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by

Simon Benninga
Aris Protopapadakis

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RODNEY L. WHITE CENTER FOR FINANCIAL RESEARCH
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6367

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Simon Benninga
School of Business
Hebrew University, Jerusalem, Israel

and

Aris Protopapadakis
School of Business Administration
University of Southern California
Los Angeles, CA 90089-1421

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ABSTRACT

We derive a closed-form expression for the differences between forward and futures prices in the framework of a Lucas (1978) equilibrium model. We calculate this difference for fixed-income securities in two ways: 1. Using historic interest rate data to calibrate the matrix of nominal state price, and 2. By testing a large sample of randomly-generated state price matrices. In both cases we find few meaningful differences between futures and forward prices. Larger differences are generated for highly diagonal state price matrices. We conclude that in most economically relevant circumstances the costs of marking to market for fixed income securities are negligible.

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1. Introduction

In three simultaneously-published papers Cox, Ingersoll and Ross (1981b), Jarrow and Oldfield (1981), and Richard and Sundaresan (1981) establish that in an equilibrium framework, futures and forward prices in general will differ under uncertainty. The source of the difference between the two prices arises from marking-to-market in the futures contract.

Empirical research finds no consistent pattern for the differences between forward and futures prices. Rendleman and Carabini (1979) track the T-Bill futures market from its inception in January 1976 through March 1978. They find significant differences between futures prices and implied forward prices for Treasury bills, although these differences tended to disappear as the market developed. In any case these differences were not significant when transactions costs were accounted for. In their examination of the Treasury bill market for 1976-1982, Elton, Gruber, and Rentzler (1984) calculate the cost of marking to market on a 91-day short position in the futures contract. They report an average cost of \$4.00 for marking to market, with 75% of all observations between -\$31 and +\$31 per contract.

Cornell and Reinganum (1981) find no significant differences between futures and forward prices in foreign exchange markets. French (1983) finds statistically significant differences in silver and copper prices. Park and Chen (1985), find no differences in foreign exchange contracts, but report significant differences for some metals prices.

These empirical findings clearly depend on the sample that is studied. Thus it is not clear whether the general lack of significant differences between futures and forward prices for financial contracts is a property specific to the data that were used, or whether this finding is due to more fundamental reasons.

In this paper we reconsider futures and forward prices in a somewhat different context. We look at a multi-state version of the Lucas (1978) consumption-based CAPM. We use this model to derive expressions for the pricing of futures and forwards. We then apply our results to the pricing of interest rate futures and forwards in the context of both historical data and simulations.

In Section 2 we show that when the consumption growth process is stationary, and when the representative consumer is endowed with a power utility function, our version of the Lucas model gives rise to simple, recursive pricing expressions for both real and nominal bonds. In Section 3 we show that this model allows for a simple matrix formulation of futures and forward prices.

We simulate the model in several different ways. In Section 4 we use historic interest rate data to construct a matrix of nominal state prices, which we use to price the forward and futures contracts on time deposits and on long-term bonds. In Section 5 we simulate the nominal state-price matrix for a variety of term structures, and again use each simulated matrix to construct a series of forward and futures prices. We then go on to experiment with the effects of diagonalization on the state price matrix, as well as the effect of increasing interest rate volatility.

In both sections we find few significant differences between the forward and futures prices of fixed income securities. It is only for the case of highly diagonal state price matrices that there are large differences between the forward and futures prices. We examine an extreme case in detail, and conclude that it cannot correspond to any meaningful economic scenario of interest rates, since it requires that the probability of a change from the current term structure is almost impossible.

2. The model

We consider a standard infinite-horizon model with a single representative consumer and a single commodity The representative consumer maximizes the infinite-horizon, time-separable, state-independent utility of consumption.[1]

We use the following notation:

- c_i is consumption in state i at time t,
- α_s is 1 + the stochastic consumption growth rate in state s,
- ω_s is the stochastic inverse of the inflation relative in state s,
- π_{is} is the probability of growth α_s given that the system is currently in state i,
- γ is the relative risk aversion of the representative consumer,
- δ is the representative consumer's pure-time preference factor,
- S is the number of states of the world at any date.

The representative consumer maximizes a time-separable expected utility function:

$$EU(c) = \sum_{t=1}^{\infty} \delta^t \; Eu(c_t) \quad \textit{where} \quad u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}. \tag{1}$$

Uncertainty in this model is generated by the random consumption endowments and inflation.[2] Suppose that at time t the system is in state i, with consumption c_i. Then at time t+1 the system will be in state s, $s = \{1,\ldots,S\}$, with consumption $\alpha_s c_i$ and inflation rate $1/\omega_s - 1$; the probability of this transition is given by π_{is}. There are no transactions costs or trading restrictions, and asset markets are complete. The representative consumer's probability-adjusted marginal rates of substitution are the real state prices which determine the prices of all real assets in the economy.

[1] The model belongs to the class of consumption-based, general equilibrium asset pricing models (Breeden 1979, Rubinstein 1976, Lucas 1978, Cox, Ingersoll, Ross 1981a and 1985, Benninga and Protopapadakis 1983).

