A TEST OF THE COX, INGERSOLL, AND ROSS MODEL OF THE TERM STRUCTURE

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8-92

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Abstract

We present a test of the theory of the term structure developed by Cox, Ingersoll, and Ross (CIR). The econometric method uses Hansen's Generalized Method of Moments and exploits the probability distribution of the single state variable that determines real bond prices. The approach avoids problems due to measurement errors in bond prices; it does not employ data on aggregate consumption; and it enables the estimation of a continuous time model based on discretely-sampled data. The tests indicate that the model for real indexed bonds that underlies all the alternative specifications in CIR performs reasonably well when confronted with short-term Treasury Bill data. The parameter estimates indicate that term premiums are positive and that the term structure of indexed bonds can admit several shapes. However, we find it difficult to rationalize the sample serial correlation in Treasury Bill returns using our estimates of the CIR parameters.

Revised: April, 1992. Comments are welcome.

†The Wharton School of The University of Pennsylvania, Philadelphia, PA 19104. We are grateful to Doug Breeden, John Cox, Phil Dybvig, Steve Heston, Andrew Lo, Craig MacKinlay, George Pennacchi, Ken Singleton, Rob Stambaugh and Suresh Sundaresan for their comments, and to Tong-sheng Sun and Robert Whitelaw for research assistance. We thank participants of seminars at Carnegie-Mellon University, Columbia University, Dartmouth College, the French Finance Association, INSEAD, London Business School, MIT, NBER, Northwestern University, Stanford University, the University of Chicago, and the University of Illinois. Financial support was provided by the Center for the Study of Futures Markets at Columbia University and the Geewax-Terker Program in Investments. This research was started while Gibbons was on a fellowship provided by Batterymarch Financial Management. We are responsible for any errors.

An earlier version of this paper was titled "The Term Structure of Interest Rates: Empirical Evidence."
RunTEX: May 8, 1992

1 Introduction

The relation between the yields on default-free loans and their maturities has long been a topic of interest to financial economists. The focus of the early work on the term structure of interest rates was on the relation between the expected interest rate to prevail at a future date and the implied forward rate embedded in the yield curve.[1] The earliest empirical studies focused on the historical shapes of the yield curves and their relation to stages of the business cycle.

The Intertemporal Capital Asset Pricing Model pioneered by Merton (1973) and the rational expectations equilibrium model due to Lucas (1978) have led researchers to consider equilibrium models of the term structure of interest rates. The term structure model developed in Cox, Ingersoll, and Ross (1985a,b) represents an equilibrium specification that is completely consistent with stochastic production and with changing investment opportunities. This model provides testable implications for the prices of bonds whose payoffs are denominated in real terms — closed-form expressions are provided for the endogenously derived real prices in terms of a single state variable (the instantaneously riskless real rate). The evolution of this variable is determined endogenously, and this permits empirical testing of the pricing implications as well as the restrictions on the dynamics of the term structure. The availability of such a model of the term structure also permits ready application to the pricing of options and futures contracts on financial assets and to the management of bond portfolios.

In this paper, we conduct an empirical test of the Cox, Ingersoll, and Ross (1985b; henceforth "CIR") model of the term structure. Our method has the following advantages. First, we formulate a test of the implications from a continuous-time model based on discretely sampled data, and this test is designed to avoid misspecification arising from this temporal aggregation. Second, while our test centers on a stochastic Euler equation similar to tests in other studies (e.g., Hansen and Singleton (1982)) that employ Hansen's (1982) Generalized Method of Moments (henceforth, "GMM"), we avoid the use of data on aggregate consumption. This enables us to avoid many of the measurement problems that accompany the use of these series. Third, our econometric procedure is such that no stochastic specification of the process for the aggregate price level is necessary. Under an assumption made in CIR concerning the effect of the aggregate price level on bond prices, we test a necessary implication for a broad class of pricing models which differ by the assumptions regarding the process

[1] For a review of traditional hypotheses regarding the term structure, see Cox, Ingersoll, and Ross (1981). Breeden (1986) also provides a synthesis of several strands in the literature. Melino (1986) provides a review of the evidence, focusing on the expectations hypothesis.

for inflation. Fourth, our econometric method is fully consistent with the underlying theory even though the investment opportunity set is not constant over time. There is increased interest in asset pricing when conditional distributions are not constant; however, much of the empirical work is based on theory that is not completely specified as to why some moments are fixed while others are changing (for example, see Gibbons and Ferson (1985), and Ferson, Kandel and Stambaugh (1987)). By contrast, our econometric method requires no additional assumption beyond those maintained in the theory.

