EXACT SOLUTIONS FOR EXPECTED RATES OF RETURN UNDER MARKOV REGIME SWITCHING: IMPLICATIONS FOR THE EQUITY PREMIUM PUZZLE

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Abstract: This paper derives simple closed-form solutions for expected rates of return on stocks and riskless one-period bills under the assumption that shocks to the growth rates of consumption and dividends are generated by a Markov regime-switching process. These closed-form solutions are used to show that the Markov regime-switching process exacerbates the equity premium puzzle and the risk-free rate puzzle. Three empirical examples illustrate the magnitude of the effects of Markov regime switching on equilibrium expected returns.

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Exact Solutions for Expected Rates of Return under Markov Regime Switching:

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In one of the best-known applications of the Lucas model of asset pricing, Mehra and Prescott (1985) examine whether this model can reasonably account for the average rates of return on stocks and short-term bills in the United States. They find that over the period 1889-1978 the average real rate of return was 6.98% per year on stocks but only 0.80% per year on short-term bills. The 6.18% per year excess return on stocks relative to bills--the equity premium--is much larger than Mehra and Prescott could account for using a simple version of the Lucas model. In fact, Mehra and Prescott found that the Lucas model predicts an average equity premium of no more than 0.35% per year when attention is confined to the range of parameter values that Mehra and Prescott deemed to be reasonable. The inability of the Lucas model to account for the average equity premium has been dubbed the "equity premium puzzle." Subsequently, Weil (1989) has emphasized a "risk-free rate puzzle" in which asset pricing models that can produce a large equity premium also produce a riskless rate that is much higher than the historically observed average riskless rate.

Several papers have attempted to resolve the equity premium puzzle by changing various aspects of the model used by Mehra and Prescott. In this paper, I will focus on modifying two aspects of the Mehra-Prescott implementation of the Lucas model. First, in the model used by Mehra and Prescott, as in Lucas's original model, dividends on unlevered equity are identically equal to capital income, which is identically equal to total income, which is identically equal to consumption. In contrast, Cecchetti, Lam, and Mark (1991) allow aggregate consumption and aggregate dividends to differ from each other.[1] They argue that aggregate consumption can deviate from aggregate dividends in a general equilibrium model if there is labor income in addition to capital income, but they do not formally include labor income in their model. The model presented below will explicitly include labor income in a competitive economy that uses capital and labor to produce output. In this model, dividends and capital income remain identically equal, and total income and consumption remain identically equal. However, the introduction of labor income breaks the equality between capital income and total income so that in equilibrium dividends and consumption are no longer identically equal.

The second aspect of the Mehra-Prescott implementation of the Lucas model that I focus on is the stochastic process generating consumption and dividends. Cecchetti, Lam, and Mark (1990,1991), and Kandel and Stambaugh (1989,1990) have assumed that the conditional growth rates of consumption and dividends depend on an underlying random state of the world according to a Markov regime-switching process of the sort discussed by Hamilton (1989). In this paper, I will derive simple closed-form solutions for the conditional and unconditional expected rates of return on stocks and short-term riskless bills in the presence of technological uncertainty that follows a Markov regime-switching process. In addition to applying to a more general class of Markov regime-switching processes, the solution procedure utilized in this

[1]In contrast, in a previous paper, Cecchetti, Lam, and Mark (1990) assume that dividends and consumption are identically equal, and use three different empirical measures of the quantity that represents both consumption and dividends: consumption, dividends, and GNP.

paper allows computation of conditional and unconditional expected rates of return without using the state transition probabilities required in the procedures used by Cecchetti, Lam, and Mark (1990) and Kandel and Stambaugh (1989,1990). Although the solution procedure delivers simpler expressions for first moments of rates of return and applies to a more general class of stochastic processes, it does not provide simple expressions for higher moments of rates of return.

An important substantive payoff from the closed-form solutions is the derivation of analytic results comparing unconditional expected rates of return under Markov regime switching and under conditional i.i.d. distributions. I show that the unconditional riskless rate is higher under Markov regime switching than under conditional i.i.d. shocks, which further exacerbates the risk-free rate puzzle pointed out by Weil (1989). In addition, I show that under conditional lognormality the added stochastic richness of the Markov regime-switching process will generally reduce the size of the equity premium predicted by the asset pricing model, and thus will exacerbate rather than resolve the equity premium puzzle. This result indicates that further exploration with different parameterizations of Markov regime-switching processes will not help solve the equity premium puzzle with a time-separable constant relative risk aversion utility function.

Section I discusses the model of production, income, and equilibrium asset returns. The model is used in section II to calculate closed-form solutions for expected rates of return under constant relative risk aversion. Then equilibrium expected rates of return are related to underlying technological shocks in section III. Section IV compares expected rates of return under the Markov regime-switching process and under conditional i.i.d. technological shocks and shows that Markov regime switching exacerbates the equity premium puzzle and the risk-free rate puzzle. Results presented in section IV include both theoretical and empirical findings. Concluding remarks are presented in section V.

I. The Model

I.A. Production

Consider an economy that produces a homogeneous non-storable consumption good according to the production function

$$y_t = f(k_t, n_t, e_t) \qquad (1)$$

where y_t is output per capita, k_t is the capital stock per capita, n_t is the amount of labor per capita, and e_t is a vector of stochastic productivity shocks in period t. The production function $f(k_t, n_t, e_t)$ is increasing, concave, and homogeneous of degree one in k_t and n_t. Output is completely perishable; it cannot be stored from one period to the next, nor can it be used to augment the capital stock. Capital is assumed to be perfectly durable so that the capital stock is fixed. The fixed capital stock is normalized to $k_t = 1$. In addition, the supply of labor is completely inelastic and the fixed labor supply is normalized to $n_t = 1$.

