MULTIFACTOR MODELS DO NOT EXPLAIN DEVIATIONS FROM THE CAPM

by

A. Craig MacKinlay

15-93

RODNEY L. WHITE CENTER FOR FINANCIAL RESEARCH
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6367

MULTIFACTOR MODELS DO NOT EXPLAIN DEVIATIONS FROM THE CAPM

A. Craig MacKinlay*

First Draft: November 1992
Current Draft: July 1993

Comments welcome.

*Department of Finance, Wharton School, University of Pennsylvania, Philadelphia, PA 19104-6367.

I thank Bruce Grundy, John Heaton, Ravi Jagannathan, Krishna Ramaswamy, and especially Andrew Lo and Robert Stambaugh for useful discussions relating to this paper. A preliminary draft was completed while I was a summer visitor at the University of British Columbia in July 1991. Thanks to the UBC finance faculty for constructive suggestions. An early draft circulated under the title "Distinguishing Among Asset Pricing Theories." I am grateful to seminar participants at Indiana University, MIT, Rutgers University, University of Illinois, Vanderbilt, and Wharton for helpful comments. Research support from the Batterymarch Fellowship and the Geewax-Terker Research Fund is gratefully acknowledged.

Abstract

Evidence of deviations from the Capital Asset Pricing Model (CAPM) has accumulated over the past decades. The source of these deviations is often assumed to be missing risk factors, leading researchers to look to multifactor asset pricing models as alternatives. The analysis in this paper suggests that consideration of different alternatives may be fruitful, since existing evidence against the CAPM is also evidence against multifactor alternatives being the whole story. Data-snooping biases, market frictions, or market inefficiencies are more likely explanations for the deviations.

The framework created for the analysis provides additional insights. These insights relate to the role of residual risk in explaining the cross-section of expected asset returns and to the formation of prior distributions for Bayesian analysis of portfolio efficiency.

1. Introduction.

One of the important problems of modern financial economics is the quantification of the tradeoff between risk and expected return. Although common sense suggests that investments free of risk will generally yield lower returns than more risky investments such as the stock market, it was only with the development of the Sharpe-Lintner Capital Asset Pricing Model (CAPM) that economists were able to say how much lower this would have to be. In particular, the CAPM shows that the cross-section of expected excess returns of financial assets must be linearly related to the market betas with an intercept of zero. Because of the practical importance of this risk/return relation, numerous studies have empirically examined this implication. Over the past fifteen years, a number of these studies have presented evidence which contradict the CAPM in that the hypothesis that the intercept of a regression of excess returns on the excess return of the market is zero is statistically rejected.

The apparent violations of the CAPM have spawned research into possible explanations.[1] For the analysis of this paper the explanations will be divided into two categories – the risk based alternatives and the non-risk based alternatives. The risk based category includes multifactor asset pricing models developed under the assumptions of investor rationality and perfect capital markets. For this category the source of deviations from the CAPM is missing risk factors. The non-risk based category includes biases introduced in the empirical methodology, the existence of market frictions, or explanations arising from the presence of irrational investors.[2] Examples are data–snooping biases, biases in computing returns, transaction costs and liquidity effects, and market efficiencies.

The findings of many empirical tests of the CAPM that the intercepts deviate statistically from zero has naturally lead to the empirical examination of multifactor asset pricing models motivated by the Arbitrage Pricing Theory (APT) due to Ross (1976) and the Intertemporal Capital Asset Pricing Model (ICAPM) due to Merton (1973). The basic approach has been to introduce additional factors in the form of excess returns on traded portfolios and then re-examine the zero-intercept hypothesis. A recent example of a paper which includes this approach is Fama and French (1993). They document that the

[1] Of course, Roll's (1977) critique is relevant here. In a strict sense, most studies reject the mean–variance efficiency of the CRSP Indexes. In the analysis of this paper, the missing risk factors analysis will apply to the case where the source of the violations is the misidentification of the market portfolio.

[2] Although this category is labelled non-risk based, clearly some of the explanations contain elements of risk.

estimates of the CAPM intercepts deviate from zero for portfolios formed on the basis of book value to market value of equity ratios as well as for portfolios formed based on market capitalization.[3] Upon finding that the intercepts for these portfolios with a three factor model are closer to zero, they conclude that missing risk factors in the CAPM are the source of the deviations.

Another line of research attributes the deviations to data–snooping. This explanation for CAPM deviations is presented in Lo and MacKinlay (1990). The argument is that on an *ex post* basis one will always be able to find deviations from the CAPM. Such deviations considered in a group will appear statistically significant. However, under this explanation they are not real but just a result of grouping assets with common disturbance terms. Since in financial economics our empirical analysis is *ex post* in nature, this problem is difficult to directly control. Further, in practice direct adjustments for potential snooping are difficult to implement and, when implemented, make it very difficult to find real deviations.

Other researchers interpret the deviations from the CAPM as indications of the presence of irrational behavior by market participants. A number of theories have been developed which are consistent with this line of thought.[4] A recent example is the work of Lakonishok, Shleifer, and Vishny (1993) where they argue that the deviations arise from investors following naive strategies including extrapolating past growth rates too far into the future, assuming a trend in stock prices, overreacting to good or bad news, or liking to invest in firms with a high level of profitability. With this alternative the possibility of non-zero intercepts arises not only from missing risk factors but from specific firm characteristics.

Conrad and Kaul (1993) consider the possibility that biases in computed returns explain the deviations. They find that the implicit portfolio rebalancing in most analyses biases measured returns upwards. This leads to overstated returns and consequently CAPM deviations. However, the biases are most relevant over long horizons and hence more applicable to results using long intervals such as that of DeBondt and Thaler (1985) than to the results of much of the recent work which uses monthly data.

Finally, market frictions and liquidity effects could induce a non-zero intercept in the CAPM tests. Since the model is developed in a perfect market, such effects are

[3] Fama and French also focus on the observation that the relation between average returns and market betas is weak. This point is not addressed in this paper but has been addressed in a number of recent papers. Some examples are Chan and Lakonishok (1992), Kandel and Stambaugh (1993), Kothari, Shanken and Sloan (1992), and Roll and Ross (1992).

[4] DeBondt and Thaler (1985) is an important early paper responsible for initiating interest in this area.

not accommodated. Amihud and Mendleson (1986) present some evidence of returns containing effects from market frictions and demands for liquidity. Also, Luttmer (1991) provides evidence on the importance of market frictions in the context of the bounds on the moments of the intertemporal marginal rate of substitution derived in Hansen and Jagannthan (1991).

The controversy of whether or not the CAPM deviations are due to missing risk factors flourishes because empirically it is hard to distinguish between the various arguments. The difficulty arises because, on an *ex post* basis, we can always find a set of risk factors that will make the asset pricing model intercept zero.[5] Given this, without specifics concerning the risk factors, we will always be able to explain the cross-section of expected returns with a multifactor asset pricing model. This will be true even if the real explanation lies in one of the non-risk based categories.

In this paper I address the issue using *ex ante* analysis. The objective is to evaluate the plausability of the argument that the deviations from the CAPM can be explained by additional risk factors. I argue that *ex ante* we should expect that CAPM deviations due to missing risk factors will be very difficult to statistically detect. Intuitively this is because the deviation in expected return is also accompanied by increased variance. I formally analyze the issue using mean–variance efficient set mathematics in conjunction with the zero- intercept F-test presented in Gibbons, Ross, and Shanken (1989) and MacKinlay (1987). Technically the difficulty exists because when deviations from the CAPM or other multifactor pricing models are the result of omitted risk factors, there is an upper limit on the distance between the null distribution of the test statistic and the alternative distribution. With the non-risk based alternatives where the source of the deviations is not missing factors, no such limit exists.

Another way of viewing the distinction between the two categories is the difference in the behavior of the maximum Sharpe measure squared as the cross-section of securities is increased.[6] For the risk based alternatives the maximum Sharpe measure squared is bounded and for the non-risk based alternatives the maximum Sharpe measure squared is a less useful construct and will in principle be unbounded.

A number of other important implications which follow from a link between the deviations from the model and the model's residual variance are also developed. These include

[5] Roll (1977) argues this point for the CAPM case.

[6] The Sharpe measure is the ratio of the mean excess return to the standard deviation of the excess return.

1) the fact that residual risk generally will matter in terms of explaining the deviations from the model; 2) prior distributions used in Bayesian analysis of pricing models which do not explicitly recognize the link generally put heavy weight on parameter values implying unrealistic risk – return tradeoffs; and 3) alternatives which ignore the link will generally not be consistent with the absence of asymptotic arbitrage opportunities.[7]

The results of the paper underscore the important role that economic analysis plays in distinguishing among different pricing models for the relation between risk and return. In the absence of a specific alternative theories, without very long time series of data, we are limited in what we can say about risk/return relations among financial securities.

The paper proceeds as follows. In section 2 the framework for the analysis is presented. In section 3 we define the *optimal orthogonal portfolio* which will play a key role in the arguments of the paper. Many of the results in the paper can be related to the values of the squared Sharpe measure for relevant portfolios. In section 4 we present the relations between the parameters of the returns and the Sharpe measures for use in subsequent sections. Section 5 develops the implications relating to the missing risk factors controversy. Section 6 presents the other implications and the paper concludes with section 7.

2. Linear Pricing Models and Mean-Variance Analysis.

We begin by specifying the distributional properties of excess returns for $\overline{N}$ primary assets in the economy. Let z_t represent the $\overline{N}$x1 vector of excess returns for period t. Assume z_t is stationary and ergodic with mean μ and covariance matrix V which is full rank. Given these assumptions for any set of factor portfolios we will have a linear relation between the excess returns and the portfolios' excess returns. Formally we have:

$$z_t \;=\; \alpha + Bz_{pt} + \epsilon_t \tag{2.1}$$

$$E[\epsilon_t] \;=\; 0 \tag{2.2}$$

[7] Although this is an asymptotic result, as we shall see, for some alternatives unrealistic payoffs can be constructed with a relatively small number of securities.

$$E[\epsilon_t \epsilon_t{}'] \;=\; \Sigma \tag{2.3}$$

$$E[z_{pt}] \;=\; \mu_p \quad , \quad E[(z_{pt} - \mu_p)(z_{pt} - \mu_p)'] \;=\; \Omega_p \tag{2.4}$$

$$Cov[z_{pt}, \epsilon_t] \;=\; 0. \tag{2.5}$$

B is the $\overline{N} \times K$-matrix of factor loadings, z_{pt} is the $K \times 1$-vector of time-t factor portfolio excess returns, and α and ϵ_t are $\overline{N} \times 1$-vectors of asset return intercepts and disturbances respectively.[8]

It is well-known that all of the elements of the vector α will be zero if a linear combination of the factor portfolios form the tangency portfolio (i.e. the mean-variance efficient portfolio of risky assets given the presence of a riskfree asset). Let z_{qt} be the excess return of the (ex ante) tangency portfolio and let x_q be the $\overline{N} \times 1$ vector of portfolio weights. Here, and throughout the paper, we let ι represent a conforming vector of ones. From mean–variance analysis we have:

$$x_q \;=\; (\iota' V^{-1} \mu)^{-1} V^{-1} \mu. \tag{2.6}$$

In the context of our previous discussion, we will consider the asset pricing model to be well-specified when the tangency portfolio can be formed from a linear combination of the K-factor portfolios.

In the next section we construct a portfolio which will be useful to characterize the asset pricing model deviations when factor portfolios are not jointly mean–variance efficient.

3. Optimal Orthogonal Portfolio.

Our interest is in formally developing the relation between the deviations from the

[8] The dependence of α, B, and Σ on the factor portfoios is suppressed for notational convenience.

asset pricing model, α, and the residual covariance matrix Σ when a linear combination of the factor portfolios do not form the tangency portfolio. To facilitate this we define the *optimal orthogonal portfolio*.[9] This is the unique portfolio which can be combined with the factor portfolios to form the tangency portfolio and is orthogonal to the factor portfolios. We next formally define this portfolio.