Other empirical research has examined the CIR model. Most of this work has focused on the nominal prices of U.S. Government Securities. Using a general framework, Stambaugh (1988) relies on nominal Treasury Bill data to reject a single latent variable model of conditional expected returns, but he finds that the data are consistent with a model with two or three latent variables. Brown and Dybvig (1986) have examined the fit of nominal Treasury Bill prices to CIR's single state formulation,[2] and Pearson and Sun (1990) extended Brown and Dybvig's method to CIR's models with explicit processes for inflation. Heston (1991) also uses the CIR model to find the nominal price of a nominal Treasury Bond; however, his statistical method is a modification of the econometric approach that we develop here.[3]

The inability of a single state variable model to find the nominal value of a nominal Government Bond has led to the development of models with multiple state variables.[4] For example, Brennan and Schwartz (1982) and Nelson and Schaefer (1983) consider some two state variable models, where the factor risk premiums are specified exogenously. Multiple state variable models of the term structure are of considerable interest, especially when one attempts to price nominal bonds, but it is not clear to us that one must abandon the study of single state variable specifications of the term structure of real rates. In fact, it might well be the case that the term structure of rates embedded in indexed bonds is adequately described by one forcing variable while the behavior of nominal bonds of various maturities is driven by a vector of forcing variables. One obvious disadvantage of multiple state variable formulations for the pricing of indexed bonds is that this makes the valuation problem complicated and often intractable — and this leads us to examine the CIR model as a parsimonious, and hopefully useful, description of the term structure.

[2]Using an econometric method similar to that in Brown and Dybvig (1986), Brown and Schaefer (1990) examine the fit of indexed gilts in the UK to CIR's single state model.

[3]Heston's (1991) specification allows him to work with bond returns in excess of the return on a short-term Treasury Bill; this procedure does not rely on inflation data. However, he approximates an instantaneous holding period with a discrete holding period, and he doesn't identify all of CIR's parameters.

[4]Constantinides (1991) models the nominal term structure in the spirit of Brown and Dybvig (1986). His theoretical development makes his "SAINTS" model amenable to the econometric framework that we suggest in later sections.

The plan of the paper is as follows. Section 2 lays out a general framework for real and nominal bond prices, and Section 3 summarizes the CIR model. In Section 4 we discuss the design of the econometric method; Section 5 describes the data. Section 6 contains the main empirical results and summarizes the successes and failures of the CIR model. Section 7 concludes the paper.

2 A General Framework for Nominal and Real Bond Prices

In this Section we describe a general framework for pricing real and nominal discount bonds of various maturities. The discussion here applies, strictly speaking, to the general treatment in CIR (1985a). The development of the empirical test relies heavily on the arguments in this Section.

The framework within which CIR develop their continuous-time valuation model can briefly be described as follows: there are infinitely-lived and identical individuals who maximize the discounted expected utility of consumption of a single good, which is produced stochastically from a finite number of technologies, each exhibiting constant stochastic returns to scale. The individuals' wealths are totally invested in these firms, and they each choose a consumption rule and an investment allocation rule in maximizing their expected utility. The values of the firms in the economy evolve continuously as a vector Itô process, whose drift rate and covariance matrix depend on the evolution of a vector of state variables. The evolution of this vector of state variables is itself governed by a system of stochastic differential equations — therefore, the future investment opportunities in this model are stochastic. The environment is competitive and frictionless; a riskless asset (which is in zero net supply) and the firms' shares are available for continuous trading with no transaction costs or taxes.

The CIR model uses additional assumptions that we discuss later; the above framework is sufficient to permit a simple exposition of the valuation model. From the first-order conditions for the representative individual's maximizing problem, it follows that the current (date t) real price of a claim that pays one unit of the consumption good at date $t+\tau$, written $P_t(\tau)$ is given by

$$P_t(\tau) = E_t\left[\delta^\tau \frac{U'(\tilde{c}(t+\tau))}{U'(c(t))}\right] . \tag{1}$$

In (1), $U(c(s))$ is the utility of the optimal consumption flow $c(s)$ at date s, δ is the rate of time preference, and $E_t[\cdot]$ denotes the conditional expectation where the t-subscript reflects

the conditioning information set. Note that this is the expected marginal rate of substitution, and it corresponds to the real price of an indexed (or real) bond that is default-free. Denoting by $\pi(s)$ the money price of one unit of the consumption good at date s, the real price at date t of a nominal bond that pays one dollar at date $t+\tau$ is

$$N_t(\tau) = E_t\left[\delta^\tau \left\{\frac{U'(\tilde{c}(t+\tau))}{U'(c(t))}\right\}\frac{1}{\tilde{\pi}(t+\tau)}\right], \tag{2}$$

which is the expected real payoff weighted by the marginal rates of substitution. Hence the nominal price of a nominal unit discount bond, $N_t^*(\tau)$, can be written

$$N_t^*(\tau) \equiv \pi(t)N_t(\tau) = E_t\left[\delta^\tau \left\{\frac{U'(\tilde{c}(t+\tau))}{U'(c(t))}\right\}\frac{\pi(t)}{\tilde{\pi}(t+\tau)}\right]. \tag{3}$$

