A GENERAL EQUILIBRIUM MODEL OF PORTFOLIO INSURANCE

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Abstract

This paper examines the effects of portfolio insurance on market and asset price dynamics in a general equilibrium continuous-time model. Portfolio insurers are modeled as expected utility maximizing agents in two alternative ways. Martingale methods are employed in solving the individual agents' dynamic consumption-portfolio problems. Comparisons are made between the optimal consumption processes, optimally invested wealth and portfolio strategies of the portfolio insurers and "normal agents". At a general equilibrium level, comparisons across economies reveal that the market volatility and risk premium are decreased, and the asset and market price levels increased, by the presence of portfolio insurance.

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Amongst both academicians and practitioners, there has recently been a surge in interest in the intertemporal behavior of market volatility (standard deviation of price return).[1] Since the market crash of October 1987, at which time market volatility was very high, much attention has focused on the factors which affect volatility. Of particular interest is the influence of (i) the rate of information flow into the market, (ii) various security trading practices and (iii) macroeconomic variables. The efficient markets hypothesis would claim that the volatility of a stock's return is solely due to the rate of arrival of new information about that stock's future payoff. However, the popular press, regulators, and numerous researchers suggest that certain types of dynamic trading strategies tend to increase stock market volatility. In particular, portfolio insurance and program trading have been accused of contributing to the stock market crash by increasing volatility (for example, the Brady Report (1988)) and in general having a destabilizing effect on the market (for example, Hull (1993, p. 332)).

A portfolio insurer is broadly defined as someone who follows a trading strategy so that at some horizon, he or she is guaranteed a minimum level of wealth (the "floor") yet is able to participate in the potential gains of some reference portfolio, e.g., S&P 500. This definition of portfolio insurance, the one adopted in this paper, agrees with the description of portfolio insurance given in both the academic literature (Grossman and Vila (1989)) and the professional literature (Luskin (1988, p.77)).

The primary objective of this paper is to study the effects of portfolio insurance on the market price level, volatility and risk premium. Our approach is to take a very standard environment (Lucas (1978)) in asset pricing theory, that is well understood and extensively studied, and introduce portfolio insurance into it. To this end we develop a continuous-time consumption-based general equilibrium model of a multi-agent pure-exchange economy. Our main tool of analysis is the martingale representation technology of Cox and Huang (1989, 1991), Karatzas, Lehoczky and Shreve (1987) and Pliska (1986).

To include portfolio insurance in the economy we postulate that trade takes place between two types of traders, "portfolio insurers" and "normal agents". We model the portfolio insurers as expected utility maximizers, in two different ways according to the above definition of portfolio insurance. We must model the portfolio insurers in ways to assure them of a wealth above the floor at the portfolio insurance horizon. In the first formulation, we explicitly apply to their expected utility maximization problem an additional constraint that the horizon wealth be above the floor. Our second method of modeling the portfolio insurers is to let them derive utility from wealth at the portfolio insurance horizon, where the utility function is chosen to implicitly inherit the condition of wealth above the floor. Specifically, we let the marginal utility go to infinity at the floor so that the optimal wealth at the horizon will automatically lie above the floor. The advantage of this second approach is that, through an appropriate choice of the utility function, the general equilibrium analysis of a multi-agent economy becomes more tractable.

A comparison of the optimal consumption processes reveals that, before the portfolio in-

[1] Although there is now a large empirical literature on price volatility (for example, Black (1976), Merton (1980), French, Schwert and Stambaugh (1987), Schwert (1989, 1990), Stoll and Whaley (1990)), theoretical work in this area is limited. Apart from this paper, recent theoretical work includes Hindy (1992), Basak (1993, §5.3).

the opposite to ours; they find the market volatility to be increased by the presence of portfolio insurance. Donaldson and Uhlig develop a single-period model in which trading decisions are made and equilibrium defined at only one point in time. They model their portfolio insurers by exogeneously specifying their demands for a risky and a riskless security. They show the existence of two equilibria and find that increasing the level of portfolio insurance in the economy increases the separation between the two equilibrium prices, which they interpret as an increase in volatility. In the continuous-time model of Brennan and Schwartz, the portfolio insurers are also not expected utility maximizers; equilibrium prices are set by the remaining normal agents and the portfolio insurers demand a convex function of the market (aggregate wealth) at the portfolio insurance horizon, which coincides with the final date. The closest model to ours is the paper by Grossman and Zhou. They expand upon the Brennan and Schwartz work by modeling portfolio insurers as expected utility maximizers, as in our first formulation of portfolio insurance. The pertinent modeling difference between the Grossman and Zhou paper and ours is that, as in the Brennan and Schwartz paper, their agents only consume at one point in time, the portfolio insurance horizon, whereas our agents consume continuously throughout their lifetime. In Section 3, we briefly discuss the implications of this difference on the results.

Other papers that have looked at the effect of portfolio insurance on market price and volatility have done so in the context of asymmetric information. Grossman (1988) argues that market volatility may increase in an economy in which the extent of portfolio insurance dynamic trading strategies is not perfectly known prior to trading, due to illiquidity problems caused by unexpected coordinated selling by the insurers. The primary objective of Jacklin, Kleidon and Pfleiderer (1992) and Gennotte and Leland (1990) is to explain the market crash of 1987 in the presence of portfolio insurance. Gennotte and Leland (in a one-period model) and Jacklin, Kleidon and Pfleiderer (in a multi-period model) posit that the extent of uninformed portfolio insurance motivated trade in the market is unknown and hence can be misinterpreted as informed trading; then when the extent of portfolio insurance gets completely revealed the market price experiences a large drop. They attribute this price drop not to portfolio insurance itself, but rather to imperfect information about the portfolio insurance based trades.

The remainder of our paper is organized as follows. Section 1 presents the multi-agent pure-exchange continuous-time formulation to be used in our models. Section 2 describes the two ways in which portfolio insurance is to be modeled in an expected utility maximizing framework. Section 3 solves the agents' optimization problems and for general equilibrium when the portfolio insurers are modeled as constrained expected utility maximizers, and Section 4 when the portfolio insurers are specified in terms of their utility function of wealth at the portfolio insurance horizon. In Section 5, we study how close our portfolio insurers' behavior is to the general description of portfolio insurance given earlier, and to portfolio insurance trading strategies employed in practice. Section 6 concludes. The Appendix provides the proofs of all lemmas and propositions.

or

$$P_i(t)+\int_0^t \delta_i(s)ds = P_i(0)\exp\left\{\int_0^t \mu_i(s)ds + \int_0^t \sum_{j=1}^{L}\sigma_{ij}(s)dW_j(s) - \frac{1}{2}\int_0^t \sum_{j=1}^{L}\sigma_{ij}^2(s)ds + q_iA(t)\right\}.$$

Since these prices are ex-dividend we have $P_i(T') = 0$, $i = 1, ..., L$. Here, the interest rate $r(\cdot)$ of the bond, the vector of drifts $\mu(\cdot) = (\mu_1(\cdot), ..., \mu_L(\cdot))^\top$ and the volatility matrix $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}$ are $\mathcal{F}_t$-measurable processes, and in particular are allowed to be path dependent. The coefficients q_0, q_1, ...,q_L are $\mathcal{F}_T$-measurable random variables, and the process $A(t)$ is the right-continuous step function defined by $A(t) \equiv 1_{\{t \geq T\}}$ so that $dA(t)$ is a measure assigning unit mass to time T and zero mass to all $t \neq T$.[3] The coefficients $\mu_i(t)$ and $\sigma_{ij}(t)$ are interpreted as the instantaneous expected return of the ith security and the instantaneous covariance of the ith security's return with the jth Brownian motion at time t. The coefficients q_i are related to the size of the price jumps at time T by

$$q_i = \ln\left(P_i(T)/P_i(T\text{-})\right),$$

where $P_i(T\text{-})$ is the left limit of $P_i(\cdot)$ at T. Since the q_i are predictable at time T (i.e., $\mathcal{F}_{T\text{-}}$-measurable), the jumps are not a surprise in the sense that their sizes are revealed immediately before the jumps occur. Hence by no arbitrage we must have $q_i = q_0 \equiv q$, for all i; otherwise buying a security with a larger relative jump and short-selling another with a smaller jump just before time T would constitute an arbitrage opportunity.[4]

In our analysis, we use the martingale representation technology, which requires the construction of the following processes, related to the price dynamics. The details are given in Cox and Huang (1989, 1991), Harrison and Kreps (1979), Karatzas, Lehoczky and Shreve (1987) and Pliska (1986). We will briefly present the required notions for our set-up and not concern ourselves with stating the regularity conditions.