[2] For simplicity, we abstract from issues involving production and investment. The Markovian endowment structures that we study can be generated by appropriately specified linear production technologies.

Since consumption growth is Markovian, the real state prices are time-independent, and can be denoted by an SxS matrix $Q \equiv [q_{ij}]$, whose typical element is given by,

$$(2) \qquad q_{ij} = \frac{\delta\pi_{ij} u'(\alpha_j c_i)}{u'(c_i)} = \delta\pi_{ij}\left[\frac{1}{\alpha_j}\right]^{\gamma}, \quad i = 1,\ldots,S; \quad j = 1,\ldots,S.$$

In a monetary economy nominal state prices can be defined from real state prices and state-contingent inflation rates. Benninga and Protopapadakis (1983) show that given real state prices q_{ij} and state-contingent changes in the money price of the consumption good from p_i in state i to p_{ij} in state j, *nominal state prices* can be defined as,

$$(3) \qquad b_{ij} = q_{ij}\omega_j.$$

The existence of recursive n-period pure nominal discount factors requires that the transition between the nominal state prices is Markovian. This amounts to assuming that both the inflation rates and the endowments are Markovian.

Let B be the matrix of nominal state prices. Then the vector that contains the S possible n-period nominal discount factors is given by,

$$(4) \qquad I(n) = B^n I(0),$$

where I(0) is a S-dimensional unit column vector.

3. The prices of futures and forwards

In this section we derive the pricing relations for forward and futures markets. Our results are similar to those of Cox, Ingersoll, and Ross (1981b) but in different form. The framework that we use allows significant economies in expressing the formulas for futures and forward prices.

We use a somewhat modified version of the CIR notation:

G(t,t+m) is the forward price at date t for delivery at date t + m,

H(t,t+m) is the futures price,

V( ) is the spot price.

In an S-state Lucas model, there are S possible futures, forward, and spot prices at any date. When necessary, we will specify the state-specific forward, futures, or spot price in state s of the world by G(t,t+m;s), H(t,t+m;s), or V(s). The vectors of forward, futures, and spot prices are represented by:

$$(5) \qquad G(t,t+m) = \begin{bmatrix} G(t,t+m;1) \\ G(t,t+m;2) \\ . \\ . \\ G(t,t+m;S) \end{bmatrix}, \quad H(t,t+m) = \begin{bmatrix} H(t,t+m;1) \\ H(t,t+m;2) \\ . \\ . \\ H(t,t+m;S) \end{bmatrix}, \quad V = \begin{bmatrix} V(1) \\ V(2) \\ . \\ . \\ V(S) \end{bmatrix}.$$

In what follows, we define a diagonal matrices $X^{(1)}$, $X^{(2)}$, $X^{(3)}$, ... whose non-zero elements are the m-period nominal accumulation factors. We write,

$$(6) \qquad X^{(1)} = \begin{bmatrix} 1/\sum_j b_{1j} & 0 & 0 & \dots & 0 \\ 0 & 1/\sum_j b_{2j} & 0 & \dots & 0 \\ . & & & & \\ . & & & & \\ 0 & 0 & 0 & \dots & 1/\sum_j b_{Sj} \end{bmatrix},$$

where the i-th diagonal element is 1 over the i-th element of the vector BI(0). Similarly, we denote by $X^{(k)}$ the matrix whose i-th diagonal element is one over the i-th element of the vector $B^k I(0)$. Multiplying a column vector V prices by $X^{(k)}$ gives the *future value* of the vector after k periods.

Proposition 1: The vector of forward prices at date t for assets deliverable at date t+m is given by:

(7) $$G(t,t+m) = X^{(m)}B^{m}V.$$

Proof:

Since the present value of a forward contract is zero, for a one-period contract we have,

(8) $$\sum_{j=1}^{S} b_{sj}[G(t,t+1;s) - V(j)] = 0, \quad for\ all\ s = 1,\dots,S,$$

so that

(9) $$G(t,t+1;s) = \frac{\sum_{j} b_{sj}V(j)}{\sum_{j} b_{sj}}.$$

Writing this in matrix notation gives G(t,t+1) = $X^{(1)}$BV. This proves the proposition for (t,t+1).

A two-period contract priced at G(t,t+2;s) today delivers, in state j of the world at t+2, a cash flow of G(t,t+2;s)-V(j). The state-i discount factor for state j of the world two periods hence is B_{ij}^{2}, where B_{ij}^{2} is the ij-th element of $B^{2} \equiv B * B$. Since the present value is zero, it follows that,

(10) $$B_{sj}^{2}[G(t,t+2;s) - V(j)] = 0, \quad for\ all\ s = 1,\dots,S.$$

From equation (10) it follows that G(t,t+2;s) can be written:

(11) $$G(t,t+2;s) = \frac{\sum_{j} B_{sj}^{2}V(j)}{\sum_{j} B_{sj}^{2}}.$$

Eliminating the "s", we see that the numerator of this expression can be written as B^2V. The denominator is an element of the matrix . Hence it follows that G(t,t+2) = $X^{(2)}B^{2}V$. The extension to more periods is clear. ||

The economic meaning of this proposition is straightforward: B^kV is the present value of the deliverable asset. The $X^{(k)}$ matrix along its diagonal contains all the possible k-period maturity future values, and hence represents the k-period carrying costs for a spot position established today. $X^{(k)}B^kV$ is the arbitrage price at which the value of a deliverable position in the future equates the value of the same position purchased today, as long as there are no intermediate payments or other carrying costs, as is the case with a forward contract on a financial asset.