Definition: *optimal orthogonal portfolio.*

We take as given K factor portfolios which cannot be combined to form the tangency portfolio or the global minimum variance portfolio. A portfolio h will be defined as the optimal orthogonal portfolio with respect to these K factor portfolios if:

$$x_q \;=\; X_p\,\omega \;+\; x_h\,(1-\iota'\omega) \tag{3.1}$$

and

$$x_h'\,V\,X_p \;=\; 0 \tag{3.2}$$

for a $(K \times 1)$ *vector* ω *where* X_p *is the* $(\overline{N} \times K)$ *matrix of asset weights for the factor portfolios,* x_h *is the* $(\overline{N} \times 1)$ *vector of asset weights for the optimal orthogonal portfolio, and* x_q *is the* $(\overline{N} \times 1)$ *vector of asset weights for the tangency portfolio. If we consider a model without any factor portfolios* $(K = 0)$ *then the optimal orthogonal portfolio will be the tangency portfolio.*

The usefulness of this portfolio comes from the fact that when added to (2.1) the intercept will vanish and the factor loading matrix B will not be altered. The optimality restriction in (3.1) leads to the intercept vanishing, and the orthogonality condition in (3.2) leads to B being unchanged. Adding in z_{ht} we have:

$$z_t \;=\; Bz_{pt} \;+\; \beta_h z_{ht} \;+\; u_t \tag{3.3}$$

$$E[u_t] \;=\; 0 \tag{3.4}$$

$$E[u_t u_t'] \;=\; \Phi \tag{3.5}$$

[9] See Roll (1980) for properties of orthogonal porfolios in a general context and see Lehmann (1987, 1988, 1992) for discussions of the role of orthogonal portfolios in asset pricing tests. Also related is the orthogonal factor employed in MacKinlay (1987) and the modifying payoff used in Hansen and Jagannathan (1992).

$$E[z_{ht}] = \mu_h \quad , \quad E[(z_{ht} - \mu_h)^2] = \sigma_h^2 \tag{3.6}$$

$$Cov[z_{pt}, u_t] = 0. \tag{3.7}$$

$$Cov[z_{ht}, u_t] = 0. \tag{3.8}$$

The link results from comparing (2.1) and (3.3). Taking the unconditional expectations of both sides we have:

$$\alpha = \beta_h \mu_h \tag{3.9}$$

and by equating the variance of ϵ_t with the variance of $\beta_h z_{ht} + u_t$ we have:

$$\begin{aligned} \Sigma &= \beta_h \beta_h' \sigma_h^2 + \Phi \\ &= \alpha\, \alpha' \frac{\sigma_h^2}{\mu_h^2} + \Phi. \end{aligned} \tag{3.10}$$

The key link between the model deviations and the residual variances and covariances emerges from (3.10). The intuition for the link is straight forward. Deviations from the model must be accompanied by a common component in the residual variance in order to prevent the formation of a portfolio with a positive deviation and a residual variance which decreases to 0 as the number of securities in the portfolio grows. In cases where the link is not present (i.e. the link is undone by Φ), asymptotic arbitrage opportunities will exist.

4. Squared Sharpe Measures.

The squared Sharpe measure is a useful construct for interpreting much of the ensuing analysis. The Sharpe measure for a given portfolio is calculated by dividing the mean excess return by the standard deviation of return. It is well-known that the tangency portfolio q will have the maximum squared Sharpe measure of all portfolios.[10] The squared Sharpe

[10] See Jobson and Korkie (1982) for a development of this point and a performance measurement application.

measure of q, s_q^2, is:

$$s_q^2 \;=\; \mu' V^{-1} \mu. \tag{4.1}$$

Since the K factor portfolios p and the optimal orthogonal portfolio h can be combined to form the tangency portfolio, it follows that the maximum squared Sharpe measure of these $K+1$ portfolios will be s_q^2. Since h is orthogonal to the portfolios p, we can express s_q^2 as the sum of the squared Sharpe measure of the orthogonal portfolio and the squared maximum Sharpe measure of the factor portfolios. We have:

$$s_q^2 \;=\; s_h^2 \;+\; s_p^2 \tag{4.2}$$

where $s_h^2 \;=\; \frac{\mu_h^2}{\sigma_h^2}$ and $s_p^2 \;=\; \mu_p' \Omega_p^{-1} \mu_p$.

In applications we will be employing subsets of the $\overline{N}$ assets. Results similar to those above will hold within a subset of N assets. For the subset analysis when considering the tangency portfolio (of the subset) and the maximum squared Sharpe measure of the assets, it is necessary to augment the N assets with the factor portfolios p. Defining $z_{t_s}^*$ as the $(N+K \times 1)$ vector $[z_t' \; z_{pt}']'$ with mean $\mu_s^{*\prime}$ and covariance matrix V_s^*, we have for the tangency portfolio of these $N+K$ assets:

$$s_{q_s}^2 \;=\; \mu_s^{*\prime} V_s^{*-1} \mu_s^*. \tag{4.3}$$

The subscript s indicates we are using a subset of the assets.

As we shall see, the analysis (with a subset of assets) will involve the quadratic $\alpha' \Sigma^{-1} \alpha$ computed using the parameters for the N assets. Gibbons, Ross, and Shanken (1989) and Lehmann (1988, 1992) provide interpretations of this quadratic term in terms of Sharpe measures. Assuming Σ is of full rank,[11] they show:

$$\alpha_s' \Sigma_s^{-1} \alpha_s \;=\; s_{q_s}^2 \;-\; s_p^2. \tag{4.4}$$

and, further, consistent with (4.2), for the subset of assets $\alpha' \Sigma^{-1} \alpha$ will be the squared Sharpe measure of the subset's optimal orthogonal portfolio h_s. Therefore for the a given subset of assets we have:

[11] If Σ is singular then one must use the generalized inverse.

$$s^2_{h_s} \;=\; \alpha_s' \Sigma_s^{-1} \alpha_s \tag{4.5}$$

and

$$s^2_{q_s} \;=\; s^2_{h_s} \;+\; s^2_p. \tag{4.6}$$

We can also relate the squared Sharpe measure of the subset's optimal orthogonal portfolio to that of the population optimal orthogonal portfolio. To see this relation we first analytically invert Σ (of the subset) in (3.10). Using Bartlett's Identity[12] we have:

$$\Sigma_s^{-1} = [\Phi_s^{-1} \;-\; \frac{\frac{\sigma_h^2}{\mu_h^2}}{1 + \frac{\sigma_h^2}{\mu_h^2}\alpha_s'\Phi_s^{-1}\alpha_s}\Phi_s^{-1}\alpha_s\alpha_s'\Phi_s^{-1}]. \tag{4.7}$$

Forming $\alpha_s'\Sigma_s^{-1}\alpha_s$, and simplifying we have:

$$s^2_{q_s} \;=\; \frac{\mu_h^2}{\sigma_h^2}[\frac{\frac{\sigma_h^2}{\mu_h^2}\alpha_s'\Phi_s^{-1}\alpha_s}{1 \;+\; \frac{\sigma_h^2}{\mu_h^2}\alpha_s'\Phi_s^{-1}\alpha_s}]. \tag{4.8}$$

The first righthand side term in (4.8) is s^2_h and the second term is both bounded between zero and one, giving us a relation between $s^2_{h_s}$ and s^2_h. We have:

$$s^2_{h_s} \;<\; s^2_h. \tag{4.9}$$

Next we use the optimal orthogonal portfolio and the Sharpe measures results to develop implications for distinguishing among asset pricing models. Hereafter will suppress the s subscript. No ambiguity will result since in the subsequent analysis we will be working only with subsets of the assets.

5. Implications for Missing Risk Factors.

Many asset pricing model tests involve testing the null hypothesis that the model

[12] See Morrison (1990) page 69.

intercept is zero using tests in the spirit of the zero-intercept F-test.[13] A common conclusion is that rejection of this hypothesis using one or more factor portfolios is an indication that more risk factors are required to explain the risk–return relation. This conclusion has lead to the inclusion of additional factors so that the null hypothesis will be accepted. A shortcoming of this approach is that, after adding factors, when all is said and done there are multiple potential interpretations of why the hypothesis is accepted. One view is that we have made genuine progress in terms of identifying the "right" asset pricing model. An alternative view is that, since the additional factors lack strong theoretical motivation, we have suceeded in finding a within sample fit through data–snooping. Certainly advocates of the market irrationality and other non-risk based positions would argue this alternative view.

In this section we employ *ex ante* analysis to attempt to discriminate between the two interpretations. The analysis integrates the link between the pricing model intercept and the residual covariance matrix of (3.10) and the squared Sharpe measures results with the distribution theory for the zero-intercept F-test. We consider two approaches. The first approach is a testing approach which compares the null hypothesis test statistic distribution with the distribution under each of the alternatives. The second approach is estimation based, drawing on the squared Sharpe measures analysis to develop estimators for the squared Sharpe measure of the optimal orthgonal portfolio. Before presenting the two approaches we present the zero-intercept F-test.

5.1. Zero Intercept F-Test.

To implement the F-test presented below we add the additional assumption that excess asset returns are jointly normal and temporally independently and identically distributed. This assumption, though restrictive, buys us exact finite sample distributional results thereby simplifying the analysis. However, it is important to note that this assumption is not central to the point; similar results will hold under much weaker assumptions. Using a Generalized Method of Moments approach, MacKinlay and Richardson (1991) present a more general test statistic which has asymptotically a chi-square distribution. Analysis

[13] Examples of tests which basically fit into this framework are those in Campbell (1987), Connor and Korajczyk (1988), Fama and French (1993), Huberman and Kandel (1987), Lehmann and Modest (1988), and MacKinlay (1987). The arguments in the paper can also be related to the zero-beta CAPM tests in Gibbons (1982), Shanken (1985), and Stambaugh (1982).

similar to that presented for the F-test holds for this general statistic.

We begin with a summary of the zero-intercept F-test of the null hypothesis that the intercept vector α from (2.1) is be 0. Let H_o be the null hypothesis and H_a be the alternative.

$$H_o: \qquad \alpha = 0$$

$$H_a: \qquad \alpha \neq 0.$$

H_o can be tested using the following test statistic:

$$\theta_1 = [(T-N-K)/N][1+\hat{\mu}_p'\hat{\Omega}_p^{-1}\hat{\mu}_p]^{-1}\hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha}. \tag{5.1}$$

where T is the number of time series observations, N is the number of assets or portfolios of assets included, and K is the number of factor portfolios. The hat superscripts indicate the maximum likelihood estimators. Under the null hypothesis, θ_1 is unconditionally distributed central F with N degrees of freedom in the numerator and $(T-N-K)$ degrees of freedom in the denominator.

We can also characterize the distribution of θ_1 in general. Conditional on the factor portfolio returns for the distribution of θ_1 we have:

$$\theta_1 \sim F_{N,T-N-K}(\lambda), \tag{5.2}$$

$$\lambda = T[1+\hat{\mu}_p'\hat{\Omega}_p^{-1}\hat{\mu}_p]^{-1}\alpha'\Sigma^{-1}\alpha \tag{5.3}$$

where λ is the noncentrality parameter of the F distribution.[14]

5.2. Testing Approach.

In this approach we consider the distribution of θ_1 under two different alternatives. The alternatives can be separated by their implications for the maximum value of the

[14] If $K=0$ then the term $[1+\hat{\mu}_p'\hat{\Omega}_p^{-1}\hat{\mu}_p]^{-1}$ will not appear in (5.1) and in (5.3) and θ_1 will be unconditionally distributed non-central F.

squared Sharpe measure. With the risk based multifactor alternative there will be an upper bound on the squared Sharpe measure, whereas with the non-risk based alternatives the maximum squared Sharpe measure in principle can be unbounded (as the number of assets increases).

First we consider the distribution of θ_1 under the alternative hypothesis when deviations are due to missing factors. Drawing on the results for the squared Sharpe measures, for the noncentrality parameter of the F distribution we have:

$$\lambda \ = \ T \ [1 + \hat{\mu}_p' \hat{\Omega}_p^{-1} \hat{\mu}_p]^{-1} \ s_{h_s}^2. \tag{5.4}$$

From (4.9), the third term in (5.4) is bounded above by s_h^2 and positive. The second term is bounded between zero and one. Thus we have an upperbound for λ.

$$\lambda < \ Ts_h^2 \ \leq \ Ts_q^2. \tag{5.5}$$

The second inequality follows from the fact that the tangency portfolio q has the maximum Sharpe measure of any asset or portfolio.[15]

Given a maximum value for the squared Sharpe measure, the upper bound on the noncentrality parameter can be important. With this bound, independent of how one arranges the assets to be included as dependent variables in the pricing model regression and for any value of N,[16] there is a limit on the distance between the null distribution and the distribution when the alternative is missing factors. Almost all the assets can be mispriced and yet the bound will still apply. As a consequence one should be cautious in interpreting rejections of the zero-intercept as evidence in favor of a model with more risk factors.

In contrast, when the alternative one has in mind is that the source of non-zero intercepts is non-risk based such as data snooping, market frictions, or market irrationalities, the notion of a maximum squared Sharpe measure is not useful. The squared Sharpe measure (and the noncentrality parameter) are in principle unbounded. When comparing alternatives with the intercepts of about the same magnitude, in general, we might expect to see larger test statistics in this non-risk based case.

[15] The first half of this bound appears in MacKinlay (1987) for the case of the Sharpe–Lintner CAPM. Related results appear in Kandel and Stambaugh (1987), Shanken (1987a), and Hansen and Jagannathan (1991).