Given the above prices, market completeness (assuming the volatility matrix $\sigma(\cdot)$ is non-singular) and the absence of arbitrage opportunities allow us to construct a unique system of Arrow-Debreu securities. Accordingly we define the *state price density process* as

$$\xi(t) \equiv \frac{1}{P_0(0)}\exp\left\{-\int_0^t r(s)ds - \int_0^t \theta(s)^\top dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds - qA(t)\right\},$$

where $\theta(\cdot)$ is the unique L-dimensional $\mathcal{F}_t$-measurable *market price of risk process*, given by

$$\theta(t) \equiv \sigma(t)^{-1}[\mu(t) - r(t)\mathbf{1}],$$

where $\mathbf{1}$ is an L-dimensional vector with every component equal to 1. $\xi(t,\omega)$ is interpreted as the Arrow-Debreu price per unit probability $\mathcal{P}$ of one unit of consumption good in state $\omega \in \Omega$

[3] The posited stock prices are discontinuous, positive semimartingales, whose bounded variation parts contain an $\mathcal{F}_T$-measurable jump, but whose local martingale parts are continuous. In particular, we are not allowing the stock prices to have discontinuous local martingale parts, as, for example, in the asset pricing models of Back (1991) or of Dothan (1990, chapter 12).

[4] The jump in a security price i is often defined as $\Delta P_i(T) \equiv P_i(T) - P_i(T\text{-})$ and is related to q_i in our set-up by $\Delta P_i(T) = P_i(T\text{-})[\exp(q_i) - 1]$. Note that this absolute jump is not necessarily the same for each security, only the relative jump (i.e., jump "return").

the paper, a symbol with a ˆ will denote the optimal quantity corresponding to $\hat{c}_n(t)$ and its associated portfolio process $\hat{\pi}_n(t)$.

The following lemma provides a result which we will need in Sections 3 and 4 in deriving explicit expressions for the market and its dynamics. It simply states that the (optimally invested) wealth at time t equals the cost of the future (optimal) consumption stream.

Lemma 2: *The optimally invested wealth of agent n at time t is given by*

$$\hat{X}_n(t) = \frac{1}{\xi(t)} E\left[\int_t^{T'} \xi(s)\hat{c}_n(s)ds \mid \mathcal{F}_t\right], \quad t \in [0, T'], \tag{4}$$

or alternatively, for $t < T$, by

$$\hat{X}_n(t) = \frac{1}{\xi(t)} E\left[\int_t^{T} \xi(s)\hat{c}_n(s)ds + \xi(T\text{-})\hat{X}_n(T\text{-}) \mid \mathcal{F}_t\right], \quad t \in [0, T). \tag{5}$$

1.3 Equilibrium

In this paper, we characterize security prices by appealing to general equilibrium restrictions. The following definition enforces market clearing in the consumption good, the risky securities and the bond, respectively. Recall that the net supply of each risky asset is one unit of share at all times and the net supply of the riskless security is zero.

Definition: *An equilibrium is a collection of $\hat{c}_n(\cdot)$, $\hat{\pi}_n(\cdot)$, $r(\cdot)$, $\mu(\cdot)$, $\sigma(\cdot)$ and q such that the $(\hat{c}_n(\cdot), \hat{\pi}_n(\cdot))$ are optimal and*

$$\sum_{n=1}^{N} \hat{c}_n(t) = \delta(t), \quad t \in [0, T'], \tag{6}$$

$$\sum_{n=1}^{N} \hat{\pi}_{ni}(t) = P_i(t), \quad t \in [0, T'], \quad i = 1, ..., L, \tag{7}$$

$$\sum_{n=1}^{N} \hat{X}_n(t) = \sum_{i=1}^{L} P_i(t), \quad t \in [0, T']. \tag{8}$$

1.4 Representative Agent

We introduce a representative agent formulation (following, for example, Huang (1987)) since it will be useful for the equilibrium analyses in Sections 3 and 4. We define the representative agent's utility of consumption by

$$U(c; \Lambda) \equiv \max_{c_1, \ldots, c_N} \sum_{n=1}^{N} \lambda_n u_n(c_n)$$

this paper, instead of restricting the portfolio insurers to possibly sub-optimal strategies we model portfolio insurers as expected utility maximizers, and simultaneously derive their optimal trading strategies.

In accordance with the above description of portfolio insurance, in Section 3 we formulate a portfolio insurer's (mth agent's) consumption-portfolio problem by adding the following additional constraint on the time T ($< T'$) wealth to his original $[0, T']$ optimization problem:

$$X_m(T) \geq K.$$

Grossman and Vila (1989) analyze portfolio insurance behavior in this way in a one period partial equilibrium framework. In Section 3 we present the solution to this problem and then derive explicitly the equilibrium market price and volatility in the presence of portfolio insurers of this type.

One possible criticism of the above formulation is that when the portfolio insurer's horizon wealth hits the floor, his marginal indirect utility of wealth suddenly jumps to infinity. It might seem more reasonable for an investor to become gradually rather than suddenly unhappy as his wealth approaches the floor. This leads us to the alternative formulation of portfolio insurance adopted in Section 4. There we model the portfolio insurers instead as unconstrained expected utility maximizers who have a *modified* power or log utility function of time T wealth, which implicitly inherits the condition $\hat{X}_m(T) \geq K$ with the marginal utility going smoothly to infinity at the floor. These utility functions are:

$$v_m(X_T) = \frac{(X_T - K)^\gamma}{\gamma},$$

$$\text{and} \quad v_m(X_T) = \log(X_T - K),$$

with the required property that $\lim_{X_T \to K} v'_m(X_T) = \infty$. Both utility functions of time T wealth exhibit decreasing relative risk aversion, and are more risk averse than their standard counterparts X_T^γ/γ and $\log(X_T)$ over the region $X_T > K$. An advantage of this formulation is that this form of utility function allows simple aggregation across multiple agents and hence heterogeneous-agent general equilibrium analysis becomes a lot more tractable. These preferences are of course distinctly different from those of the constrained agents. However, we will see that the two types of preferences lead to equilibria with similar qualitative properties. Note that the utility of time T wealth must be consistent with the utility functions of future (i.e. $[T, T']$) consumption, so in effect choosing a specific utility function of time T wealth corresponds to restricting the $[T, T']$ utility functions of consumption.

In both Sections 3 and 4 we model the portfolio insurers as agents who maximize their expected utility from consumption in $[0, T']$ and let the portfolio insurance horizon, T, be before the end of period T'. In our formulation of the model we must have $T < T'$ because the portfolio insurers require $X_m(T) \geq K$ and since all end of period security prices $P_i(T')$ are zero, this wealth condition could not be attained in equilibrium if $T = T'$. An alternative formulation of the model might be to define the portfolio insurers as agents who have a floor on a lump sum consumption rather than their wealth at time $T = T'$. However, this formulation is less appealing both from an economic and a modeling point of view. Economically, investors in portfolio

$$\text{subject to} \quad E\left[\int_0^T \xi(t)c_m(t)dt + \xi(T\text{-})X_m(T\text{-})\right] \leq \xi(0)x_{mo},$$

$$E\left[\int_T^{T'} \xi(t)c_m(t)dt \mid \mathcal{F}_T\right] \leq \xi(T\text{-})X_m(T\text{-}), \quad \text{almost surely,}$$

$$X_m(T\text{-}) \geq K \quad \text{almost surely.}$$

Recall that the wealth is allowed to have an $\mathcal{F}_T$-measurable jump at T; $X_m(T\text{-})$ denotes the left limit of $X_m(\cdot)$ at T. We apply the constraint to the left limit simply to preserve the standard convention of right continuity of the price and wealth processes. We will often refer to this left limit at T as "time T". In order to ensure the existence of a solution to this optimization problem, we need to impose the condition that $K \leq x_{m0}\exp\{\int_0^T r(s)ds\}$ almost surely. In equilibrium this imposes a condition on x_{m0}, as will be seen later. The following lemma presents the optimal solutions to the above static problems.