We can rewrite (3)

$$\begin{aligned} N_t^*(\tau) &= P_t(\tau)E_t\left(\frac{\pi(t)}{\tilde{\pi}(t+\tau)}\right) \\ &\quad + \delta^\tau Cov_t\left(\frac{U'(\tilde{c}(t+\tau))}{U'(c(t))}, \frac{\pi(t)}{\tilde{\pi}(t+\tau)}\right), \end{aligned} \tag{4}$$

where $Cov_t(\cdot,\cdot)$ is the covariance operator conditional on information at time t. Relations (1) and (4) give the prices of real and nominal discount bonds as a function of maturity. From these equations we can readily deduce conventional yield curves in real and nominal terms. It is important to note from (1) that the real yield

$$y_t(\tau) \equiv -\ln\left(P_t(\tau)\right)/\tau \tag{5}$$

is observable and achievable in a τ period strategy only if there is an indexed bond available to investors. The availability of a nominally riskless pure discount bond ensures, however, that the nominal yield

$$y_t^*(\tau) = -\ln\left(N_t^*(\tau)\right)/\tau \tag{6}$$

is observable and achievable in a τ period strategy.

Every model of the nominal term structure must specify the conditional moments in (4). One way to achieve this is to put sufficient structure on the model to specify the joint, conditional distribution of the marginal rate of substitution $U'(\tilde{c}(t+\tau))/U'(c(t))$ and the inverse of the inflation rate $\pi(t)/\tilde{\pi}(t+\tau)$. The specification of the joint distribution (between the marginal rate of substitution and the inverse of the inflation rate) calls for an explanation of the precise way in which money enters the economic environment.

CIR (1985b) implicitly assume[5] in their nominal bond pricing examples that the covariance in (4) is zero. Since our work is a test of the CIR framework, we follow CIR in

[5]See CIR (1985b, page 402). Benninga and Protopapadakis (1983) discuss this assumption in the context of the Fisherian hypothesis in a discrete time framework.

assuming that changes in the price level have no effect on the real variables in the model.[6] The resulting expression for the nominal discount bond price is therefore

$$N_t^*(\tau) = P_t(\tau) E_t \left[\frac{\pi(t)}{\tilde{\pi}(t+\tau)} \right] , \tag{7}$$

which readers will recognize can be transformed (by taking logarithms) into a version of the Fisherian hypothesis on interest rates.

We can employ relation (7) to specify the real return on a nominal bond over any holding period. The gross real return on a nominal discount bond, held from date t to date $t+u$, is defined by

$$\begin{aligned} {}_t\tilde{R}_{t+u}(\tau) &\equiv \frac{\tilde{N}^*_{t+u}(\tau-u)/\tilde{\pi}(t+u)}{N_t^*(\tau)/\pi(t)} \quad \text{for } u \leq \tau \\ &= \frac{\tilde{P}_{t+u}(\tau-u)}{P_t(\tau)} \times \frac{E_{t+u}(1/\tilde{\pi}(t+\tau))}{E_t(1/\tilde{\pi}(t+\tau))} , \end{aligned} \tag{8}$$

where the first fraction is the gross return on an *indexed* discount bond with maturity τ, held from t to $t+u$. Relation (8) is the object that is at the heart of our computations. The numerators of the two fractions on the right hand side (RHS) of relation (8) depend on the information set at date $t+u$. By taking the date t conditional expectation of (8) and recalling the CIR approach where the stochastic process for inflation is exogenous and independent of the pricing of indexed bonds, we can see that the expected gross, real, holding period return on a nominal discount bond is equal to the conditional expectation of the return on its (hypothetical) indexed counterpart:

$$E_t \left({}_t\tilde{R}_{t+u}(\tau) \right) = E_t \left(\frac{\tilde{P}_{t+u}(\tau-u)}{P_t(\tau)} \right), \quad \text{for } u \leq \tau. \tag{9}$$

From relation (8) we can also compute the products of the gross, real returns to discount bonds of various maturities. These computations lead to specifications of comoments, which are closely related to autocovariances and serial cross-covariances of returns. From (8) we can write, for $0 < u \leq v < w \leq v + \tau_2$ and $u \leq \tau_1$,

$$\begin{aligned} {}_t\tilde{R}_{t+u}(\tau_1) \times {}_{t+v}\tilde{R}_{t+w}(\tau_2) &= \frac{\tilde{P}_{t+u}(\tau_1-u)}{P_t(\tau_1)} \cdot \frac{\tilde{P}_{t+w}(v+\tau_2-w)}{P_{t+v}(\tau_2)} \\ &\times \frac{E_{t+u}(1/\tilde{\pi}(t+\tau_1))}{E_t(1/\tilde{\pi}(t+\tau_1))} \cdot \frac{E_{t+w}(1/\tilde{\pi}(t+v+\tau_2))}{E_{t+v}(1/\tilde{\pi}(t+v+\tau_2))} . \end{aligned} \tag{10}$$

If $v \geq u$, then the holding periods $[t, t+u]$ and $[t+v, t+w]$ are nonoverlapping. We now combine nonoverlapping holding periods with the CIR approach where the stochastic

[6] An alternative model is developed and tested in Pennacchi (1991), where the instantaneous real rate and expected inflation are found to be correlated.

