once $\delta(t)$ has passed δ_{crit} it will on average be above δ_{crit} in the future. For $e_m < 1/2$, c_m^* will be expected to be lower than $\bar{c}_m$ on average in the future.

Note that the results of Corollary 2 do not necessarily hold at the beginning of the period, since $\delta(0)$ may not satisfy the required condition. In fact, since δ drifts upwards, $\delta(0)$ will be relatively low compared with the average level of the process, so it is unlikely that $\delta(0)$ will be higher than δ_{crit}. Hence the comparisons will not tend to hold for the expected consumption integrated over the agent's lifetime.

5.2 The Equilibrium State Prices, Interest Rate and Market Price of Risk

We now begin to investigate the effect of the presence of the non-price-taker on the equilibrium prices. In the benchmark economy, $\bar{\xi}(t)$ is given by (23) and an application of Itô's Lemma yields the coefficients of its dynamics, the interest rate and the market price of risk, as

$$\bar{r}(t) = \frac{a}{2}\mu_\delta(t) - \frac{a^2}{8}\sigma_\delta(t)^2,$$
$$\bar{\theta}(t) = \frac{a}{2}\sigma_\delta(t).$$

In the non-price-taker economy, $\xi^*(t)$ satisfies equation (13). By applying Itô's Lemma for the case of exponential utility to (13) and making use of (29)-(30), we derive the following results.

[3] I thank Steve Shreve for bringing to my attention these mean comparison theorems of solutions to stochastic differential equations.

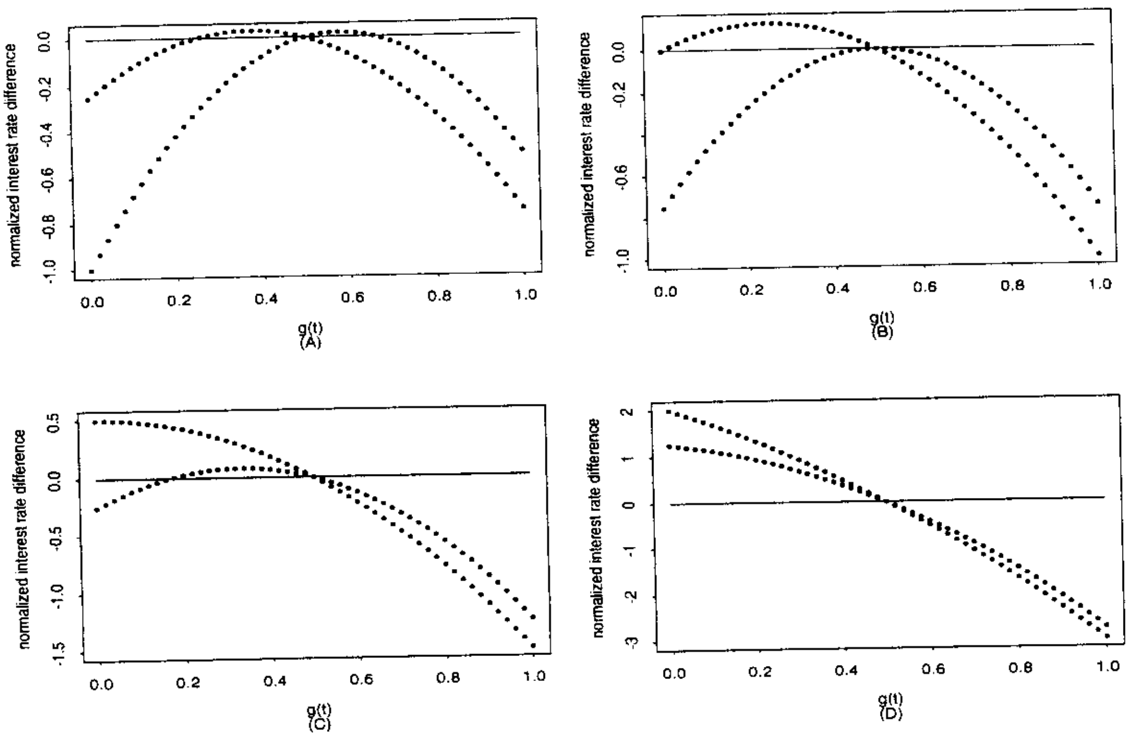

Figure 1: Normalized difference between interest rate in non-price-taking and price-taking economies, $r^*(t) - \bar{r}(t)$. $r^*(t) - \bar{r}(t)$ must lie between the dotted lines. Plotted for various choices of the exogeneous parameter $2\mu_\delta / a\sigma_\delta^2$: (A) 0.5, (B) 1, (C) 2, (D) 5.

Proposition 9: *In the non-price-taking economy, the equilibrium interest rate and market price of risk are given by*

$$r^*(t) = a(1 - g(t))\mu_\delta(t) - \frac{2f(t)}{1 + f(t)} a \left[g(t)^2 + g(t) - \frac{3}{4} \right] \sigma_\delta(t)^2 - \frac{a^2}{2}(1 - g(t))^2 \sigma_\delta(t)^2, \qquad (33)$$

$$\theta^*(t) = a(1 - g(t))\sigma_\delta(t), \qquad (34)$$

where $f(t)$ and $g(t)$ are defined as in equations (31) and (32).

As a consequence, if $e_m > 1/2$, $|\theta^*(t)| > |\bar{\theta}(t)|$; if $e_m < 1/2$, $|\theta^*(t)| < |\bar{\theta}(t)|$.

Proposition 9 reveals that, when the non-price-taker initially owns more than half of the market, he causes the market price of risk to be higher. In other words he makes the Arrow-Debreu state prices (the price of consumption) riskier. This follows from a result of previous section that he chooses a riskier consumption stream. If he initially owns less than half of the market he causes the Arrow-Debreu state prices to be less risky.

The interest rate is driven by both the drift and the volatility of agent m's consumption. Since an increase or a decrease in these two quantities affect the interest rate in opposite directions, comparisons between $r^*(t)$ and $\bar{r}(t)$ are not unambiguous as they are for $\theta(t)$. In

Figure 1 we plot ranges within which $[r^*(t) - \bar{r}(t)]/a$ must fall as a function of $g(t)$, for a series of choices of the exogeneous parameter $2\mu_\delta/a\sigma_\delta^2$. For example, in the price-taking economy, this exogeneous parameter is given by $2\mu_\delta/a\sigma_\delta^2 = r\sigma^2/(\mu - r)^2 + 1/2$. Matching "normal" market conditions of $r = 8\%, \sigma = 20\%, \mu = 15\%$, to the exogeneous parameters of the price-taking economy yields $2\mu_\delta/a\sigma_\delta^2 = 1.15$. In general, the higher the interest rate or the volatility of the market, the higher this parameter. From the definition of $g(t)$ we have that $g(t) > 1/2$ corresponds to $e_m > 1/2$ and $g(t) < 1/2$ corresponds to $e_m < 1/2$. So from the graphs it appears that for most market conditions, for $e_m > 1/2$ the interest rate in the non-price-taking economy is lower than in the benchmark economy. For large enough $2\mu_\delta/a\sigma_\delta^2$, $r^*(t) < \bar{r}(t)$ for $e_m > 1/2$ and $r^*(t) > \bar{r}(t)$ for $e_m < 1/2$.

Using the mean comparison theorems we can deduce conclusions about the expected growth of state prices across economies. We will see later that this comparison is relevant for asset price volatility and portfolio strategy comparisons across economies.

Corollary 3: *Assume A1. Then for* $e_m < 1/2$

$$E\left[\frac{\xi^*(s)}{\xi^*(t)} \mid \mathcal{F}_t\right] \leq E\left[\frac{\bar{\xi}(s)}{\bar{\xi}(t)} \mid \mathcal{F}_t\right], \quad s > t,$$

and so

$$E\left[\int_t^T \frac{\xi^*(s)}{\xi^*(t)} ds \mid \mathcal{F}_t\right] \leq E\left[\int_t^T \frac{\bar{\xi}(s)}{\bar{\xi}(t)} ds \mid \mathcal{F}_t\right].$$

Note that the first terms compared correspond to the price at time t of a bond paying a certain payout of one unit at time s; the second terms compared correspond to the price at time t of an annuity paying a certain payout of one unit at all times until the end of the horizon.

We mentioned before that, with $\delta(t)$ drifting upwards, when $e_m < 1/2$ the non-price-taker on average consumes more earlier on in his lifetime compared with when he is a price-taker. Hence he puts a lower value on a bond providing sure future consumption than if he were a price-taker. Although we cannot yet verify mathematically since the required mean comparison theorem does not exist, we anticipate that the opposite result holds for the reverse case: when $e_m > 1/2$ the non-price-taker on average postpones consumption to later in his lifetime compared with when he is a price-taker, hence valuing a bond that pays him sure future consumption more highly than if he were a price-taker.

5.3 Agents' Wealths and Portfolio Strategies

It is well-known (e.g., Cox and Huang (1989)) that, by the construction of the state price density process, the equilibrium optimally invested wealths (satisfying (3)) of the agents in either economy are given in terms of their equilibrium consumptions by

$$X_k(t) = \frac{1}{\xi(t)} E\left[\int_t^T \xi(s) c_k(s) ds \mid \mathcal{F}_t\right].$$

We can draw some conclusions about the agents' wealth from our discussions of their consumptions in Section 5.1. We have said in general that the non-price-taker's consumption tends to deviate towards his endowment, $\delta_m(t)$. From the above equation this might suggest that the non-price-taking agent's wealth tends to deviate towards the value of "his" dividend stream, $e_m P(t)$. This in turn suggests that the non-price-taker deviates his portfolio strategy towards not trading at all.

In the benchmark economy, the equilibrium portfolio strategies are

$$\bar{\alpha}_m(t) = \frac{1}{2}; \quad \bar{\alpha}_n(t) = \frac{1}{2}.$$

Regardless of their initial endowments, identical CARA utility agents share the risk equally in the economy, and hence each agent holds half of the risky asset. As long as $e_m \neq 1/2$ there is net riskless lending and borrowing in equilibrium in the benchmark economy. If $e_m > 1/2$, the initially wealthier agent m is a lender; if $e_m < 1/2$, agent m is a borrower.

Proposition 10 presents the trading strategies of the agents in the non-price-taking economy. These expressions are derived by appealing to the Clark formula and some properties of Malliavin derivatives. This use of Malliavin calculus has also been employed by Ocone and Karatzas (1991). We say more about Malliavin calculus in the next subsection.

Proposition 10: *Assume A1. The equilibrium portfolio strategies in the non-price-taking economy can be expressed as*

$$\alpha_m^*(t) = \frac{1}{2} + \frac{\sigma_\delta}{P^*(t)^2} E\left[\int_t^T \frac{\xi^*(s)}{\xi^*(t)}[g(s) - 1/2]ds \mid \mathcal{F}_t\right]$$
$$+\frac{\sigma_\delta}{P^*(t)^2} E\left[\int_t^T \frac{\xi^*(s)}{\xi^*(t)}[g(s) - g(t)](c_m^*(s) - \delta(s)/2)ds \mid \mathcal{F}_t\right],$$
$$\alpha_n^*(t) = \frac{1}{2} - \frac{\sigma_\delta}{P^*(t)^2} E\left[\int_t^T \frac{\xi^*(s)}{\xi^*(t)}[g(s) - 1/2]ds \mid \mathcal{F}_t\right]$$
$$-\frac{\sigma_\delta}{P^*(t)^2} E\left[\int_t^T \frac{\xi^*(s)}{\xi^*(t)}[g(s) - g(t)](c_m^*(s) - \delta(s)/2)ds \mid \mathcal{F}_t\right].$$

There are additional terms in the agents' portfolio strategies in the non-price-taking economy. Agents do not simply hold half each of the risky asset as they do in the benchmark economy.