[16] In practice when using the F-test it will be necessary for N to be less than $T - K$ so that $\hat{\Sigma}$ will be of full rank.

We can examine the potential informativeness of the above analysis by considering alternatives with realistic parameter values. We construct the distribution of the test statistic for three cases: the null hypothesis, the missing risk factors alternative, and the non-risk based alternative. For the risk based alternative, we draw on a framework designed to be similar to that in Fama and French (1993). For the non-risk based alternative we use a setup that is consistent with the analysis of Lo and MacKinlay (1990) and the work of Lakonishok, Shleifer, and Vishny (1993).

We study a one factor asset pricing model using a time series of the excess returns for 32 portfolios for the dependent variable. The one factor (independent variable) is the excess return of the market so that the zero–intercept null hypothesis is the CAPM. The length of the time series is 342 months. This setup corresponds to that of Fama and French (1993) Table 9 regression (ii). For the null distribution of the test statistic θ_1 we have:

$$\theta_1 \quad \sim \quad F_{32,309}(0). \tag{5.6}$$

To define the distribution of θ_1 under the alternatives of interest we need to specify the parameters necessary to calculate the noncentrality parameter. First for the risk based alternative, given a value for the Sharpe measure of the optimal orthogonal portfolio, we can consider the distribution corresponding to the upper bound of the noncentrality parameter. Acknowledging that specifying the Sharpe measure is subjective, I posit that *ex ante* a reasonable value for the Sharpe measure squared of the optimal orthogonal portfolio for an interval of one month is 0.021. This value corresponds to a portfolio with an annual expected excess return of 8% and a standard deviation of 16% and, for example, would imply a maximum squared Sharpe measure of 0.042 if the maximum squared Sharpe measure of the included factor portfolios is the same. The monthly value of 0.0416 would be consistent with an annualized mean excess return of 10% and annualized standard deviation of 14% for the tangency portfolio.

Historical Sharpe measures are presented for a number of broad based indices to allow the reader to gauge the reasonableness of the value selected for the analysis. These measures are reported in Table 1. For each index two estimates are presented, the *ex post* measure and an unbiased squared Sharpe measure estimate. For the July 1963 through December 1991 period the squared Sharpe measures are presented for the CRSP value weighted index, the CRSP small stock (tenth decile) portfolio, and the *ex post* optimal portfolio of the two above indices plus the long term government index and the corporate bond index distributed by CRSP in the SBBI file. The small stock portfolio has a monthly squared Sharpe measure of 0.013 (or 0.010 using the unbiased estimate) substantially below the value used for the optimal orthogonal portfolio. The *ex post* optimal porfolio's measure is only slightly higher at 0.014.

Table 1 also contains results for the period from January 1981 through June 1992 results for the S&P 500 Index, a growth index and a value index.[17] The source of the return statistics used to calculate the measures is Capaul, Rowley, and Sharpe (1993). These results provide a useful perspective on the maximum magnitudes of Sharpe ratios since it is generally acknowledged that the 1980's is a period of strong stock market performance especially for value-based investment strategies. Given this characterization, one would expect these results to provide a high estimate of possible Sharpe measures. We can see that the Sharpe measures from this period are very much in line with (and lower than) the value used in the analysis of the risk based alternative. The highest *ex post* estimate is 0.021 for the value index. Generally, I interpret the evidence in this table as supporting the measure selected to calibrate the risk based analysis.

Proceeding with a squared Sharpe measure of 0.021 to calculate λ, for the distribution of θ_1 we have:

$$\theta_1 \sim F_{32,309}(7.1). \tag{5.7}$$

We will use this distribution to characterize the risk based alternative.

We specify the distribution for two non-risk based alternatives by specifying values of α, Σ, and $\hat{\mu}_p' \hat{\Omega}_p^{-1} \hat{\mu}_p$ and the calculating λ from (5.3). To specify the intercepts we assume the elements of α are normally distributed with a mean of zero. We consider two values for the standard deviation, 0.0007 and 0.001. When the standard deviation of the elements of α is 0.001 about 95% of the alphas will lie between - 0.002 and +0.002, a annualized spread of about 4.8%. A standard deviation of 0.0007 for the alphas would correspond to an annual spread of about 3.4%. These spreads are consistent with spreads that could arise from data-snooping and also plausible and somewhat conservative given the contrarian strategy returns presented in Lakonishok, Shleifer, and Vishny. For Σ we use a sample estimate based on portfolios sorted by market capitalization for the period

[17] The growth index contains the S&P 500 stocks with high price to earnings ratios and the value index is constructed from stocks with low price to earnings ratios.

1963 to 1991 inclusive. $\hat{\mu}_p' \hat{\Omega}_p^{-1} \hat{\mu}_p$ will typically be small, so we set it to zero. To get an idea of a reasonable value for the noncentrality parameter given this alternative, we calculate the expected value of λ given the distributional assumption for the elements of α conditional upon $\Sigma = \hat{\Sigma}$. The expected value of the noncentrality parameter is 39.4 for a standard deviation of 0.0007 and 80.3 for a standard deviation of 0.001. Using these values for the noncentrality parameter of the distribution of θ_1 we have:

$$\theta_1 \sim F_{32,309}(39.4). \tag{5.8}$$

when $\sigma_\alpha = 0.0007$ and

$$\theta_1 \sim F_{32,309}(80.3). \tag{5.9}$$

when $\sigma_\alpha = 0.001$.

A plot of the four distributions from (5.6), (5.7), (5.8), and (5.9) is in Figure 1. The vertical bar on the plot represents the value 1.91 which Fama and French calculate for the test statistic. From this figure notice that the null hypothesis distribution and the risk based alternative distribution are quite close together. This reflects the impact of the upperbound on the noncentrality parameter. In contrast the non-risk based alternatives' distributions are far to the right of the other two distributions consistent with the noncentrality parameter being unbounded for these alternatives.

What do we learn from this plot? I would claim two things. First, if we want to distinguish among risk based linear asset pricing models the zero-intercept test is not particularly useful because the null distribution and the alternative distribution have substantial overlap. Second, if we want to compare a risk based pricing model with a non-risk based alternative in mind the zero-intercept test can be very useful since the distributions of the test statistic for these alternatives has little overlap.

What can we say about the risk based multifactor alternatives versus the non-risk based arguments for deviations from the Sharpe–Lintner CAPM? The results suggest that the risk based missing risk factors argument is not the whole story. From Figure 1 we can see that the test statistic is still in the upper tail when we tabulate the distribution of θ_1 in the presence of missing risk factors. The p-value using this distribution is 0.03. Hence there is a lack of support for the view that missing factors completely explain the deviations. The fact that, when Fama and French increase the number of factors to three, the significance of the test statistic only decreases marginally is also consistent with the argument that missing risk factors is not the whole story.

On the other hand, given the parametrization considered, there is some support for the non-risk based alternative views. The test statistic falls almost in the middle of the non-risk based alternative with the lower standard deviation of the elements of alpha. Several of the non-risk based alternatives could equally well explain the results. Different non-risk based views can give the same noncentrality parameter and test statistic distribution. For example, the results are consistent with the data snooping alternative of Lo and MacKinlay (1990) but are also consistent with the market inefficiencies view. Nonetheless, the analysis does suggest that more than missing risk factors is needed to explain the empirical results.

5.3. Estimation Approach.

In this section we use an estimation approach to make inferences about possible values for Sharpe measures. An estimator for the squared Sharpe measure of the optimal orthogonal portfolio for a given subset of assets is presented. Using this estimator and its variance, confidence intervals for the squared Sharpe measure can be constructed facilitating judgements on the question of the value implied by the data and reasonable alternatives given this value. An unbiased estimator of the squared Sharpe measure is presented.[18] This estimator corrects for the bias that is introduced by searching over N assets to find the maximum. For the estimator we have:

$$\tilde{s}^2_{h_s} = [\theta_1 - \frac{(T-N-K)}{(T-N-K-2)}]\,[\frac{N(T-N-K-2)}{T(T-N-K)}]\,[1+\hat{\mu}_p'\hat{\Omega}_p^{-1}\hat{\mu}_p] \tag{5.10}$$

$$var(\tilde{s}^2_{h_s}|\hat{\mu}_p'\hat{\Omega}_p^{-1}\hat{\mu}_p) = [\frac{2(1+\hat{\mu}_p'\hat{\Omega}_p^{-1}\hat{\mu}_p)^2}{T^2}] \times$$

$$[\frac{(N+T[1+\hat{\mu}_p'\hat{\Omega}_p^{-1}\hat{\mu}_p]^{-1}s^2_{h_s})^2 + (N+2T[1+\hat{\mu}_p'\hat{\Omega}_p^{-1}\hat{\mu}_p]^{-1}s^2_{h_s})(T-N-K-2)}{(T-N-K-4)}] \tag{5.11}$$

[18] This estimator is derived using the fact that θ_1 is distributed as a non-central F variate. Its moments follow from the moments of the non-central F distribution.

Conditional on the factor portfolio returns, the estimator of $s^2_{h_s}$ in (5.10) is unbiased.

$$E[\tilde{s}^2_{h_s}|\hat{\mu}'_p\hat{\Omega}_p^{-1}\hat{\mu}_p] = s^2_{h_s} \tag{5.12}$$

Recall that when $K = 0$ the optimal orthogonal portfolio is the tangency portfolio and hence $s^2_{h_s} = s^2_{q_s}$. The estimator can be applied when $K = 0$ by setting $\hat{\mu}'_p\hat{\Omega}_p^{-1}\hat{\mu}_p = 0$.

The estimation approach is illustrated using the above estimator for the Fama and French (1993) portfolios. We consider the case of $K = 0$ and therefore we are estimating the maximum squared Sharpe measure from 33 assets – the value-weighted CRSP index, the 25 stock portfolios and 7 bond portfolios. (Recall, with $K = 0$, $s^2_{h_s} = s^2_{q_s}$.) The estimator of $s^2_{h_s}$ can be readily calculated, but the variance of $\tilde{s}^2_{h_s}$ cannot since it depends on $s^2_{h_s}$. To calculate the variance we use a consistent estimator, $\tilde{s}^2_{h_s}$, and then asymptotically (as T increases) we have:

$$\tilde{s}^2_{h_s} \sim N(s^2_{h_s}, \widehat{var}(\tilde{s}^2_{h_s})) \tag{5.13}$$

Using monthly data from July 1963 through December 1991, the estimate of $s^2_{h_s}$ is 0.092 and the asymptotic standard error is 0.044. Thus using this data set, for a two-sided centered 90% confidence interval we have (0.020, 0.163) and for a one-sided 90% confidence interval we have (0.036, ∞). It is worth noting the upward bias of the ex post maximum squared Sharpe measure as an estimator. For the above case the *ex post* maximum is 0.209 substantially higher than the unbiased estimate of 0.092. The bias is particular severe when N is large (relative to T).

In terms of an annualized Sharpe measure, the two-sided interval corresponds to a lower value of 0.49 and an upper value of 1.40, and the one-sided interval corresponds to a lower value of 0.65. Given that the tangency portfolio and the optimal orthogonal portfolio are the same, we can use this interval to provide an indication of the magnitude of the maximum Sharpe measure needed for a set of factor portfolios to explain the cross section of excess returns. Consistent with rejections of the mean- variance efficiency of the index, the *ex post* Sharpe measure of the value weighted market index lies well outside the intervals with an annualized value of 0.33. In general one can use the confidence intervals to decide on promising alternatives. For example, if one believes that *ex ante*

Sharpe measures in the 90% confidence interval are unlikely in a risk based world, then the non-risk based alternatives may provide an attractive area for future study.

6. Other Implications.

The link between an asset expected return deviations from the asset pricing model prediction and the residual variance of the model has additional implications for asset pricing research. We look at some of these implications in this section.

6.1. Residual Risk and Pricing Deviations.

Past research has examined the ability of residual risk to explain the deviations from the CAPM. The general approach is to include residual standard deviation or variance as a risk measure in a linear model and see if residual risk matters. Fama and MacBeth (1973), use a cross-sectional regression approach to examine the importance of residual risk.[19] Defining σ^2 as a $(N \times 1)$ vector of the assets' residual variances, Fama and MacBeth estimate the following regression for each time period t

$$z_t \;=\; \iota\,\gamma_{0t} \;+\; \beta\,\gamma_{1t} \;+\; \sigma^2\,\gamma_{2t} \;+\; \eta_t \tag{6.1}$$

and examine the time series of γ_{2t} to see if residual risk can explain the deviations form the CAPM. This approach is motivated by diversification effects arguments, where the residual risk proxies for degree of diversification.