Lemma 3: *The optimal consumption processes and the optimal time T wealths of the N agents are given by:*

$$\hat{c}_n(t) = I(y_n\xi(t)), \quad t \in [0, T'], \quad n = M+1, ..., N, \tag{9}$$
$$\hat{c}_m(t) = I(y_{m1}\xi(t)), \quad t \in [0, T), \quad m = 1, ..., M, \tag{10}$$
$$\hat{c}_m(t) = I(y_{m2}\xi(t)), \quad t \in [T, T'], \quad m = 1, ..., M, \tag{11}$$

$$\hat{X}_n(T\text{-}) = \frac{1}{\xi(T\text{-})}E\left[\int_T^{T'} \xi(t)I(y_n\xi(t))dt \mid \mathcal{F}_T\right] \quad n = M+1, ..., N, \tag{12}$$

$$\hat{X}_m(T\text{-}) = \max\left\{K, \frac{1}{\xi(T\text{-})}E\left[\int_T^{T'} \xi(t)I(y_{m1}\xi(t))dt \mid \mathcal{F}_T\right]\right\}, \quad m = 1, ..., M, \tag{13}$$

where $I(\cdot)$ is the inverse of $u'(\cdot)$, and where the y_n and y_{m1} are the unique nonnegative numbers and the y_{m2} the unique nonnegative $\mathcal{F}_T$-measurable random variables, such that the budget constraints hold with equality at the optimal consumption and wealth, i.e., y_n, y_{m1} and y_{m2} satisfy

$$E\left[\int_0^{T'} \xi(t)I(y_n\xi(t))dt\right] = \xi(0)x_{no}. \tag{14}$$

$$E\left[\int_0^T \xi(t)I(y_{m1}\xi(t))dt + \xi(T\text{-})\max\left\{K, \frac{1}{\xi(T\text{-})}E\left[\int_T^{T'} \xi(t)I(y_{m1}\xi(t))dt \mid \mathcal{F}_T\right]\right\}\right] = \xi(0)x_{mo}, \tag{15}$$

$$E\left[\int_T^{T'} \xi(t)I(y_{m2}\xi(t))dt \mid \mathcal{F}_T\right] = \xi(T\text{-})\max\left\{K, \frac{y_{m2}}{y_{m1}}\frac{1}{\xi(T\text{-})}E\left[\int_T^{T'} \xi(t)I(y_{m2}\xi(t))dt \mid \mathcal{F}_T\right]\right\}, \tag{16}$$

We have the following properties: (i) if $x_{m0} = x_{n0}$ then $y_{m1} \geq y_n$; and (ii) when m's portfolio insurance constraint is binding $y_{m1} = y_{m2}$, and when not binding $y_{m1} \geq y_{m2}$.

The optimal consumption processes are given by the inverse marginal utility evaluated at the properly normalized state price density process $\xi(\cdot)$, where the normalization factors, ys,

$(1/y_{11},...1/y_{M1},1/y_{M+1},...1/y_N)$ and $\Lambda_2 = (\lambda_{12},...,\lambda_{N2}) = (1/y_{12},...1/y_{M2},1/y_{M+1},...1/y_N)$, conditions (17) and (18) are equivalent to the familiar condition that the state prices are given by the marginal utility of the representative agent:

$$\xi(t) = U'(\delta(t);\Lambda_1), \quad t \in [0,T), \tag{19}$$

$$\xi(t) = U'(\delta(t);\Lambda_2), \quad t \in [T,T'], \tag{20}$$

where (Λ_1, Λ_2) is determined up to a multiplicative constant from (14) - (16) with the equilibrium $\xi(t)$ substituted in. $\xi(t)$, $t \in [0,T']$ is determined up to a multiplicative constant from (19) and (20). The existence of equilibrium is established by showing the existence of the weights (Λ_1, Λ_2) solving (14)-(16).

The interest rate process $r(t)$, the market price of risk process $\theta(t)$ and the jump parameter q can be derived in equilibrium from (17)-(18) or (19)-(20). The parameter q is given by

$$q = \ln\left(\xi(T\text{-})/\xi(T)\right) = \ln\left(U'(\delta(T);\Lambda_1)/U'(\delta(T);\Lambda_2)\right).$$

We see that whenever $\Lambda_1 \neq \Lambda_2$, i.e., at least one of the y_{m2} is greater than y_{m1} (at least one of the portfolio insurers' wealth constraints is binding), there is a discontinuous (upwards) jump in the state price density at T. The intuition for this is as follows. If ξ were continuous at the horizon T, the normal agents' consumption demands would be continuous whereas the demands of the portfolio insurers with their constraints binding would jump upwards, giving rise to a discontinuous aggregate consumption demand. This is not possible since the aggregate supply $\delta(t)$ is continuous. Hence the price of consumption, ξ, jumps up at T to counteract the upward jump in aggregate consumption demand. This in turn leads to discontinuous (downwards) jumps in the asset prices at T.

Remark 1: As an aside we now mention the case of power utility of consumption and derive a result which will be important for the remainder of Section 3. This result is also true for log and negative exponential utilities of consumption. For power utility ($u(c) = c^\gamma/\gamma$) the equilibrium state price density is given from (19) and (20) by

$$\xi(t) = \left(\sum_{m=1}^{M} y_{m1}^{\frac{1}{\gamma-1}} + \sum_{n=M+1}^{N} y_n^{\frac{1}{\gamma-1}}\right)^{1-\gamma} \delta(t)^{\gamma-1}, \quad t \in [0,T),$$

$$\xi(t) = \left(\sum_{m=1}^{M} y_{m2}^{\frac{1}{\gamma-1}} + \sum_{n=M+1}^{N} y_n^{\frac{1}{\gamma-1}}\right)^{1-\gamma} \delta(t)^{\gamma-1}, \quad t \in [T,T'],$$

with the individual agents' weights only appearing in a multiplying factor. This is the well-known situation for power utility that the representative agent's utility function is essentially not affected by the weights assigned to each agent. Since ξ follows the multiplicative dynamics in equation (1), we conclude that $r(t)$ and $\theta(t)$ are unaffected by these weights, and hence by portfolio insurance since it is these weights that capture the effects of portfolio insurance.

3.3 Equilibrium Asset and Market Prices, Market Volatility and Risk Premium

To analyze the asset and market prices in a heterogeneous multi-agent economy containing more than one constrained portfolio insurer at the present level of generality is rather complicated

Proposition 2: *The equilibrium market and asset prices in economy 1 are*

$$X^{(1)}_{em}(t) = \frac{1}{u'(\delta(t)/N)} E\left[\int_t^{T'} u'(\delta(s)/N)\delta(s)ds \mid \mathcal{F}_t\right], \quad t \in [0, T'], \tag{24}$$

$$P^{(1)}_i(t) = \frac{1}{u'(\delta(t)/N)} E\left[\int_t^{T'} u'(\delta(s)/N)\delta_i(s)ds \mid \mathcal{F}_t\right], \quad t \in [0, T'], \quad i = 1, ..., L,$$

and before the portfolio insurance horizon in economy 2 are

$$X^{(2)}_{em}(t) = X^{(1)}_{em}(t) + \frac{1}{u'(\delta(t)/N)} E\left[u'(\delta(T)/N) \max\left\{NK - X^{(1)}_{em}(T), 0\right\} \mid \mathcal{F}_t\right], \tag{25}$$

$$P^{(2)}_i(t) = P^{(1)}_i(t) + \frac{1}{u'(\delta(t)/N)} E\left[u'(\delta(T)/N) \frac{P^{(1)}_i(T)}{X^{(1)}_{em}(T)} \max\left\{NK - X^{(1)}_{em}(T), 0\right\} \mid \mathcal{F}_t\right] \tag{26}$$

After the portfolio insurance horizon the prices in both economies are identical.

Equation (24) simply states that the market price in the normal agent economy is the present value of the future aggregate consumption with the discount factor as the marginal rate of substitution evaluated at equilibrium allocations. Moving to economy 2, equation (25) reveals that, before the portfolio insurance horizon, the market price in the portfolio insurance economy is equal to that in the normal agent economy plus the value of a European put option, with exercise price NK and maturity T, on the market in the normal agent economy. (This result relates to the partial equilibrium result in subsection 3.1, that a portfolio insurer's time T wealth is equal to that of a normal agent plus a put option on the normal agent's time T wealth.) An important implication is that before the portfolio insurance horizon, the level of the market in economy 2 is always higher than in economy 1. The higher market value arises from the different valuation of dividend streams by the portfolio insurers as compared with the normal agents. The portfolio insurance constraint is on time T wealth which is equal to the present value of consumption after T. Hence, consumption after time T not only provides the portfolio insurer with utility, but also helps him meet his time T wealth constraint. Consumption before time T, however, hinders him from meeting his constraint. As a result, relative to current consumption, the portfolio insurers value consumption (dividends) before time T the same, but consumption (dividends) after time T more highly, than the normal agents do. Thus the before-horizon value of the equity market, which is a claim against all the dividend streams after the horizon, is higher in the presence of portfolio insurance.

Proposition 2 also allows us to address how individual asset prices are affected by portfolio insurance. Equation (26) reveals that, as with the equity market, before the portfolio insurance horizon the price of each asset is higher in the presence of portfolio insurance, for the reason discussed above. We can rearrange (26) as

$$P^{(2)}_i(t) = P^{(1)}_i(t) + \frac{1}{u'(\delta(t)/N)} E\left[P^{(1)}_i(T) \mid \mathcal{F}_t\right] E\left[u'(\delta(T)/N) \max\left\{\frac{NK}{X^{(1)}_{em}(T)} - 1, 0\right\} \mid \mathcal{F}_t\right]$$
$$+ \frac{1}{u'(\delta(t)/N)} \text{cov}_t\left(P^{(1)}_i(T), u'(\delta(T)/N) \max\left\{\frac{NK}{X^{(1)}_{em}(T)} - 1, 0\right\}\right).$$

where

$$a(t) \equiv \frac{1}{\eta}\left[\exp\{\eta(T-t)\}-1\right]; \quad a(t) \geq 0,$$

$$b(t) \equiv \frac{1}{\eta}\left[\exp\{\eta(T'-T)\}-1\right]\exp\{\eta(T-t)\}; \quad b(t) \geq 0,$$

$$d_1 \equiv \frac{\ln\left(b(t)\delta(t)/NK\right)+\left(r+\frac{1}{2}\|\sigma_\delta\|^2\right)(T-t)}{\|\sigma_\delta\|\sqrt{T-t}}, \quad d_2 \equiv d_1 - \|\sigma_\delta\|\sqrt{T-t},$$

$\mathcal{N}(\cdot)$ is the distribution function for a standard normal random variable, and r and η are as given in Lemma 4. After the portfolio insurance horizon, the equilibria in both economies are identical.