Recall that $g(\cdot)$ captures how the non-price-taker reacts to changes in the dividend stream. If $e_m > 1/2$, we always have $g(s) > 1/2$, so the second term in the non-price-taker's portfolio strategy is positive. The presence of this term tends to make the non-price-taker hold more than half of the risky asset and the price-taker hold less than half of the risky asset in the non-price-taking economy. For $e_m < 1/2$, we have $g(s) < 1/2$, and hence the second term is negative, tending to make the non-price-taker hold less than half of the risky asset and the price-taker holds more than half. The sign of this term is independent of the assumptions made on the aggregate dividend process; its effect does not depend on the dividend drifting upwards. The sign of the third term on the other hand is not unambiguous and does depend on the drift of δ because of the $g(s) - g(t)$ term. This term measures the changes in the non-price-taker's reaction to dividend stream changes. It can be shown that the mapping $g(\delta(t,\omega), t, \omega)$ is increasing (decreasing) in $\delta(t,\omega)$ if $e_m > 1/2$ (if $e_m < 1/2$). So, if the economy is expanding, one would expect the quantity $g(s) - g(t)$ to be positive (negative) on average. The other quantity appearing in the third term, $c_m^*(s) - \delta(s)/2$, measures how much the non-price-taker is deviating from one half of the aggregate dividend. Since this is a highly state-dependent term, its sign is ambiguous.

In summary, without the contribution of this third term, we would conclude that if the non-price-taker is initially wealthier, he holds more of the risky asset than when he is a price-taker. For $e_m < 1/2$ he holds less than when he is a price-taker. Our intuition also suggests this result. Suppose the non-price-taker's initial endowment is 3/4 of the market. Since he tends to deviate from his price-taking strategy towards holding his initial endowment in the non-price-taking economy, his non-price-taking portfolio process will tend to deviate from 1/2 towards 3/4 suggesting he tends to increase his holding of the market. Further, since it appears that the non-price-taker deviates towards holding onto his endowment and not trading, there will be less net riskless lending and borrowing in a non-price-taking economy than in a price-taking economy.

5.4 Market Price, Volatility and Risk Premium in Equilibrium

We have so far discussed how the non-price-taker affects state prices. As for the value of the market, when agent m, the non-price-taker, initially owns (sufficiently) more than half of the market, he will be a net seller of the market, and so we would expect him to raise the market level in the non-price-taking economy as compared with the price-taking economy. The reverse situation would be expected when he initially owns less than half of the market. In summary, we would expect $P^*(t) > \bar{P}(t)$ for $e_m > 1/2$, and $P^*(t) < \bar{P}(t)$ for $e_m < 1/2$. A modified mean comparison theorem applied to the price formula of Lemma 1, might prove applicable to this comparison, but the intuition has still to be formally proved.

To compare market volatility across economies, we make use of some techniques of Malliavin calculus (Ikeda and Watanabe (1989), Ocone (1988)), in particular the Clark-Ocone formula. These techniques can be used to write representations of, and in some cases evaluate, the dynamics of conditional expectations. In particular we may derive the following representation for the market volatility when $\xi(t)$ and $\delta(t)$ are driven by general Itô processes.

Lemma 2: *When $\xi(t)$ and $\delta(t)$ follow the processes*

$$d\xi(t) = -\xi(t)\left[r(t)dt + \theta(t)dW(t)\right],$$

and

$$d\delta(t) = \mu_\delta(t)dt + \sigma_\delta(t)dW(t),$$

the market price volatility in a one risky asset economy may be expressed as

$$P(t)\sigma(t) = \sigma_\delta(t) E\left[\int_t^T \frac{\xi(s)}{\xi(t)} ds \mid \mathcal{F}_t\right] + E\left[\int_t^T \frac{\xi(s)}{\xi(t)} \left\{\int_t^s \mathcal{D}_t \mu_\delta(u) du + \int_t^s \mathcal{D}_t \sigma_\delta(u) dW(u)\right\} \mid \mathcal{F}_t\right]$$

$$- E\left[\int_t^T \frac{\xi(s)}{\xi(t)} \delta(s) \left\{\int_t^s \mathcal{D}_t r(u) du + \int_t^s \mathcal{D}_t \theta(u) dW(u) + \frac{1}{2}\int_t^s \mathcal{D}_t \theta(u)^2 du\right\} \mid \mathcal{F}_t\right], \qquad (35)$$

where $\mathcal{D}_t F$ is the Malliavin derivative of the functional F, as described in the Appendix.

The Malliavin derivative of a Brownian functional represents the change in that functional due to a perturbation in the path of $W(t)$. Now, $\sigma(t)$ is the volatility of the conditional expectation given in Lemma 1. Equation (35) shows that in an arbitrage-free economy, the market price (rather than return) volatility, $P(t)\sigma(t)$, is equal to the aggregate dividend risk times the price of an annuity paying one unit of sure consumption till the terminal date, plus two additional terms arising from the stochastic nature of the coefficients of the processes in the conditional expectation. Individual contributions arise due to shocks in the market price of risk, the interest rate, and the drift and volatility of the aggregate dividend process.

We now apply Lemma 2 to our two economies, for the case when $\delta(t)$ is driven by an arithmetic Brownian motion.

Proposition 11: *Assume A1. The equilibrium market volatility and risk premia in the two economies are given by*

$$\bar{P}(t)\bar{\sigma}(t) = \sigma_\delta E\left[\int_t^T \frac{\bar{\xi}(s)}{\bar{\xi}(t)} ds \mid \mathcal{F}_t\right], \qquad (36)$$

$$\bar{P}(t)(\bar{\mu}(t) - \bar{r}) = a\sigma_\delta^2 E\left[\int_t^T \frac{\bar{\xi}(s)}{\bar{\xi}(t)} ds \mid \mathcal{F}_t\right],$$

and

$$P^*(t)\sigma^*(t) = \sigma_\delta E\left[\int_t^T \frac{\xi^*(s)}{\xi^*(t)} ds \mid \mathcal{F}_t\right] + a\sigma_\delta E\left[\int_t^T \frac{\xi^*(s)}{\xi^*(t)} \delta(s)[g(s) - g(t)] ds \mid \mathcal{F}_t\right], \qquad (37)$$

$$P^*(t)(\mu^*(t) - r^*(t)) = a(1-g(t))\sigma_\delta^2 E\left[\int_t^T \frac{\xi^*(s)}{\xi^*(t)} ds \mid \mathcal{F}_t\right] + a^2(1-g(t))\sigma_\delta^2 E\left[\int_t^T \frac{\xi^*(s)}{\xi^*(t)} \delta(s)[g(s) - g(t)] ds \mid \mathcal{F}_t\right].$$

There are extra terms in the volatility of the non-price-taking economy introduced because $\xi^*(t)$ is no longer driven by a geometric Brownian motion. The volatility has an extra contribution due to the stochastic nature of the market price of risk, $\theta^*(t) = a(1 - g(t))\sigma_\delta$, and of the interest rate, in the non-price-taking economy.

We argued previously from Corollary 3 that if $e_m > 1/2$ ($e_m < 1/2$) the price of the annuity given in the first term of the market volatility expressions will be higher (lower) in a non-price-taking economy than in a price-taking economy. The second term in equation (37) does not have an unambiguous sign. However, in the previous subsection we argued that in an expanding economy, on average the quantity $g(s) - g(t)$ is positive (negative) for $e_m > 1/2$ ($e_m < 1/2$). In conclusion, the market price volatility appears to be higher in the non-price-taking economy than in the price-taking, when the non-price-taker initially owns more than half of the market, and lower when the non-price-taker initially owns less than half of the market. The intuition for this result is as follows. Suppose $e_m > 1/2$. If the non-price-taker holds more than half of the wealth initially, from the results of the previous subsection, he will tend to hold more than half of the market (his price-taking portfolio strategy). Hence, moving from a price-taking to a non-price-taking economy, to clear the markets, the non-price-taker has to make the risky asset less attractive for the other agent, i.e., he has to persuade the other agent to hold less than half of the market. A way for the non-price-taker to achieve this "less attractiveness" in this set-up is to increase the market price volatility.

As the market price volatility and the market price of risk (as defined by $\theta = (\mu - r)/\sigma$) are both higher in the non-price-taking economy with $e_m > 1/2$, the excess drift of the market price, $(\mu - r)P$ is also higher in the non-price-taking economy than in the benchmark economy. Correspondingly, $(\mu - r)P$ is lower in the non-price-taking economy when $e_m < 1/2$.

6 Summary

In this paper we have included a non-price-taking agent into a continuous-time, pure-exchange, general equilibrium model. We have analyzed the equilibrium consumption-portfolio choice of the non-price-taking agent for general utility functions and in more detail for the special case of CARA utility. In addition we have studied the effect of the presence of the non-price-taker on the asset, market and state price dynamics.

We have formulated the problem by positing that the consumption choice of the non-price-taker affects the state price density process. The advantage of this method is that the problem can be analyzed using martingale techniques, which makes the problem highly tractable in continuous time. A drawback is that we can only model agents who are non-price-taking in the market as a whole (represented by the state prices), and not agents who are non-price-taking in individual risky assets.

A main conclusion of this work is that an extra factor, the non-price-taking-taker's endowment stream, drives the equilibrium allocations and prices in the non-price-taking economy. Compared with a price-taking equilibrium in which the equilibrium consumptions of the agents depend only on the aggregate consumption process, the equilibrium consumptions of the two agents now also depend on this extra factor. We have observed that, relative to when he is a price-taker, the consumption of the non-price-taker tends to deviate towards his endowment. The non-price-taker's endowment process also appears in the determination of the equilibrium interest rate process and as an extra factor in the consumption CAPM, which becomes a two-beta consumption CAPM. We have explored the consequences of these main results on the consumption behavior and asset prices in the non-price-taking economy.

Further work related to this paper may include the following. We have only described an economy in which there is one non-price-taking agent; an extension to multiple non-price-taking agents would be of further interest. One could initially formulate this extension as a one-shot Cournot game played in the Arrow-Debreu securities market. The main results of the single non-price-taker economy would extend qualitatively to this case, with multiple additional factors now driving the economy, the endowment streams of each non-price-taking agent.

One could extend the results of Section 5 as a starting point towards investigating the case when there are multiple risky assets. More than one risky asset is needed for the non-price-taker's endowment to not be perfectly correlated with the aggregate dividend, and for the two-beta consumption CAPM to apply. This general case would be expected to yield further results, such as comparisons of the correlations between state prices, consumptions and dividends across economies. Studying the effect of the non-price-taker on individual asset prices would then also appear to be possible.

APPENDICES

A General Formulation

Proof of Lemma 1: Define

$$\bar{P}_i(t) \equiv P_i(t) + \frac{1}{\xi(t)} \int_0^t \xi(s)\delta_i(s)ds, \quad i = 1, ..., d; t \in [0, T],$$

i.e., $\bar{P}_i(t)$ is the current price plus the current value of accumulated dividends.