process for inflation is exogenous and independent of the pricing of indexed bonds. These two assumptions allow us to write the conditional expectation of the product of the gross, real returns on the nominal discount bonds as the conditional expectation of the product of the gross, real returns on their indexed counterparts:

$$E_t\left({}_t\tilde{R}_{t+u}(\tau_1)\cdot {}_{t+v}\tilde{R}_{t+w}(\tau_2)\right) = E_t\left(\frac{\tilde{P}_{t+u}(\tau_1-u)}{P_t(\tau_1)}\cdot\frac{\tilde{P}_{t+w}(v+\tau_2-w)}{P_{t+v}(\tau_2)}\right). \tag{11}$$

Equation (11) followed from equation (10) because the first two factors are uncorrelated with the last two factors in (10). Furthermore, the last two fractions in (10) involve the same random variable in the numerator and the denominator — leading to cancellation when iterating expectations over a coarser information set. We can compute conditional expectations of the product of three (or more) gross, real, holding period returns by extending the above arguments, and keeping the holding periods nonoverlapping.[7]

If we knew the relevant information upon which the expectations in (9) and (11) are based, then these equations provide a natural basis for econometric work.[8] However, if the state variables in the relevant information set are unobservable, then we need to pursue an alternative path to develop the econometric framework, to which we now turn.

It is easy to see in relations (9) and (11) that, by the law of iterated expectations, the unconditional expectation of the corresponding quantities would also be equal. First, take the unconditional expectation of (9):

$$E\left\{E_t\left({}_t\tilde{R}_{t+u}(\tau)\right)\right\} = E\left\{E_t\left(\frac{\tilde{P}_{t+u}(\tau-u)}{P_t(\tau)}\right)\right\}, \quad \text{for } u\le\tau \tag{12}$$

$$\equiv \Phi_1(u,\tau;\beta)\,. \tag{13}$$

In the RHS of (12) the real indexed bond prices $P_t(\tau)$ and $P_{t+u}(\tau-u)$ depend on conditioning information (the state variables) at dates t and $t+u$ respectively. Knowledge of the functional form of these real indexed bond prices, together with knowledge of the probability densities of the state variables allows us to pass to relation (13), where the unconditional expectation has been taken. The resulting function $\Phi_1(u,\tau;\beta)$ is the unconditional first moment of the real holding period return on a nominal bond, and β is a vector of parameters.

[7] If $v<u$, then the holding periods overlap. This will lead to nonzero correlation between $E_{t+u}(1/\tilde{\pi}(t+\tau_1))$ and $E_{t+w}(1/\tilde{\pi}(t+v+\tau_2))$. Without an explicit process for inflation, we cannot calculate this correlation. Thus, restricting our attention to nonoverlapping holding periods is an important ingredient in our econometric modeling.

[8] If the relevant information were known, then equations (9) and (11) could generate a set of orthogonality conditions much like that in Hansen and Singleton (1982). Of course, this presumes that the relevant information is observed and that the proper specification for the impact of this information on bond returns and products of bond returns is available.

Next, take the unconditional expectation of (11):

$$E\left\{E_t\left({}_t\tilde{R}_{t+u}(\tau_1)\cdot\ {}_{t+v}\tilde{R}_{t+w}(\tau_2)\right)\right\} = E\left\{E_t\left(\frac{\tilde{P}_{t+u}(\tau_1-u)}{P_t(\tau_1)}\cdot\frac{\tilde{P}_{t+w}(v+\tau_2-w)}{P_{t+v}(\tau_2)}\right)\right\} \quad (14)$$

$$\equiv \Phi_2(u,v,w,\tau_1,\tau_2;\beta)\,. \quad (15)$$

In the RHS of (14) the real indexed bond prices $P_s(\cdot)$ depend on conditioning information (the state variables) at dates $s=t,t+u,t+v$ and $t+w$. Again, knowledge of the functional form of these real indexed bond prices, together with knowledge of the probability densities of the state variables, allows us to pass to relation (15), where the unconditional expectation has been taken. The resulting function $\Phi_2(u,v,w,\tau_1,\tau_2;\beta)$ is the unconditional second moment of the product of two nonoverlapping, real, holding period returns on nominal bonds. It is easy to see that we can extend these calculations to compute unconditional moments of higher order.[9]

The functions $\Phi_1(\cdot)$ and $\Phi_2(\cdot)$ do not depend on the unobservable state variables, because these variables were integrated out as part of the transition from conditional to unconditional expectations.[10] In models where explicit formulas are available for these functions, relations (13) and (15) provide a basis for empirical tests. This is the method pursued in this paper.

Before we study the exact specification of the CIR model, it is useful to recognize that all the examples of nominal discount bill valuation in CIR (1985b) employ the same model for the real price $P_t(\cdot)$ and treat the process for inflation as exogenous and independent of $P_t(\cdot)$. These examples all share the common testable implication for the real price of a riskless real bond. We derive a test of the central implication of the CIR model, which is about the term structure of real prices of bonds with real payoffs, in a way that is robust to misspecifications of the process on the price level. Of course, our test is a necessary implication for any model of the term structure which uses the same real price $P_t(\cdot)$ as in CIR even if the alternative model differs from CIR in the process for inflation.