I.B. Competitive Factor Markets

The services of the two factors of production, capital and labor, are rented in competitive factor markets. Each factor is paid the value of its marginal product. Let d_t be the dividend, or capital income, per capita and observe that

$$d_t = f_k(1,1,e_t) \tag{2}$$

Similarly, labor income per capita, w_t, is given by

$$w_t = f_n(1,1,e_t) \tag{3}$$

Because the production function is homogeneous of degree one in capital and labor, total factor payments, $d_t + w_t$, equal total output y_t.

I.C. Consumers

The economy is populated by a large number of identical infinitely-lived consumers, each of whom maximizes

$$E_t\{\Sigma_{j=0}^{\infty} \beta^j u(c_{t+j})\} \tag{4}$$

where c_t is per capita consumption in period t, $\beta > 0$ is the time preference discount factor, and the utility function u() is strictly increasing and strictly concave. The operator $E_t\{\}$ denotes the expectation conditional on Ω_t, the information set at the beginning of period t. Notice that leisure does not enter the utility function, which is consistent with the assumption of completely inelastic labor supply.

The representative consumer chooses how much output to consume, how much stock to hold, and how much of riskless one-period bills to hold. A share of stock pays a dividend d_t in period t to a holder of the stock at the beginning of period t and then sells for an ex-dividend price of p_t. The well-known first-order condition for a consumer considering buying a share of stock (ex-dividend) in period t is

$$p_t u'(c_t) = \beta E_t\{(p_{t+1}+d_{t+1})u'(c_{t+1})\} \tag{5}$$

The interpretation of eq. (5) is straightforward: A consumer can buy a share of stock in period t by giving up p_t units of consumption in period t, thereby reducing utility by $p_t u'(c_t)$ in period t. In the following period, the consumer receives the dividend d_{t+1} and can sell the share of stock at a price p_{t+1}. Therefore, the consumer can increase consumption in period t+1 by $p_{t+1}+d_{t+1}$ units, thereby increasing utility by $(p_{t+1}+d_{t+1})u'(c_{t+1})$. The expected value of this increase in utility, discounted one period to account for time preference, appears on the right hand side of eq. (5) and is set equal to the reduction in utility in period t which appears on the left hand side of eq. (5).

Each unit of output that is used to buy a riskless one-period bill in period t will be worth $R_{F,t+1}$ in period t+1. We refer to $R_{F,t+1}$ as the riskless rate of return. The well-known first-order condition for a consumer considering buying a riskless one-period bill in period t is

$$u'(c_t) = \beta E_t\{R_{F,t+1}u'(c_{t+1})\} \tag{6}$$

The left hand side of eq. (6) is the reduction in utility in period t caused by reducing consumption by one unit to purchase the bill. The right hand side of eq. (6) is the discounted expected increase in utility in period t+1 due to increasing c_{t+1} by $R_{F,t+1}$ units.

I.D. Equilibrium

Because the homogeneous good is non-storable, all output is consumed in each period. Therefore, goods market equilibrium implies

$$c_t = y_t = f(1,1,e_t) \tag{7}$$

Note that with the capital stock and labor supply fixed exogenously, output y_t depends only on the realization of the productivity shock e_t and thus is exogenous with respect to the decisions of consumers and firms. Substituting eq. (7) into eqs. (5) and (6) yields equations characterizing asset market equilibrium

$$p_t = \beta E_t\{(p_{t+1}+d_{t+1})u'(y_{t+1})/u'(y_t)\} \tag{8}$$

$$R_{F,t+1} = [\beta E_t\{u'(y_{t+1})/u'(y_t)\}]^{-1} \tag{9}$$

In the Lucas model of an exchange economy used by Mehra and Prescott, all income is capital income so that the dividend d_t would be set equal to y_t in eq. (8). However, in the general equilibrium model presented here, there is labor income in addition to capital income so that the dividend is not equal to y_t. Nevertheless, both the dividend and output (which equals consumption) have the convenient property that they are exogenous with respect to the decisions of consumers and firms.

II. Expected Rates of Return Under Constant Relative Risk Aversion

Many of the studies of the equity premium puzzle, including the original study by Mehra and Prescott, have assumed that (1) the utility function displays constant relative risk aversion; and (2) the process for (the logarithms of) consumption and dividends has a unit root. I will now adopt both of these assumptions. Assuming that the coefficient of relative risk aversion is constant and equal to α yields

$$u'(c) = c^{-\alpha} \qquad \alpha > 0 \tag{10}$$

To describe the unit root process for (the logarithms of) consumption and dividends, define the gross growth rates of consumption and dividends as $g_{c,t+1} \equiv c_{t+1}/c_t = y_{t+1}/y_t$ and $g_{d,t+1} \equiv d_{t+1}/d_t$. Recall from (7) and (2) that c_{t+1} and d_{t+1} depend only on the technological shock e_{t+1}. Therefore, the vector of gross growth rates, $g_{t+1} \equiv (g_{c,t+1}, g_{d,t+1})$, depends only on the technological shocks e_t and e_{t+1}.