Itô's Lemma implies:

$$d(\xi(t)\bar{P}_i(t)) = \xi(t)P_i(t)(\sigma_i(t) - \theta(t)^\top)dW(t),$$

where $\sigma_i(t)$ denotes the ith row of $\sigma(t)$. Hence $\xi(t)\bar{P}_i(t)$ is a $\mathcal{P}$-martingale, that is

$$\bar{P}_i(t) = \frac{1}{\xi(t)} E[\xi(T)\bar{P}_i(T)|\mathcal{F}_t],$$

which, using the fact that $P_i(T) = 0$, implies the desired result. *Q.E.D.*

B Agents' Optimization and Equilibrium

Proof of Proposition 1:

Necessity: The optimality of $c_m^*(t)$ given by (7) and (8) and the relationship between $\xi(t)$ and $c_m(t)$ in (13) imply (14) and (15).

Sufficiency: (4) and (12) imply (9). The proof that the optimally invested wealths and portfolio strategies associated with these consumptions automatically clear the securities markets is a variation on Karatzas, Lehoczky and Shreve (1990) to include non-redundant positive net supply securities, and appears in Basak (1993, Proposition 1).

To prove the existence and uniqueness of equilibrium for HARA utility we first show that for any given time t and state ω, for any $y_m \in (0, \infty)$ there exists a unique solution $c_m^*(y_m; t, \omega)$ to equation (14). Secondly, we show that there exists a unique y_m^* so that the process $c_m^*(y_m^*; \cdot)$ solves (15).

At each time and state define the mappings

$$L(c) \equiv u_m'(c)$$

and

$$R(c; y_m) \equiv y_m \left[U'(\delta(t, \omega) - c) - U''(\delta(t, \omega) - c)(c - \delta_m(t, \omega))\right].$$

(In this proof from now on we will use the shorthand δ for $\delta(t,\omega)$ and δ_m for $\delta_m(t,\omega)$.) By the assumption on $u_m(\cdot)$, $L(c) > 0$, $L'(c) < 0$ and $L(c)$ is continuous in (c_∞, ∞), the domain over which L is defined, and $\lim_{c\to c_\infty} L(c) = \infty$, $\lim_{c\to\infty} L(c) = 0$.

We now determine the domain over which R is defined. The HARA family includes the negative exponential utility function ($\gamma = -\infty$, $\eta = 1$), since $\lim_{\gamma\to-\infty} \frac{1-\gamma}{\gamma}\left(\frac{\beta c}{1-\gamma}+1\right)^\gamma = -\lim_{\epsilon\to\infty}\left(\frac{\beta c}{\epsilon}+1\right)^{-\epsilon} = -\exp\{-\beta c\}$, by the property of exponential. It is convenient to consider this case somewhat separately. Substituting for $U(c)$ into R we obtain

$$R(c;y_m) = y_m\beta^2\gamma(1-\gamma)\left(\frac{\beta(\delta-c)}{1-\gamma}+\eta\right)^{\gamma-2}\left[\delta_m + \frac{1-\gamma}{\gamma\beta}\left(\frac{\beta(\delta-\delta_m)}{1-\gamma}+\eta\right) - c\right]; \quad \gamma\in(-\infty,\infty), \tag{38}$$

$$R(c;y_m) = -y_m\beta^2\exp\{-\beta(\delta-c)\}\left[\delta_m - \frac{1}{\beta} - c\right]; \quad \gamma=-\infty, \eta=1. \tag{39}$$

Because of the restricted domain of $U(c)$, R is only defined for $\left(\frac{\beta(\delta-c)}{1-\gamma}+\eta\right) > 0$. For $\gamma < 1$ R is defined and continuous in $(-\infty, c_{crit})$; for $\gamma > 1$ R is defined and continuous in (c_{crit}, ∞), where $c_{crit} \equiv \delta + (1-\gamma)\eta/\beta$. (For negative exponential $c_{crit} = \infty$, so R is defined everywhere.)

We now determine the domain over which R is greater than zero. (38) and (39) show that $R(c;y_m)$ crosses zero at $c_0 \equiv \delta_m + \frac{1-\gamma}{\gamma\beta}\left(\frac{\beta(\delta-\delta_m)}{1-\gamma}+\eta\right) = c_{crit} + \frac{(1-\gamma)^2}{\gamma\beta}\left(\frac{\beta(\delta-\delta_m)}{1-\gamma}+\eta\right)$. Since $\delta - \delta_m = \sum_{n=1}^N \delta_n$, by the assumptions on dividends we have $\left(\frac{\beta(\delta-\delta_m)}{1-\gamma}+\eta\right) > 0$ and $\delta_m < c_\infty$. For $\gamma < 0$ (including $\gamma = -\infty$), within the allowed domain of R, $R(c;y_m) > 0$ if and only if $c > c_0$. Since $c_0 < c_{crit}$, we have $R(c;y_m) > 0$ for $c \in (c_0, c_{crit})$. For $\gamma = 0$, we have $R(c;y_m) = y_m\beta(\beta(\delta-c)+\eta)^{-2}(\beta(\delta-\delta_m)+\eta)$, which is greater than zero for the whole of the allowed domain, $(-\infty, c_{crit})$. For $0 < \gamma < 1$, within the allowed domain, $R(c;y_m) > 0$ if and only if $c < c_0$, but $c_0 > c_{crit}$ so $R(c;y_m) > 0$ in the whole allowed domain, $(-\infty, c_{crit})$. For $\gamma > 1$, $R(c;y_m) > 0$ if and only if $c > c_0$, and $c_0 > c_{crit}$ so $R(c;y_m) > 0$ for the restricted domain $c \in (c_0, \infty)$.

We now determine the domain over which $R(c;y_m)$ and $R'(c;y_m)$ are both greater than zero. We have

$$R'(c;y_m) \equiv y_m\left[U'''(\delta-c)(c-\delta_m) - 2U''(\delta-c)\right],$$

and substituting for HARA,

$$R'(c;y_m) = y_m\beta^3\gamma(1-\gamma)\left(\frac{\beta(\delta-c)}{1-\gamma}+\eta\right)^{\gamma-3}\left[\delta_m + \frac{2(1-\gamma)}{\gamma\beta}\left(\frac{\beta(\delta-\delta_m)}{1-\gamma}+\eta\right) - c\right]; \quad \gamma\in(-\infty,\infty), \tag{40}$$

$$R'(c;y_m) = -y_m\beta^3\exp\{-\beta(\delta-c)\}\left[\delta_m - \frac{2}{\beta} - c\right]; \quad \gamma=-\infty, \eta=1. \tag{41}$$

(40) and (41) show that $R'(c;y_m)$ crosses zero at $c_{ex} \equiv \delta_m + \frac{2(1-\gamma)}{\gamma\beta}\left(\frac{\beta(\delta-\delta_m)}{1-\gamma}+\eta\right) = c_{crit} + \frac{(1-\gamma)(2-\gamma)}{\gamma\beta}\left(\frac{\beta(\delta-\delta_m)}{1-\gamma}+\eta\right) = c_0 + \frac{(1-\gamma)}{\gamma\beta}\left(\frac{\beta(\delta-\delta_m)}{1-\gamma}+\eta\right)$. For $\gamma < 0$, for allowed c, $R'(c;y_m) > 0$ if and only if $c > c_{ex}$. But since $c_{ex} < c_0$, we have $R'(c;y_m) > 0$ in (c_0, c_{crit}). For $\gamma = 0$, $R'(c;y_m) = 2y_m\beta^3(\beta(\delta-c)+\eta)^{-3}(\beta(\delta-\delta_m)+\eta)$, which is greater than zero for all of $(-\infty, c_{crit})$. For $0 < \gamma < 1$, $R'(c;y_m) > 0$ if and only if $c < c_{ex}$, and $c_{ex} > c_{crit}$ so $R'(c;y_m) > 0$ in $(-\infty, c_{crit})$.

For $\gamma > 1$, $R'(c; y_m) > 0$ if and only if $c > c_{ex}$, and $c_{ex} < c_0$ so $R'(c; y_m) > 0$ in $c \in (c_0, \infty)$. Hence in all cases $R'(c; y_m) > 0$ over the whole domain for which $R(c; y_m) > 0$.

Let us compare the values of $L(c)$ and $R(c; y_m)$ at the lower and upper bounds of these domains. For $\gamma < 1$, inspection of (38) and (39) reveals that $\lim_{c \to c_{crit}} R(c; y_m) = \infty$. Now for $\gamma < 1$ $c_{crit} = \delta + \frac{(1-\gamma)\eta}{\beta} = \delta_m + \frac{(1-\gamma)}{\beta}\left(\frac{\beta(\delta-\delta_m)}{1-\gamma} + \eta\right) > \delta_m > c_\infty$. Since $c_{crit} > c_\infty$, $L(c_{crit}) < \infty$. For $\gamma > 1$, inspection of (38) reveals that $\lim_{c\to\infty} R(c; y_m) = \infty$, and we know $\lim_{c\to\infty} L(c) = 0$. So if we let $c_u \equiv c_{crit}$ for $\gamma < 1$ and $c_u \equiv \infty$ for $\gamma > 1$, we have $L(c_u) < R(c_u; y_m)$ for all γ. For the lower bounds of these domains, $L(c)$ may not be defined if c_∞ is larger than the lower bound. So instead we define $c_l \equiv \max\{c_0, c_\infty\}$ for $\gamma < 0$ or $\gamma > 1$, and $c_l \equiv c_\infty$ for $0 \le \gamma < 1$. Then $L(c_l) = L(c_0) > 0 = R(c_0; y_m)$ or $L(c_l) = L(c_\infty) = \infty > R(c_\infty)$. So there exists $-\infty \le c_l < c_u \le \infty$ (independent of y_m) such that R and L are continuous, $R(c; y_m) > 0$, $R'(c; y_m) > 0$, $L(c) > 0$, $L'(c) < 0$ in (c_l, c_u), and $R(c_l; y_m) < L(c_l)$ and $R(c_u; y_m) > L(c_u)$. Hence by the intermediate value theorem, there exists a unique $c_m^*(y_m) \in (c_l, c_u)$ such that $L(c_m^*(y_m)) = R(c_m^*(y_m); y_m)$, i.e. $c_m^*(y_m)$ is a solution to (14) for a given y_m. Since the domain of (c_l, c_u) is the whole domain for which $L(c)$ is defined and for which $R(c; y_m) > 0$ there cannot be another solution to (14) outside (c_l, c_u).

We now show that $c_m^*(y_m)$ is strictly decreasing in y_m for $y_m \in (0, \infty)$ and that $\lim_{y_m \to 0} c_m^*(y_m) = c_u$ and $\lim_{y_m\to\infty} c_m^*(y_m) = c_l$. We have $L(c_m^*(y_m)) = R(c_m^*(y_m); y_m)$. Taking derivatives with respect to y_m we obtain $L'(c_m^*(y_m)) = R'(c_m^*(y_m); y_m) c_m^{*'}(y_m) + R(c_m^*(y_m); y_m)/y_m$, implying $c_m^{*'}(y_m) = R(c_m^*(y_m); y_m)/\{y_m[L'(c_m^*(y_m)) - R'(c_m^*(y_m); y_m)]\} < 0$ for $c_m^* \in (c_l, c_u)$ (and hence for $y_m \in (0, \infty)$), since $R > 0$, $L' < 0$, $R' > 0$. So $c_m^*(y_m)$ is strictly decreasing in y_m.