We see that the portfolio insurance market price is equal to the normal agent market price plus the value (adjusted for dividends) of a Black-Scholes put option with maturity T on the normal agent market. Moving from the base economy 1, where the market volatility and risk premium are constants, in the portfolio insurance economy the market risk premium and volatility are stochastically time-varying. The volatility and risk premium are seen to be driven by the level of the market and by the additional amount of wealth invested in the bond to synthetically create the Black-Scholes put option. The explicit analytical expressions in Proposition 3 lead to the following comparative statics.

Corollary 1: *Before the portfolio insurance horizon we have*
(i) $\mu^{(2)}_{em}(t) - r \leq \mu^{(1)}_{em}(t) - r$ and $\|\sigma^{(2)}_{em}(t)\| \leq \|\sigma^{(1)}_{em}(t)\|$;
(ii) $\mu^{(2)}_{em}(t) - r$ and $\|\sigma^{(2)}_{em}(t)\|$ are decreasing in K;
(iii) $\mu^{(2)}_{em}(t) - r$ and $\|\sigma^{(2)}_{em}(t)\|$ are increasing in $X^{(2)}_{em}(t)$ while $\mu^{(1)}_{em}(t) - r$ and $\|\sigma^{(1)}_{em}(t)\|$ are independent of $X^{(1)}_{em}(t)$;
(iv) $\mu^{(2)}_{em}(t) - r$ and $\|\sigma^{(2)}_{em}(t)\|$ are decreasing in N while $\mu^{(1)}_{em}(t) - r$ and $\|\sigma^{(1)}_{em}(t)\|$ are independent of N.

The most important comparisons are the first two, stating that the market volatility and risk premium are lower in the presence of portfolio insurance and are decreasing in the level of the floor. We provide detailed intuition for these results in subsection 3.3.3. The third result of Corollary 1 shows that the effect of portfolio insurance in decreasing the volatility and risk premium is less pronounced in states where the market is high. These are the "good" states in which the portfolio insurers have to strive less to meet their wealth constraint and hence their effect on prices is less. Item (iv) further reveals that the more portfolio insurers there are in economy 2, the greater the effect on price dynamics. This number of agents effect is special to this economy of identical agents. More generally, this result represents that the "poorer" a portfolio insurer is relative to the whole economy, the greater his effect, since he must strive all the harder to meet his constraint.[6]

[6]Comparing the volatility of the individual asset returns across economies has proved to be a hard exercise since we can only consistently assume that either the $\delta_i(t)$ or $\delta(t)$ follows geometric Brownian motion. Without the assumption of geometric Brownian motion of all these processes, the analysis of the conditional expectation terms in the price formula leads to multiple terms whose signs cannot be compared unambiguously across economies.

where

$$d_{m1} \equiv \frac{\ln\left((T'-T)\delta(t)/y_{m1}K\right) + \left(r + \frac{1}{2}\|\sigma_\delta\|^2\right)(T-t)}{\|\sigma_\delta\|\sqrt{T-t}}, \quad d_{m2} \equiv d_1 - \|\sigma_\delta\|\sqrt{T-t}, \quad r = \mu_\delta - \|\sigma_\delta\|^2,$$

and $\mathcal{N}(\cdot)$ is the distribution function for a standard normal random variable. After the portfolio insurance horizon, the presence of portfolio insurance does not affect the equilibrium.

Remark 2 (Extensions):
(i) In this heterogeneous-agent economy, we could easily allow portfolio insurers to have differing floors K_m, and differing portfolio insurance horizons T_m. Then in each of the M put options in the above expressions K and T would be replaced by their respective K_m and T_m. The prices could now exhibit M possible jumps, one at each horizon T_m. Market volatility and risk premium would still be decreased by the presence of portfolio insurance.
(ii) Further extension to portfolio insurers with repeated portfolio insurance horizons would lead to a nesting of put options on put options; in this case the effect on market and asset prices is the same, but we do not obtain explicit formulae for the market dynamics.

3.3.3 Discussion of Main Results

To provide some intuition for the decrease in market volatility and risk premium in the presence of porfolio insurers, we consider the optimal portfolio strategies of the log utility normal agents and portfolio insurers of subsection 3.3.2, where r and θ are constants.

Proposition 5: *Assume r and θ are constants. Before the portfolio insurance horizon, the fractions of wealth $\hat{\phi}(t)$ optimally invested in the risky assets by a normal agent (n) and a portfolio insurer (m) with log utility of consumption are*

$$\hat{\phi}_n(t) = \left[\sigma(t)^\top\right]^{-1} \sigma(t)^{-1}\left(\mu(t) - r\mathbf{1}\right), \tag{29}$$

$$\hat{\phi}_m(t) = \left[1 - \frac{K\exp\{-r(T-t)\}\mathcal{N} - d_{m2})}{\hat{X}_m(t)}\right]\left[\sigma(t)^\top\right]^{-1} \sigma(t)^{-1}\left(\mu(t) - r\mathbf{1}\right), \tag{30}$$

where

$$d_{m2} \equiv \frac{\ln\left((T'-T)/y_{m1}\xi(t)K\right) + \left(r - \frac{\|\theta\|^2}{2}\right)(T-t)}{\|\theta\|\sqrt{T-t}}.$$

Hence, $\hat{\phi}_{mi}(t) \leq \hat{\phi}_{ni}(t), \quad t \in [0,T), \quad i = 1, ..., L.$

The above two demand functions differ only by a multiplying factor less than 1. Hence, at the same prices a portfolio insurer demands less in the risky assets and more in the bond than a normal agent; in a sense the portfolio insurer behaves as more risk averse. This is because, the distinguishing characteristic of a portfolio insurer's portfolio strategy, a synthetic put option, consists of a short position in the risky securities and a long position in the bond.

4 An Economy with Portfolio Insurers Modeled in Terms of a Specific Utility Function of Time T Wealth

In this section we model the agents as ones who maximize their expected utility from consumption in $[0, T)$ and from wealth at time T, subject to their budget constraints. The portfolio insurers and the normal agents have the same utility function of consumption, but the portfolio insurers' utility function of time T wealth is chosen to implicitly inherit the condition $X_m(T\text{-}) \geq K$. Although the world goes on beyond T until T', most of our discussion will focus on $[0, T)$ since the time T wealth and agents' utility for that wealth summarize the consumption and agents' utility for consumption beyond time T. We again assume that the first M agents are portfolio insurers and the remaining $N - M$ are "normal" agents. We first solve the individual agents' optimization problems, then characterize equilibrium in a heterogeneous multi-agent economy.

4.1 Individual Agents' Preferences and Optimization Problems

We assume that all agents have power utility of consumption in $[0, T)$, i.e., $u_n(c) = c^\gamma/\gamma$, $\gamma < 1$, $\gamma \neq 0$ for $n = 1, ..., N$. The utilities of time T wealth are assumed to be

$$v_m(X) = \frac{(X - K)^\gamma}{\gamma}, \quad m = 1, ..., M,$$

$$v_n(X) = \frac{X^\gamma}{\gamma}, \quad n = M + 1, ..., N,$$

where $\gamma < 1$, $\gamma \neq 0$, and $m = 1, ..., M$ refer to the portfolio insurers. Note that the normal agents here are also modeled differently from those in Section 3, to make them comparable with the portfolio insurers and to be able to aggregate over all agents.

Each agent $n = 1, ..., N$ solves the following optimization problem

$$\max_{(c_n(.), X_n(T\text{-}))} E\left[\int_0^T u_n(c_n(t))dt + v_n(X_n(T\text{-}))\right]$$

$$\text{subject to} \quad E\left[\int_0^T \xi(t)c_n(t)dt + \xi(T\text{-})X_n(T\text{-})\right] \leq \xi(0)x_{no}.$$

Lemma 5 presents the solution to these problems.