Now assume that the underlying technological shocks are governed by a Markov regime-switching process, such as the process described in Hamilton (1989). Specifically, there is an underlying state vector s_t that governs the conditional distribution of g_{t+1}. The conditional distribution of g_{t+1} is denoted by $F_g(g_{t+1}|s_t)$. The state vector s_t

evolves according to a Markov process with conditional distribution function $F_s(s_{t+1}|s_t)$. The information set at the beginning of time t, Ω_t, includes s_{t-j} and g_{t-j} for j = 0,1,2,.... Following Cecchetti, Lam, and Mark (1990) and Kandel and Stambaugh (1989,1990), assume that (1) the joint distribution of s_{t+1} and g_{t+1} conditional on Ω_t depends only on s_t, and (2) conditional on s_t, the vectors s_{t+1} and g_{t+1} are independent of each other.[2,3] That is,

$$F_{g,s}(g_{t+1},s_{t+1}|\Omega_t) = F_{g,s}(g_{t+1},s_{t+1}|s_t) = F_g(g_{t+1}|s_t)F_s(s_{t+1}|s_t) \quad (11)$$

The Markov regime-switching process in eq. (11) contains as special cases the processes used by Cecchetti, Lam, and Mark (1990) and by Kandel and Stambaugh (1989,1990), but is considerably more general. Those papers assume that the state vector s_t is a discrete random vector with a finite number of possible values. However, the process in eq. (11) allows the state vector s_t to be either a discrete random vector or a continuous random vector.

[2] Although the notation in Cecchetti, Lam, and Mark (1991) makes it appear as if this conditional independence is not assumed, the stochastic process in that paper actually does display this conditional independence. However, the information set upon which asset prices are based in Cecchetti, Lam, and Mark (1991) is different from the information sets used in Cecchetti, Lam, and Mark (1990), Kandel and Stambaugh (1989,1990), and in this paper. See footnote 4.

[3] Wizman and Fullenkamp (1992) analyze asset pricing under Markov regime switching. They introduce the concept of "surety" to mean that s_t is in the information set Ω_t of investors/consumers at time t. Surety is distinct from "certainty" which Wizman and Fullenkamp define to mean that g_{t+1} is in the information set Ω_t of investors/consumers at time t. Wizman and Fullenkamp analyze asset prices both in the absence of surety and in the presence of surety. In this paper, as in Cecchetti, Lam, and Mark (1990) and Kandel and Stambaugh (1989,1990), we confine our attention to the case of surety.

Under the assumptions in (10) and (11), it is easy to calculate the expected rates of return on stocks and bills. First, define the stock price-dividend ratio $v(\Omega_t)$ as

$$v(\Omega_t) \equiv p_t/d_t \tag{12a}$$

Under the assumptions in this paper, all of the information about future distributions of consumption growth and dividend growth is contained in the current state vector s_t so that the price-dividend ratio depends only on s_t. Thus, $v(\Omega_t) = v(s_t)$ so the price-dividend ratio in eq. (12a) can be rewritten as

$$v(s_t) \equiv p_t/d_t \tag{12b}$$

Substituting the marginal utility function from eq. (10) and the price-dividend ratio from eq. (12b) into eq. (8) yields

$$v(s_t) = \beta E_t\{(v(s_{t+1})+1)g_{d,t+1}g_{c,t+1}{}^{-\alpha}\} \tag{13}$$

Equation (13) is a fairly standard equation for the price-dividend ratio $v(s_t)$ of a share of unlevered equity. For example, it is the same as eq. (9) in Cecchetti, Lam, and Mark (1991).[4] It is also the same as equation (8) in Cecchetti, Lam, and Mark (1990) if the distinction between dividends and consumption is ignored as in that paper. After

[4]There is an important difference between eq. (13) in this paper and eq. (9) in Cecchetti, Lam, and Mark (1991). In contrast to this paper (and in contrast to Cecchetti, Lam, and Mark (1990) and Kandel and Stambaugh (1989,1990)), the information set upon which asset prices are based in Cecchetti, Lam, and Mark (1991) includes s_{t-i}, $i = 1,2,3,...$ and e_{t-j}, $j=0,1,2,...$, but does not include s_t.

deriving this equation, the standard approach when using a finite state Markov regime-switching process is to solve for the price-dividend ratio $v(s_t)$. Then the solution for $v(s_t)$ can be used to compute the conditional and unconditional expected rates of return on equity. Below I present an alternative procedure for calculating conditional and unconditional expected rates of return on equity without having to solve for $v(s_t)$. In addition to its simplicity, this alternative procedure has the important advantage that it can be used for both discrete and continuous distributions of the state vector s_t. Furthermore, this procedure requires only the unconditional distribution of the state vector s_t, but not the state transition probabilities $F_s(s_{t+1}|s_t)$. Avoiding the use of transition probabilities has the advantage of simplifying the computation of expected rates of return. In addition, estimated unconditional probabilities may have more precision than estimated transition probabilities. For a discrete state vector that takes one of N possible values, there are N-1 independent unconditional probabilities to be estimated, but N(N-1) state transition probabilities to be estimated.

To calculate the conditional expected rate of return on equity, first observe that the conditional independence of s_{t+1} and g_{t+1} in eq. (11) implies that conditional on Ω_t, $v(s_{t+1})$ is independent of $g_{d,t+1}$ and $g_{c,t+1}$ so that eq. (13) can be written as

$$v(s_t) = \beta E_t\{v(s_{t+1})+1\} E_t\{g_{d,t+1} g_{c,t+1}{}^{-\alpha}\} \tag{14}$$

Equation (14) along with the fact that $v(s_t)$ is in Ω_t implies that

$$E_t\{[v(s_{t+1})+1]/v(s_t)\} = [\beta E_t(g_{d,t+1}g_{c,t+1}^{-\alpha})]^{-1} \tag{15}$$

Let $R_{S,t+1}$ denote the gross realized rate of return on stocks between period t and period t+1, and observe that

$$R_{S,t+1} = (p_{t+1}+d_{t+1})/p_t = [(v(s_{t+1})+1)/v(s_t)]g_{d,t+1} \tag{16}$$

Using the conditional independence of s_{t+1} and $g_{d,t+1}$, and using eq. (15), eq. (16) implies that the conditional expectation of the return on stocks is

$$E_t(R_{S,t+1}) = E_t(g_{d,t+1})/[\beta E_t(g_{d,t+1}g_{c,t+1}^{-\alpha})] \tag{17}$$

Notice that the conditional expected rate of return on stocks in eq. (17) depends only on the conditional moments of the exogenous joint stochastic process for consumption growth and dividend growth. That is, the expression for the conditional expected rate of return depends only on the stochastic process of the underlying fundamentals in the economy.