However, if the source of deviations is missing factors, from the deviation - residual variance link of (3.10), we can see that residual risk must matter, but in a nonlinear way.[20] For a typical element of Σ we have:

$$\sigma_{ij} \;=\; \alpha_i \alpha_j\, \frac{\sigma_h^2}{\mu_h^2} \;+\; \phi_{ij}. \tag{6.2}$$

Hence the linear relation is between $\alpha_i \alpha_j$ and σ_{ij} and not between α_i and σ_i^2.

[19] See Lakonishok and Shapiro (1986) and Lehmann (1990) for similar examples.

[20] Lehmann (1990) notes that with a multifactor model the factor loadings squared will appear in the residual variance but does not use this observation in his empirical design.

In Figure 2 we illustrate the linear versus the nonlinear relation. Given the presence of negative and positive alphas, the coefficient associated with σ_i or σ_i^2 as a regressor in a linear pricing model will not in general be non-zero (which is consistent with much of the existing evidence) even though residual risk matters. In order to exploit the fact that residual risk matters when deviations from the pricing model exist we need to accommodate the relation in (6.2). If residual risk does not matter (in the sense of (6.2)) and we let the number of assets increase, there will be asymptotic arbitrage opportunities. We look at one such example in section **6.3**.

6.2. Prior Distributions for Bayesian Analysis.

Recent research has employed a Bayesian approach to study the zero–intercept implication of the CAPM and multifactor pricing models. McCulloch and Rossi (1990, 1991) and Harvey and Zhou (1990), and Kandel, McCulloch, and Stambaugh (1993) are examples of such research.[21] These papers have generally concluded that the probability that the market portfolio is mean–variance efficient is low and hence the CAPM is not supported. However, as Kandel, McCulloch, and Stambaugh (1993) show, such a conclusion should be interpreted with caution as it to some extent is driven by the prior distribution on α and Σ. In general the distributions used give heavy prior weight to non–risk based alternatives and consequently heavy prior weight to the market being an mean–variance inefficient portfolio or equivalently to the Sharpe–Lintner CAPM being misspecified. To avoid this result using the priors of previous research, the distribution for α needs to be very concentrated about zero. This is the approach that Kandel, McCulloch, and Stambaugh adopt. The results are very sensitive to the prior distribution of α.

The deviation – residual variance link developed in this paper suggests an alternative approach to forming the prior distributions. From the link in (3.10), when forming the prior distribution of α and Σ, we can see the need to accommodate the dependence of Σ on α. This will be necessary to allow for deviations due to missing factors in the prior distributions. Previously employed priors have not had this feature. To illustrate the importance of the link we study three different prior distributions for α and Σ. Then for each prior, we form the prior distribution of s_h^2. To facilitate comparability with other

[21] Another paper in this area is Shanken (1987b). He does not put prior distributions on α and Σ and hence the discussion does not apply directly to his work.

papers we also form the prior distribution of ρ, where ρ is the correlation between the market portfolio return and the tangency portfolio return. If the market is efficient then ρ is one. Kandel and Stambaugh (1987) and Shanken (1987a) development the theory underlying ρ. Given α, Σ, and the squared Sharpe measure of the market portfolio, s_m^2, for ρ^2 we have:

$$\rho^2 = \frac{s_m^2}{s_m^2 + \alpha'\Sigma^{-1}\alpha} \tag{6.3}$$

ρ is restricted to be non-negative and hence we consider the positive square root of ρ^2.

A simulation approach is used to generate the various prior distributions. The priors of s_h^2 and ρ are based on the simulated realizations of (α, Σ). For ρ the prior results presented are conditioned on $s_m^2 = 0.02$. For all three frameworks the prior distribution of Σ takes the same form. The priors differ in the formulation of the prior for α. We express Σ in terms of two components,

$$\Sigma = \delta\delta' c^{-1} + \Phi \tag{6.4}$$

and specify prior distributions for δ, c, and Φ:

$$\delta \sim N(0, \sigma_\delta^2 I) \tag{6.5}$$

$$c \sim a\chi^2_{\nu_1}(0) \tag{6.6}$$

$$\Phi^{-1} \sim Wishart(\frac{1}{\nu_2\sigma^2} I, \nu_2, N). \tag{6.7}$$

We consider three priors for α:

$$\alpha|\delta \sim N(\delta, 0) \tag{6.8}$$

$$\alpha \sim N(0, \sigma_\delta^2 I) \tag{6.9}$$

$$\alpha \sim N(0, \frac{1}{\nu_3}\Sigma) \tag{6.10}$$

The first prior for α in (6.8) is proposed to reflect the deviation - residual variance link. The dependence of Σ on α is built in. The second prior in (6.9) corresponds to that employed by Kandel, McCulloch, and Stambaugh (1993) and the third prior in (6.10) corresponds to that used by McCulloch and Rossi (1991). The second and third priors do not accommodate the deviation - residual variance link.

We choose the prior parameters to correspond to reasonable values for monthly stock return data and to be consistent across priors. The prior parameters we choose are $N = 10$, $\nu_1 = 60$, $a = \frac{0.0208}{\nu_1}$, $\nu_2 = 60$, and $\sigma = 0.02$. The prior mean of the deviations α is 0. σ_δ controls the magnitudes of the deviations. We choose three values for σ_δ, 0.001, 0.002 and 0.005. For the third prior ν_3 is selected such that the prior dispersion of the elements of α will be comparable to that of the first two priors.

The resulting priors for s_h^2 are presented in Figure 3a-c and for ρ are presented in Figure 4a-c. The impact of accommodating (3.10) in the first prior is apparent as the distribution of s_h^2 is shifted more towards 0 than in the other priors and the distribution of ρ is shifted towards 1. The difference is especially apparent in Figures 3c and 4c where the standard deviation of the elements of α is set at the larger value. By incorporating the deviation - residual variance link in the formulation of the priors the sensitivity of the priors to the size of the deviations is dramatically reduced. One would expect that, in a complete analysis, using such a prior would lead to less evidence against the efficiency of the market portfolio.

6.3. Asymptotic Arbitrage in Finite Economies.

In the absence of the link between the model deviation and the residual variance expressed in (3.10) asymptotic arbitrage opportunities can arise. However, given the results are asymptotic, the question of the importance of the opportunities in a finite economy needs to be addressed. In this section we address this point with a simple example where the link is absent. We begin with analysis of the case where the number of assets increases to infinity and then follow with analysis when there is a fixed number of assets.

Consider a one factor pricing model with N assets and a factor portfolio p. We include the intercept α to permit deviations from the model.[22]

$$z_t \;=\; \alpha \;+\; \beta_p \; z_{pt} \;+\; \epsilon_t \tag{6.11}$$

$$E[z_{pt}] \;=\; \mu_p \quad , \quad E[(z_{pt}-\mu_p)^2] \;=\; \sigma_p^2 \tag{6.12}$$

$$E[\epsilon_t|z_{pt}] \;=\; 0 \tag{6.13}$$

$$E[\epsilon_s {\epsilon_t}'] \;=\; \begin{cases} \Sigma & \text{for } s=t. \\ 0 & \text{otherwise.} \end{cases} \tag{6.14}$$

$$\Sigma \;=\; [(1-\rho)\; I_N \;+\; \rho \;\iota\; \iota']\omega^2 \tag{6.15}$$

$$0 \;<\; \rho \;<\; 1 \qquad 0 \;<\; \omega \;<\; \infty$$

$$\alpha_i \;\sim\; IID(0,\sigma_\alpha^2) \qquad i=1,....,N. \tag{6.16}$$

Equation (6.16) characterizes the cross-sectional distribution of the true alphas. The model is misspecified since the alphas are not all identically zero.

Given this model of asset prices, consider a portfolio of N assets where the investment in each asset is proportional to its own alpha. Asymptotically, as the number of assets increases, this portfolio (with appropriately scaled weights) will be a zero investment (arbitrage) portfolio. By considering the properties of this portfolio, we can show the presence of asymptotic arbitrage opportunities.

Specifically, consider the following portfolio formation strategy. For asset i set its weight to $\frac{\alpha_i}{N\sigma_\alpha^2}$. Defining x_a as the $(N \times 1)$ portfolio weight vector, then for x_a we have:

$$x_a \;=\; \frac{1}{N\sigma_\alpha^2}\;\alpha$$

[22] This example is presented in Gibbons, Ross, and Shanken (1989).

For this portfolio we can determine the sum of the weights, the alpha α_a and the residual variance σ_a^2.

$$\iota' x_a = \frac{1}{N\sigma_\alpha^2}\iota'\alpha \tag{6.17}$$

$$\alpha_a = \alpha' x_a$$

$$= \frac{1}{N\sigma_\alpha^2}\alpha'\alpha \tag{6.18}$$

$$var(\epsilon_{at}) = x_a'\Sigma\, x_a$$

$$= x_a'[(1-\rho)\, I_N \;+\; \rho\, \iota\, \iota']\omega^2\, x_a$$

$$= [(1-\rho)(\frac{1}{N\sigma_\alpha^2})^2\; \alpha'\alpha \;+\; \rho\; (\frac{\iota'\alpha}{N\sigma_\alpha^2})^2]\omega^2. \tag{6.19}$$

For large N the sum of the portfolio weights in (6.17) will converge to 0, the value of α_a in (6.18) will converge to 1, and the residual variance in (6.19) will converge to 0.[23] Therefore we have constructed a portfolio with zero investment, a positive payoff, and no residual uncertainity. Given weak assumptions about the cross-sectional distribution of the betas, the β of the portfolio will converge to 0 and the arbitrage opportunity is immediate. If the β of this portfolio is not 0 then an arbitrage opportunity can be created by mixing the portfolio with the model factor and the risk free asset to create a zero beta portfolio and an arbitrage opportunity.

The above analysis is conducted for large N. However, the basic point is also relevant for relatively small portfolios of stocks as we now show. Consider the following strategy. Allocate the N stocks into two portfolios, the positive-α stocks in the long portfolio and the negative-α stocks in the short portfolio. Within each portfolio weigh each stock in proportion to its own α with the weight normalized so that the portfolio weights sum to one. Given the long portfolio and the short portfolio, sell $1 of the short portfolio and

[23] This analysis for large N is obviously related to the APT literature. See, for example, Ross (1976) or Chamberlain and Rothschild (1983).

invest \$1 in the long portfolio. Define β_L and β_S as the betas of the long and short portfolios respectively. Borrow $\$(\beta_L - \beta_S)$ of the risk free asset and invest the proceeds in the factor portfolio. We have a zero-beta, zero-investment strategy for which we simulate the payoff distribution using the framework in equations (6.11)–(6.16).

We present results for six simulations. For all cases we choose the following parameters: $\sigma_\alpha = 0.005$, $\mu_p = 0.007$, $\sigma_p = 0.05$, $\beta_p' = (0.5, \ldots, 1.5)$ (equally spaced), $\rho = 0.25$, and $\omega = 0.05$. The results are presented in Table 2 and Figures 5 through 10. For the first three cases we assume the alphas and betas are known and consider a universe of 1000, 500, and 100 stocks, respectively. The empirical payoff distributions for 1000 realizations are plotted in Figures 5, 6, and 7 and the payoff summary statistics are in the first three rows of Table 2. For case one with a 1000 stocks 99.8% of the payoffs are positive and the mean payoff is 1.24% of the long position. The largest negative payoff is small at only 0.18% of the long position. As the size of the stock universe is reduced from 1000 to 500 to 100, the payoff distribution's mean remains about the same but the variance increases because of the reduced diversification effect with fewer stocks. However, even with as few as 100 stocks 90.0% of the payoffs are positive and, while such a distribution cannot be completely ruled out on economic grounds, it does appear unrealistic.

The cases described above assume knowledge of the alphas and betas in constructing the position. In the final three cases we dispose of this assumption and estimate the alphas and betas by using OLS on past data. These cases are presented in the last three rows of Table 2 and in Figures 8, 9, and 10. Cases four and five, presented in Figures 8 and 9, are comparable to the third cases since a universe of 100 stocks is employed. For case four the OLS regressions are based on 120 observations and for case five the regressions are based on 60 observations. For these cases the majority of the payoffs are still positive with over 80% positive for case four and nearly 75% positive for case five. Even in the absence of knowledge about the parameters positions with unrealistic payoff distributions can be created. Case six, presented in Figure 10, is comparable to case two as a universe of 500 stocks is considered. While, with the alphas unknown, we have more negative payoffs, with a larger number of stocks the flavor of the results remains the same.

We see that, given the above one factor model, unrealistic investment opportunites can be constructed with a small number of stocks. The bottom line is that alternatives which ignore the link between the model deviations and the residual variance will be inconsistent with omitted risk factors explaining the deviations and hence are not informative as far as the zero-intercept F-test's properties against such alternatives is concerned.