Lemma 5: *The optimal* $[0, T)$ *consumption processes and optimal time* T *wealths of the* N *agents are given by*

$$\begin{aligned}
\hat{c}_n(t) &= [y_n\xi(t)]^{\frac{1}{\gamma-1}}, \quad t \in [0, T), \quad n = M + 1, ..., N, \\
\hat{c}_m(t) &= [y_m\xi(t)]^{\frac{1}{\gamma-1}}, \quad t \in [0, T), \quad m = 1, ..., M, \\
\hat{X}_n(T\text{-}) &= [y_n\xi(T\text{-})]^{\frac{1}{\gamma-1}}, \quad n = M + 1, ..., N, \\
\hat{X}_m(T\text{-}) &= [y_m\xi(T\text{-})]^{\frac{1}{\gamma-1}} + K, \quad m = 1, ..., M,
\end{aligned}$$

and if

$$U(t,c) = \frac{D}{\gamma}\left[c - MK_c(t)\right]^\gamma, \quad t \in [T, T'],$$

where $D \equiv \frac{1}{\xi(T\text{-})^\gamma}\left(E\left[\int_T^{T'} \xi(t)^{\frac{\gamma}{\gamma-1}} dt \mid \mathcal{F}_T\right]\right)^{\gamma-1}$
and $K_c(t)$ *is such that* $K = \frac{1}{\xi(T\text{-})} E\left[\int_T^{T'} \xi(t) K_c(t) dt \mid \mathcal{F}_T\right]$, *then*

$$V(X_{T\text{-}}) = \frac{(X_{T\text{-}} - MK)^\gamma}{\gamma}.$$

The utilities of consumption are such that the representative agent wants to consume at least a subsistence level of $MK_c(t)$ at all times $t \in [T, T']$. MK is exactly the amount of wealth he needs at time T to be able to afford $MK_c(t)$ at all future times $t \in [T, T']$ so he wants at least this much wealth at time T. The future utilities of consumption for the individual agents are given analogously by $u_n(c) = \frac{D}{\gamma} c^\gamma$ and $u_m(c) = \frac{D}{\gamma}[c - K_c(t)]^\gamma$.

Using the representative agent constructed above, analogously to equations (19) and (20) we obtain the following equilibrium expressions for the state price density process and establish the existence of equilibrium:

$$\xi(t) = \delta(t)^{\gamma-1}, \quad t \in [0, T), \tag{31}$$

$$\xi(t) = \frac{\delta(T)^\gamma}{E\left[\int_T^{T'} (\delta(s) - MK_c(s))^\gamma ds \mid \mathcal{F}_T\right]} \left[\delta(t) - MK_c(t)\right]^{\gamma-1}, \quad t \in [T, T']. \tag{32}$$

Again a discontinuity in $\xi(\cdot)$ occurs at time T in equilibrium, which in turn leads to discontinuities in the asset prices. The interest rate process $r(t)$ and the market price of risk process $\theta(t)$ can be derived by applying Itô's Lemma to (31) and (32). Before T, $r(t)$ and $\theta(t)$ are the same across economies (i.e., are independent of M) because of the power utility of consumption, as explained in Remark 2 of Section 3. However, after T $r(t)$ and $\theta(t)$ differ across economies, in contrast to the result in Section 3.

4.3 Equilibrium Market Price, Volatility and Risk Premium

In this case we are able to calculate the market and its dynamics in a power utility heterogeneous multi-agent economy. As in Section 3 we assume that the aggregate dividend process follows geometric Brownian motion. Then, $\xi(\cdot)$ in equilibrium follows a geometric Brownian motion until time T when it jumps, as summarized by Lemma 6.

Lemma 6: *Under assumption A1, before the portfolio insurance horizon, T, the equilibrium r and θ are constants given by*

$$r = (1-\gamma)\mu_\delta - \frac{(1-\gamma)(2-\gamma)}{2}\|\sigma_\delta\|^2 \quad \textit{and} \quad \theta_j = (1-\gamma)\sigma_{\delta j}, \quad j = 1, \ldots, L,$$

Results (ii) and (iii) are comparable with those of Section 3 with similar intuition. In this case when $\gamma > 0$, the effect of portfolio insurance on the market volatility and risk premium becomes monotonically more pronounced as the time approaches the portfolio insurance horizon, as stated in item (iv). After the portfolio insurance horizon, unlike in Section 4, the market price and its dynamics are still affected by the presence of portfolio insurance. However, the particular dependence will be sensitive to the choice of future utility of consumption functions, and hence not due to the presence of portfolio insurance *per se*.

Remark 3:
(i) If agents had log utility of consumption and final wealth, $u_m(c) = u_n(c) = \log(c), v_m(c) = \log(X - K)$ and $v_n(X) = \log(X)$ we would get the results of this Section 4 with $\gamma = 0$.
(ii) In this Section 4, we could easily allow portfolio insurers to have differing floors, K_m. Then MK would simply be replaced by $\sum_{m=1}^{M} K_m$.

To provide the intuition for the volatility and risk premium results, in Proposition 8 we compare the agents' asset demands as we did in subsection 3.3.3. We assume that $\xi(\cdot)$ follows geometric Brownian motion up to time T, which anticipates the equilibrium result.

Proposition 8: *Assume r and θ are constants in $[0,T)$. Before the portfolio insurance horizon, the fractions of wealth optimally invested in the risky assets by a normal agent (n) and a portfolio insurer (m) are*

$$\hat{\phi}_n(t) = \frac{1}{1-\gamma}\left[\sigma(t)^{\top}\right]^{-1}\sigma(t)^{-1}\left(\mu(t) - r\mathbf{1}\right), \tag{35}$$

$$\hat{\phi}_m(t) = \frac{1}{1-\gamma}\left[1 - \frac{K\exp\{-r(T-t)\}}{\hat{X}_m(t)}\right]\left[\sigma(t)^{\top}\right]^{-1}\sigma(t)^{-1}\left(\mu(t) - r\mathbf{1}\right). \tag{36}$$

Hence, $\hat{\phi}_{mi}(t) \leq \hat{\phi}_{ni}(t), \quad t \in [0,T), \quad i = 1, ..., L.$

Again, the portfolio insurers invest a smaller fraction in each risky asset than the normal agents. Hence to clear the markets, the risky assets must become more favorable as we increase the fraction of portfolio insurers across economies. Since the interest rate and the market price of risk are unchanged, the only way to make the risky assets more favorable is for their volatility to decrease. Hence the market volatility, and in turn the market risk premium, must also decrease. This intuition is identical to that in Section 3 except that the extra demand for bond by the portfolio insurers now arises due to a riskless term in their wealth rather than a put option.

5 A Closer Look at the Behavior of the Portfolio Insurers

In this section we question whether the ways we model portfolio insurance in Sections 3 and 4 are a satisfactory choice from two viewpoints. Firstly, we ask whether our optimizing portfolio insurers do fit the definition of portfolio insurance described in Section 2; we already know that they achieve a minimum level of wealth, but do they really participate in the gains of

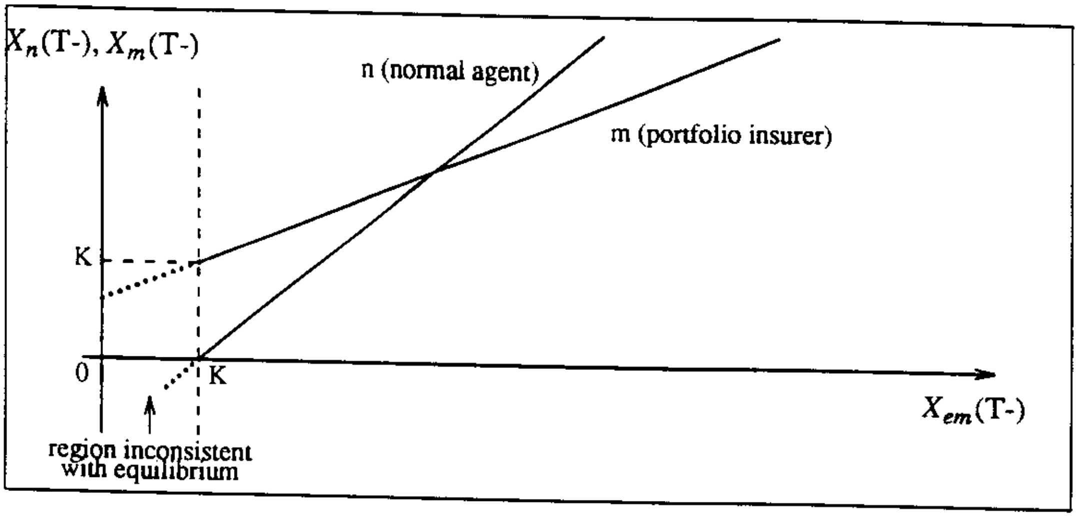

Figure 2: **Agents' horizon wealths in the specific utility function portfolio insurance economy.** The solid lines are the agents' time T wealths (Lemma 5) plotted against the time T level of the equity market of a two-agent economy. The dotted lines are extrapolations into the region inconsistent with equilibrium.