Proposition 2: Futures contracts are priced by the following relation:

(12)
$$\begin{aligned} H(t,t+1) &= G(t,t+1) - XBV \\ H(t,t+2) &= \left[X^{(1)}B\right]^2 V \\ &\dots \\ H(t,t+m) &= \left[X^{(1)}B\right]^m V \end{aligned}.$$

Proof:

First consider a one-period futures contract. Since the present value of a futures contract is zero, we have,

(13)
$$\sum b_{sj}[H(t,t+1;s) - V(j)] = 0, \quad \textit{for all}\ s = 1,\dots,S,$$

so that

(14)
$$H(t,t+1;s) = \frac{\sum_j b_{sj} V(j)}{\sum_j b_{sj}}.$$

Writing this in matrix notation gives $H(t,t+1) = X^{(1)}BV$. This proves the proposition for (t,t+1). It also shows that the prices of one-period futures and forward contracts are identical; i.e., $H(t,t+1,s) = G(t,t+1,s)$ for all s.

Now consider a two-period contract. We start by noting that marking-to-market is equivalent to closing out the futures contract in every period. Thus all present values of the cash flows from a futures contract must be zero. This means that in particular the cash flow obtained by closing out the position one period hence must have zero value. Writing this out gives.

(15) $$\sum_j b_{sj}[H(t,t+2;s) - H(t+1,t+2;j)] = 0.$$

Collecting terms gives,

(16) $$H(t,t+2;s) = \frac{\sum_j b_{sj} H(t+1,t+2;j)}{\sum_j b_{sj}},$$

which, when written in matrix notation gives $H(t,t+2) = X^{(1)}BH(t+1,t+2)$. Since $H(t+1,t+2) = H(t,t+1) = X^{(1)}BV$, this proves that $H(t,t+2) = [X^{(1)}B]^2V$. The extension to more than two periods is immediate. ||

Because of marking-to-market, a futures contract is priced as if it is a sequence of rolled-over one-period forward contracts. Thus a k-period futures contract is priced by k multiples of $X^{(1)}B$, the pricing matrix for a one-period forward contract.

From Propositions 1 and 2 it follows that the difference between a futures and a forward price can be expressed by,

(17) $$H(t,t+m) - G(t,t+m) = \left\{\left[X^{(1)}B\right]^m - X^{(m)}B^m\right\}V.$$

(Note that $X^m \ddagger X^{(m)}$, since we have used $X^{(m)}$ to stand for the diagonal matrix with the m-period accumulation factors.)

Proposition 3 below says that differences between forward and futures prices stem from uncertainty in interest rates and in asset prices.

Proposition 3: The following three conditions are sufficient for the equality of futures and forward prices:

(3.1) The B matrix is diagonal.

(3.2) One-period rates are the same in each state.

(3.3) The vector of asset prices V is non-stochastic; i.e., V(s) = constant for all $s = 1,\ldots,S$.

Proof:

If the B matrix is diagonal, the $X^{(1)}$ has diagonal elements $1/b_{ii}$. Similarly B^m diagonal elements $(b_{ii})^m$ and $X^{(m)}$ has diagonal elements $(1/b_{ii})^m$. Thus both $X^{(1)}B$ and $X^{(m)} B^m$ are equal to the identity matrix. This proves (3.1).

If the one-period rates are the same in each state, then the row sums of matrix B are equal. Denote these row sums by R. It follows that $[X^{(1)}B]^m = [B/R]^m = B^m/R^m$, where the division of a matrix by a scalar denotes the division of each of the matrix's elements by the scalar. We complete the proof of (3.2) by noting that $B^m I(0)$, the m-period discount factors, are equal to $1/R^m$.

To prove (3.3), write $V = cI(0)$, where c is a scalar and I(0) is the S-dimensional unit vector. First note that $B^m V = c.[X^{(m)}]^{-1}$; this follows since $X^{(m)}$ is a diagonal matrix. Thus the forward price for this case is given by cI(0). To see that the futures price for this case is the same, note that $X^{(1)}BV = cI(0)$, from which it follows that $[X^{(1)}B]^m V = cI(0)$.

This completes the proof of the proposition.[3] ||

[3] (3.2) was first proved, in a different framework, by Margrabe (1976).

The most appropriate assumption for the B matrix --the matrix of state prices-- is that each state is reachable from any other state of the world. This assumption implies that B is ergodic. Proposition 4 below shows that in this case, the difference between the forward and futures prices approaches a constant as the maturity of the contracts increases.