7. Conclusion.

Empirical work in economics in general and in finance in particular is *ex post* in nature. Given this, it is often difficult to discriminate between various explanations for observed phenomena. A partial solution to this difficulty is to examine the alternatives and make judgements from an *ex ante* point of view. The current explanations of asset pricing empirical results are particularly well suited to *ex ante* analysis. This paper presents a framework based on the economics of mean-variance analysis to address and reinterpret prior empirical results.

It has been common to look to multifactor asset pricing models as an alternative to the Sharpe-Lintner CAPM. However, the results in this paper suggest that looking at other alternatives may be more fruitful because the evidence against the CAPM can also be interpreted as evidence against multifactor alternatives. It is unlikely that multifactor models on their own will explain the deviations from the CAPM.

Other implications of the *ex ante* analysis for empirical research are also developed. These relate to studies considering the role of residual risk in expected asset returns and to Bayesian analysis of portfolio efficiency. Generally, in my view, the results suggest that we can learn quite a lot by considering the likelihood of various existing empirical results under differing specific economic models.

References

Amihud, Y., and H. Mendleson, 1986, "Asset Pricing and the Bid-Ask Spread," *Journal of Financial Economics* 17, 223–250.

Campbell, J., 1987, "Stock Returns and the Term Structure," *Journal of Financial Economics* 18, 373–399.

Capaul, C., Rowley, I. and W. Sharpe, 1993, "International Value and Growth Stock Returns," *Financial Analysts Journal* January–February, 27–36.

Chamberlain, G. and M. Rothschild, 1983, "Arbitrage, Factor Structure, and Mean-Variance Analysis on Large Asset Markets," *Econometrica* 51, 1281–1304.

Chan, Louis K. C. and J. Lakonishok, 1992, "Are Reports of Beta's Death Premature?" Faculty Working Paper 92-0168, Bureau of Economic and Business Research, University of Illinois at Urbana-Champaign.

Chen, N., Roll, R. and S. Ross, 1986, "Economic Forces and the Stock Market," *Journal of Business* 59, 383–403.

Connor, G. and R. Korajczyk, 1988, "Risk and Return in an Equilibrium APT: Application of a New Test Methodology," *Journal of Financial Economics* 21, 255–290.

Conrad, J., and G. Kaul, 1993, "Long-term Market Overreaction or Biases in Computed Returns?" *Journal of Finance* 48, 39–63.

DeBondt, W., and R. Thaler, 1985, "Does the Stock Market Overreact?" *Journal of Finance* 40, 793–805.

Fama, E. and K. French, 1993, "Common Risk Factors in the Returns on Stocks and Bonds," *Journal of Financial Economics* 33, 3–56.

Fama, E. and J. MacBeth, 1973, "Risk, Return, and Equilibrium: Empirical Tests," *Journal of Political Economy* 71, 607–636.

Gibbons, M. R., 1982, "Multivariate Tests of Financial Models: A New Approach," *Journal of Financial Economics* 10, 3–27.

Gibbons, M., Ross, S. and J. Shanken, 1989, "A Test of the Efficiency of a Given Portfolio," *Econometrica* 57, 1121–1152.

Hansen, L., and R. Jagannathan, 1991, "Implications of Security Market Data for Models of Dynamic Economies," *Journal of Political Economy* 99, 225–262.

Hansen, L., and R. Jagannathan, 1992, "Assessing Specification Errors in Stochastic Discount Factor models," working paper, University of Minnesota.

Harvey, C. and G. Zhou, 1990, "Bayesian Inference in Asset Pricing Tests," *Journal of Financial Economics* 26, 221–254.

Huberman, G. Kandel, S. and R. Stambaugh, 1987, "Mimicking Portfolios and Exact Arbitrage Pricing," *Journal of Finance* 42, 1–10.

Jobson, D., and R. Korkie, 1982, "Potential Performance and Tests of Portfolio Efficiency," *Journal of Financial Economics* 10, 433–466.

Kandel, S. McCulloch, R. and R. Stambaugh, 1993, "Bayesian Inference and Portfolio Efficiency," *Rodney White Center Working Paper* 4–93.

Kandel, S. and R. Stambaugh, 1987, "On Correlations and Inferences About Mean-Variance Efficiency," *Journal of Financial Economics* 18, 61–90.

Kandel, S. and R. Stambaugh, 1990, "A Mean-Variance Framework for Tests of Asset Pricing Models," *Review of Financial Studies* 2, 125–156.

Kandel, S. and R. Stambaugh, 1993, "Portfolio Inefficiency and the Cross-Section of Mean Returns," Rodney White Center for Financial Research Paper 3-93, Wharton School, University of Pennsylvania.

Kothari, S., J. Shanken, and R. Sloan, 1992, "Another Look at the Cross-Section of Expected Returns," manuscript, Wharton School, University of Pennsylvania.

Lakonishok, J., A. Shleifer, and R. Vishny, 1993, "Contrarian Investment, Extrapolation, and Risk," unpublished working paper, University of Illinois, University of Chicago, and Harvard University.

Lakonishok, J. and A. Shapiro, 1986, "Systematic Risk, Total Risk and Size as Determinants of Stock Market Returns," *Journal of Banking and Finance* 10, 115–132.

Lehmann, B., 1987, "Orthogonal Frontiers and Alternative Mean Variance Efficiency Tests," *Journal of Finance* 42, 601–619.

Lehmann, B., 1989, "Mean-Variance Efficiency Tests in Large Cross-Sections," working paper, Columbia University and National Bureau of Economic Research.

Lehmann, B., 1990, "Residual Risk Revisited," *Journal of Econometrics* 45, 71–98.

Lehmann, B., 1992, "Empirical Testing of Asset Pricing Models," in *The New Palgrave Dictionary of Money and Finance*, Peter Newman, Murray Milgate, and John Eatwell eds., Stockton Press (New York), 749–759.

Lehmann, B. N. and D. Modest, 1988, "The Empirical Foundations of the Arbitrage Pricing Theory," *Journal of Financial Economics* 21, 213–254.

Lo, A. and A. C. MacKinlay, 1990, "Data-Snooping Biases in Tests of Financial Asset Pricing Models," *Review of Financial Studies* 3, 431–467.

Luttmer, E., 1991, "Asset Pricing in Economies with Frictions," working paper, University of Chicago, November.

MacKinlay, A. C., 1987, "On Multivariate Tests of the CAPM," *Journal of Financial Economics* 18, 341–372.

MacKinlay, A. C., and M. Richardson, 1991, "Using Generalized Methods of Moments to Test Mean-Variance Efficiency," *Journal of Finance* 46, 511–527.

McCulloch, R. and P. Rossi, 1990, "Posterior, Predictive, and Utility Based Approaches to Testing the Arbitrage Pricing Theory, "Journal of Financial Economics 28, 7–38.

McCulloch, R. and P. Rossi, 1989, "A Bayesian Approach to Testing the Arbitrage Pricing

Theory, "Journal of Econometrics 49, 141-168.

Merton, R., 1973, "An Intertemporal Capital Asset Pricing Model," *Econometrica* 41, 867–887.

Morrison, D., 1990, *Multivariate Statistical Methods*, McGraw Hill, New York.

Roll, R., 1977, "A Critique of the Asset Pricing Theory's Tests: Part I," *Journal of Financial Economics* 4, 129–176.

Roll, R., 1980, "Orthogonal Portfolios," *Journal of Financial and Quantitative Analysis* 15, 1005–1023.

Roll, R., and S. Ross, 1992, "On the Cross-Sectional Relation Between Expected Returns and Betas," Working Paper #21 (Series F), Yale School of Organization and Management.

Ross, S., 1976, "The Arbitrage Theory of Capital Asset Pricing," *Journal of Economic Theory* 13, 341–360.

Shanken, J., 1985, "Multivariate Tests of the Zero-Beta CAPM," *Journal of Financial Economics* 14, 327–348.

Shanken, J., 1987a, "Multivariate Proxies and Asset Pricing Relations: Living With the Roll Critique," *Journal of Financial Economics* 18, 91–110.

Shanken, J., 1987b, "A Bayesian Approach to Testing Portfolio Efficiency" *Journal of Financial Economics* 19, 195–215.

Stambaugh, R. F., 1982, "On the Exclusion of Assets from Tests of the Two Parameter Model," *Journal of Financial Economics* 10, 235–268.

Table 1

Historical Sharpe measures for selected stock indices

Time Period	Index	$\hat{s}_h^2$	$\hat{s}_h(\text{ann})$	$\tilde{s}_h^2$	$\tilde{s}_h(\text{ann})$
6307 - 9112	CRSP VW Index	0.0091	0.33	0.0061	0.27
6307 - 9112	Small Stock Decile	0.013	0.40	0.010	0.35
6307 - 9112	Portfolio of four Indices	0.014	0.41	0.0021	0.16
8101 - 9206	S&P 500 Index	0.016	0.44	0.0085	0.32
8101 - 9206	S&P/BARRA Value Index	0.021	0.50	0.013	0.40
8101 - 9206	S&P/BARRA Growth Index	0.011	0.36	0.0033	0.20

$\hat{s}_h^2$ is the monthly expost Sharpe measure squared and $\hat{s}_h(\text{ann})$ is the positive square root of this measure annualized. $\tilde{s}_h^2$ is an unbiased estimate of the monthly Sharpe measure squared and $\tilde{s}_h(\text{ann})$ is the positive square root of this measure annualized. The portfolio of four indices is the portfolio with the maximum expost Sharpe measure squared . The four indices are CRSP value-weighted index, CRSP small stock decile, CRSP longterm government bond index and CRSP corporate bond index. The bond indices are from the CRSP SBBI file. All indices are based on total return.

Table 2

Summary statistics for payoffs on one month zero investment positions with $1 invested long and $1 invested short. The six cases are: case 1 - 1000 stocks, alphas known; case 2 - 500 stocks, alphas known; case 3 - 100 stocks, alphas known; case 4 - 100 stocks, alphas estimated using OLS with 120 observations; case 5 - 100 stocks, alphas estimated using OLS with 60 observations; and case 6 - 500 stocks, alphas estimated using OLS with 60 observations. The results in each case are based on 1000 replications.

case	% > 0	mean	std dev	minimum	maximum
1	99.8	0.0124	0.0034	-.0018	0.0240
2	99.6	0.0127	0.0050	-.0021	0.0295
3	90.0	0.0129	0.0105	-.0230	0.0476
4	80.7	0.0101	0.0117	-.0256	0.0577
5	74.5	0.0081	0.0122	-.0295	0.0526
6	93.9	0.0086	0.0058	-.0099	0.0285

Figure 1

Distributions for the CAPM zero-intercept test statistic for three alternatives -- 1) the CAPM is true; 2) the risk based alternative (deviations from the CAPM are from missing risk factors); and 3) and 4) the non-risk based alternatives (deviations form the CAPM are unrelated to risk). The distributions are $F_{32,309}(0)$, $F_{32,309}(7.1)$, $F_{32,309}(39.4)$, and $F_{32,309}(80.3)$ for alternatives 1, 2, 3, and 4 respectively.