Figure 3 reveals that there is a probability mass build-up at the floor and that the density loses its left tail. The higher the floor the more pronounced the build-up. In Figure 4, the probability density of the portfolio insurer with a specific utility of horizon wealth is shifted to the right of that of the normal agent. The probability mass build-up is broader and lies above the floor K.

5.2 Comparison with Portfolio Insurance Employed in Practice

We now briefly compare the portfolio insurance trading strategies of Sections 3 and 4 to those followed in practice. It is clear that the behavior of the portfolio insurers with the explicit horizon wealth constraint is very similar to the synthetic put approach to portfolio insurance, since their strategy involves replicating synthetic put options to meet the wealth constraint.

As for the portfolio insurers of Section 4 with a specific utility of horizon wealth, we may compare their optimal portfolio strategy with the constant proportion portfolio insurance (CPPI) trading strategy. This strategy invests in the risky asset a constant "multiple" of the the difference between the agent's wealth and some specified "floor", and the remainder in the bond. (The floor is typically not a constant but grows at the risk-free rate.) Hence the proportion of wealth invested under the CPPI trading strategy is given by

$$\phi_{m1}(t) = \psi \left(1 - \frac{k \exp\{-r(T-t)\}}{X_m(t)}\right),$$

102.2 units of consumption in Figure 3 and 143 in Figure 4, and for $K = 75$ it was 112.9 and 135. We simulated random realizations of the aggregate dividend process at T, computed the equilibrium time T wealths for each realization and nonparametrically estimated the probability density function of each agent's time T wealth using a Gaussian kernel.

where ψ is the multiple which must satisfy $\psi > 1$, and $k \exp\{-r(T-t)\}$ is the "floor". There is a close resemblance between the optimal portfolio strategy, for the case of one risky asset, of our optimizing portfolio insurers of Section 4 (Proposition 8) and the CPPI strategy; the main difference is that in our formulation the exogeneous parameter ψ gets replaced by $\theta/(1-\gamma)\sigma$.

An issue related to portfolio insurance strategies employed in practice, which we do not pursue here, is the fact that these strategies typically involve the transfer of funds to the risky assets as their prices go up and away from the risky assets as their prices go down. It is this type of "trend-chasing" behavior that is often argued to increase market volatility. However, Basak (1993, Chapter 1, §6.2) finds that in our type of general equilibrium model, there is no clear-cut relationship between the effect of the presence of a portfolio insurer on the market volatility and the extent of his trend-chasing behavior.

6 Summary and Conclusion

Attention has recently focused on market volatility as a measure of market instability, particularly since the volatility was observed to be high during the stock market crash of 1987. This paper focuses on the effects of portfolio insurance trading strategies on the market dynamics. We construct a consumption-based general equilibrium model of a continuous-time economy that does capture the effects of portfolio insurance. These effects are represented by explicit formulae for the quantities of interest, derived under common assumptions made in finance.

One major result we find is that the market price before the portfolio insurance horizon is increased by the presence of portfolio insurance. Then at the portfolio insurance horizon the price jumps down discontinuously. This is an important point in the paper that, because of the effective discontinuity of portfolio insurers' consumption preferences, the market, security and state prices may all jump discontinuously at the horizon in order to attain equilibrium.

Our main conclusion is that the market volatility and risk premium before the portfolio insurance horizon are decreased in the presence of portfolio insurance. The intuition for these results is that the portfolio insurers in striving to meet their objectives demand less than the normal agents in the risky assets and more in the bond. The constrained portfolio insurers behave like the normal agents but purchase an additional synthetic put option, consisting of a long position in the bond and a short position in the risky assets. The portfolio insurers described by a specific utility of time T wealth function behave like the normal agents but purchase an additional amount of certain wealth by buying extra bond. Since the supply of each asset is unchanged across economies, to achieve equilibrium in a portfolio insurance economy, the risky assets must become more favorable and the riskless asset less favorable than in an economy with no portfolio insurers. With preferences over continuous consumption exhibiting CRRA, the interest rate and the market price of risk are the same across economies. Hence we find that the only way to make the equity market more favorable is to decrease its volatility, and that the risk premium must decrease simultaneously. Our result is in sharp contrast to the popular belief that portfolio insurance increases market volatility. The striking point about our conclusions is that this popular belief breaks down even in one of the most standard well understood set-ups in finance (Lucas (1978) and CRRA preferences).

APPENDIX: Proofs

Proof of Lemma 1: Define

$$\bar{P}_i(t) \equiv P_i(t) + \frac{1}{\xi(t)} \int_0^t \xi(s)\delta_i(s)ds, \quad t \in [0, T'], \quad i = 1, ..., L,$$

i.e., $\bar{P}_i(t)$ is the current price plus the current value of accumulated dividends. For all $t \neq T$, $dA(t) = 0$, and Itô's Lemma implies

$$d(\xi(t)\bar{P}_i(t)) = \xi(t)P_i(t)(\sigma_i(t) - \theta(t)^\top)dW(t), \quad \forall t \neq T$$

where $\sigma_i(t)$ denotes the ith row of $\sigma(t)$. The above result implies that

$$\bar{P}_i(t) = \frac{1}{\xi(t)} E[\xi(T')\bar{P}_i(T') \mid \mathcal{F}_t] = \frac{1}{\xi(t)} E\left[\int_0^{T'} \xi(s)\delta_i(s)ds \mid \mathcal{F}_t\right], \quad t \in [T, T'],$$

using the fact that $P_i(T') = 0$, giving the desired result for $t \in [T, T']$. Similarly

$$\bar{P}_i(t) = \frac{1}{\xi(t)} E[\xi(T\text{-})\bar{P}_i(T\text{-}) \mid \mathcal{F}_t], \quad t \in [0, T).$$

From the dynamics of P_i and ξ we have $\exp(q) = P_i(T)/P_i(T\text{-}) = \xi(T\text{-})/\xi(T)$, and so

$$\bar{P}_i(t) = \frac{1}{\xi(t)} E[\xi(T)\bar{P}_i(T) \mid \mathcal{F}_t] = \frac{1}{\xi(t)} E[\xi(T')\bar{P}_i(T') \mid \mathcal{F}_t], \quad t \in [0, T),$$

which provides the desired result for $t \in [0, T)$. *Q.E.D.*

Proof of Lemma 2: See, for example, Lemma 2.4, Cox and Huang (1989). *Q.E.D.*

Proof of Lemma 3: We solve both types of agent's optimization problems (heuristically) by the Lagrangian method. Letting y_n, y_{m1}, y_{m2} and y_{m3} be the Lagrange multipliers associated, respectively, with the normal agent's budget constraint, the portfolio insurer's budget constraint for $[0, T)$, the portfolio insurer's budget constraint for $[T, T']$, and the portfolio insurer's wealth constraint, and given that all budget constraints will hold with equality since agents prefer more to less, we obtain the following Kuhn-Tucker conditions:

$$u'(\hat{c}_n(t)) = y_n\xi(t), \quad t \in [0, T'] \tag{37}$$

$$E\left[\int_0^{T'} \xi(t)\hat{c}_n(t)dt\right] = \xi(0)x_{no} \tag{38}$$

$$u'(\hat{c}_m(t)) = y_{m1}\xi(t), \quad t \in [0, T) \tag{39}$$

$$u'(\hat{c}_m(t)) = y_{m2}\xi(t), \quad t \in [T, T'] \tag{40}$$

$$0 = y_{m1}\xi(T\text{-}) - y_{m3} - y_{m2}\xi(T\text{-}) \tag{41}$$

$$E\left[\int_0^{T} \xi(t)\hat{c}_m(t)dt + \xi(T\text{-})\hat{X}_m(T\text{-})\right] = \xi(0)x_{mo} \tag{42}$$

$$E\left[\int_T^{T'} \xi(t)\hat{c}_m(t)dt \mid \mathcal{F}_T\right] = \xi(T\text{-})\hat{X}_m(T\text{-}) \tag{43}$$

$$y_n, y_{m1}, y_{m2}, y_{m3} \geq 0 \tag{44}$$

$$y_{m3}\left[\hat{X}_m(T\text{-}) - K\right] = 0. \tag{45}$$

Sufficiency: $\hat{c}_m(t)$ and $\hat{c}_n(t)$ given by (9)-(11) together with (17) and (18) imply (6). Using Lemma 2, we obtain (8) as follows:

$$\sum_{n=1}^{N} \hat{X}_n(t) = \frac{1}{\xi(t)} E\left[\int_t^{T'} \xi(s) \sum_{n=1}^{N} \hat{c}_n(s) ds \mid \mathcal{F}_t\right] = \frac{1}{\xi(t)} E\left[\int_t^{T'} \xi(s) \sum_{i=1}^{L} \delta_i(s) ds \mid \mathcal{F}_t\right] = \sum_{i=1}^{L} P_i(t),$$

where we have used (6) and Lemma 1.