3 The CIR Model of the Term Structure

In the principal model presented in their paper, CIR (1985b) derive the formula for the real price of an indexed bond assuming a single state variable $x(t)$ and logarithmic preferences.

[9]The functions $\Phi_1(\cdot)$ and $\Phi_2(\cdot)$ are computed in the Appendix using the specific distributional results in CIR. Throughout the rest of the paper we will use Φ_i to represent an expectation computed under the CIR framework.

[10]Note that the arguments of $\Phi_1(\cdot)$ and $\Phi_2(\cdot)$ depend only on the maturities of the nominal bonds being considered, and not on calendar time.

In their framework, $x(t)$ follows an autoregressive process with a conditional variance of the instantaneous change proportional to x. Further, the means, variances and covariances of the rates of return on the production technologies are proportional to the level x.

CIR then show that the instantaneous riskless rate of interest, $r(t)$, which corresponds to the expected rate of change of the marginal utility of wealth, has a one-to-one correspondence with the state variable $x(t)$.[11] Hence, the stochastic process for $r(t)$ inherits the properties of the process for $x(t)$; its process can be written as

$$dr = \kappa(\theta - r)\,dt + \sigma\sqrt{r}\,dz\,, \tag{16}$$

where $\{z(t), t > 0\}$ is a standard Wiener process; κ is the speed of adjustment of r to its long-run mean, θ; and σ is a positive scalar. The stochastic differential equation for the instantaneous riskless rate implies the date t conditional distribution of $r(s)$, $s > t$, is a transform of a noncentral χ^2 and the steady state distribution is a gamma (see Feller (1951)). The CIR pricing formula for the real unit discount bond is

$$P_t(\tau) = A(\tau)\,\exp\{-B(\tau)r(t)\}\,, \tag{17}$$

where $A(\cdot)$ and $B(\cdot)$ are given by

$$A(\tau) \equiv \left[\frac{2\gamma\exp\{(\kappa+\lambda+\gamma)\tau/2\}}{D(\tau)}\right]^{2\kappa\theta/\sigma^2}\,, \tag{18}$$

$$B(\tau) \equiv \frac{2\left[\exp(\gamma\tau)-1\right]}{D(\tau)}\,, \tag{19}$$

$$D(\tau) \equiv \{\kappa+\lambda+\gamma\}\{\exp(\gamma\tau)-1\} + 2\gamma\,, \tag{20}$$

and

$$\gamma \equiv \sqrt{(\kappa+\lambda)^2 + 2\sigma^2}\,. \tag{21}$$

λ is the parameter which determines the risk premium — this follows from the fact that the instantaneous expected return on any default free bond in the CIR model is

$$r + \frac{\lambda r}{P_t(\tau)}\frac{\partial P_t(\tau)}{\partial r} = r - \lambda B(\tau) r\,. \tag{22}$$

The risk premium is positive whenever $\lambda < 0$. Other comparative statics properties of the discount bond price are given in CIR (1985b, p. 393).

While the parameters θ, κ, λ and σ have a natural role to play in the context of CIR's pricing model, we have adopted an alternative parametrization that we find more intuitive

[11] In what follows we use the term "state" variable for r, even though that applies strictly to x.

and more convenient for the numerical work that follows. We transform κ to a parameter that has a natural interpretation from discrete time autoregressive models. We eliminate λ by focusing on a parameter describing the asymptote of the term structure. The scalar parameter σ is replaced by a parameter which measures the standard deviation of the steady state distribution for r. The transformation will allow the reader to interpret the model for discrete time intervals. Furthermore, these parameters are related to the yield curve, which is a more familiar object. It is noteworthy that we do estimate the parameters using the implications of the continuous time process for discrete sampling intervals — we do not rely on approximations of instantaneous holding periods for returns.

Here is a brief description of the transformed parameters. We define an autoregressive parameter for the interest rate process, ρ, given by $\rho \equiv \exp(-\kappa/12)$, instead of working with κ directly. The parameter ρ is the coefficient of a regression of the intercept of the yield curve for indexed bonds on the intercept of last month's yield curve. It is easier to interpret the unconditional standard deviation of the intercept of the yield curve, σ_U,

$$\sigma_U \equiv \sqrt{\frac{\sigma^2\theta}{2\kappa}}\,, \tag{23}$$

rather than σ. Instead of using λ directly, we focus on the effect of λ on the long-run yield (y_∞), which is independent of the level of the state variable and is the asymptote of the CIR yield curve as maturity increases. The transformation to this long-run yield is:

$$y_\infty = \frac{2\kappa\theta}{\kappa + \lambda + \gamma}\,. \tag{24}$$

We find the long-run mean, θ, of the intercept of the yield curve easy to interpret, and we have not transformed this parameter. In summary, the vector of parameters that we estimate for the CIR model is β, where

$$\beta \equiv (\ \theta \quad \rho \quad y_\infty \quad \sigma_U\ )\,. \tag{25}$$