For $c_u < \infty$, take any $\epsilon > 0$ and any $\tilde{y}_m \in (0, \infty)$. Let $\delta_u \equiv \tilde{y}_m L(c_u - \epsilon/2)/R(c_u - \epsilon/2; \tilde{y}_m) > 0$, since $L(c) > 0$, $R(c; \tilde{y}_m) > 0$ in (c_l, c_u). Then $R(c_u - \epsilon/2; \delta_u) = \delta_u R(c_u - \epsilon/2; \tilde{y}_m)/\tilde{y}_m$, since R is linear in y_m and hence $R(c_u - \epsilon/2; \delta_u) = L(c_u - \epsilon/2)$, implying $c_m^*(\delta_u) = c_u - \epsilon/2$. Then since $c_m^*(y_m)$ is decreasing in y_m, $c_u > c_m^*(y_m) > c_u - \epsilon/2$ for all $y_m < \delta_u$. Hence for every $\epsilon > 0$ there exists $\delta_u > 0$ such that $y_m < \delta_u$ implies $|c_m^*(y_m) - c_u| < \epsilon$. Hence $\lim_{y_m\to 0} c_m^*(y_m) = c_u$. Similarly, for $c_u = \infty$, take any $\tilde{c} < \infty$ and any $\tilde{y}_m \in (0, \infty)$. Let $\delta_u \equiv \tilde{y}_m L(\tilde{c})/R(\tilde{c}; \tilde{y}_m) > 0$. Then $R(\tilde{c}; \delta_u) = L(\tilde{c})$, and so $c_m^*(\delta_u) = \tilde{c}$ and $c_m^*(y_m) > \tilde{c}$ for all $y_m < \delta_u$. Hence for every $\tilde{c} < \infty$ there exists $\delta_u > 0$ such that $y_m < \delta_u$ implies $c_m^*(y_m) > \tilde{c}$. Hence $\lim_{y_m \to 0} c_m^*(y_m) = \infty = c_u$. For $c_l > -\infty$, take any $\epsilon > 0$ and any $\tilde{y}_m \in (0, \infty)$. Let $\delta_l \equiv \tilde{y}_m L(c_l + \epsilon/2)/R(c_l + \epsilon/2; \tilde{y}_m) \in (0, \infty)$. Then $R(c_l + \epsilon/2; \delta_l) = L(c_l + \epsilon/2)$, and so $c_m^*(\delta_l) = c_l + \epsilon/2$. Then, since $c_m^*(y_m)$ is decreasing in y_m, $c_l < c_m^*(y_m) < c_l + \epsilon/2$ for all $y_m > \delta_l$. Hence for every $\epsilon > 0$ there exists $\delta_l > 0$ such that $y_m > \delta_l$ implies $|c_m^*(y_m) - c_l| < \epsilon$. Hence $\lim_{y_m\to\infty} c_m^*(y_m) = c_l$. Similarly, for $c_l = -\infty$, take any $\tilde{c} > -\infty$ and any $\tilde{y}_m \in (0, \infty)$. Let $\delta_l \equiv \tilde{y}_m L(\tilde{c})/R(\tilde{c}; \tilde{y}_m)$. Then $R(\tilde{c}; \delta_l) = L(\tilde{c})$, and so $c_m^*(\delta_l) = \tilde{c}$ and $c_m^*(y_m) < \tilde{c}$ for all $y_m < \delta_l$. Hence for every $\tilde{c} > -\infty$ there exists $\delta_l < \infty$ such that $y_m > \delta_l$ implies $c_m^*(y_m) < \tilde{c}$. Hence $\lim_{y_m\to\infty} c_m^*(y_m) = -\infty = c_l$.

Now, for $\gamma < 1$, $c_u = c_{crit} = \delta_m + \frac{(1-\gamma)}{\beta}\left(\frac{\beta(\delta-\delta_m)}{1-\gamma} + \eta\right) > \delta_m$, and for $\gamma > 1$, $c_u = \infty > \delta_m$. So, since $U'(\cdot) > 0$

$$\lim_{y_m \to 0} E\left[\int_0^T U'(\delta(t) - c_m^*(y_m; t))(c_m^*(y_m; t) - \delta_m(t))dt\right] = E\left[\int_0^T U'(\delta(t) - c_u)(c_u - \delta_m(t))dt\right] > 0.$$

Similarly, either $c_l = c_\infty < \delta_m$, or (for $\gamma < 0$ or $\gamma > 1$) $c_l = c_0 = \delta_m + \frac{(1-\gamma)}{\gamma\beta}\left(\frac{\beta(\delta-\delta_m)}{1-\gamma} + \eta\right) < \delta_m$.

Hence

$$\lim_{y_m \to \infty} E\left[\int_0^T U'(\delta(t) - c_m^*(y_m;t))(c_m^*(y_m;t) - \delta_m(t))dt\right] = E\left[\int_0^T U'(\delta(t) - c_l)(c_l - \delta_m(t))dt\right] < 0.$$

Therefore by the intermediate value theorem, there exists $y_m^* \in (0, \infty)$ such that (15) is satisfied, i.e.

$$E\left[\int_0^T U'(\delta(t) - c_m^*(y_m^*;t))(c_m^*(y_m^*;t) - \delta_m(t))dt\right] = 0.$$

Now,

$$\frac{\partial}{\partial y_m} E\left[\int_0^T U'(\delta(t) - c_m^*(y_m;t))(c_m^*(y_m;t) - \delta_m(t))dt\right]$$

$$\begin{aligned} &= E\left[\int_0^T \left\{U'(\delta(t) - c_m^*(y_m;t)) - U''(\delta(t) - c_m^*(y_m;t))(c_m^*(y_m;t) - \delta_m(t))\right\} c_m^{*\prime}(y_m;t)dt\right] \\ &= E\left[\int_0^T R(c_m^*(y_m;t); y_m, t) c_m^{*\prime}(y_m;t)dt\right]. \end{aligned}$$

Since $R(c;y_m) > 0$ for $c \in (c_l, c_u)$ and $c_m^{*\prime}(y_m;t) < 0$ for $y_m \in (0,\infty)$, the expectation is strictly decreasing in y_m so the solution y_m^* is unique. *Q.E.D.*

Proof of Proposition 2:

(a) By assumption, $\bar{c}_m(t) = \delta_m(t)$ is a solution to equilibrium in the price taking economy. This implies there exists a constant $\bar{y}_m$ such that

$$u_m'(\delta_m(t)) = \bar{y}_m \xi(t) = \bar{y}_m U'(\delta(t) - \delta_m(t)), \quad \text{a.s.}$$

We can add a term equaling zero to the right hand side of this equation, as follows:

$$u_m'(\delta_m(t)) = \bar{y}_m \left[U'(\delta(t) - \delta_m(t)) - U''(\delta(t) - \delta_m(t))(\delta_m(t) - \delta_m(t))\right], \quad \text{a.s.}$$

This equation corresponds to the sufficient condition for equilibrium in the non-price-taking economy, equation (14), for $c_m^*(t) = \delta_m(t)$ and $y_m = \bar{y}_m$. Clearly agent m's budget constraint holds with equality for $c_m^*(t) = \delta_m(t)$. So $c_m^*(t) = \delta_m(t)$ $t \in [0,T]$, a.s., is also an equilibrium in the non-price-taking economy.

(b) By assumption, $c_m^*(t) = \delta_m(t)$ is a solution to equilibrium in the non-price-taking economy. So, there exists y_m such that

$$u_m'(\delta_m(t)) = y_m \left[U'(\delta(t) - \delta_m(t)) - U''(\delta(t) - \delta_m(t))(\delta_m(t) - \delta_m(t))\right] = y_m U'(\delta(t) - \delta_m(t)), \quad \text{a.s.}$$

Together with m's budget constraint (which must obviously hold at $\bar{c}_m(t) = \delta_m(t)$), this condition is sufficient for $\bar{c}_m(t) = \delta_m(t) = c_m^*(t)$, $t \in [0,T]$, a.s., to be an equilibrium in the price-taking economy.

(c) By assumption, $\bar{c}_m(t)$ is a solution to equilibrium in the price-taking economy, where $\bar{c}_m(t) \neq \delta_m(t)$ for some interval of t, with probability > 0. Hence there exists a $\bar{y}_m$ such that

$$u'_m(\bar{c}_m(t)) = \bar{y}_m U'(\delta(t) - \bar{c}_m(t)), \quad \text{a.s.}$$

Assume $\bar{c}_m(t)$ is also a solution to equilibrium in the non-price taking economy. Then there exists a constant y_m such that

$$u'_m(\bar{c}_m(t)) = y_m \left[U'(\delta(t) - \bar{c}_m(t)) - U''(\delta(t) - \bar{c}_m(t))(\bar{c}_m(t) - \delta_m(t))\right], \quad \text{a.s.} \quad t \in [0, T]$$

and $\bar{c}_m(t)$ satisfies m's budget constraint with equality. The above expressions imply

$$\bar{y}_m U'(\delta(t) - \bar{c}_m(t)) = y_m \left[U'(\delta(t) - \bar{c}_m(t)) - U''(\delta(t) - \bar{c}_m(t))(\bar{c}_m(t) - \delta_m(t))\right], \quad \text{a.s.} \quad t \in [0, T]$$

or

$$\frac{\bar{y}_m - y_m}{\bar{y}_m} = \frac{U''(\delta(t) - \bar{c}_m(t))(\bar{c}_m(t) - \delta_m(t))}{U'(\delta(t) - \bar{c}_m(t))} \quad \text{a.s.} \quad t \in [0, T].$$

Since $U''(\cdot) < 0$ and $U'(\cdot) > 0$, and $\bar{c}_m(t) - \delta_m(t) \neq 0$ for some interval of t with probability > 0, for the right hand side of the above expression to be a constant, we must have either (i) $\bar{c}_m(t) - \delta_m(t) > 0$ a.s., $t \in [0, T]$ or (ii) $\bar{c}_m(t) - \delta_m(t) < 0$ a.s., $t \in [0, T]$, either of which contradicts m's budget constraint holding with equality. So we get a contradiction to the assumption that $\bar{c}_m(t)$ is also a solution to equilibrium in the non-price taking economy. *Q.E.D.*

Proof of Proposition 3: We may define $\mathcal{A}_m$, the set of price-taking equilibrium agent-m consumption processes, as follows:

$$\mathcal{A}_m \equiv \{c_m(\cdot); \quad \text{there exists } \bar{y}_m, \bar{y}_1, ..., \bar{y}_N, \quad \text{and} \quad \bar{\xi}(\cdot) \quad \text{such that}$$

$$\sum_{k=1}^{N+1} I_k(\bar{y}_k \bar{\xi}(t)) = \delta(t), \quad t \in [0, T], \text{a.s.}, \tag{42}$$

$$E\left[\int_0^T (I_k(\bar{y}_k \bar{\xi}(t)) - \delta_k(t))\bar{\xi}(t)dt\right] = 0, \quad k = m, 1, ..., N, \tag{43}$$

$$\text{and} \quad c_m(t) = I_m(\bar{y}_m \bar{\xi}(t)), \quad t \in [0, T]\}. \tag{44}$$