The riskless rate of return is calculated by substituting the marginal utility function from eq. (10) into eq. (9) to obtain

$$R_{F,t+1} = [\beta E_t(g_{c,t+1}^{-\alpha})]^{-1} \tag{18}$$

Now use eqs. (17) and (18) to write the conditional expected equity premium in ratio form as

$$E_t(R_{S,t+1})/R_{F,t+1} = E_t(g_{d,t+1})E_t(g_{c,t+1}^{-\alpha})/E_t(g_{d,t+1}g_{c,t+1}^{-\alpha}) \tag{19}$$

Note that if dividend growth $g_{d,t+1}$ and consumption growth $g_{c,t+1}$ were conditionally independent of each other, the right hand side of eq. (19) would equal 1, and the conditional expected rates of return on stocks and riskless bills would be equal. However, if dividend growth and consumption growth are conditionally positively correlated, then the right hand side of eq. (19) is greater than one and there is a positive conditional expected equity premium on stocks relative to riskless bills.

III. Relating Asset Returns to Underlying Technological Shocks: A Parametric Example

I have derived expressions for the conditional expected rates of return on stocks and bills in terms of the conditional moments of the distributions of consumption growth and dividend growth. As discussed in section II, these growth rates depend only on the technological shocks e_t and e_{t+1}. In this section I present a parametric example and derive expressions for expected rates of return directly in terms of the distributions of these technological shocks.

Suppose that output is produced according to the following CES production function

$$y_t = A_t[\gamma_t k_t^{\rho} + (1-\gamma_t)n_t^{\rho}]^{1/\rho} \qquad \rho \leq 1 \tag{20}$$

where A_t and γ_t are random productivity shocks comprising the vector e_t in eq. (1). The shocks A_t and γ_t may be correlated with each other. Recall the normalization that $k_t \equiv n_t \equiv 1$. Thus, in equilibrium

$$c_t = y_t = A_t \tag{21a}$$

$$d_t = \gamma_t y_t = \gamma_t A_t \tag{21b}$$

Equations (21a,b) suggest the interpretation of A_t as a scale shock and γ_t as a share shock.

Let $g_{A,t} \equiv A_t/A_{t-1}$ and $g_{\gamma,t} \equiv \gamma_t/\gamma_{t-1}$ and suppose that, conditional on Ω_t, the vector $(\ln g_{A,t+1}, \ln g_{\gamma,t+1})$ is normally distributed.[5] It follows directly from eq. (21a,b) that

$$\ln g_{c,t+1} = \ln g_{A,t+1} \tag{22a}$$

$$\ln g_{d,t+1} = \ln g_{A,t+1} + \ln g_{\gamma,t+1} \tag{22b}$$

Because $\ln g_{c,t+1}$ and $\ln g_{d,t+1}$ are linear combinations of $\ln g_{A,t+1}$ and $\ln g_{\gamma,t+1}$, it follows that, conditional on Ω_t, $\ln g_{c,t+1}$ and $\ln g_{d,t+1}$ are jointly normally distributed.[6] In this case, the expression for the conditional equity premium in ratio form in eq. (19) can be rearranged to yield

[5]The capital share γ_t must lie in [0,1]. With a normal distribution for $\ln(\gamma_{t+1}/\gamma_t)$, γ_{t+1} will not be confined to [0,1]. However, by making the conditional variance of $\ln(\gamma_{t+1}/\gamma_t)$ sufficiently small, the conditional probability that γ_{t+1} is not in [0,1] can be made arbitrarily small.

[6]Suppose that, conditional on Ω_t, $(\ln g_{A,t+1}, \ln g_{\gamma,t+1})'$ is normal with mean μ_t and variance Σ_t where μ_t is a 2 x 1 vector and Σ_t is a 2 x 2 matrix. It follows from eq. (22a,b) that, conditional on Ω_t, $(\ln g_{c,t+1}, \ln g_{d,t+1})$ is $N(M\mu_t, M\Sigma_t M')$ where

$$M = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$\ln \left[E_t(R_{S,t+1})/R_{F,t+1}\right] = \alpha \mathrm{Cov}_t(\ln g_{c,t+1}, \ln g_{d,t+1}) \tag{23}$$

It will be convenient to refer to the expression on the left-hand side of eq. (23) as the conditional equity premium. To motivate the focus on eq. (23) as the conditional equity premium, I will show that the expression in the right-hand side of eq. (23) is equivalent to an expression for the equity premium derived by Grossman and Shiller (1982). First observe from eq. (16) that

$$\ln R_{S,t+1} = \ln \left[(v(s_{t+1})+1)/v(s_t)\right] + \ln g_{d,t+1} \tag{24}$$