This transformation of the parameters also has the property that given β we can invert to find the original CIR parameters.[12]

4 The Econometric Method

Recognizing the definition of a yield on an indexed bond given in equation (5), equation (17) implies:

$$y_t(\tau) = -\frac{\log(A(\tau))}{\tau} + \frac{B(\tau)}{\tau}\, r(t)\,, \tag{26}$$

[12]Straightforward algebra will verify that the transformation is one-to-one.

which is linear in the unobserved variable. This implies that the correlation between the yields of indexed bonds of different maturities are all unity. Therefore, applying the CIR model (viz., relation (17)) to nominal data on nominal bonds leads to a rejection of the model, for casual empiricism (ignoring the effects of measurement error) suggests that nominal yields are not perfectly correlated.

A sufficient history of properly measured prices of indexed bonds would, however, enable a direct test of the CIR model. Brown and Schaefer (1990) test the CIR model with data on indexed bonds in the UK.[13] Although the state variable is not observed in this case, nonlinear cross-sectional regressions employing (26) permits the estimation of some, but not all, of the underlying parameters. Their procedure "inverts" the CIR formula for r and some of the other parameters from a cross-section of prices, just as if we backed out the stock price and the implied volatility using the Black-Scholes model.[14]

While these nonlinear cross-sectional regressions are tractable, they cannot connect directly the estimated parameters with the time series properties of the bond prices. For example, θ, ρ, and σ_U are not linked to the sample mean, autocorrelation, and standard deviation of r, which is estimated over time from each cross-sectional regression. We will integrate the dynamic properties of the CIR model with its cross-sectional implications for bonds of differing maturities. The use of CIR's specification of the stochastic evolution of the state variable lends the test considerable sharpness, for we are able to exploit this information in testing the overidentifying restrictions and arriving at parameter estimates.

Our objective is to test the CIR model of indexed bond prices from data on nominal bonds. While our test will be robust to measurement error, we do not incorporate an explicit model of measurement error to rationalize why real bond prices are not perfectly dependent — unanticipated inflation will preclude perfect correlation in our view of the data.

In Section 4.1 below we discuss an econometric procedure which allows us to compare the implications of the CIR model with the sample characteristics. This method has certain distinguishing features that are outlined in Section 4.2.

[13]Also see Brown and Dybvig (1986) who apply the CIR model to nominal prices of nominally riskless bonds.

[14]The methods in Brown and Schaefer (1990) and in Brown and Dybvig (1986) are similar; however, the former paper analyzes indexed bonds while the latter focuses on nominal bonds. Both papers can only identify the following functions of the CIR parameters: $\kappa + \lambda$, $\kappa\theta$, and σ.

4.1 Comparing Population and Sample Moments

The econometric technique corresponds to the GMM procedure developed by Hansen (1982) and employed in Hansen and Singleton (1982), Brown and Gibbons (1985), and elsewhere. The differences in our procedure are (1) that we avoid the use of consumption data or data on aggregate wealth (the "market") and (2) that we exploit the availability of a functional form within the CIR model for the relevant densities of the unobserved state variable, r.

Before we can apply the GMM approach, we must calculate some population moments for real returns on nominal bonds. These population moments are characteristics that we expect to see in the data if the CIR model were true. Our procedure involves a comparison of the implied population moments with the corresponding sample moments as a way to estimate the CIR parameters and to judge the model's descriptive validity.

Recall from relation (12) that the expectation of the gross real return from owning a nominal discount bond of maturity τ from t to $t+u$ is given by

$$
\begin{aligned}
E\left\{E_t\left({}_t\tilde{R}_{t+u}(\tau)\right)\right\} &= E\left\{E_t\left(\frac{\tilde{P}_{t+u}(\tau-u)}{P_t(\tau)}\right)\right\}, \quad \text{for } u \le \tau && (27)\\
&= E\left\{E_t\left(\frac{A(\tau-u)}{A(\tau)}\exp\left(-B(\tau-u)\tilde{r}(t+u)+B(\tau)\tilde{r}(t)\right)\right)\right\} && (28)\\
&= \frac{A(\tau-u)}{A(\tau)}E\left\{\exp\left(-B(\tau-u)\tilde{r}(t+u)+B(\tau)\tilde{r}(t)\right)\right\} && (29)\\
&\equiv \Phi_1(u,\tau;\beta)\,, && (30)
\end{aligned}
$$

where the CIR formula has been used to pass to (28). Note that the expectation in the RHS of relation (29) is taken using the joint, unconditional distribution of the random variables $\{\tilde{r}(t),\tilde{r}(t+u)\}$. CIR's model specifies this joint density from relation (16). The conditional distribution of $\tilde{r}(t+u)$ given $r(t)$ is noncentral chi-square, and the unconditional distribution of $\tilde{r}(t)$ is a gamma. Clearly, the expectation in (29) defines the moment generating function for this bivariate distribution. The appendix provides an explicit calculation of $\Phi_1(u,\tau;\beta)$, and it shows how the unobservable variables $\tilde{r}(t)$ and $\tilde{r}(t+u)$ have been integrated out.