Proposition 4: If the B matrix is ergodic, then the difference between forward and futures prices converges. That is,

$$\lim_{m\to\infty}\left[G(t,t+m)-H(t,t+m)\right]\to \mathrm{cI}(0). \tag{18}$$

Proof:

If B is ergodic $X^{(1)}B$ is ergodic $[X^{(1)}B]^m$ has a limit. From the ergodicity of B it follows that $X^{(m)}B^m$ also has a limit. This proves the proposition. ||

4. Price differences using historic interest rate data

We explore two approaches to calculating the differences between forward and futures prices. The approaches differ in their derivation of the nominal state price matrix B; as Section 3 shows, this matrix is necessary in order to price forward and futures contracts.

In this section we use historic bond yields to derive an empirical nominal state price matrix B, and we use these state prices to evaluate the differences between forward and futures prices. In Section 5 we use a large number of arbitrary B matrices to derive an upper bound for this difference. In both sections we find that the differences in these prices are negligible, except for the case when the state price matrix is highly diagonal..

4.a. Calibration of the Model

Equation (4), $I(n) = B^n I(0)$, and the associated model suggest a way to calculate B under certain assumptions. Combine equations (2) and (3) and write the typical element of B,

(19) $$b_{ij} = \delta \pi_{ij} \left[\frac{1}{\alpha_j} \right]^{\gamma} \omega_j .$$

It is clear that the matrix of state prices B can be decomposed into a Markov transition probability matrix Π (whose typical element is π_{ij}), and into a diagonal matrix A, whose jth diagonal element is $a_{jj} = \delta\, \omega_j [1/\alpha_j]^{\gamma}$. We may therefore write:

(20) $$B = \Pi A .$$

By defining a fixed number of states based on T-bill yield data, it is possible to calculate the transition probability matrix Π and from that, matrix B.[4]

For 1-period bonds equation (4) becomes I(1) = BI(0), and substituting this into equation (20) gives, I(1) = ΠAI(0). Since both I(1) (the 1-period discount bond prices) and Π are available from the data, we can calculate A by,

(21) $$AI(0) \equiv C = \Pi^{-1} I(1) .$$

Since A is diagonal, the elements of the vector C are the diagonal elements of A. We sort the historic 3-month T-Bill rates into 12 possible cells and compute the transition probability matrix from the frequency with which interest rates jump from one cell to another. Table 1 shows the average interest rates and the ranges for each of the 12 states, as well as the empirical transition probability matrix based on these state definitions.[5]

Using equation (21), we calculate the matrix A. Equation (20) then gives the empirical matrix B, displayed below.

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[4] What follows below is a synopsis of the procedure. A more complete explanation is given in Benninga and Protopapadakis (1991).

[5] The data are 3-month Treasury bill rates from Citibase for 59:1 through 86:2.

(22)
$$B = \begin{bmatrix} 0.895 & 0.099 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.110 & 0.441 & 0.220 & 0.109 & 0.111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.099 & 0.595 & 0.197 & 0.100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.110 & 0.220 & 0.328 & 0.221 & 0.109 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.090 & 0 & 0.269 & 0.271 & 0.179 & 0.179 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.271 & 0.447 & 0.179 & 0.089 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.111 & 0 & 0.328 & 0.437 & 0.109 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.179 & 0.179 & 0.268 & 0.358 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.098 & 0 & 0.197 & 0.197 & 0.293 & 0.197 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.089 & 0.269 & 0.532 & 0.090 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.108 & 0.549 & 0.320 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.123 & 0.122 & 0.123 & 0.600 \end{bmatrix}.$$

Note that both the transition probability and the state price matrices are highly diagonal. This shows that the one-step transitions in interest rates tend to move slowly through time.

4.b The Differences Between Forward and Futures Prices

We use the B matrix calculated above to price forward and futures on CDs and coupon bonds.[6] Propositions 1 and 2 of Section 3 give us the basic pricing results, and it remains only to determine the spot price vector V of the instruments themselves.

The price of a one-period CD or time deposit with a face value of \$1000 and a coupon rate of h is:

(23)
$$V_{CD}(s) = \sum_j b_{sj}(1+h)1000.$$

The $[V_{CD}(s)]$ is the price vector for which we calculate the futures and forward prices.

Figure 1 shows the difference between forward and futures prices for a 3-month time deposit of \$1000. The x-axis of the figure shows the period (in quarters) at which the forward and futures contracts mature (i.e., the 3-month time deposit is delivered), and the y-axis shows the

[6] Note that in the framework used here, the pricing of a CD and a deposit is the same, so that our results apply equally well to CD futures contracts and to the Eurodollar futures contracts.

current forward minus the futures price. The arresting feature of Figure 1 is that the differences between the forward and the futures prices, while varying, are extremely small, with the largest difference being less than four cents (fewer than 0.4 basis points).