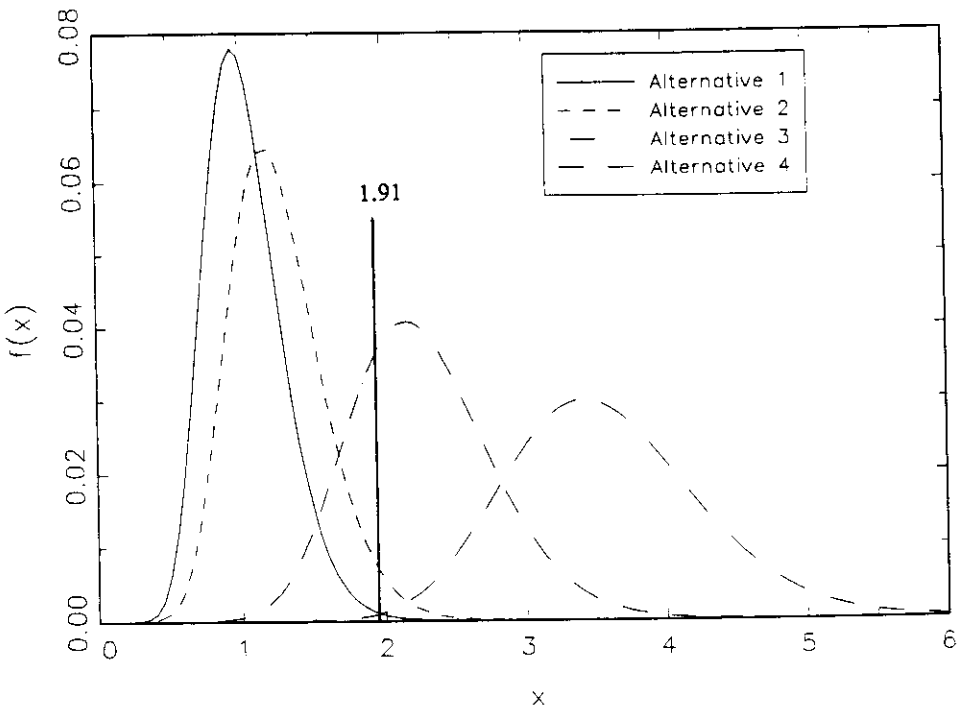

Figure 2

Residual Variance - Alpha Relations

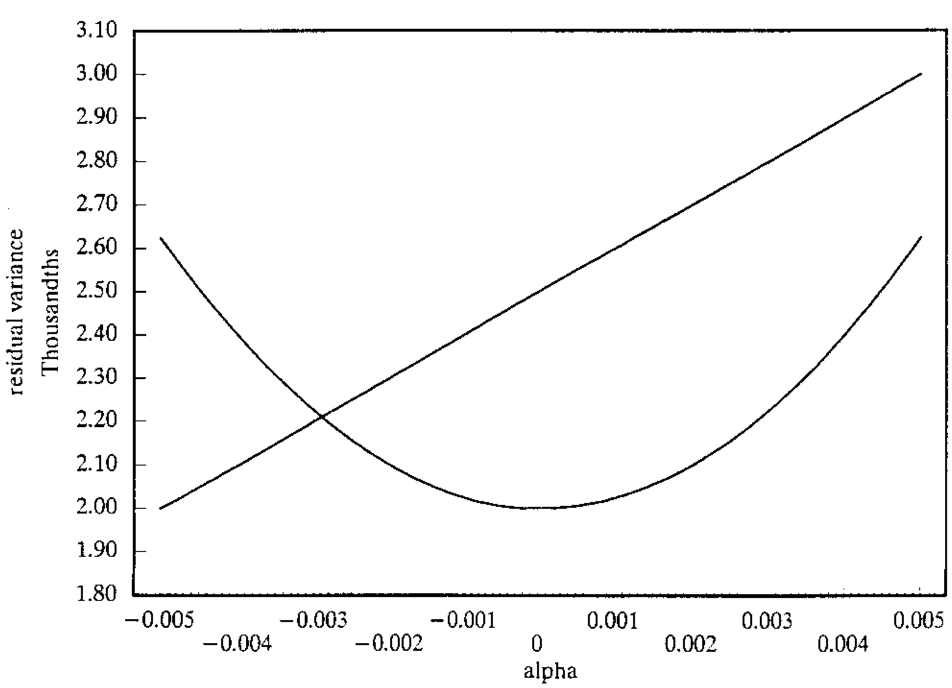

Figure 3a

Bayesian Prior Distributions for s_h^2, $\sigma_\delta = 0.001$

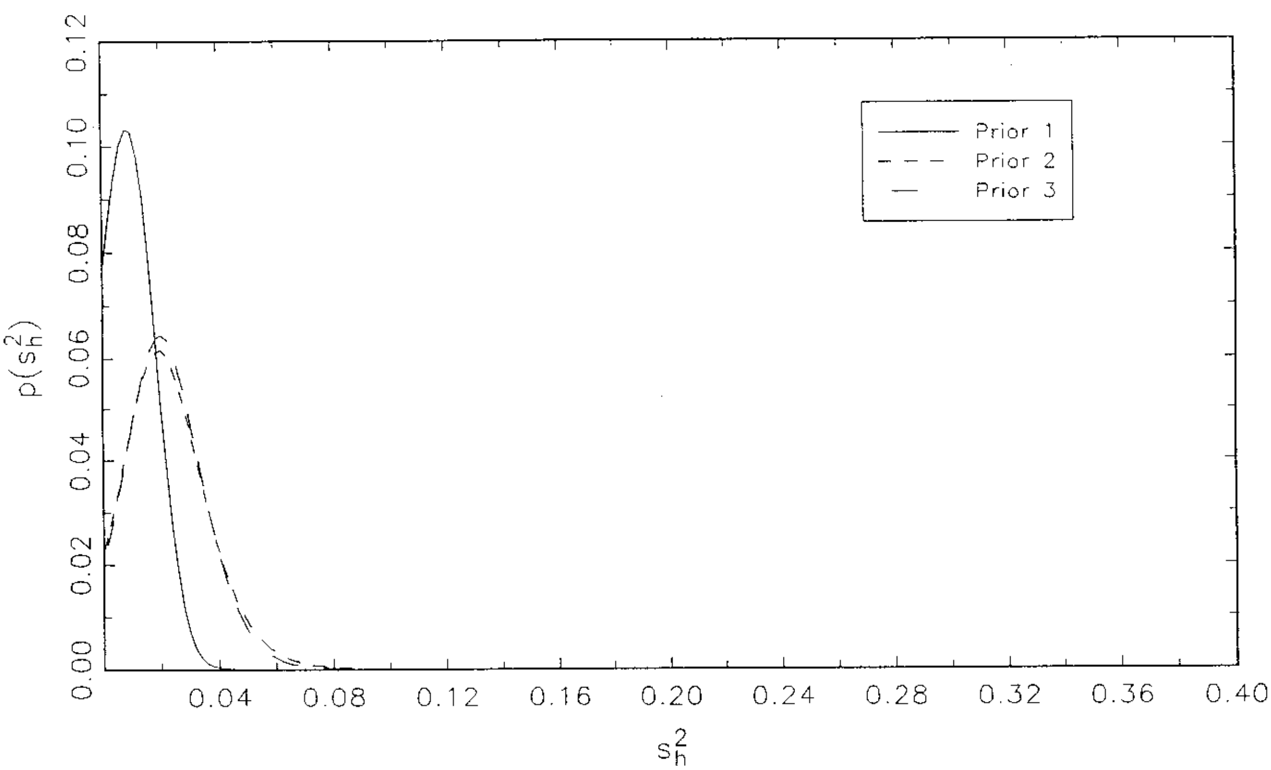

Figure 3b

Bayesian Prior Distributions for s_h^2, $\sigma_\delta = 0.002$

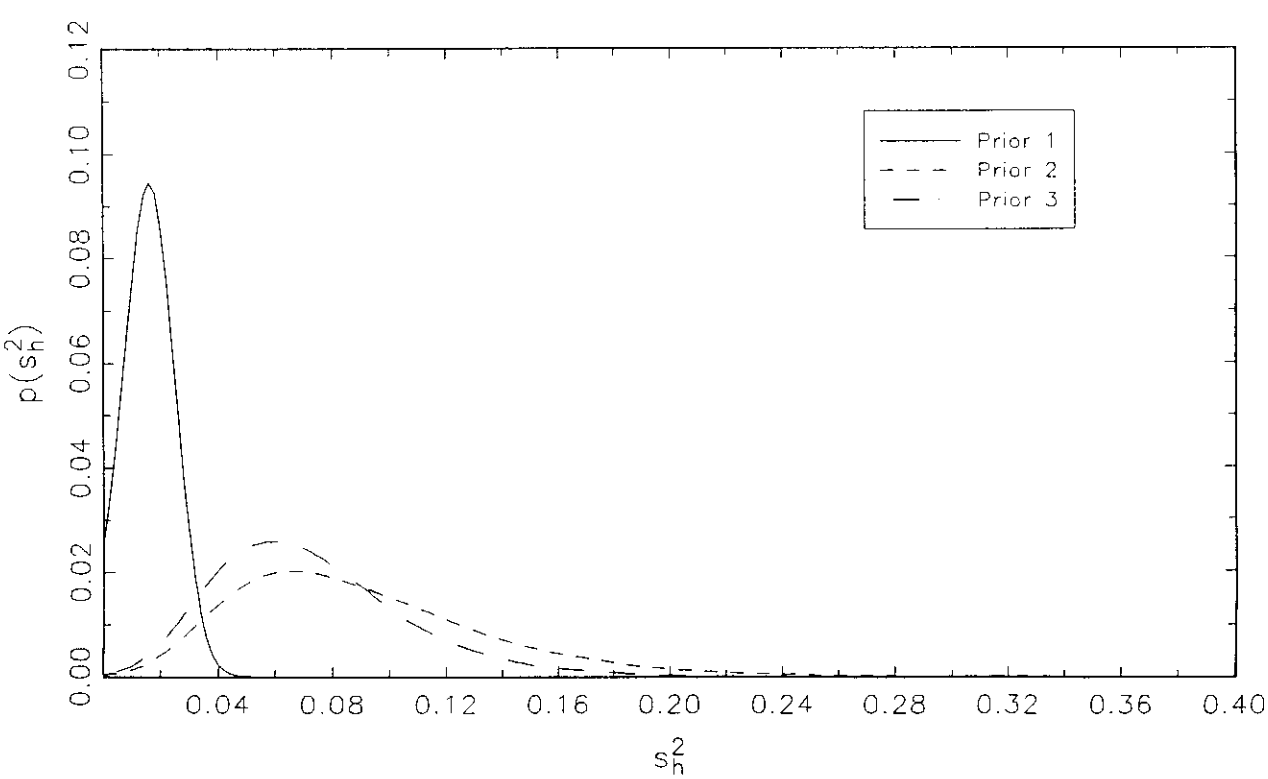

Figure 3c

Bayesian Prior Distributions for s_h^2, $\sigma_\delta = 0.005$

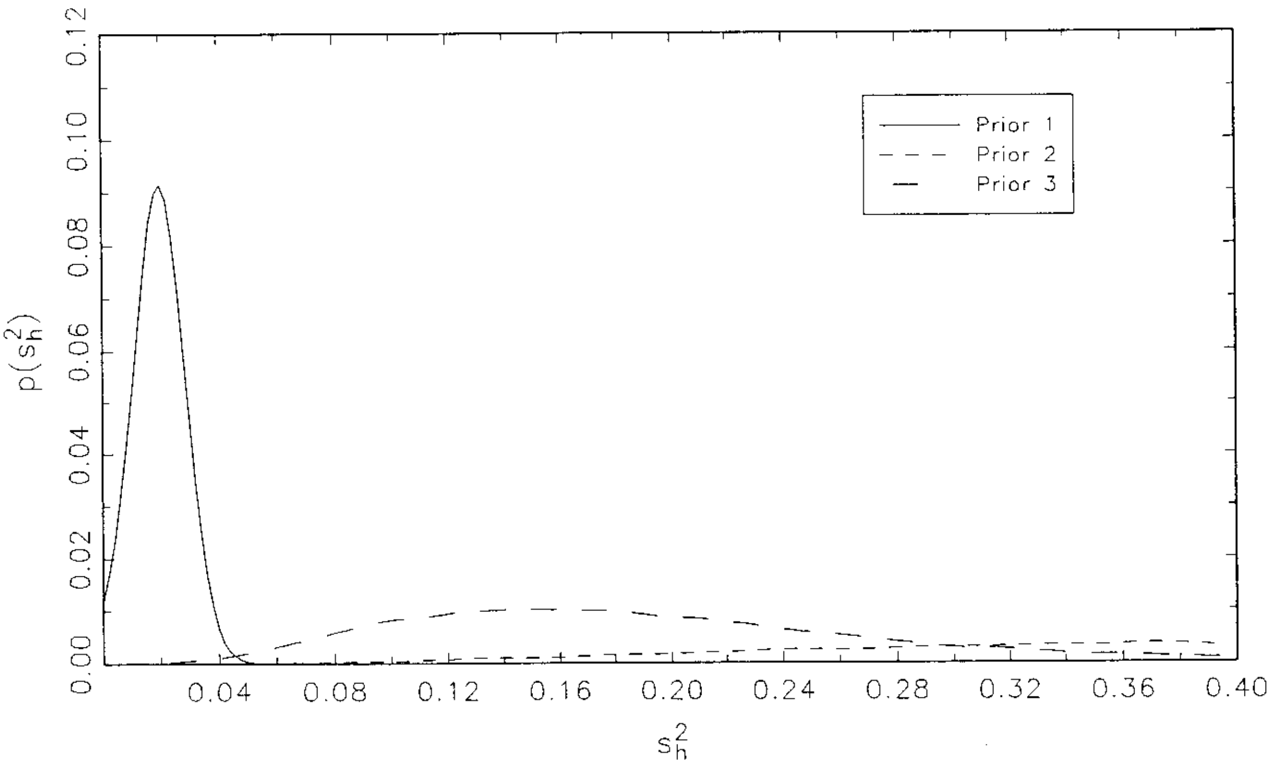

Figure 4a