Applying Itô's Lemma to $\xi(t)\hat{X}_n(t)$, using (3) and summing over all agents, we obtain

$$d\left(\xi(t) \sum_{n=1}^{N} \hat{X}_n(t)\right) = \xi(t) \left[\sum_{n=1}^{N} \hat{\pi}_n(t)^\top \sigma(t) - \sum_{n=1}^{N} \hat{X}_n(t)\theta(t)^\top\right] dW(t) - \xi(t) \sum_{n=1}^{N} \hat{c}_n(t) dt,$$

which using (6) and (8) can be written as

$$d\left(\xi(t) \sum_{i=1}^{L} P_i(t)\right) = \xi(t) \left[\sum_{n=1}^{N} \hat{\pi}_n(t)^\top \sigma(t) - \sum_{i=1}^{L} P_i(t)\theta(t)^\top\right] dW(t) - \xi(t) \sum_{i=1}^{L} \delta(t) dt.$$

Applying Itô's Lemma to $\xi(t)P_i(t)$ and summing over all risky securities i we obtain

$$d\left(\xi(t) \sum_{i=1}^{L} P_i(t)\right) = \xi(t) \left[\sum_{i=1}^{L} P_i(t)\sigma_i(t) - \sum_{i=1}^{L} P_i(t)\theta(t)^\top\right] dW(t) - \xi(t) \sum_{i=1}^{L} \delta(t) dt.$$

Equating the coefficients of the $dW_j(t)$ terms in the above two dynamics of $\xi(t) \sum_{i=1}^{L} P_i(t)$ yields

$$P(t)^\top \sigma(t) = \sum_{n=1}^{N} \hat{\pi}_n(t)^\top \sigma(t),$$

which using the nonsingularity of $\sigma(t)$ implies (7). *Q.E.D.*

Proof of Proposition 2: For economy 1, substituting (21) into (9) and (12), each normal agent's equilibrium consumption and time T wealth are

$$\hat{c}(t) = \delta(t)/N \quad t \in [0, T']$$

$$\hat{X}(T\text{-}) = \frac{1}{u'(\delta(T)/N)} E\left[\int_T^{T'} u'(\delta(s)/N) \frac{\delta(s)}{N} ds \mid \mathcal{F}_T\right].$$

In this identical-agent economy, $X_{em}(t) = N\hat{X}(t)$, so the equity market in economy 1 is given, using Lemma 2, by equation (24). Substituting (21) into Lemma 1 we also obtain the expression for $P_i^{(1)}(T)$.

For economy 2, substituting (22), (23) into (10), (11), (46) each portfolio insurer's equilibrium consumption and time T wealth are

$$\hat{c}(t) = \delta(t)/N \quad t \in [0, T']$$

$$\hat{X}(T\text{-}) = \max\left\{K, \frac{1}{u'(\delta(T)/N)} E\left[\int_T^{T'} u'(\delta(s)/N) \frac{\delta(s)}{N} ds \mid \mathcal{F}_T\right]\right\}.$$

Since $\delta(t)$ is geometric Brownian motion (for $s > t$, $\ln \delta(s)$ is normally distributed with mean $\ln \delta(t) + \mu_\delta(s-t) - \frac{\|\sigma_\delta\|^2}{2}(s-t)$ and variance $\|\sigma_\delta\|^2(s-t)$) the conditional expectations can be evaluated to derive

$$X_{em}^{(1)}(t) = \frac{1}{\eta}\left[\exp\{\eta(T-t)\} - 1\right]\delta(t) + \frac{1}{\eta}\left[\exp\{\eta(T'-T)\} - 1\right]\exp\{\eta(T-t)\}\delta(t)$$

and

$$\begin{aligned} X_{em}^{(2)}(t) &= \frac{1}{\eta}\left[\exp\{\eta(T-t)\} - 1\right]\delta(t) + \frac{1}{\eta}\left[\exp\{\eta(T'-T)\} - 1\right]\exp\{\eta(T-t)\}\delta(t) \\ &+ \delta(t)^{1-\gamma} E\left[\delta(T)^{\gamma-1}\max\left\{NK - \frac{1}{\eta}\left[\exp\{\eta(T'-T)\} - 1\right]\delta(T), 0\right\} \mid \mathcal{F}_t\right], \end{aligned}$$

where $\eta \equiv \gamma\mu_\delta - \gamma(1-\gamma)\|\sigma_\delta\|^2/2$. Substituting back for the equilibrium $\xi^{(2)}(t) = (\delta(t)/N)^{\gamma-1}/y_1$, $t \in [0,T)$ (from (22)), the third term in the portfolio insurance economy market can be written as

$$\frac{1}{\xi(t)} E\left[\xi(T\text{-})\max\left\{NK - \frac{1}{\eta}\left[\exp\{\eta(T'-T)\} - 1\right]\delta(T), 0\right\} \mid \mathcal{F}_t\right]$$

We define the unique *equivalent martingale measure* $\tilde{\mathcal{P}}$, given by

$$\tilde{\mathcal{P}}(A) = E[z(T')1_A], \quad A \in \mathcal{F}_{T'},$$

where

$$z(t) \equiv \exp\left\{-\int_0^t \theta(s)^\top dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right\} = \xi(t)P_0(t).$$

Changing the measure to the equivalent measure $\tilde{\mathcal{P}}$, the conditional expectation becomes

$$\frac{P_0(t)}{P_0(T\text{-})}\tilde{E}\left[\max\left\{NK - \frac{1}{\eta}\left[\exp\{\eta(T'-T)\} - 1\right]\exp\{\eta(T-t)\}\delta(T), 0\right\} \mid \mathcal{F}_t\right].$$

This can be integrated out explicitly yielding

$$-b(t)\delta(t)\mathcal{N}(\text{-}d_1) + NK\exp\{\text{-}r(T\text{-}t)\}\mathcal{N}(\text{-}d_2)$$

where d_1, d_2 and $b(t)$ are as given in the proposition. Hence we derive the required expressions for $X_{em}^{(1)}(t)$ and $X_{em}^{(2)}(t)$. We observe that the defined (deterministic) processes $a(t)$ and $b(t)$ are nonnegative since $[\exp\{\eta(T-t)\} - 1]$ and $[\exp\{\eta(T'-T)\} - 1]$ have the same sign as η.

Finally applying Itô's Lemma to $X_{em}^{(1)}(t)$ and $X_{em}^{(2)}(t)$ leads to the market volatilities and risk premia as presented. *Q.E.D.*

Proof of Corollary 1: By taking derivatives of (27) and (28) and straightforward algebraic manipulations. *Q.E.D.*

Proof of Proposition 4: Since the ys are only determined up to a multiplicative constant from (14)-(16), we can set $\left(\sum_{m=1}^{M} 1/y_{m1} + \sum_{n=M+1}^{N} 1/y_n\right) = 1$ in the expression for ξ in $[0,T)$.

where

$$d_{m1} \equiv \frac{\ln\left((T'-T)/y_{m1}\xi(t)K\right) + \left(r + \frac{\|\theta\|^2}{2}\right)(T-t)}{\|\theta\|\sqrt{T-t}}, \quad d_{m2} \equiv d_{m1} - \|\theta\|\sqrt{T-t}.$$

Applying Itô's Lemma to the products $\xi(t)\hat{X}_n(t)$ and $\xi(t)\hat{X}_m(t)$, we get

$$\begin{aligned} d\left(\xi(t)\hat{X}_n(t)\right) &= -\frac{1}{y_n}dt \\ d\left(\xi(t)\hat{X}_m(t)\right) &= -\left[-\frac{(T'-T)}{y_{m1}}\mathcal{N}(\text{-}d_{m1}) + \xi(t)K\exp\{\text{-}r(T\text{-}t)\}\mathcal{N}(\text{-}d_{m2})\right]\theta(t)^\top dW(t) \\ &- \frac{(T'-T)}{y_{m1}}\mathcal{N}(\text{-}d_{m1})\theta(t)^\top dW(t) + dt \text{ terms} \\ &= -\xi(t)K\exp\{\text{-}r(T\text{-}t)\}\mathcal{N}(\text{-}d_{m2})\theta(t)^\top dW(t) + dt \text{ terms}, \end{aligned}$$

where for the first equality we used the fact that

$$\partial\left[\frac{(T'-T)}{y_{m1}\xi(t)}\mathcal{N}(\text{-}d_{m1}) + K\exp\{\text{-}r(T\text{-}t)\}\mathcal{N}(\text{-}d_{m2})\right] / \partial\left[\frac{(T'-T)}{y_{m1}\xi(t)}\right] = -\mathcal{N}(\text{-}d_{m1}).$$

Equating these coefficients of $dW(t)$ with those of equation (50) yields the optimal portfolio strategies $\hat{\pi}_n(t)$ and $\hat{\pi}_m(t)$, which upon dividing by $\hat{X}_n(t)$ $\hat{X}_m(t)$ leads to (29) and (30).