It is notable in Figure 1 that for all maturities the forward price is larger than the futures price. As shown by Cox, Ingersoll, and Ross (1981b), the difference between forward and futures prices depends on the correlation of the price of the deliverable asset with the one-period interest rate. If this correlation is positive, then an increase in the short-term interest rate is associated both with a decrease in the price of the deliverable asset and with an increase in the cost of marking to market. This will cause the futures price to be lower than the forward price. Figure 2 shows the term structures associated with this empirical B matrix.[7] It is obvious that for most of the possible term structures the correlation between the one-period interest rate and longer-term interest rates is positive and large. This leads to forward prices that are larger than the corresponding futures prices.

We can use the same technology to price coupon bonds and forwards and futures on these bonds. The price of a coupon bond is given by:

$$V(N,h,s)=\sum_{k=1}^{N-1} hI_s(k)+(1+h)I_s(N). \tag{24}$$

where h is the coupon on the bond, N is the maturity of the bond, and s is the state. The results for the differences between forward and futures prices for long-term coupon bonds are very similar.

[7] We calculate the term structures from powers of the B matrix. Figure 2 shows the resulting term structures for each state, and reports its unconditional probability.

5. An upper bound for the differences

5.a. Investigating the differences with random matrices

It is possible that the results of the previous section are a special result of the calibrated B matrix. In order to investigate this possibility, we repeat the above calculations for a large number of randomly drawn B matrices. Each run of this experiment proceeds as follows:

1. We take one period to be *one quarter*, the same as in our empirical matrix. We draw from a uniform distribution over a range of 0.0% to 5.0% (0% to 20% annualized) a vector of quarterly interest rates, I(1). We also draw from a uniform distribution over a range of 0.0 to 1.0 a 12x12 matrix B of nominal state prices.[8] We normalize this matrix so that it produces the one-period interest rates given by I(1). That is,

(25) $$\sum_{j=1}^{N} b_{ij} = 1/(1+r_i),$$

where r_i is the i'th random interest rate.

2. Forward and futures prices for one-period $1000 time deposits are derived for 1 to 25 periods.[9]

In 8,500 runs of this experiment, the maximum absolute difference between the forward and futures prices were recorded, as well as the matrix B which gave rise to these prices. The results of this experiment are dramatic. The maximum absolute difference found is $0.62. This maximum deviation of a forward from a futures price is less than 6 basis points.

The experiment was repeated for coupon bonds. In all cases the results were qualitatively the same.

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[8] We experimented with matrices of different sizes and found that the number of states did not meaningfully affect the nature of the results.

[9] The longest maturity futures contract we are aware of is 5 years for Eurodollars.

5.b. The effects of diagonalization

In the previous section, we discuss how the correlation of long-term and short-term interest rates affects the difference between forward and futures prices. This implies that the diagonality of the state price matrix is likely to affect the size of the difference between forward and futures prices because nearly diagonal matrices produce very flat term structures; flat term structures result in large positive correlations between short- and long-term rates. The empirical state price matrix we calculate in Section 4 is fairly diagonal with high values on the diagonal.[10] In constrast, the typical random matrices we produce are full matrices. We investigate the effect of making some of our random state price matrices B more diagonal by defining a functional transformation B(h) as follows:

(26)
$$b_{ij}(h) = \begin{cases} \dfrac{b_{ij} h^{|i-j|}}{(1+r_i)\sum_j b_{ij} h^{|i-j|}}, & 0 \le h < 1 \\ \dfrac{1}{(1+r_i)}, & h = 1 \end{cases} .$$

Thus $B(h_1)$ is more nearly diagonal than $B(h_2)$ if $h_1 < h_2$, and B(0) is exactly diagonal, a matrix for which we know by Proposition 3 that there are zero differences between forward and futures prices. Starting with a random matrix B(h=1), we normalize all matrices $B(h_i)$ to have the same one-period interest rates. Thus the variable h controls the degree of diagonality of a given state price matrix, while keeping one-period rates the same.

We experiment with the effect of these diagonal transformations on the difference between forward and futures prices, using B matrices which gave large deviations in the previous

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[10] Our terminology is impressionistic. The formal definition of a diagonal matrix is that all off-diagonal elements are zero. We have found no terminology to describe a matrix which is *almost diagonal*, in the sense that the larger elements are on the diagonal and that value of the off-diagonal elements tends to decline with their distance from the diagonal. We refer to such matrices as *nearly diagonal* and use freely nomenclature such as *more nearly diagonal*, *highly diagonal*, or *almost diagonal.*

experiment, as well as with additional random matrices. We find that for very nearly diagonal matrices, differences between forward and futures prices can be as much as 10-15 times larger than the largest deviations obtained in the experiment described in Section 5a. Figure 3 plots (for all 12 states) the differences between forward prices and futures prices for a particular random matrix as functions of the diagonality of the matrix as measured by h, for contracts that mature in 25 periods.[11] To obtain the results of Figure 3, we experimented with several hundred matrices and chose one which exhibited a large difference between forward and futures prices; most of the matrices experimented with produced maximal differences of only one or two dollars (i.e., less than 20 basis points). The patterns in Figure 3 are typical for matrices which produced large differences between forward and futures prices. The diferences are zero when the matrix is exactly diagonal (h=0), as proved in Proposition 3. The differences between the forward and futures prices tend to increase dramatically as the matrix becomes near-diagonal ($h \approx 0$), and then they fall as the off-diagonal elements become larger ($h \rightarrow 1$).