Bayesian Prior Distributions for ρ, $\sigma_\delta = 0.001$

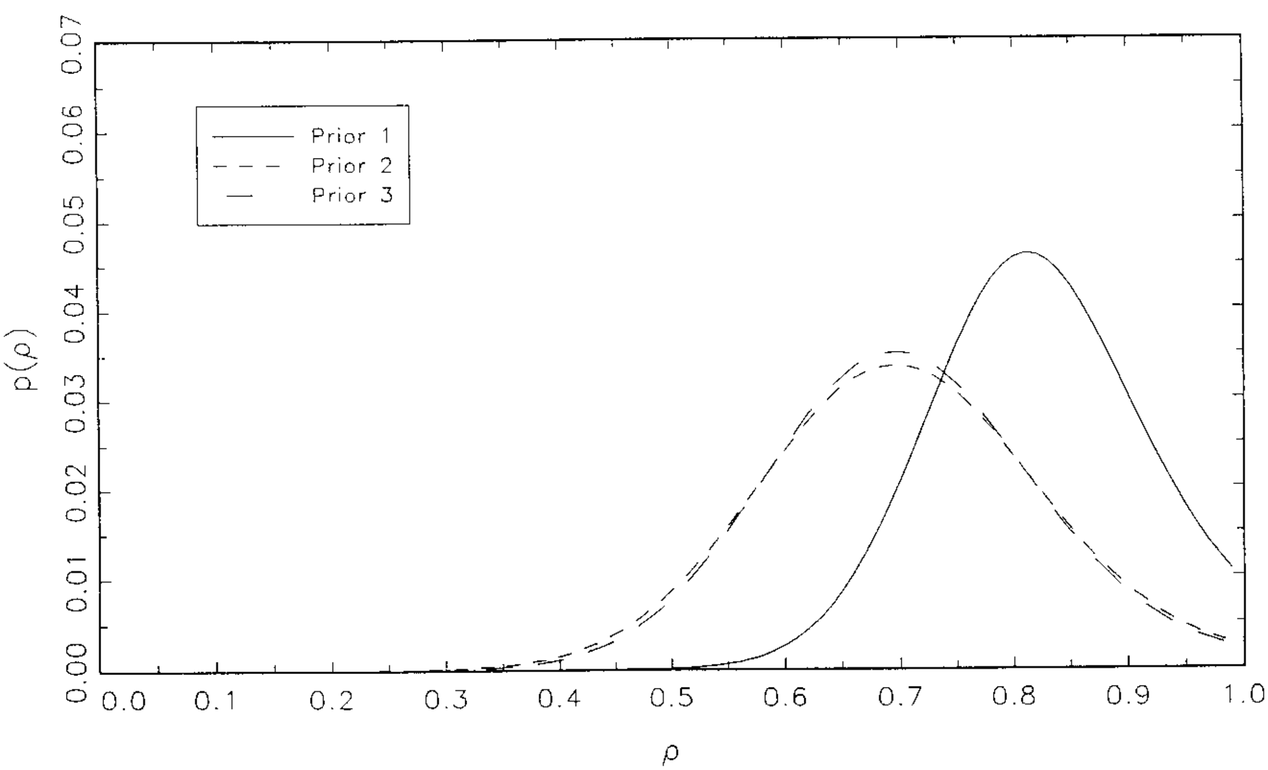

Figure 4b

Bayesian Prior Distributions for ρ, $\sigma_\delta = 0.002$

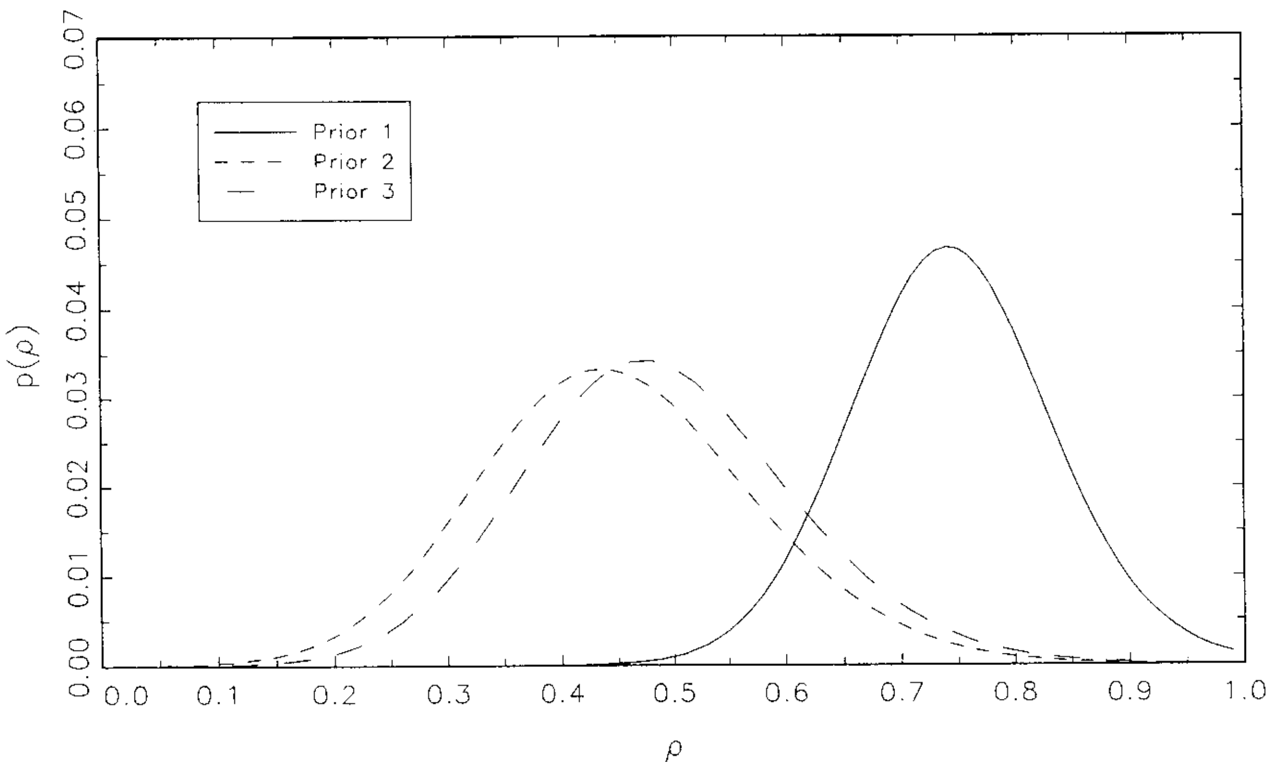

Figure 4c

Bayesian Prior Distributions for ρ, $\sigma_\delta = 0.005$

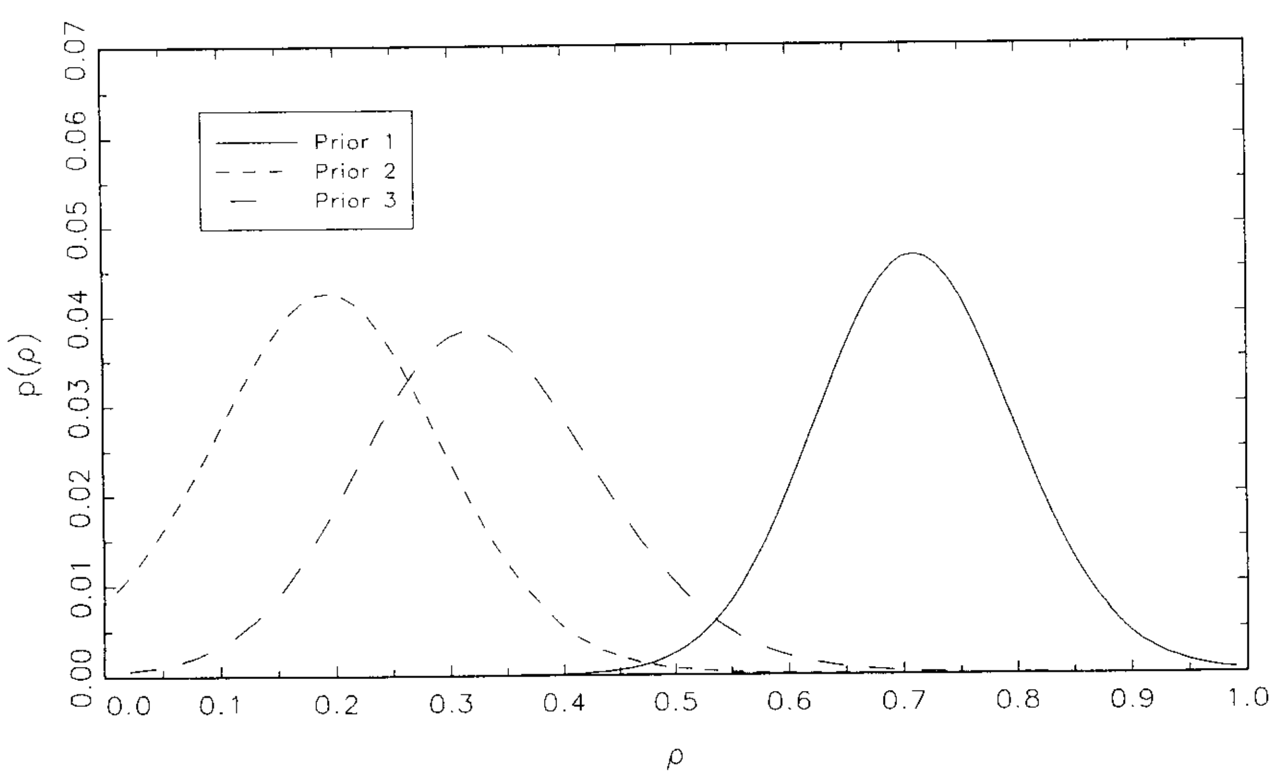

FIGURE 5
Payoff Distribution
(1000 stocks; alphas known)

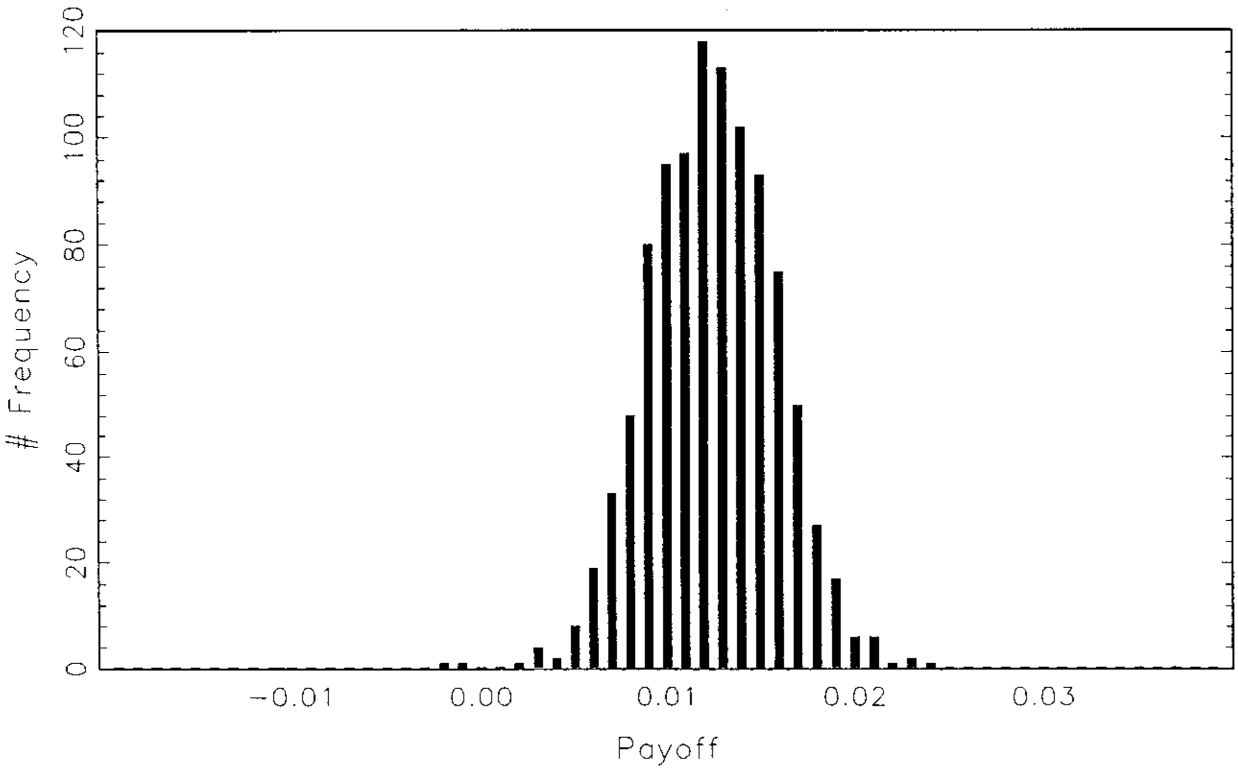

FIGURE 6
Payoff Distribution
(500 stocks; alphas known)