In a non-price-taking economy, agent m solves

$$\max_{c_m(\cdot) \in \mathcal{B}_m} E\left[\int_0^T u_m(c_m(t))dt\right],$$

where $\mathcal{B}_m$ is the set of all agent-m consumption precesses such that the consumption good market clears with all price-taking agents n following their optimal consumption processes at some state price process and all budget constraints are satisfied, i.e.,

$$\mathcal{B}_m \equiv \{c_m(\cdot); \quad \text{there exists } \tilde{y}_m, \tilde{y}_1, ..., \tilde{y}_N, \quad \text{and} \quad \tilde{\xi}(\cdot) \quad \text{such that}$$

$$c_m(t) = \delta(t) - \sum_{n=1}^{N} I_n(\tilde{y}_n \tilde{\xi}(t)), \quad t \in [0, T],$$

$$E\left[\int_0^T (I_n(\tilde{y}_n \tilde{\xi}(t)) - \delta_n(t))\tilde{\xi}(t)dt\right] = 0, \quad n = 1, ..., N,$$

$$\text{and} \quad E\left[\int_0^T (c_m(t) - \delta_m(t))\tilde{\xi}(t)dt\right] = 0\}.$$

To prove the proposition we need to show that $\mathcal{A}_m$ is a subset of $\mathcal{B}_m$ so that the non-price-taker could always have chosen any price-taking equilibrium consumption and hence must do at least this well in the non-price-taking equilibrium. Take any $\bar{c}_m(\cdot) \in \mathcal{A}_m$. Then there exists $\bar{y}_m, \bar{y}_m, \ldots, \bar{y}_N, \bar{\xi}(\cdot)$ such that $\bar{c}_m(\cdot)$ satisfies (from (42) and (44))

$$\bar{c}_m(t) = \delta(t) - \sum_{n=1}^{N} I_n(\bar{y}_n \bar{\xi}(t)) \quad t \in [0,T], \text{a.s.}$$

and (from (43) and (44))

$$E\left[\int_0^T (\bar{c}_m(t) - \delta_m(t))\bar{\xi}(t)dt\right] = 0.$$

So clearly $\bar{c}_m(\cdot) \in \mathcal{B}_m$ with $\tilde{y}_m = \bar{y}_m, \tilde{y}_1 = \bar{y}_1, \ldots \tilde{y}_N = \bar{y}_N, \tilde{\xi}(\cdot) = \bar{\xi}(\cdot)$. So $\mathcal{A}_m \subset \mathcal{B}_m$.

Proof of Proposition 4: We first argue that if the non-price-taking equilibrium differs with probability > 0 from the price-taking equilibrium, then there exist subsets $A, B \subset \Omega$ and time intervals (t_{a1}, t_{a2}), $(t_{b1}, t_{b2}) \subset [0,T]$ such that

$$c_m^*(t,\omega) > \delta_m(t,\omega), \quad t \in (t_{a1}, t_{a2}); \quad \omega \in A$$

and

$$c_m^*(t,\omega) < \delta_m(t,\omega), \quad t \in (t_{b1}, t_{b2}); \quad \omega \in B.$$

Assume this is not the case and that $c_m^*(t) = \delta_m(t)$ a.s., $t \in [0,T]$. Then, from (14) we have

$$u_m'(\delta_m(t)) = y_m U'(\delta(t) - \delta_m(t)), \quad \text{a.s.}, \quad t \in [0,T],$$

and $c_m^*(t)$ clearly satisfies m's budget constraint, implying that $c_m^*(t)$ is also the solution to equilibrium in the price-taking economy. Hence, by the contrapositive to this argument there must exist a finite time and probability interval such that $c_m^*(t) \neq \delta_m(t)$. Furthermore, by the continuity of $c_m^*(t)$ and $\delta_m(t)$ and since the budget constraint must hold with equality, there must exist finite time and probability intervals in which $c_m^*(t) > \delta_m(t)$ and in which $c_m^*(t) < \delta_m(t)$.

From equations (13) and (14) we have then

$$\frac{u_m'(c_m^*(t,\omega))}{y_m} > \xi^*(t,\omega) = U'(\delta(t,\omega) - c_m^*(t,\omega)), \quad t \in (t_{a1}, t_{a2}); \quad \omega \in A$$

and

$$\frac{u_m'(c_m^*(t,\omega))}{y_m} < \xi^*(t,\omega) = U'(\delta(t,\omega) - c_m^*(t,\omega)), \quad t \in (t_{b1}, t_{b2}); \quad \omega \in B.$$

Define a process $\phi(t)$ by

$$\phi(t) \equiv \frac{1}{2}\left(\frac{u_m'(c_m^*(t))}{y_m} + \xi^*(t)\right).$$

Then by concavity of $u_m(\cdot)$ and $U(\cdot)$, and continuity of $u_m'(\cdot)$ and $U'(\cdot)$, there exists $\Upsilon > 0$ such that, for all $v \in (0, \Upsilon)$,

$$\frac{u_m'(c_m^*(t,\omega))}{y_m} > \frac{u_m'(c_m^*(t,\omega)+v)}{y_m} > \phi(t,\omega) > U'(\delta(t,\omega)-c_m^*(t,\omega)-v) > U'(\delta(t,\omega)-c_m^*(t,\omega)), t \in (t_{a1}, t_{a2}), \tag{45}$$

$$\frac{u_m'(c_m^*(t,\omega))}{y_m} < \frac{u_m'(c_m^*(t,\omega)-v)}{y_m} < \phi(t,\omega) < U'(\delta(t,\omega)-c_m^*(t,\omega)+v) < U'(\delta(t,\omega)-c_m^*(t,\omega)), t \in (t_{a1}, t_{a2}). \tag{46}$$

Now, let us perturb $c_m^*(t)$ to $\tilde{c}_m(t) = c_m^*(t) + \epsilon\psi(t)$, where $\psi(t,\omega) > 0$ for $t \in (t_{a1}, t_{a2})$ and $\omega \in A$; $\psi(t,\omega) < 0$ for $t \in (t_{b1}, t_{b2})$ and $\omega \in B$; $\psi(t,\omega) = 0$ otherwise; and $\psi(t)$ satisfies

$$E\left[\int_0^T \phi(t)\psi(t)dt\right] = 0.$$

Since $\psi(t,\omega) = 0$ elsewhere we have

$$E_B\left[\int_{t_{b1}}^{t_{b2}} \phi(t)\psi(t)dt\right] = E_A\left[\int_{t_{a1}}^{t_{a2}} \phi(t)\psi(t)dt\right], \tag{47}$$

where E_A and E_B denote expectations over the subsets A and B, respectively. Then we choose $\epsilon > 0$ such that $\epsilon|\psi(t)| < \Upsilon$, a.s., $t \in [0,T]$.

We finally show that both the non-price-taker and the price-taker representative agent are better off when we perturb $c_m^*(t)$ to $\tilde{c}_m(t)$ and $c_n^*(t)$ to $\delta(t) - \tilde{c}_m(t)$, which is clearly feasible. Agent m's expected lifetime utility becomes

$$E\left[\int_0^T u_m(\tilde{c}_m(t))dt\right] = E\left[\int_0^T \left\{u_m(c_m^*(t)) + \int_{c_m^*(t)}^{c_m^*(t)+\epsilon\psi(t)} u_m'(c)dc\right\}dt\right]$$

$$= E\left[\int_0^T u_m(c_m^*(t))dt\right] + E_A\left[\int_{t_{a1}}^{t_{a2}}\left\{\int_{c_m^*(t)}^{c_m^*(t)+\epsilon\psi(t)} u_m'(c)dc\right\}dt\right] + E_B\left[\int_{t_{b1}}^{t_{b2}}\left\{-\int_{c_m^*(t)+\epsilon\psi(t)}^{c_m^*(t)} u_m'(c)dc\right\}dt\right]$$

$$> E\left[\int_0^T u_m(c_m^*(t))dt\right] + E_A\left[\int_{t_{a1}}^{t_{a2}} \epsilon\phi(t)\psi(t)dt\right] - E_B\left[\int_{t_{b1}}^{t_{b2}} \epsilon\phi(t)\psi(t)dt\right] > E\left[\int_0^T u_m(c_m^*(t))dt\right],$$

where we have made use of (45)-(47). Similarly, the price-taker representative agent's expected lifetime utility is

$$E\left[\int_0^T U(\delta - \tilde{c}_m(t))dt\right] = E\left[\int_0^T \left\{U(\delta - c_m^*(t)) - \int_{c_m^*(t)}^{c_m^*(t)+\epsilon\psi(t)} U'(\delta(t)-c)dc\right\}dt\right]$$

$$= E\left[\int_0^T U(\delta(t) - c_m^*(t))dt\right] - E_A\left[\int_{t_{a1}}^{t_{a2}} \int_{c_m^*(t)}^{c_m^*(t)+\epsilon\psi(t)} U'(\delta(t)-c)dcdt\right] + E_B\left[\int_{t_{b1}}^{t_{b2}} \int_{c_m^*(t)+\epsilon\psi(t)}^{c_m^*(t)} U'(\delta(t)-c)dcdt\right]$$

$$> E\left[\int_0^T U(\delta(t) - c_m^*(t))dt\right] - E_A\left[\int_{t_{a1}}^{t_{a2}} \epsilon\phi(t)\psi(t)dt\right] + E_B\left[\int_{t_{b1}}^{t_{b2}} \epsilon\phi(t)\psi(t)dt\right] > E\left[\int_0^T U(\delta(t) - c_m^*(t))dt\right].$$

Hence there exists some feasible consumption allocation such that both m and the price-taker representative agent are better off than at the non-price-taking equilibrium allocation. Hence the non-price-taking equilibrium is not pareto optimal. *Q.E.D.*

Proof of Proposition 5:

Sufficiency: As in the proof of Proposition 1, with (6) and (17) now implying (9).

Necessity: The optimality of $c_n^*(t)$ given by (4), the expression for $\xi(t)$ in (17), and the $N+1$ agents' budget constraints holding with equality imply (19)-(20).

Substituting for $\xi(t)$ using (17), agent m solves

$$\max_{c_m(\cdot)} E\left[\int_0^T u_m(c_m(t))dt\right]$$

$$\text{subject to} \quad E\left[\int_0^T g(t)dt\right] = 0 \quad \text{and} \quad E\left[\int_0^T h_n(t)dt\right] = 0, \quad n = 1, ..., N,$$

where the processes $g(t)$ and $h_n(t)$ are defined by

$$g(t) \equiv U'(\delta(t) - c_m(t); \Lambda)(c_m(t) - \delta_m(t))$$

and

$$h_n(t) \equiv U'(\delta(t) - c_m(t); \Lambda)(I_n(y_n U'(\delta(t) - c_m(t); \Lambda)) - \delta_n(t))$$

and represent, respectively, the cost of m's and of n's "net" consumption (his consumption minus the dividend from his initial endowment) at time t.