Now use eq. (24) and the fact that s_{t+1} is conditionally independent of $\ln g_{c,t+1}$ (so that $\mathrm{Cov}_t(\ln [(v(s_{t+1})+1)/v(s_t)], \ln g_{c,t+1}) = 0$) to calculate the conditional covariance of $\ln R_{S,t+1}$ and $\ln g_{c,t+1}$ as

$$\mathrm{Cov}_t(\ln g_{c,t+1}, \ln R_{S,t+1}) = \mathrm{Cov}_t(\ln g_{c,t+1}, \ln g_{d,t+1}) \tag{25}$$

Substituting eq. (25) into eq. (23) yields

$$\ln \left[E_t(R_{S,t+1})/R_{F,t+1}\right] = \alpha \mathrm{Cov}_t(\ln g_{c,t+1}, \ln R_{S,t+1}) \tag{26}$$

The expression on the right-hand side of eq. (26) is equivalent to eq. (11b) in Grossman and Shiller (1986, p. 202).[7] However, the derivation

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[7]Equation (11b) in Grossman and Shiller (1982) shows the conditional expected excess return between any two assets. Applying that equation to stock returns and the riskless rate of return gives an expression for the conditional equity premium that is equivalent to the right hand side of eq. (26).

in Grossman and Shiller assumes that asset returns and consumption growth are conditionally i.i.d., whereas the derivation in this paper allows for variation in the underlying conditional distributions and hence allows for variation in the conditional distributions of consumption and asset returns.

Now we can express the conditional equity premium directly in terms of the distribution of technological shocks by observing from (22a,b) that

$$\mathrm{Cov}_t(\ln g_{c,t+1}, \ln g_{d,t+1}) = \mathrm{Var}_t(\ln g_{A,t+1}) + \mathrm{Cov}_t(\ln g_{A,t+1}, \ln g_{\gamma,t+1}) \qquad (27)$$

Substituting eq. (27) into eq. (23) yields

$$\ln [E_t(R_{S,t+1})/R_{F,t+1}] = \alpha \mathrm{Var}_t(\ln g_{A,t+1}) + \alpha \mathrm{Cov}_t(\ln g_{A,t+1}, \ln g_{\gamma,t+1}) \qquad (28)$$

Notice that if the scale shock A_{t+1} and the share shock γ_{t+1} are conditionally independent, the conditional equity premium is simply $\alpha \mathrm{Var}_t(\ln g_{A,t+1}) = \alpha \mathrm{Var}_t(\ln g_{c,t+1})$ which is identical to the conditional equity premium that would prevail in the absence of labor income, as in the Mehra-Prescott implementation of the Lucas model.

IV. Comparing the Equity Premium under Markov Regime Switching and Conditional i.i.d Shocks

The Markov regime-switching process was introduced into asset pricing models by Cecchetti, Lam, and Mark (1990,1991) and Kandel and

Stambaugh (1989,1990) in an attempt to improve the ability of these models to account for the empirically observed behavior of asset returns, including second moments of rates of return in addition to first moments of rates of return. In this section I will focus on the first moments of asset returns and will compare expected rates of return under Markov regime switching with expected rates of return under conditionally i.i.d. shocks.

IV.A. The Riskless Rate

In a large sample, the average observed riskless rate equals the unconditional expectation of the conditional riskless rate in eq. (18). Letting $E(R_F)$ denote this unconditional riskless rate yields

$$E(R_F) \quad = \quad E\{[\beta E_t(g_{c,t+1}{}^{-\alpha})]^{-1}\} \tag{29}$$

Now suppose that instead of taking account of the Markov regime-switching process, the riskless rate is calculated under the assumption that all shocks are conditionally i.i.d. so that the conditional distribution of shocks is the same as the unconditional distribution. In this case, the conditional and unconditional riskless rate is

$$E(R_F{}^*) \quad = \quad [\beta E(g_{c,t+1}{}^{-\alpha})]^{-1} \tag{30}$$

where the asterisk denotes a rate of return under the assumption of conditionally i.i.d. growth rates.

It is straightforward to compare the unconditional riskless rates in eqs. (29) and (30) using Jensen's inequality and the law of iterated projections to obtain

$$E(R_F) = E\{[\beta E_t(g_{c,t+1}{}^{-\alpha})]^{-1}\}$$
$$> [\beta E(E_t(g_{c,t+1}{}^{-\alpha}))]^{-1} = [\beta E(g_{c,t+1}{}^{-\alpha})]^{-1} = E(R_F{}^*) \quad (31)$$

Thus, taking account of Markov regime switching increases the riskless rate relative to the case of conditionally i.i.d. shocks. Although this result is derived for the case of constant relative risk aversion, it holds for any utility function $u(c_t)$ with $u' > 0$ and $u'' < 0$.[8]

IV.B. The Equity Premium

In a large sample, the average conditional equity premium defined in eq. (23) will equal the unconditional expected value of eq. (23). Taking the unconditional expectation of both sides of eq. (23) yields an expression for what I will call the unconditional equity premium

$$E\{\ln [E_t(R_{S,t+1}/R_{F,t+1})]\} = \alpha E\{Cov_t(\ln g_{c,t+1}, \ln g_{d,t+1})\} \quad (32)$$

Notice that the unconditional equity premium defined here is not $E(R_{S,t+1} - R_{F,t+1})$ but is the expression on the left-hand side of eq. (32). As explained above, this expression for the unconditional equity premium is consistent with the expression for the equity premium in Grossman and Shiller (1982).

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[8]To prove this more general result, replace $g_{c,t+1}{}^{-\alpha}$ in eq. (31) by $u'(c_{t+1})/u'(c_t)$ to obtain $E(R_F) = E\{[\beta E_t\{u'(c_{t+1})/u'(c_t)\}]^{-1}\} > [\beta E(E_t\{u'(c_{t+1})/u'(c_t)\})]^{-1} = [\beta E(u'(c_{t+1})/u'(c_t))]^{-1} = E(R_F{}^*)$.