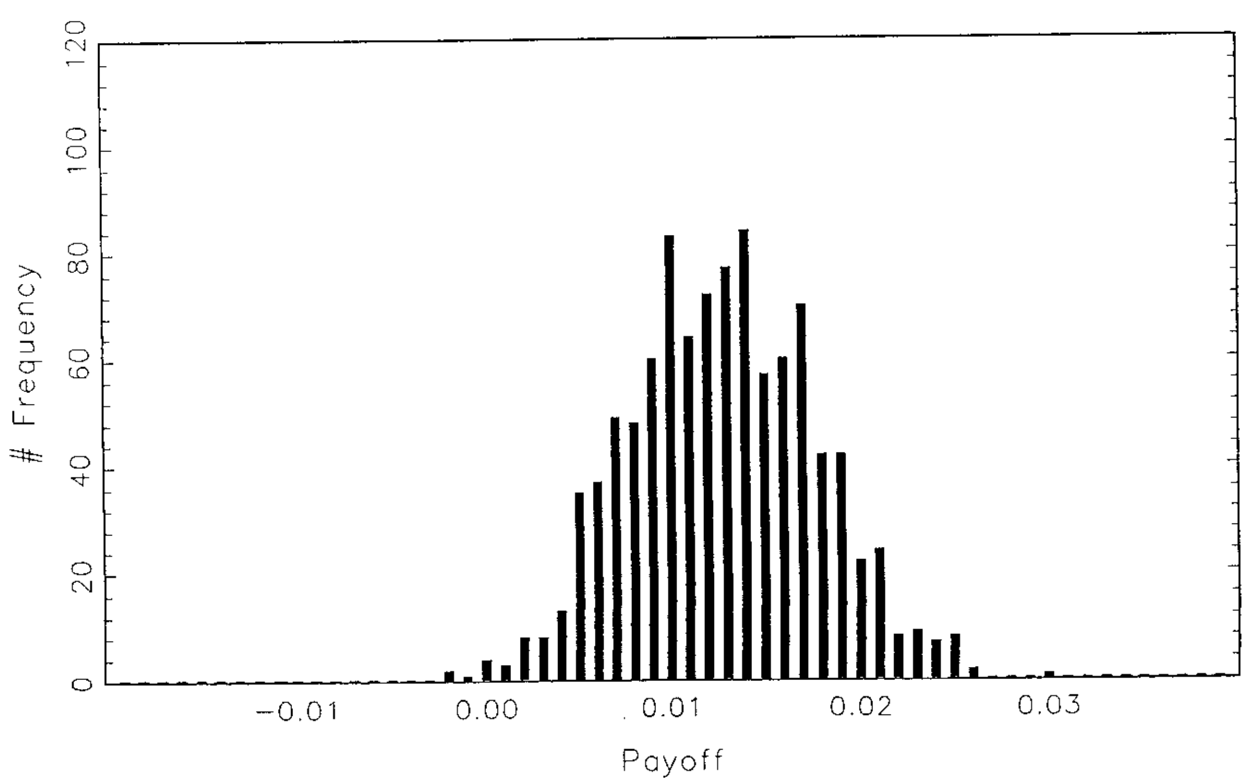

FIGURE 7
Payoff Distribution
(100 stocks; alphas known)

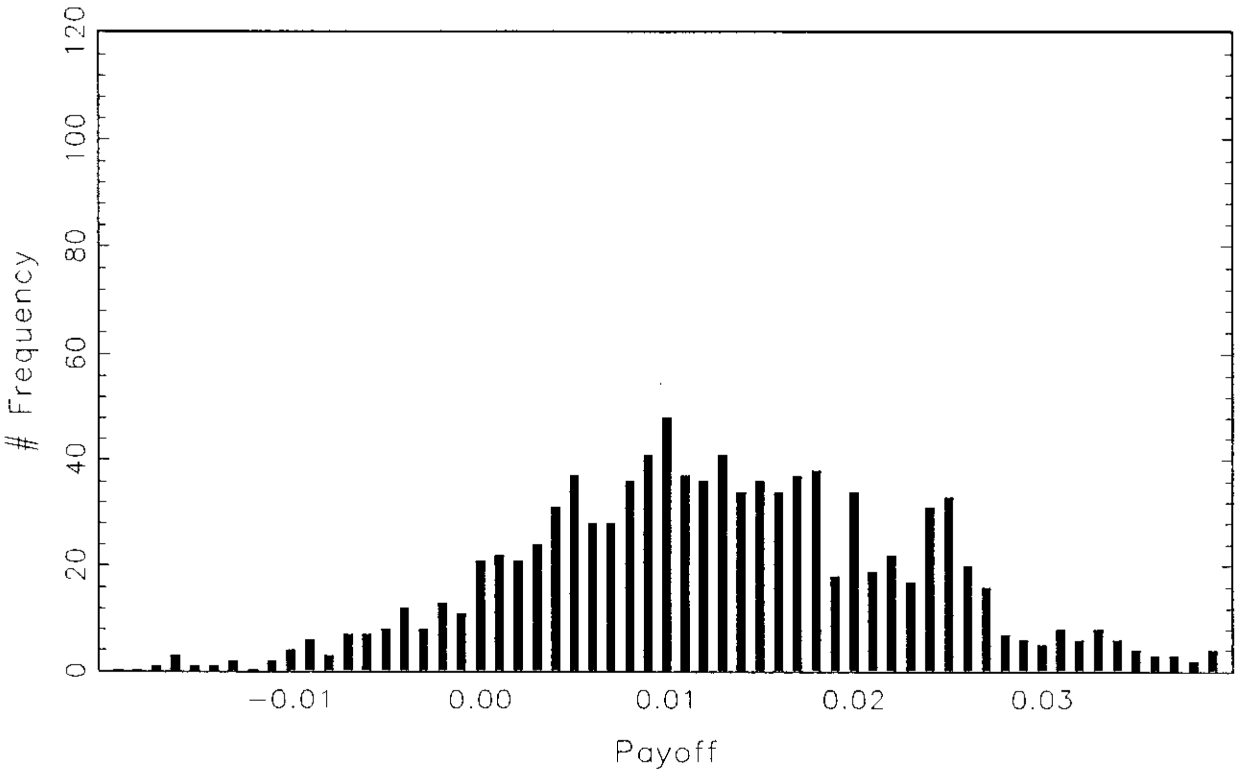

FIGURE 8
Payoff Distribution
(100 stocks; alphas estimated using 120 months)

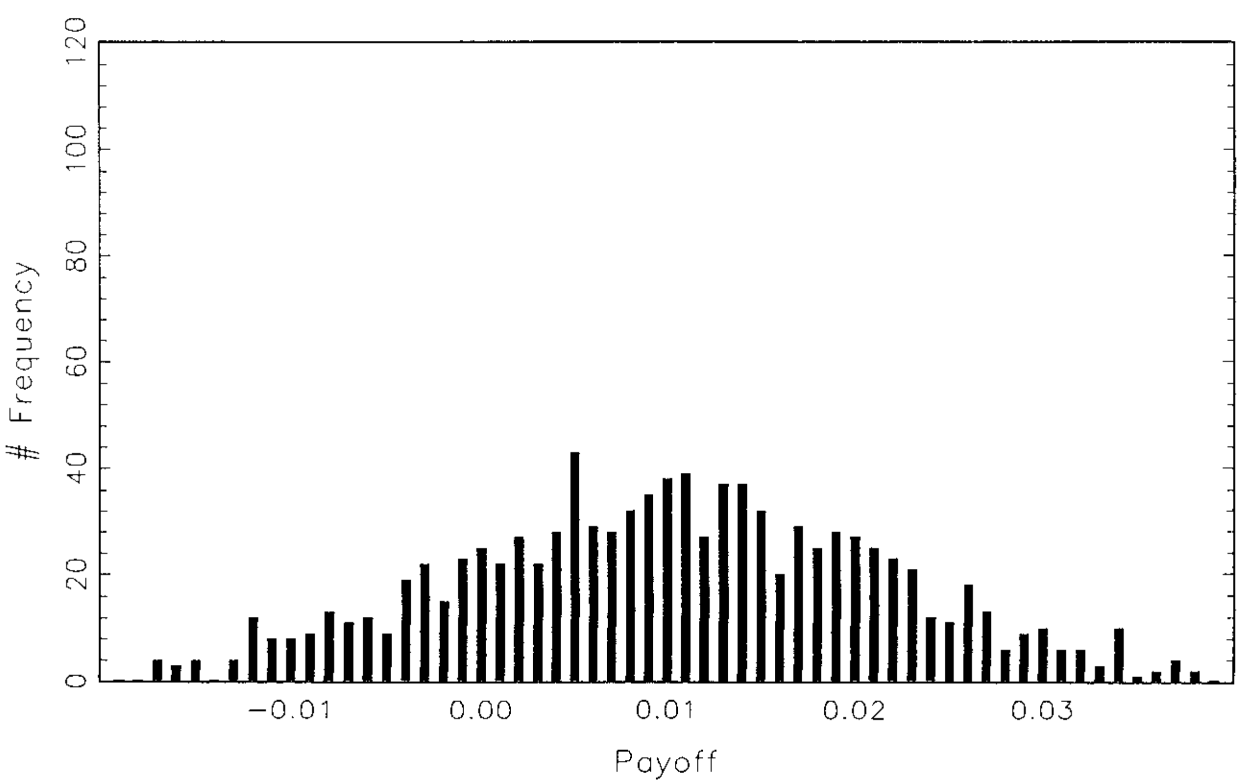

FIGURE 9

Payoff Distribution

(100 stocks; alphas estimated using 60 months)

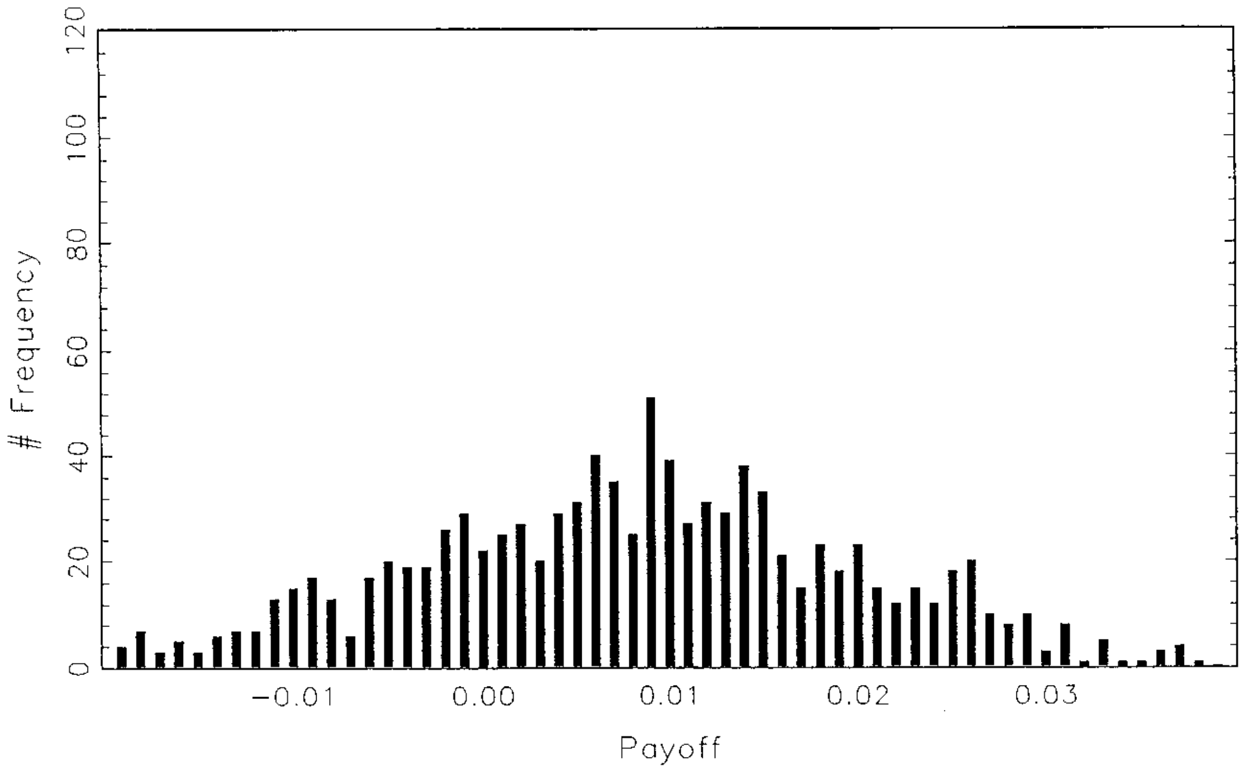

FIGURE 10

Payoff Distribution

(500 stocks; alphas estimated using 60 months)