In the particular case plotted in Figure 3, the largest differences are found for h=0.0045; that is, for a very nearly diagonal matrix. This matrix is shown in Table 2.

We draw three conclusions from this experiment. The first is that even these much larger differences are economically very small; we did not find any differences that exceeded 60 basis points. The second conclusion is that extreme diagonality is necessary for these large premia. For instance, the degree of diagonality of our empirical matrix only produces approximately 4 basis points of difference, and the degree of diagonality of the matrix in Table 2 makes it highly unlikely that the term structure will ever change from one state to another. The largest off-diagonal element in this matrix is 0.00368, and the probability of this off-diagonal change in states must be even smaller than this element. The third observation is a statistical one: Even though our procedure of generating random matrices in principle produces all types of matrices, it did not in fact produce highly diagonal matrices in these 8,500 tries. This observation may suggest that

[11] We chose a maturity of 25 periods (6 1/4 years), because typically the largest differences between the forward and futures prices occur at the longer maturities.

highly diagonal cases are pathological, and that they need not concern us. However, the empirical matrix that we discuss earlier in the paper, coupled with the wide-spread phenomenon that U.S. and most foreign interest rates are highly correlated suggests that reasonably diagonal matrices are economically relevant.

5.c. The effect of the price variability of the deliverable asset

A determinant of the potential differences between forward and future prices is the variability of price of the deliverable asset. In the case of deliverable CDs, discount bonds or T-Bonds, the variability of the prices of these assets is directly related to the underlying state prices. In particular, the sum of the rows of the B matrix gives all the possible prices of a discount bond with one period to maturity; the row sums of B^n give the discount bond prices with n periods to maturity. Even so, we can investigate the effect of changes in the variability of the one-period interest rates

In Table 3 we show the effect of changing the volatility of interest rates on the differences between forward and futures prices. We experiment with the same underlying 12x12 matrix of random numbers, normalized to be the state prices for a vector of one-period interest rates drawn from a uniform distribution from (0%,20%), (0%,10%), (0%,5%). Table 3 gives statistics for the maximum and minimum deviation of forward minus futures prices for futures contract maturities of 2,3,4,5,10,15,20,25 periods. The table also gives the mean and standard deviation of the difference. The experiment was repeated 8,500 times.

As can be seen from Table 3, while greater interest rate volatility (which is, of course, directly correlated with the volatility of the asset prices considered) affects the differences between forward and futures prices, they are still extremely small.

7. Conclusion

A well-known literature shows that forward and futures prices may be different in equilibrium. We derive closed-form expressions for these differences in the context of Lucas's (1978) asset-pricing model. After investigating several different numerical implementations of these expressions for fixed income securities, we conclude that for economically relevant term structures, there are few meaningful differences between futures and forward prices for such securities. We thus conclude that the costs of marking-to-market on these securities are minimal. These results are consistent with the empirical findings to date.

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TABLE 1

Some Data for the Empirical Matrix

A. Interest rate ranges

state	Interest Rate Range	Average Rate
1	2.0 through 2.8	2.50%
2	2.8 through 3.5	3.15%
3	3.5 through 4.0	3.73%
4	4.0 through 4.6	4.34%
5	4.6 through 5.1	4.88%
6	5.1 through 5.6	5.39%
7	5.6 through 6.2	5.91%
8	6.2 through 7.0	6.52%
9	7.0 through 7.9	7.37%
10	7.9 through 9.0	8.35%
11	9.0 through 11.0	9.86%
12	11.0 through 17.0	13.83%

B. The transition probability matrix

	(2,2.8]	(2.8,3.5]	(3.5,4]	(4,4.6]	(4.6,5.1]	(5.1,5.6]	(5.6,6.2]	(6.2,7]	(7,7.9]	(7.9,9]	(9,11]	(11,17]
(2,2.28]	0.900	0.100	0	0	0	0	0	0	0	0	0	0
(2.8,3.5]	0.111	0.444	0.222	0.111	0.111	0	0	0	0	0	0	0
(3.5,4]	0	0.100	0.600	0.200	0.100	0	0	0	0	0	0	0
(4,4.6]	0	0.111	0.222	0.333	0.222	0.111	0	0	0	0	0	0
(4.6,5.1]	0	0.091	0	0.273	0.273	0.182	0.182	0	0	0	0	0
(5.1,5.6]	0	0	0	0	0.273	0.455	0.182	0.091	0	0	0	0
(5.6,6.2]	0	0	0	0	0.111	0	0.333	0.444	0.111	0	0	0
(6.2,7]	0	0	0	0	0	0.182	0.182	0.273	0.364	0	0	0
(7,7.9]	0	0	0	0	0	0.100	0	0.200	0.200	0.300	0.200	0
(7.9,9]	0	0	0	0	0	0	0	0.091	0.273	0.545	0.091	0
(9,11]	0	0	0	0	0	0	0	0	0	0.111	0.556	0.333
(11,17]	0	0	0	0	0	0	0	0	0.125	0.125	0.125	0.625