To examine the impact of Markov regime switching, it is useful to calculate the unconditional equity premium under the assumption that ($\ln g_{c,t+1}, \ln g_{d,t+1}$) is conditionally i.i.d. normal. A direct application of eq. (32) yields

$$E\{\ln [E_t\{R^*_{S,t+1}/R^*_{F,t+1}\}]\} = \alpha Cov(\ln g_{c,t+1}, \ln g_{d,t+1}) \qquad (33)$$

where Cov(,) denotes the unconditional covariance, and, as earlier, an asterisk denotes a rate of return under the assumption of conditionally i.i.d. growth rates.

To compare the unconditional equity premia on the right hand sides of eqs. (32) and (33), use the fact that for any random variables x_{t+1} and z_{t+1},

$$E\{Cov_t(x_{t+1}, z_{t+1})\} = Cov(x_{t+1}, z_{t+1}) - Cov(E_t\{x_{t+1}\}, E_t\{z_{t+1}\}) \qquad (34)$$

Subtracting eq. (33) from eq. (32) and using eq. (34) yields

$$E\{\ln [E_t\{R_{S,t+1}\}/R_{F,t+1}\}]\} - E\{\ln [E_t\{R^*_{S,t+1}\}/R^*_{F,t+1}\}]\}$$
$$= \alpha Cov(E_t\{\ln g_{d,t+1}\}, E_t\{\ln g_{c,t+1}\}) \qquad (35)$$

Equation (35) shows that allowing for variation in the conditional distribution of ($\ln g_{c,t+1}, \ln g_{d,t+1}$) reduces the unconditional equity premium by $\alpha Cov(E_t\{\ln g_{d,t+1}\}, E_t\{\ln g_{c,t+1}\})$. Provided that the conditional expected values $E_t\{\ln g_{d,t+1}\}$ and $E_t\{\ln g_{c,t+1}\}$ have a positive covariance, allowing for variation in conditional distributions reduces the average equity premium and thus exacerbates the equity premium puzzle. If consumption and dividends are identically equal, as in the Mehra-Prescott application of the Lucas model, then

$$\text{Cov}(E_t(\ln g_{d,t+1}),E_t(\ln g_{c,t+1})) = \text{Var}(E_t(\ln g_{c,t+1})) > 0 \qquad (36)$$

In this case, a Markov regime-switching process reduces the unconditional equity premium relative to the case with conditionally i.i.d. growth rates.

IV.C. **Empirical Examples**

I have derived analytic results in subsections IV.A and IV.B comparing unconditional expected rates of return under Markov regime switching and under conditionally i.i.d shocks. In this subsection, I present three empirical examples taken from the literature to gauge the magnitude of the effect of Markov regime switching on expected rates of return. All three examples are based on lognormal shocks and can be exposited using the notation presented below.

Suppose that there are N possible realizations of the state vector s_t, and let π_t be an N-element vector containing the unconditional probabilities of each state. The state vector $s_t = (\mu_{c,t}, \mu_{d,t}, \sigma_{c,t}, \sigma_{d,t}, \rho_t)$ where

$$\mu_{c,t} = E_t(\ln (c_{t+1}/c_t))$$

$$\mu_{d,t} = E_t(\ln (d_{t+1}/d_t))$$

$$\sigma_{c,t} = [\text{Var}_t(\ln (c_{t+1}/c_t))]^{0.5}$$

$$\sigma_{d,t} = [\text{Var}_t(\ln (d_{t+1}/d_t))]^{0.5}$$

$$\rho_t = \text{correlation}_t (\ln(c_{t+1}/c_t), \ln(d_{t+1}/d_t))$$

Example I uses the Markov regime-switching model for the bivariate process for consumption growth and dividend growth in Cecchetti, Lam, and Mark (1991). Example II uses the Markov regime-switching process for consumption growth in Cecchetti, Lam, and Mark (1990), and Example III uses the Markov regime-switching process for consumption growth in Kandel and Stambaugh (1989). In Examples II and III, only consumption data were used and, following Cecchetti, Lam, and Mark (1990) and Kandel and Stambaugh (1989) the sequence of dividends is assumed to be identical to the sequence of consumption. Table I specifies π_t and s_t for each state in each of the three examples.

I have shown in eq. (35) that the difference between the equity premium under Markov regime switching and the equity premium under conditional i.i.d. lognormality is $\alpha\mathrm{Cov}(E_t\{\ln g_{d,t+1}\}, E_t\{\ln g_{c,t+1}\})$. Table II illustrates the size of this covariance for each of the three empirical examples. In particular, Table II illustrates three statistics for each empirical example. The statistic A is defined as $E\{\mathrm{Cov}_t[\ln(c_{t+1}/c_t), \ln(d_{t+1}/d_t)]\}$; observe from eq. (32) that the unconditional equity premium under Markov regime switching equals αA. The statistic B is defined as the unconditional covariance $\mathrm{Cov}[(\ln(c_{t+1}/c_t), \ln(d_{t+1}/d_t)]$; observe from eq. (33) that the unconditional equity premium under conditional i.i.d. lognormality is αB. The statistic C is defined as $\mathrm{Cov}[E_t\{\ln(c_{t+1}/c_t)\}, E_t\{\ln(d_{t+1}/d_t)\}]$; note that $C = B - A$ so that the difference in the equity premium under Markov regime switching and conditional i.i.d. lognormality is αC.