Following an identical argument, we can use the CIR formula to compute the expectation of the product of two gross, real, nonoverlapping returns from nominal discount bonds. In the following expression we examine this product, where the first return (from a bond with maturity τ_1) is from t to $t+u$, and the second return (from a bond with maturity τ_2) from

$t+v$ to $t+w$:[15]

$$E\left\{E_t\left({}_t\tilde{R}_{t+u}(\tau_1)\,{}_{t+v}\tilde{R}_{t+w}(\tau_2)\right)\right\} = E\left\{E_t\left(\frac{\tilde{P}_{t+u}(\tau_1-u)}{P_t(\tau_1)}\cdot\frac{\tilde{P}_{t+w}(\tau_2-(w-v))}{P_{t+v}(\tau_2)}\right)\right\} \tag{31}$$

$$= \quad A(\tau_1-u)\quad A(\tau_2-(w-v))\quad A^{-1}(\tau_1)\quad A^{-1}(\tau_2)\quad \times$$

$$E\left\{\exp\left(-B(\tau_1-u)\tilde{r}(t+u)+B(\tau_1)\tilde{r}(t)-B(\tau_2-(w-v))\tilde{r}(t+w)+B(\tau_2)\tilde{r}(t+v)\right)\right\} \tag{32}$$

$$\equiv \Phi_2(u,v,w,\tau_1,\tau_2;\beta)\,. \tag{33}$$

The expectation in the RHS of (32) involves the calculation of the moment generating function of a joint distribution of four random variables, which is tedious but straightforward to compute (see the appendix for details).

Relations (30) and (33) serve as restrictions on the first moment and the second comoments of the gross real returns on nominal bonds. These moments are expressed solely as functions of the maturities of the bonds and of the vector of parameters, given the CIR model.

We are now in a position to apply Hansen's GMM. Suppose we have data on the real gross returns on nominal discount bonds of maturity $\tau_i, i=1,2,\ldots,n$. Define the following functions of the data and the moments and nonoverlapping comoments:

$$h_{1t}(u,\tau_i;\beta) \equiv {}_tR_{t+u}(\tau_i)-\Phi_1(u,\tau_i;\beta), \tag{34}$$

$$h_{2t}(u,v,w,\tau_i,\tau_j;\beta) \equiv {}_tR_{t+u}(\tau_i)\,{}_{t+v}R_{t+w}(\tau_j)-\Phi_2(u,v,w,\tau_i,\tau_j;\beta)\,. \tag{35}$$

Now stack these into a vector:

$$g_T(\beta)\equiv\begin{pmatrix} \frac{1}{T}\sum_t h_{1t}(u,\tau_1;\beta) \\ \frac{1}{T}\sum_t h_{1t}(u,\tau_2;\beta) \\ \vdots \\ \frac{1}{T}\sum_t h_{1t}(u,\tau_n;\beta) \\ \frac{1}{T}\sum_t h_{2t}(u,v,w,\tau_1,\tau_1;\beta) \\ \frac{1}{T}\sum_t h_{2t}(u,v,w,\tau_1,\tau_2;\beta) \\ \frac{1}{T}\sum_t h_{2t}(u,v,w,\tau_i,\tau_j;\beta) \\ \vdots \end{pmatrix} \qquad i,j=1,2,\ldots,n\,, \tag{36}$$

where n is the number of maturities for the available bills. Alternatively, $g_T(\beta)\equiv\frac{1}{T}\sum_t h_t(\beta)$, where $h_t(\beta)$ is a vector built by stacking $h_{1t}(\cdot)$ and $h_{2t}(\cdot)$ in the obvious way given equation (36). The vector $g_T(\beta)$ has dimension $l\times 1$, and we assume that $l>4$ so that the number

[15] We assume that $t<t+u\le t+\tau_1$, $t+v<t+w\le t+v+\tau_2$, and $u\le v$. If $u=v$, then the left hand side (LHS) of relation (31) represents the expected real growth in a strategy of rolling over discount instruments.

of restrictions exceeds the number of parameters to be estimated. The model's implications (from relations (30) and (33)) are

$$E\left(g_T(\beta)\right) = 0\,, \tag{37}$$

so we choose β to make the sample counterparts to these moments close to zero. Hansen's procedure involves choosing β from a feasible[16] region $\mathcal{B}$:

$$\min_{\beta \in \mathcal{B}} \quad T\, g_T(\beta)'\, \Omega^{-1}\, g_T(\beta)\,, \tag{38}$$

where the weighting matrix Ω is the asymptotic covariance matrix of the vector of sample moment conditions. Given the CIR model and mild regularity conditions, the minimand in (38) has, asymptotically, a χ^2 distribution with $(l-4)$ degrees of freedom; this is the test employed below. Hansen (1982) provides the sufficient conditions for the consistency and asymptotic normality of $\hat{\beta}$ as well.