We may define the mappings G, G^{-1} and H_n, between the time t, state ω costs of net consumption, agent m's consumption and the price-takers' weights, as follows:

$$G(c, y_1, ..., y_N; t, \omega) = U'(\delta(t, \omega) - c; (1/y_1, ..., 1/y_N))(c - \delta_m(t, \omega)),$$

$$g = U'\left(\delta(t, \omega) - G^{-1}(g, y_1, ..., y_N; t, \omega); 1/y_1, ..., 1/y_N\right)\left(G^{-1}(g, y_1, ..., y_N; t, \omega) - \delta_m(t, \omega)\right)$$

and

$$H_n(g, y_1, ..., y_N; t, \omega) = U'\left(\delta(t, \omega) - G^{-1}(g, y_1, ..., y_N; t, \omega); (1/y_1, ..., 1/y_N)\right) *$$
$$\left(I_n\left(y_n U'(\delta(t, \omega) - G^{-1}(g, y_1, ..., y_N; t, \omega); (1/y_1, ..., 1/y_N))\right) - \delta_n(t, \omega)\right).$$

We will use the notation G_k, G_k^{-1} and H_{nk} to denote the derivatives of these mappings with respect to their kth argument. Then agent m's optimization problem can be written as

$$\max_{g(\cdot)} E\left[\int_0^T u_m\left(G^{-1}(g(t), y_1, ..., y_N; t, \omega)\right) dt\right]$$

$$\text{subject to} \quad E\left[\int_0^T g(t)dt\right] = 0 \quad \text{and} \quad E\left[\int_0^T H_n(g(t), y_1, ..., y_N; t, \omega)dt\right] = 0, \quad n = 1, ..., N.$$

Now, let us suppose that $c_m^*(\cdot)$ is an equilibrium agent-m consumption process with associated weights $y_1^*, ..., y_N^*$ and that

$$g^*(t) = G(c_m^*(t), y_1^*, ..., y_N^*; t, \omega), \quad t \in [0, T].$$

Then perturb the process $g^*(\cdot)$ to $g^\epsilon(t) = g^*(t) + \epsilon\eta(t)$. Since by assumption $g^*(\cdot)$ is the solution to the above optimization problem we must have

$$\lim_{\epsilon \to 0} \frac{d}{d\epsilon} E\left[\int_0^T u_m\left(G^{-1}(g^\epsilon(t), y_1(\epsilon), ..., y_N(\epsilon); t, \omega)\right) dt\right]$$

$$= E\left[\int_0^T u_m'(c_m^*(t)) G_1^{-1}(g^*(t), y_1^*, ..., y_N^*; t, \omega)\eta(t)dt\right]$$

$$+ \sum_{j=1}^N y_j'(0) E\left[\int_0^T u_m'(c_m^*(t)) G_{j+1}^{-1}(g^*(t), y_1^*, ..., y_N^*; t, \omega)dt\right] = 0 \qquad (48)$$

for all processes $\eta(t)$ satisfying $E\left[\int_0^T \eta(t)dt\right] = 0$, and where the functions $y_j(\epsilon)$ are determined from

$$E\left[\int_0^T H_n(g^\epsilon(t), y_1(\epsilon), ..., y_N(\epsilon); t, \omega)dt\right] = 0, \quad n = 1, ..., N.$$

We can take derivatives and evaluate at $\epsilon = 0$ to derive

$$E\left[\int_0^T H_{n1}(g^*(t), y_1^*, ..., y_N^*; t, \omega)\eta(t)dt\right] + \sum_{j=1}^N E\left[\int_0^T H_{n(j+1)}(g^*(t), y_1^*, ..., y_N^*; t, \omega)dt\right] y_j'(0) = 0.$$

If we define a matrix $\mathcal{H}$ by

$$\mathcal{H}_{nj} \equiv E\left[\int_0^T H_{jn}(g^*(t), y_1^*, ..., y_N^*; t, \omega)dt\right], \quad n, j = 1, ..., N,$$

then we solve for the $y_j'(0)$ as

$$y_j'(0) = -\sum_{j=1}^N \left(\mathcal{H}^{-1}\right)_{jn} E\left[\int_0^T H_{n1}(g^*(t), y_1^*, ..., y_N^*; t, \omega)\eta(t)dt\right], \quad n = 1, ..., N.$$

Substituting into (48) we obtain the condition

$$E\left[\int_0^T \left\{u_m'(c_m^*(t))G_1^{-1}(g^*(t), y_1^*, ..., y_N^*; t, \omega) - \sum_{j=1}^N E\left[\int_0^T u_m'(c_m^*(s))G_{j+1}^{-1}(g^*(s), y_1^*, ..., y_N^*; s, \tilde{\omega})ds\right]\right.\right.$$

$$\left.\left. * \sum_{n=1}^N \left(\mathcal{H}^{-1}\right)_{jn} H_{n1}(g^*(t), y_1^*, ..., y_N^*; t, \omega)\right\}\eta(t)dt\right] = 0,$$

for all $\eta(\cdot)$ satisfying $E\left[\int_0^T \eta(t)dt\right] = 0$. Hence we must have

$$u_m'(c_m^*(t))G_1^{-1}(g^*(t), y_1^*, ..., y_N^*; t, \omega) - \sum_{n=1}^N \sum_{j=1}^N E\left[\int_0^T u_m'(c_m^*(s))G_{j+1}^{-1}(g^*(s), y_1^*, ..., y_N^*; s, \tilde{\omega})ds\right]$$

$$* \left(\mathcal{H}^{-1}\right)_{jn} H_{n1}(g^*(t), y_1^*, ..., y_N^*; t, \omega) = y_m,$$

where y_m is some constant. Evaluating the G_1^{-1} and H_{n1} terms and rearranging we arrive at the condition in the proposition with

$$K_n \equiv \sum_{j=1}^{N} E\left[\int_0^T u'_m(c_m^*(s))G_{j+1}^{-1}(g^*(s), y_1^*, ..., y_N^*; s, \tilde{\omega})ds\right]\left(\mathcal{H}^{-1}\right)_{jn}.$$

The extra terms in equation (18) compared with equation (14) are $-K_n H_{n1}/G_1^{-1}$. We here argue that each term represents the indirect marginal disutility to m of an extra unit of $c_m(t,\omega)$ via agent n's budget constraint. Each expectation term in the K_n expression is the marginal (expected, lifetime) utility to m due to a change in agent j's weight y_j. The elements $\mathcal{H}_{nj}$ of the matrix $\mathcal{H}$ represent the sensitivity of agent n's budget constraint to agent j's weight y_j. Hence K_n is the sensitivity of agent m's expected lifetime utility to agent n's budget constraint (via the weights y). Then H_{n1} is the sensitivity of agent n's budget constraint to agent m's cost of "net" consumption, and G_1^{-1} is the sensitivity of agent m's current consumption to his cost of net consumption. Hence H_{n1}/G_1^{-1} is the sensitivity of agent n's budget constraint to agent m's time t, state ω consumption. therefore we conclude that each extra term $-K_n H_{n1}/G_1^{-1}$ is the sensitivity of agent m's expected lifetime utility to time t, state ω consumption, via agent n's budget constraint.

Finally, we need to show that this collapses to the simpler expression (14) when $U(c;\Lambda) = h(\Lambda)U(c)$. We do this by showing

$$E\left[\int_0^T u'_m(c_m^*(t))G_{j+1}^{-1}(g^*(t), y_1^*, ..., y_N^*; t, \omega)dt\right] = 0, \quad j = 1, ..., N,$$

and hence $K_n = 0$, $n = 1, ..., N$. Evaluating the G_{j+1}^{-1} term yields the term

$$E\left[\int_0^T \frac{u'_m(c_m^*(t))U'_{yj}(\delta(t) - c_m(t);\Lambda)(c_m(t) - \delta_m(t))}{U'(\delta(t) - c_m(t);\Lambda) - U''(\delta(t) - c_m(t);\Lambda)(c_m(t) - \delta_m(t))}dt\right],$$

where $U'_{yj}(c;\Lambda) \equiv \frac{\partial^2 U(c;\Lambda)}{\partial y_j \partial c} = \frac{\partial h(\Lambda)}{\partial y_j}U'(c)$. Using this fact and supposing that the simpler result (14) does hold we simplify to

$$y_m \frac{\partial h(\Lambda)}{\partial y_j} E\left[\int_0^T U'(\delta(t) - c_m(t);\Lambda)(c_m(t) - \delta_m(t))dt\right],$$

which must equal zero from m's budget constraint. *Q.E.D.*

C The Equilibrium Interest Rate and the Consumption-Based CAPM

Proof of Proposition 6: We apply Itô's Lemma to both sides of (13), and make use of the dynamics of $\xi(t)$ in (1) to derive

$$-\xi(t)r(t)dt - \xi(t)\theta(t)^\top dW(t) =$$

$$\left[\left(\mu_\delta(t) - \mu^*_{c_m}(t)\right) U''(\delta(t) - c^*_m(t)) + \left(\|\sigma_\delta(t)\|^2 + \|\sigma^*_{c_m}(t)\|^2 - 2\sigma_\delta(t)^\top \sigma^*_{c_m}(t)\right) \frac{U'''(\delta(t) - c^*_m(t))}{2}\right] dt$$
$$+U''(\delta(t) - c^*_m(t)) \left(\sigma_\delta(t)^\top - \sigma^*_{c_m}(t)^\top\right) dW(t),$$

where $\mu^*_{c_m}(t)$ and $\|\sigma^*_{c_m}(t)\|$ denote respectively the drift and volatility of $c^*_m(t)$. Equating terms we obtain

$$r(t) = -\frac{U''(\delta(t) - c^*_m(t))}{U'(\delta(t) - c^*_m(t))} \left(\mu_\delta(t) - \mu^*_{c_m}(t)\right) - \frac{U'''(\delta(t) - c^*_m(t))}{2U'(\delta(t) - c^*_m(t))} \|\sigma_\delta(t) - \sigma^*_{c_m}(t)\|^2 \qquad (49)$$

and

$$\theta(t) = -\frac{U''(\delta(t) - c^*_m(t))}{U'(\delta(t) - c^*_m(t))} \left(\sigma_\delta(t) - \sigma^*_{c_m}(t)\right). \qquad (50)$$

Next we would like to replace the endogeneous parameters $\mu^*_{c_m}(t)$ and $\sigma^*_{c_m}(t)$ by the exogeneous parameters of $\delta(t)$ and $\delta_m(t)$. To do this we apply Itô's Lemma to both sides of equation (14) and equate terms, yielding

$$\begin{aligned}
\sigma^*_{c_m}(t) &= \frac{y_m U''(\delta(t) - c^*_m(t)) - y_m U'''(\delta(t) - c^*_m(t))(c^*_m(t) - \delta(t))}{u''_m(c^*_m(t)) - y_m U'''(\delta(t) - c^*_m(t))(c^*_m(t) - \delta_m(t)) + 2y_m U''(\delta(t) - c^*_m(t))} \sigma_\delta(t) \\
&- \frac{y_m U''(\delta(t) - c^*_m(t))}{U''(\delta(t) - c^*_m(t)) - y_m u'''_m(c^*_m(t))(c^*_m(t) - \delta_m(t)) + 2y_m U''(\delta(t) - c^*_m(t))} \sigma_{\delta_m}(t).
\end{aligned}$$

The corresponding expression for $\mu^*_{c_m}(t)$ in terms of $\mu_\delta(t)$, $\mu_{\delta_m}(t)$, $\|\sigma_\delta(t)\|^2$, $\|\sigma_{\delta_m}(t)\|^2$ and $\sigma_\delta(t)^\top \sigma_{\delta_m}(t)$ is unweildy and will not be written out here.