Table II shows that for Example I the covariance of $E_t\{\ln(c_{t+1}/c_t)\}$ and $E_t\{\ln(d_{t+1}/d_t)\}$ is large enough so that there is a substantial difference between the equity premium under Markov regime switching and under conditionally i.i.d. growth rates. As indicated in the final column, the equity premium under Markov regime switching is only 46.5% of the equity premium under conditional i.i.d lognormality. In contrast, the equity premium is virtually the same under Markov regime switching and under conditional i.i.d. lognormality in Example III. The results for Example II are intermediate between those of Examples I and III.

Under a Markov regime-switching process with conditionally lognormal shocks, the unconditional distribution will not be lognormal. If one were to calculate expected rates of return under the assumption that the shocks are conditionally i.i.d., then the (non-normal) unconditional distribution should be used to calculate the moments in eqs. (17) and (18). These calculations for unconditional expected rates of return are shown in Table III along with the unconditional rates of return using Markov regime switching. The four panels of Table III present unconditional expected rates of return for different pairs of preference parameters (α,β). The bottom panel of Table III uses the values of the preference parameters $\alpha = 28.55$ and $\beta = 0.997$ used by Kandel and Stambaugh (1989).[9] With these values of the preference parameters Kandel and Stambaugh compute an unconditional riskless rate of return $E\{r_F\} = 0.791\%$ and an unconditional rate of return on aggregate wealth $E\{r_S\} = 4.426\%$; these unconditional rates of return

[9] As noted in Table III, Kandel and Stambaugh (1989) use $\beta = 0.99975$ for monthly data which is equivalent to $\beta = 0.997$ on annual data.

are very close the values of $E(r_F) = 0.844\%$ and $E(r_S) = 4.480\%$ reported in Table III.[10,11]

The expected rates of return in Table III illustrate the theoretical results derived earlier in this section. The unconditional riskless rate under Markov regime switching ($E(r_F)$) is always higher than the unconditional riskless rate calculated using the unconditional distribution ($E(r_F^*)$). The unconditional equity premium using the unconditional distribution is always greater than the unconditional equity premium using the Markov regime-switching process.[12] Notice that, consistent with the results in Table II, using a Markov regime-switching process rather than the unconditional distribution leads to a

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[10] $E(r_F)$ is the unconditional *net* riskless rate which equals $E(R_F) - 1$. Similarly, $E(r_S)$ is the unconditional *net* rate of return on stock, which $E(R_S) -1$.

[11] There is no check on the calculation of the unconditional expected rates of return for Examples I and II. Cecchetti, Lam, and Mark (1991), on which Example I is based, assumes that s_t is not in the information set Ω_t, and thus expected rates of return calculated in that paper will differ from those presented in Table III. Cecchetti, Lam, and Mark (1990), on which Example II is based, does not report unconditional expected rates of return.

[12] For all of the parameter values in Table III, $E(R_S) > E(R_S^*)$. However, this result does not hold in general. Consider the special case in which there is no labor income so that $c_t \equiv d_t$ and hence $g_{c,t+1} \equiv g_{d,t+1}$. Suppose that conditional on s_t, $\ln g_{c,t+1}$ is $N(\mu_t, \sigma^2)$ with the a time-varying conditional mean μ_t and a constant conditional variance σ^2. Under these assumptions, equation (17) implies $E_t(R_{S,t+1}) = \beta^{-1}\exp[\alpha(1-\alpha)\sigma^2][E_t(g_{c,t+1})]^\alpha$, and the unconditional expected rate of return is $E(R_S) = \beta^{-1}\exp[\alpha(1-\alpha)\sigma^2]E([E_t(g_{c,t+1})]^\alpha)$. Alternatively, if $\ln g_{c,t+1}$ is conditionally i.i.d., the unconditional expected rate of return on stocks is $E(R_S^*) = \beta^{-1}\exp[\alpha(1-\alpha)\sigma^2][E(g_{c,t+1})]^\alpha$. It follows directly from Jensen's inequality that $E(R_S) > (=) (<) E(R_S^*)$ as $\alpha > (=) (<) 1$, so that Markov regime switching can raise, leave unchanged, or reduce the unconditional expected rate of return on stocks relative to the unconditional expectation under the assumption of conditionally i.i.d. growth rates of consumption.

substantial difference in expected rates of return using the stochastic process in Example I but makes a rather trivial difference using the stochastic process in Example III.

V. Conclusion

This paper has presented an alternative method for calculating conditional and unconditional expected rates of return in the presence of shocks generated by a Markov regime-switching process. The method is attractive because it provides simple closed-form expressions for expected rates of return and applies to a more general class of Markov regime-switching processes than has been examined in the asset pricing literature. The numerical application of these closed-form expressions does not require calculation of equilibrium equity prices. Nor, given the unconditional distribution of the state vector, does the method require use of the state transition probabilities.

Analysis of the closed-form expressions for expected rates of return indicates that taking account of Markov regime switching in an asset pricing model exacerbates both the equity premium puzzle and the risk-free rate puzzle. Specifically, for given preferences, a Markov regime-switching process increases the unconditional riskless rate of return, and--under conditional lognormality and constant relative risk aversion--tends to reduce the unconditional equity premium relative to the values that would emerge from an asset pricing model using the unconditional distribution of shocks. More precisely, if the conditional expected growth rates of consumption and dividends are positively correlated with each other (as they must be in models that don't distinguish between consumption and dividends), then introducing a Markov regime-switching process reduces the unconditional equity premium under conditional lognormality and constant relative risk aversion. Empirical examples in the literature illustrate that the effects of introducing a Markov regime-switching process may be either large or small depending on the stochastic process generating shocks.