TABLE 2

A Diagonal Matrix which Produces Large Differences of Forward Minus Futures Prices

0.991	0.000678	0	0	0	0	0	0	0	0	0	0
0.00192	0.949	0.00237	0	0	0	0	0	0	0	0	0
0	0.00073	0.974	0.00223	0	0	0	0	0	0	0	0
0	0.0000263	0.00165	0.947	0.00368	0.000015	0	0	0	0	0	0
0	0	0	0.00202	0.992	0.00238	0	0	0	0	0	0
0	0	0	0.000011	0.000862	0.984	0.00166	0.0000117	0	0	0	0
0	0	0	0	0.0000315	0.00353	0.954	0.0108	0.0000222	0	0	0
0	0	0	0	0	0	0.00169	0.963	0.00238	0	0	0
0	0	0	0	0	0	0	0.000755	0.987	0.00303	0	0
0	0	0	0	0	0	0	0	0.00232	0.951	0.0041	0
0	0	0	0	0	0	0	0	0.000013	0.00772	0.977	0.00919
0	0	0	0	0	0	0	0	0	0.0000176	0.00358	0.965

Note: The above matrix produced large differences between forward and futures prices for the diagonalization procedure described in Section 5b. All elements of the matrix are given with 3 decimal-place accuracy, and we have rounded off all elements $< 10^{-5}$ to 0.

TABLE 3

The Effect of Changing Interest Rate Volatility

maturity	1	2	3	4	5	10	15	20	25
U[0,20%]									
max	0.505	0.485	0.47	0.462	0.459	0.458	0.458	0.458	0.458
min	-0.622	-0.388	-0.383	-0.366	-0.365	-0.364	-0.364	-0.364	-0.364
mean	0.00604	0.00105	0.00141	0.00162	0.00164	0.00165	0.00165	0.00165	0.00165
standard error	0.126	0.117	0.118	0.118	0.118	0.118	0.118	0.118	0.118
U[0,10%]									
max	0.147	0.142	0.138	0.136	0.135	0.134	0.134	0.134	0.134
min	-0.177	-0.122	-0.110	-0.105	-0.105	-0.105	-0.105	-0.105	-0.105
mean	-0.00181	0.000208	0.000311	0.000373	0.000379	0.000383	0.000383	0.000383	0.000383
standard error	0.0361	0.0335	0.038	0.0338	0.0338	0.0338	0.0338	0.0338	0.0338
U[0,5%]									
max	0.0398	0.0386	0.0375	0.0369	0.0367	0.0366	0.0366	0.0366	0.0366
min	-0.0473	-0.0302	-0.0297	-0.0284	-0.0283	-0.0283	-0.0283	-0.0283	-0.0283
mean	-0.000497	0.0000431	0.000071	0.0000875	0.0000893	0.0000901	0.0000901	0.0000901	0.0000901
standard error	0.00971	0.00901	0.00908	0.00909	0.0091	0.0091	0.0091	0.0091	0.0091

Note: The table shows statistics for forward minus futures prices (in dollars) of a $1000 time deposit for cases where the annual interest rates were drawn from a uniform distribution (0%,20%), from (0%,10%), and from (0%,5%). For each case the state price matrix was derived from the same 12x12 matrix of random numbers, normalized to correspond to each particular interest rate distribution. The experiment was repeated 8,500 times.