To show that $\hat{\phi}_{mi}(t) \leq \hat{\phi}_{ni}(t)$ it is sufficient to show that

$$0 \leq \left[1 - \frac{K\exp\{\text{-}r(T\text{-}t)\}\mathcal{N}(\text{-}d_{m2})}{\hat{X}_m(t)}\right] \leq 1.$$

The second equality is obvious since both the numerator and the denominator of the second term in the square brackets are nonnegative. Using the above expressions for $\hat{X}_m(t)$ we rearrange to give

$$\left[1 - \frac{K\exp\{\text{-}r(T\text{-}t)\}\mathcal{N}(\text{-}d_{m2})}{\hat{X}_m(t)}\right] = \frac{1}{\hat{X}_m(t)}\left[\frac{(T'-T)}{y_{m1}\xi(t)} + \frac{(T'-T)}{y_{m1}\xi(t)}(1-\mathcal{N}(\text{-}d_{m1}))\right]$$

which is nonnegative as required. *Q.E.D.*

Proof of Lemma 5: The optimal consumption and wealth expressions are derived by standard static optimization techniques. To prove (i) we compare the behavior of a portfolio insurer and a normal agent with identical initial wealths. y_m must be greater than y_n in order to satisfy the budget constraints since there is an extra $E[\xi(T\text{-})K]$ term on the left hand side of the portfolio insurer's budget constraint, and since the left hand sides are decreasing in their respectice ys. *Q.E.D.*

Proof of Proposition 6: It is straightforward to show that $\hat{c}(t) = J(y\xi(t))$ where $J(\cdot)$ is the inverse of $U'(\cdot)$. Hence we have

$$V(X_{T\text{-}}; \mathcal{F}_{T\text{-}}) = E\left[\int_T^{T'} U\left(J(y\xi(t))\right)dt \mid \mathcal{F}_T\right]$$

Proof of Corollary 2: By taking derivatives of (33) and (34) and straightforward algebraic manipulations. *Q.E.D.*

Proof of Proposition 8: From Lemma 2 and the optimal consumptions and horizon wealths in Lemma 5, evaluating expectations as in the proof of Proposition 5, the optimally invested wealths of the agents are

$$\hat{X}_n(t) = a_n(t)\xi(t)^{\frac{1}{\gamma-1}} + b_n(t)\xi(t)^{\frac{1}{\gamma-1}}, \quad n = M+1, ..., N,$$

$$\hat{X}_m(t) = a_m(t)\xi(t)^{\frac{1}{\gamma-1}} + b_m(t)\xi(t)^{\frac{1}{\gamma-1}} + K\exp\{-r(T-t)\}, \quad m = 1, ..., M,$$

where

$$a_l(t) \equiv \frac{y_l^{\frac{1}{\gamma-1}}}{\eta}\left[\exp\{\eta(T-t)\} - 1\right], \quad b_l(t) \equiv y_l^{\frac{1}{\gamma-1}}\exp\{\eta(T-t)\}, \quad l = m, n, \quad \eta \equiv \frac{\gamma}{1-\gamma}r + \frac{\gamma}{2(1-\gamma)^2}\|\theta\|^2.$$

Applying Itô's Lemma to $\xi(t)\hat{X}_n(t)$ and $\xi(t)\hat{X}_m(t)$, then yields

$$\begin{aligned} d\left(\xi(t)\hat{X}_n(t)\right) &= a_n(t)\frac{1}{1-\gamma}\xi(t)^{\frac{\gamma}{\gamma-1}}\theta(t)^\top dW(t) + b_n(t)\frac{1}{1-\gamma}\xi(t)^{\frac{\gamma}{\gamma-1}}\theta(t)^\top dW(t) + dt \text{ terms} \\ &= \xi(t)\hat{X}_n(t)\theta(t)^\top dW(t) + dt \text{ terms}, \end{aligned}$$

$$\begin{aligned} d\left(\xi(t)\hat{X}_m(t)\right) &= a_m(t)\frac{\gamma}{1-\gamma}\xi(t)^{\frac{\gamma}{\gamma-1}}\theta(t)^\top dW(t) + b_m(t)\frac{1}{1-\gamma}\xi(t)^{\frac{\gamma}{\gamma-1}}\theta(t)^\top dW(t) \\ &- K\exp\{-r(T-t)\}\xi(t)\theta(t)^\top dW(t) + dt \text{ terms} \\ &= \xi(t)\left[\frac{\gamma}{1-\gamma}\hat{X}_m(t) - \frac{1}{1-\gamma}K\exp\{-r(T-t)\}\right]\theta(t)^\top dW(t) + dt \text{ terms}, \end{aligned}$$

Equating the $dW(t)$ terms with those of equation (50) yields the portfolio strategies $\hat{\pi}_n(t)$ and $\hat{\pi}_m(t)$. Then dividing by the optimal wealths gives (35) and (36). Furthermore, we have

$$1 > \left[1 - \frac{K\exp\{-r(T-t)\}}{\hat{X}_m(t)}\right] = \left[\frac{a_m(t)\xi(t)^{\frac{1}{1-\gamma}} + b_m(t)\xi(t)^{\frac{1}{1-\gamma}}}{\hat{X}_m(t)}\right] > 0,$$

since the numerators and denominators of both fractions are positive. Hence $\hat{\phi}_{mi}(t) \leq \hat{\phi}_{ni}(t)$. *Q.E.D.*

REFERENCES

Back, K., 1991, "Asset Pricing for General Processes", *Journal of Mathematical Economics*, 20, 371-395.

Basak, S., 1993, "Dynamic Consumption-Portfolio Choice and Asset Pricing with Non-Price-Taking Agents", Working Paper, The Wharton School, University of Pennsylvania.

Basak, S., 1993, Ph.D. Thesis, Carnegie Mellon University.

Black, F. and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy*, 81, 637-654.

Black, F., 1976, "Studies of Stock Price Volatility Changes", in *Proceedings of the 1976 Meetings of the Business and Economics Section*, American Statistical Association, pp177-181.

Black, F. and R. Jones, 1987, "Simplifying Portfolio Insurance", *Journal of Portfolio Management*, 14, 48-51.

Brady, Nicholas et al., 1988, *Report of the Presidential Task Force on Market Mechansims*, U.S. Government Printing Office.

Brennan, M. J. and E. S. Schwartz, 1989, "Portfolio Insurance and Financial Market Equilibrium", *Journal of Business*, 62, 455-472.

Cox, J. C. and C. F. Huang, 1989, "Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process", *Journal of Economic Theory*, 49, 33-83.

Cox, J. C. and C. F. Huang, 1991, "A Variational Problem Arising in Financial Economics", *Journal od Mathematical Economics*, 20, 465-487.

Donaldson, R. G. and H. Uhlig, 1993, "The Impact of Large Portfolio Insurers on Asset Prices", *Journal of Finance*, forthcoming.

Dothan, M. U., 1990, *Prices in Financial Markets*, Oxford University Press, New York.

Duffie, D., and C. F. Huang, 1985, "Implementing Arrow-Debreu Equilibria by Continuous Trading of a Few Long-Lived Securities", *Econometrica*, 53, 1337-1356.

Duffie, D., 1986, "Stochastic Equilibria: Existence, Spanning Number, and the 'No Expected Financial Gain from Trade' Hypothesis", *Econometrica*, 54, 1161-1383.

Duffie, D., and W. Zame, 1989, "The Consumption-Based Capital Asset Pricing Model", *Econometrica*, 57, 1279-1297.

French, K. R., G. W. Schwert and R. F. Stambaugh, 1987, "Expected Stock Returns and Volatility", *Journal of Financial Economics*, 19, 3-29.

Genotte, G. and H. Leland, 1990, "Market Liquidity, Hedging and Crashes", *American Economic Review*, 80, 999-1021.

Merton, R. C., 1980, "On Estimating the Expected Return on the Market", *Journal of Financial Economics*, 8, 323-361.

Merton, R. C., 1989, *Continuous-Time Finance*, Blackwell, Oxford.

Perold, A. F., 1986, "Constant Proportion Portfolio Insurance", Manuscript, Harvard University.

Pliska, S. R., 1986, " A Stochastic Calculus Model of of Continuous Trading: Optimal Portfolios", *Mathematics of Operations Research*, 11, 371-382.

Rubinstein, M. and H. E. Leland, 1981, "Replicating Options with Positions in Stock and Cash", *Financial Analysts Journal*, 37, 63-72.

Schwert, G. W., 1989, "Why Does Market Volatility Change over Time?", *Journal of Finance*, 44, 1115-1155.

Schwert, G. W., 1990, "Stock Market Volatility and the Crash of '87", *Review of Financial Studies*, 3, 77-102.

Stoll, H. R. and R. E. Whaley, 1990, "Stock Market Structure and Volatility ", *Review of Financial Studies*, 3, 37-71.