Should one conclude that Markov regime-switching processes are to be avoided in asset pricing models because they exacerbate the equity premium puzzle and the risk-free rate puzzle? No, if the underlying shocks are well described by a Markov regime-switching process, then such a process should be incorporated in asset pricing models. However, the results of this paper provide advance notice that such models will be less able to explain empirical first moments of rates of return for conventionally accepted values of the preference parameters.

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Table I
Specification of Stochastic Process

Example I
Consumption and Dividends from Cecchetti, Lam, and Mark (1991)

state	π_t	elements of state vector s_t				
		$\mu_{c,t}$	$\mu_{d,t}$	$\sigma_{c,t}$	$\sigma_{d,t}$	ρ_t
1	0.95788	0.0218	0.0255	0.0333	0.1030	0.2732
2	0.04212	-0.0614	-0.2953	0.0333	0.1030	0.2732

Example II
Consumption from Cecchetti, Lam, and Mark (1990)

state	π_t	elements of state vector s_t				
		$\mu_{c,t}$	$\mu_{d,t}$	$\sigma_{c,t}$	$\sigma_{d,t}$	ρ_t
1	0.95181	0.0228	0.0228	0.0320	0.0320	1.0
2	0.04819	-0.0698	-0.0698	0.0320	0.0320	1.0

Example III*
Consumption from Kandel and Stambaugh (1989)

state	π_t	elements of state vector s_t				
		$\mu_{c,t}$	$\mu_{d,t}$	$\sigma_{c,t}$	$\sigma_{d,t}$	ρ_t
1	0.08374	0.00111	0.00111	0.00915	0.00915	1.0
2	0.08899	0.00153	0.00153	0.00915	0.00915	1.0
3	0.08374	0.00194	0.00194	0.00915	0.00915	1.0
4	0.15899	0.00111	0.00111	0.01023	0.01023	1.0
5	0.16908	0.00153	0.00153	0.01023	0.01023	1.0
6	0.15899	0.00194	0.00194	0.01023	0.01023	1.0
7	0.08374	0.00111	0.00111	0.01144	0.01144	1.0
8	0.08899	0.00153	0.00153	0.01144	0.01144	1.0
9	0.08374	0.00194	0.00194	0.01144	0.01144	1.0

*The parameters of the stochastic process for Example III are based on monthly data.

Table II
Decomposition of Unconditional Expectation of Conditional Covariance: Implications for the Unconditional Equity Premium

$A = E\{Cov_t[\ln(c_{t+1}/c_t), \ln(d_{t+1}/d_t)]\}$

$B = Cov[(\ln(c_{t+1}/c_t), \ln(d_{t+1}/d_t)]$

$C = Cov[E_t\{\ln(c_{t+1}/c_t)\}, E_t\{\ln(d_{t+1}/d_t)\}]$

Note: $A = B - C$

Unconditional Equity Premium

Markov regime switching: $E\{\ln[E_t\{R_{S,t+1}\}/R_{F,t+1}]\} = \alpha A$

Conditional i.i.d. Shocks: $E\{\ln[E_t\{R_{S,t+1}\}/R_{F,t+1}]\} = \alpha B$

	A	B	C	A/B
Example I	0.0937%	0.2014%	0.1077%	46.5%
Example II	0.1024%	0.1417%	0.0393%	72.3%
Example III*	0.1272%	0.1273%	0.0001%	99.9%

*The entries for A, B, and C for Example III have been multiplied by 12 to annualize monthly returns.

Table III

Unconditional Rates of Return
under
Markov Regime Switching and Unconditional Distribution

$E(r_F) \equiv E(R_F) - 1$ = unconditional riskless rate under Markov regime switching

$E(r_S) \equiv E(R_S) - 1$ = unconditional stock return under Markov regime switching

$E(r_F^*) \equiv E(R_F^*) - 1$ = unconditional riskless rate under unconditional distribution

$E(r_S^*) \equiv E(R_S^*) - 1$ = unconditional stock return under unconditional distribution

	preference parameters: α = 5.0 and β = 1.0			
	$E(r_F)$	$E(r_S)$	$E(r_F^*)$	$E(r_S^*)$
Example I	8.405%	8.914%	7.642%	8.763%
Example II	8.674%	9.232%	7.597%	8.402%
Example III[a]	7.596%	8.236%	7.592%	8.233%

	preference parameters: α = 10.0 and β = 1.0			
	$E(r_F)$	$E(r_S)$	$E(r_F^*)$	$E(r_S^*)$
Example I	14.852%	15.934%	11.553%	14.177%
Example II	15.867%	17.059%	11.173%	12.971%
Example III[a]	12.027%	13.313%	12.013%	13.300%

	preference parameters: α = 20.0 and β = 1.0			
	$E(r_F)$	$E(r_S)$	$E(r_F^*)$	$E(r_S^*)$
Example I	19.660%	21.924%	4.966%	11.480%
Example II	23.334%	25.886%	2.120%	6.077%
Example III[a]	11.286%	13.855%	11.218%	13.792%

	preference parameters: α = 28.55 and β = 0.997			
	$E(r_F)$	$E(r_S)$	$E(r_F^*)$	$E(r_S^*)$
Example I	14.398%	17.500%	-15.696%	-6.141%
Example II	21.024%	24.615%	-22.253%	-17.243%
Example III[a,b]	0.845%	4.481%	0.677%	4.324%

[a] The entries for Example III have been multiplied by 12 to express monthly returns on an annual basis.
[b] For this example, β = 0.99975 on monthly data, which is equivalent to β = 0.997 on annual data.