FIGURE 1: Forward Minus Futures Prices

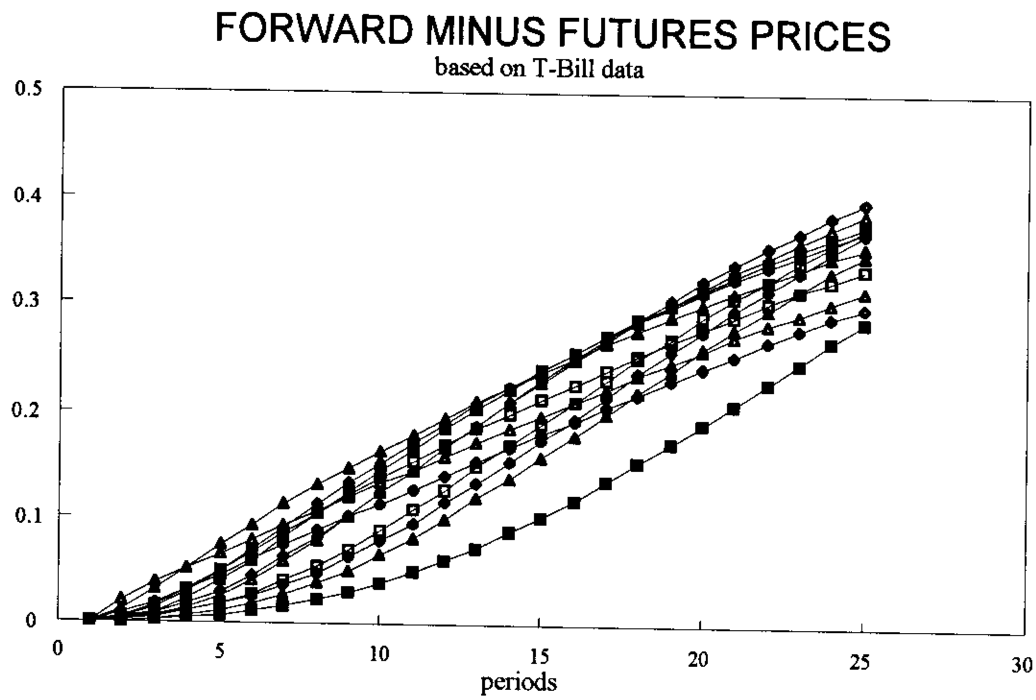
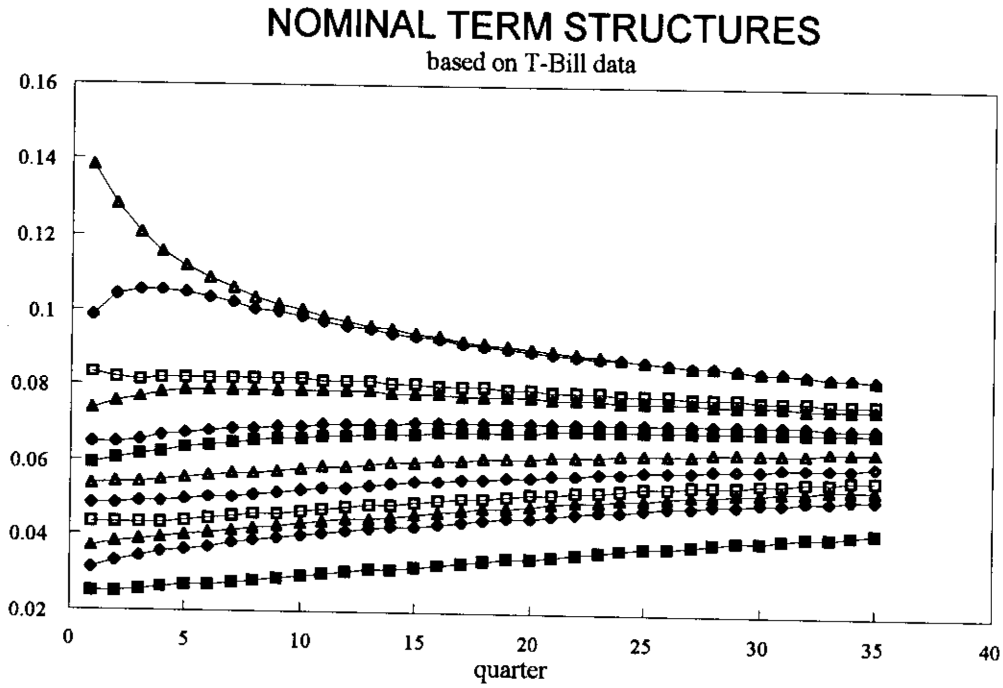
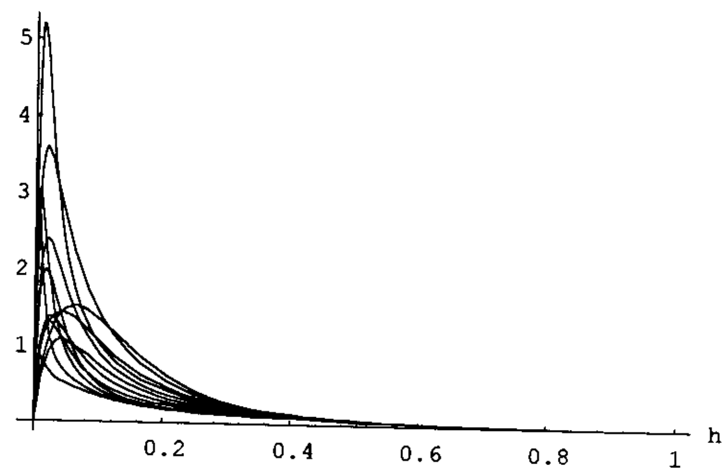


FIGURE 2: Term Structure Based on T-Bill Data



Note: The unconditional probabilities of the various term structures (as graphed, from bottom to top): 0.046, 0.041, 0.053, 0.055, 0.077, 0.094, 0.076, 0.107, 0.118, 0.130, 0.107, 0.095.

FIGURE 3: Forward Minus Futures Prices for Highly Diagonal Matrices



Note: As explained in Section 5b, h is a measure of the diagonalization of the state price matrix. The graph exhibits the differences between forward and futures prices for a given state price matrix and for $0 \leq h \leq 1$. The example shown here produced very large maximal differences.