Substitution of the above expression for $\sigma^*_{c_m}(t)$ into the expression for $\theta(t)$ in (50) and recalling that $\mu(t) - r(t)1 = \sigma(t)\theta(t)$ yields the desired result for the risk premia. We have rewritten $\sigma(t)\sigma_\delta(t)$ and $\sigma(t)\sigma_{\delta_m}(t)$ as $\text{cov}(dP(t)/P(t), d\delta(t))$ and $\text{cov}(dP(t)/P(t), d\delta_m(t))$, respectively. Similar substitution into (49) would yield an expression for $r(t)$ in terms of $\mu_\delta(t)$, $\mu_{\delta_m}(t)$, $\sigma_\delta(t)$, and $\sigma_{\delta_m}(t)$ as shown. *Q.E.D.*

D The CARA Utility and One Risky Asset Case

Proof of Proposition 7: From (24) we conclude that $\bar{c}_m(t) - \delta(t)/2$ is a constant, independent of $\delta(t)$. Equation (27) can be rearranged as

$$\exp\{-2a(c^*_m(t) - \delta(t)/2)\} = y_m \left[1 + a(c^*_m(t) - \delta(t)/2) - a\delta(t)(e_m - 1/2)\right] \qquad (51)$$

and differentiated implicitly (state by state) with respect to $\delta(t, \omega)$ to give

$$\frac{\partial(c^*_m(t,\omega) - \delta(t,\omega)/2)}{\partial\delta(t,\omega)} = \frac{a y_m}{a y_m + 2a \exp\{-2a(c^*_m(t,\omega) - \delta(t,\omega)/2)\}} (e_m - 1/2).$$

We conclude that $(c^*_m(t,\omega) - \delta(t,\omega)/2)$ is strictly monotonically increasing in $\delta(t,\omega)$ if $e_m > 1/2$ and strictly monotonically decreasing in $\delta(t,\omega)$ if $e_m < 1/2$.

Let us now show there exists a δ_{crit} such that $c_m^*(t) - \delta(t)/2 = \bar{c}_m(t) - \delta(t)/2$. At δ_{crit} we have $c_m^*(t) = \bar{c}_m(t)$, so we substitute (24) into (27) to obtain

$$\frac{1}{2}\delta(t) - \frac{1}{2a}\ln(\bar{y}_m) = \frac{1}{2}\delta(t) - \frac{1}{2a}\ln\left[1 + a\left(\frac{1}{a}\delta_{crit} - \frac{1}{2a}\ln(\bar{y}_m) - \delta_m(t)\right)\right] - \frac{1}{2a}\ln(y_m)$$

yielding

$$\delta_{crit} = \frac{1}{a(e_m - 1/2)}\left[1 - \frac{\bar{y}_m}{y_m} - \frac{1}{2}\ln(\bar{y}_m)\right].$$

In case (a), since $(\bar{c}_m(t) - \delta(t)/2)$ is a constant and $(c_m^*(t) - \delta(t)/2)$ is monotonically increasing in $\delta(t)$, we conclude that, for $\delta(t) > \delta_{crit}$

$$c_m^*(t) - \delta(t)/2 > \bar{c}_m(t) - \delta(t)/2$$

and for $\delta(t) < \delta_{crit}$

$$c_m^*(t) - \delta(t)/2 < \bar{c}_m(t) - \delta(t)/2,$$

providing the required result in part 1. Similarly in case (b), since $(c_m^*(t) - \delta(t)/2)$ is monotonically decreasing in $\delta(t)$, we derive part 1.

For the proof of parts 2 of the proposition, again we use the fact that $(c_m^*(t) - \delta(t)/2)$ is monotonically increasing (for $e_m > 1/2$) or decreasing (for $e_m < 1/2$). First let us define $\bar{\delta}$ to be such that $c_m^*(t) - \delta(t)/2 = 0$, i.e., from (51),

$$1 = y_m\left[1 - a\bar{\delta}(e_m - 1/2)\right],$$

or

$$\bar{\delta} = \frac{y_m - 1}{a y_m(e_m - 1/2)}.$$

Since $\bar{\delta}$ exists and $(c_m^*(t) - \delta(t)/2)$ is monotonic, we conclude that for $\delta(t) < \bar{\delta}$ we have $c_m^*(t) < \delta(t)/2$ for $e_m > 1/2$, and $c_m^*(t) > \delta(t)/2$ for $e_m < 1/2$. Since $c_n^*(t) = \delta(t) - c_m^*(t)$, we deduce parts (a) 2. and (b) 2.

Proof of Proposition 8:

Equation (27) can be rewritten as

$$\frac{1}{y_m}\exp\{-2ac_m^*(t)\}\exp\{a\delta(t)\} = [1 + a(c_m^*(t) - \delta_m(t))] \equiv f(t). \tag{52}$$

Applying Itô's Lemma to both sides yields

$$af(t)\left[\mu_\delta(t) - 2\mu_{c_m}^*(t) + 2a\sigma_{c_m}^*(t)^2 + \frac{a}{2}\sigma_\delta(t)^2 - 2a\sigma_{c_m}^*(t)\sigma_\delta(t)\right]dt + af(t)\left[\sigma_\delta(t) - 2\sigma_{c_m}^*(t)\right]dW(t)$$

$$= a\left[\mu_{c_m}^*(t) - e_m\mu_\delta(t)\right]dt + a\left[\sigma_{c_m}^*(t) - e_m\sigma_\delta(t)\right]dW(t).$$

Matching coefficients yields the required expressions for $\mu_{c_m}^*(t)$ and $\sigma_{c_m}^*(t)$. Taking the implicit derivative (state by state) of (17) shows that $g(t,\omega) = \partial c_m^*(t,\omega)/\partial\delta(t,\omega)$.

When $e_m > 1/2$, $g(t) > (1/2 + f(t))/(1+2f(t)) > 1/2$, and when $e_m < 1/2$, $g(t) < 1/2$, yielding the comparisons between $\bar{\sigma}_{c_m}(t)$ and $\sigma^*_{c_m}(t)$.

When $g(t) > 1/2$, $g(t)^2 + g(t) - 3/4 > 0$; when $g(t) < 1/2$, $g(t)^2 + g(t) - 3/4 < 0$, yielding the required comparisons between $\bar{\mu}_{c_m}(t)$ and $\mu^*_{c_m}(t)$. *Q.E.D.*

Proof of Corollary 1: In the benchmark economy,

$$| \bar{\sigma}_{c_m}(t) - \sigma_{\delta_m}(t) | = |1/2 - e_m||\sigma_\delta(t)|,$$

and in the non-price-taking economy,

$$| \sigma^*_{c_m}(t) - \sigma_{\delta_m}(t) | = |g(t) - e_m||\sigma_\delta(t)|,$$

which is always lower by the conclusions about $g(t)$ in the proof of Proposition 8.

Proof of Corollary 2:

We will need the following Lemma.

Lemma (Hajek 1985)

Let x and y be semimartingales with representations

$$dx(s) = \mu(s)ds + \sigma(s)dw(s),$$

$$dy(s) = mds + \rho dv(s),$$

where w and v are Wiener processes and m and ρ are constants. Suppose that $\mu(s) \leq m$ and $|\sigma(s)| \leq \rho$ and that $x(0) \leq y(0)$. Then for any nondecreasing convex function ϕ on $\mathcal{R}$

$$E\phi(x(s)) \leq E\phi(y(s)).$$

We let $x(s) \equiv c^*_m(s+t)$ and $y(s) \equiv \bar{c}_m(s+t)$. Then $x(0) = c^*_m(t)$ and $y(0) = \bar{c}_m(t)$. We have by Proposition 7, $\bar{c}_m(t) > c^*_m(t)$. By Proposition 8 and assumption A1 we also have $\mu^*_{c_m}(t) \leq \bar{\mu}_{c_m} = \mu_\delta/2$ and $\sigma^*_{c_m}(t) \leq \bar{\sigma}_{c_m} = \sigma_\delta/2$. So we can apply Hajek's Lemma with ϕ as the identity function and derive

$$E\left[c^*_m(s+t) \mid \mathcal{F}_t\right] \leq E\left[\bar{c}_m(s+t) \mid \mathcal{F}_t\right], \quad s > 0,$$

yielding the desired results.

Hajek's Lemma cannot be used for proving the reverse case for $e_m > 1/2$ since the dominating drift and volatility are not constants, and to our knowledge the necessary comparison theorem does not exist in the current literature. Nevertheless, we anticipate this reverse case to hold since our earlier results seem to suggest opposite effects for the cases $e_m < 1/2$ and $e_m > 1/2$, although mathematically it still needs to be verified. *flushright Q.E.D.*

Proof of Proposition 9: Substituting for $U(c) = -\exp\{-ac\}/a$ into equations (49) and (50) in the proof of Proposition 6, we obtain

$$r^*(t) = a\left[\mu_\delta(t) - \mu^*_{c_m}(t)\right] - \frac{a^2}{2}|\sigma_\delta(t) - \sigma^*_{c_m}(t)|^2$$

and

$$\theta^*(t) = a\left(\sigma_\delta(t) - \sigma^*_{c_m}(t)\right).$$

Then substitute for $\mu^*_{c_m}(t)$ and $\sigma^*_{c_m}(t)$ from equations (29) and (30) to obtain the required results.

The properties of $g(t)$ mentioned in the proof of Proposition 8 imply the comparisons between $\theta^*(t)$ and $\bar{\theta}(t)$. *Q.E.D.*

Proof of Corollary 3: Applying Itô's Lemma at s to the process $\ln(\xi(s)/\xi(t))$, $s > t$, yields

$$d\left[\ln\left(\frac{\xi(s)}{\xi(t)}\right)\right] = d(\ln(\xi(s))) = -\left(r(s) + \frac{1}{2}\theta(s)^2\right)ds - \theta(s)dW(s), \quad s > t.$$

From Proposition 9 we have

$$r^*(s) + \frac{1}{2}\theta^*(s)^2 = a(1 - g(s))\mu_\delta - \frac{2f(s)}{1 + f(s)}a\left[g(s)^2 + g(s) - \frac{3}{4}\right]\sigma_\delta^2.$$

Further we have

$$\bar{r} + \frac{1}{2}\bar{\theta}^2 = \frac{a}{2}\mu_\delta.$$

So if $e_m < 1/2$, $g(t) < 1/2$, $g(t)^2 + g(t) - 3/4 < 0$ implying $-\left(r^*(s) + \frac{1}{2}\theta^*(s)^2\right) < -\left(\bar{r} + \frac{1}{2}\bar{\theta}^2\right)$. We also have $|-\theta^*(s)| < |-\bar{\theta}| = a\sigma_\delta/2$.

We apply Hajek's Lemma to the processes $x(s) \equiv \ln\left(\xi^*(s+t)/\xi^*(t)\right)$ and $y(s) \equiv \ln\left(\bar{\xi}(s+t)/\bar{\xi}(t)\right)$. We have $x(0) = y(0) = 0$. Now define the function $\phi(x) \equiv \exp(x)$ and we conclude

$$E\left[\phi\left(\frac{\xi^*(s+t)}{\xi^*(t)}\right) \mid \mathcal{F}_t\right] = E\left[\frac{\xi^*(s+t)}{\xi^*(t)} \mid \mathcal{F}_t\right] \leq E\left[\frac{\bar{\xi}(s+t)}{\bar{\xi}(t)} \mid \mathcal{F}_t\right], \quad s > 0,$$

obtaining the desired result. *Q.E.D.*

Proof of Proposition 10: Before proving the proposition, we will briefly state the required notions and results form Malliavin calculus. For more details see Ikeda and Watanabe (1989) or Ocone (1988).

Malliavin Calculus of Smooth Brownian Functionals

Suppose F is a smooth Brownian functional, i.e., a functional of a finite dimensional Brownian motion W at a number of points in time:

$$F = F(W(t_1), ..., W(t_n)),$$

such that the function F is bounded and has bounded derivaives of all orders. Then the Malliavin derivative of the functional F is defined by

$$\mathcal{D}_t^i F \equiv \sum_{j=1}^{n} \frac{\partial}{\partial W_i(t_j)} F(W(t_1), ..., W(t_n)) 1_{[0,t_j]}(t), \quad i = 1, ..., d,$$

and can be interpreted as the change in F due to a perturbation in the path of $W_i(t)$.

The Malliavin derivative of a continuously differentiable function $\phi\left(F^1, ..., F^M\right)$ of a finite number of Brownian functionals, with bounded partial derivatives is given by

$$\mathcal{D}\phi\left(F^1, ..., F^M\right) = \sum_{i=1}^{M} \frac{\partial \phi(\cdot)}{\partial F^i} \mathcal{D} F^i.$$

The Malliavin derivative defined above is a special case of the more general Malliavin derivative defined on Brownian functionals satisfying certain smoothness and integrability conditions.

The Malliavin derivative of an integral is given by

$$\mathcal{D}_t \int_0^T \psi(s) ds = \int_t^T \mathcal{D}_t \psi(s) ds.$$

The Malliavin derivative of a stochastic integral is given by

$$\mathcal{D}_t \int_0^T \psi(s) dW(s) = \int_t^T \mathcal{D}_t \psi(s) dW(s) + \psi(t).$$

The Clark-Ocone Formula: *A Brownian functional F can be represented by*

$$F = E[F] + \int_0^T E\left[\mathcal{D}_s F \mid \mathcal{F}_s\right] dW(s),$$

and hence

$$E[F \mid \mathcal{F}_t] = E[F] + \int_0^t E\left[\mathcal{D}_s F \mid \mathcal{F}_s\right] dW(s),$$

or

$$dE[F \mid \mathcal{F}_t] = E\left[\mathcal{D}_t F \mid \mathcal{F}_t\right] dW(t). \tag{53}$$

We rearrange the wealth expression stated in this subsection to give

$$\xi(t) X_k(t) = E\left[\int_0^T \xi(s) c_k(s) ds \mid \mathcal{F}_t\right] - \int_0^t \xi(s) c_k(s) ds,$$

or equivalently

$$d\left(\xi(t)X_k(t)\right) = dE\left[\int_0^T \xi(s)c_k(s)ds \mid \mathcal{F}_t\right] - \xi(t)c_k(t)dt.$$

Then by the Clark-Ocone formula, equation (53), we have

$$d\left(\xi(t)X_k(t)\right) = E\left[\mathcal{D}_t \int_t^T \xi(s)c_k(s)ds \mid \mathcal{F}_t\right] dW(t) - \xi(t)c_k(t)dt.$$

We can also apply Itô's Lemma directly to $\xi(t)X_k(t)$ and use the dynamics of $\xi(t)$ in (1) and $X_k(t)$ in (3) to obtain

$$d(\xi(t)X_k(t)) = \xi(t)\left[\alpha_k(t)P(t)\sigma(t) - X_k(t)\theta(t)\right] dW(t) - \xi(t)c_k(t)dt.$$

Equating the $dW(t)$ terms on the right hand sides of the two last equations yields

$$\alpha_k(t) = X_k(t)\frac{\theta(t)}{P(t)\sigma(t)} + \frac{1}{P(t)\sigma(t)\xi(t)}E\left[\mathcal{D}_t \int_t^T \xi(s)c_k(s)ds \mid \mathcal{F}_t\right].$$

Using the above stated properties of Malliavin derivatives of integrals we may expand this as

$$\alpha_k(t) = \frac{X_k(t)\theta(t)}{P(t)\sigma(t)} + \frac{1}{P(t)\sigma(t)\xi(t)}E\left[\int_t^T \xi(s)c_k(s)\left\{-\int_s^t \mathcal{D}_t(r(u)+\frac{1}{2}\theta(u)^2)du - \int_s^t \mathcal{D}_t\theta(u)dW(u) - \theta(t)\right\} ds \mid \mathcal{F}_t\right]$$

$$+\frac{1}{P(t)\sigma(t)\xi(t)}E\left[\int_t^T \xi(s)\left\{\int_s^t \mathcal{D}_t\mu_{c_k}(u)du + \int_s^t \mathcal{D}_t\sigma_{c_k}(u)dW(u) + \sigma_{c_k}(t)\right\} ds \mid \mathcal{F}_t\right].$$

Now, from Proposition 8 and 9, we have

$$r^*(u) + \frac{1}{2}\theta^*(u)^2 = a\left(\mu_\delta - \mu^*_{c_m}(u)\right),$$

and

$$\theta^*(u) = a\left(\sigma_\delta - \sigma^*_{c_m}(u)\right),$$

so

$$\int_t^s \mathcal{D}_t\left(r^*(u) + \frac{1}{2}\theta^*(u)^2\right)du + \int_t^s \mathcal{D}_t\theta^*(u)dW(u) = -a\left\{\int_t^s \mathcal{D}_t\mu^*_{c_m}(u)du + \int_t^s \mathcal{D}_t\sigma^*_{c_m}(u)dW(u)\right\}.$$

Now let us look at the Malliavin derivative of $c^*_m(s)$. Note from equation (27) that $c^*_m(s,\omega) = c^*_m(\delta(s,\omega); e_m, y_m)$, so

$$\mathcal{D}_t c^*_m(s,\omega) = \frac{\partial c^*_m(s,\omega)}{\partial\delta(s,\omega)}\mathcal{D}_t\delta(s,\omega) = \frac{\partial c^*_m(s,\omega)}{\partial\delta(s,\omega)}\sigma_\delta \equiv g(s,\omega) = \sigma^*_{c_m}(s,\omega).$$

Also

$$\mathcal{D}_t c^*_m(s) = \int_t^s \mathcal{D}_t\mu^*_{c_m}(u)du + \int_t^s \mathcal{D}_t\sigma^*_{c_m}(u)dW(u) + \sigma^*_{c_m}(t),$$

implying

$$\int_t^s \mathcal{D}_t\mu^*_{c_m}(u)du + \int_t^s \mathcal{D}_t\sigma^*_{c_m}(u)dW(u) = \sigma^*_{c_m}(s) - \sigma^*_{c_m}(t)$$
$$= [g(s) - g(t)]\sigma_\delta.$$

Using these results the above expression for $\alpha_k(t)$ becomes, for the non-price-taker

$$\alpha^*_m(t) = \frac{\sigma_\delta}{\xi^*(t)P^*(t)^2}E\left[\int_t^T \xi^*(s)\left\{g(t) + (1 + ac^*_m(s))(g(s) - g(t))\right\}ds \mid \mathcal{F}_t\right].$$

Anticipating and rearranging the result of Proposition 11 (equation 37) yields

$$P^*(t) = \frac{\sigma_\delta}{\sigma^*(t)\xi^*(t)}E\left[\int_t^T \xi^*(s)ds \mid \mathcal{F}_t\right] + \frac{a\sigma_\delta}{\sigma^*(t)\xi^*(t)}E\left[\int_t^T \xi^*(s)\delta(s)[g(s) - g(t)]ds \mid \mathcal{F}_t\right],$$

which when substituted into with the previous expression and rearranged gives the required result for $\alpha^*_m(t)$. Then $\alpha^*_n(t)$ is found from $\alpha^*_n(t) = 1/2 - \alpha^*_m(t)$. *Q.E.D.*

Proof of Lemma 2: From Lemma 1, we have

$$\begin{aligned}
\xi(t)P_i(t) &= E\left[\int_t^T \xi(s)\delta_i(s)ds \mid \mathcal{F}_t\right] \\
&= E\left[\int_0^T \xi(s)\delta_i(s)ds \mid \mathcal{F}_t\right] - \int_0^t \xi(s)\delta_i(s)ds \\
&= E\left[\int_0^T \xi(s)\delta_i(s)ds\right] + \int_0^t E\left[\mathcal{D}_s\int_0^T \xi(u)\delta_i(u)du \mid \mathcal{F}_s\right]dW(s) - \int_0^t \xi(s)\delta_i(s)ds,
\end{aligned}$$

where the last equality follows from the Clark-Ocone formula.

Then

$$\begin{aligned}
d(\xi(t)P_i(t)) &= E\left[\mathcal{D}_t\int_0^T \xi(s)\delta_i(s)ds \mid \mathcal{F}_t\right]dW(t) - \xi(t)\delta_i(t)dt \\
&= E\left[\int_t^T \mathcal{D}_t(\xi(s)\delta_i(s))ds \mid \mathcal{F}_t\right]dW(t) - \xi(t)\delta_i(t)dt.
\end{aligned}$$

Now, applying Itô's Lemma directly to $\xi(t)P_i(t)$ we obtain

$$d(\xi(t)P_i(t)) = \xi(t)P_i(t)\left[\sigma_i(t) - \theta(t)^\top\right]dW(t) - \xi(t)\delta_i(t)dt,$$

and equating coefficients with the Clark-Ocone representation yields the ith row of the volatility matrix $\sigma(\cdot)$ as

$$\sigma_i(t) = \theta(t)^\top + \frac{1}{\xi(t)P_i(t)}E\left[\int_t^T \mathcal{D}_t(\xi(s)\delta_i(s))ds \mid \mathcal{F}_t\right]. \tag{54}$$

By the properties of the Malliavin derivatives (stated earlier), we have

$$\mathcal{D}_t(\xi(s)\delta_i(s)) = \xi(s)\mathcal{D}_t\delta_i(s) + \delta_i(s)\mathcal{D}_t\xi(s),$$

$$\begin{aligned}
\mathcal{D}_t^j\delta_i(s) &= \mathcal{D}_t^j\left\{\delta_i(0) + \int_0^s \mu_{\delta_i}(u)du + \int_0^s \sigma_{\delta_i}(u)dW(u)\right\} \\
&= \int_t^s \mathcal{D}_t^j\mu_{\delta_i}(u)du + \int_t^s \sum_l \mathcal{D}_t^j\sigma_{\delta_i l}(u)dW_l(u) + \sigma_{\delta_i j}(t),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{D}_t^j\xi(s) &= \mathcal{D}_t^j\left\{\delta(0)\exp\left\{-\int_0^s r(u)du - \int_0^s \sum_l \theta_l(u)dW_l(u) - \frac{1}{2}\int_0^s \|\theta(u)\|^2 du\right\}\right\} \\
&= \xi(s)\left\{-\int_t^s \mathcal{D}_t^j r(u)du - \theta_j(u) - \int_t^s \sum_l \mathcal{D}_t^j\theta_l(u)dW_l(u) - \frac{1}{2}\int_t^s \mathcal{D}_t^j\|\theta(u)\|^2 du\right\}.
\end{aligned}$$

Substituting into (54) yields the desired result. *Q.E.D.*

Proof of Proposition 11: We make use of Lemma 2. By assumption A1, $\mathcal{D}_t\mu_\delta(u) = \mathcal{D}_t\sigma_\delta(u) = 0$ so the second term in (35) is zero in both economies. In the price-taking economy, $\bar{r}(t)$ and $\bar{\theta}(t)$ are constants as given in Section 5.2, so $\mathcal{D}_t\bar{r}(u) = \mathcal{D}_t\bar{\theta}(u) = \mathcal{D}_t\bar{\theta}(u)^2 = 0$. Hence the third term in (35) is zero and the volatility is as quoted. In the proof of Proposition 10, we evaluated the $\{\}$ term in the third term of equation (35) for the non-price-taking economy. Substitution of that expression into Lemma 2 gives the required result for the market volatility. To find the risk premia we use

$$\mu(t) - r(t) = \theta(t)\sigma(t)$$

and substitute for $\theta(t)$ from Section 5.2. *Q.E.D.*

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