We now turn out attention to the appropriate way to construct Ω in equation (38) so as to account for the serial dependence in the observations.[17] In many applications of GMM, $h_t(\beta)$ is orthogonal to all past information, including information in the lagged values of $h_t(\beta)$. This orthogonality follows directly from the rationality assumption that agents use all past information in setting market prices — this was the appropriate assumption in the context of the models investigated by Hansen and Singleton (1982) and Brown and Gibbons (1985).[18] In our application $h_t(\beta)$ is not the deviation of the realized return from its *conditional* expectation, but it is the deviation of the realization from its unconditional expectation, $\Phi_1(\cdot)$ or $\Phi_2(\cdot)$. The CIR model predicts that the deviations of bond returns (or of the products of these returns) from the unconditional expectations *will* be serially correlated because these deviations depend on $r(t)$. Our inability to observe $r(t)$ precludes us from constructing $h_t(\beta)$ as a deviation from an expectation conditional on $r(t)$, so we cannot remove this source of serial correlation in the data.

The maintained assumptions from the CIR model permit us to specify some elements of the weighting matrix. However, to determine other elements (for example, the variances along the diagonal), we need additional — and for our purposes unnecessary — assumptions

[16]The restrictions from the CIR model are ρ, θ, $\sigma_U > 0$ and $\rho < 1$.

[17]White and Domowitz (1984) have extended Hansen's (1982) asymptotic justification for GMM to handle such cases.

[18]In this earlier work, as in Hansen and Hodrick (1980), serial correlation in $h_t(\beta)$ could only be present because overlapping observations are used. If the only problem were overlapping observations, Hansen's (1982) analysis suggests a way to handle such a case. In our application we rely on overlapping observations when we focus on *ex post* real yields for bonds held till maturity. For example, if we had examined on a monthly basis the returns on three month bill held till maturity, there would be two months of overlap in consecutive observations.

about the process on inflation and the variance of any measurement error. Therefore, it is not possible to specify the exact form of the weighting matrix although we expect serial correlation to be present in the observations.

To account for general forms of serial dependence and heteroscedasticity (at least asymptotically), we adopt the Newey-West procedure. The asymptotic justification for GMM requires only that the weighting matrix be a consistent estimator[19] of the asymptotic covariance matrix; the following weighting matrix, given in Newey and West (1987), is a consistent estimator which is always positive definite:

$$\Omega = \hat{\Omega}_0 + \sum_{j=1}^{m} \omega(j,m) \left[\hat{\Omega}_j + \hat{\Omega}_j'\right], \tag{39}$$

$$\omega(j,m) \equiv 1 - \left[\frac{j}{m+1}\right], \tag{40}$$

$$\hat{\Omega}_j \equiv \frac{1}{T} \sum_{t=j+1}^{T} \left(h_t - \bar{h}\right)\left(h_{t-j} - \bar{h}\right)'. \tag{41}$$

Asymptotic justification for the Newey-West procedure relies on m growing at least at the rate $T^{0.25}$. The covariance matrix of the asymptotic distribution of the GMM estimator for β is consistently estimated by:

$$Var(\hat{\beta}) = \left[D'(\hat{\beta})\Omega^{-1}D(\hat{\beta})\right]^{-1}, \tag{42}$$

where

$$D(\hat{\beta}) \equiv \frac{1}{T}\sum_{t} \left.\frac{\partial h_t}{\partial \beta}\right|_{\beta=\hat{\beta}}. \tag{43}$$

4.2 The Econometric Procedure: Some Additional Features

Part of the motivation for our particular method should be clear. We have developed a procedure which determines the implications of a continuous time model for discretely sampled data. Further, we are not required to observe state variables or measures of aggregate consumption in comparing the theory with the data. However, there are additional reasons which encouraged us to follow this approach. These are outlined in this subsection.

First, our moment conditions are robust to the usual forms of measurement error. Even if the gross real returns are measured in error, the measurement error has no impact on the

[19] Many applications of GMM require a two-step procedure to find the optimal set of estimates for β. The first step involves minimizing the objective in equation (38) setting Ω equal to the identity matrix. The resulting set of estimates for β are then used to construct a second Ω matrix, not equal to the identity matrix. However, in our case the first step can be avoided due to the special structure of our orthogonality conditions, $h_t(\beta)$. Our orthogonality conditions can be written as a set of sample moments that do not depend on β and functions of β that do not depend on the sample.

expectation of equation (34) as long as the error has a mean equal to zero. Furthermore, assuming the measurement error is serially uncorrelated, the expectations of the comoments in equation (35) also remain valid in the presence of such error.[20] Similarly, serial cross-comoments implicit in equation (35) should not be affected by measurement error that is uncorrelated across bonds of different maturities. This robustness to measurement error increases our confidence in the point estimates for β that are provided in Section 6.

Second, we have avoided a moment condition that is related to the variance of the real returns. If we had specified a process for inflation, we could calculate a moment condition corresponding to the population variance of the gross return on a bond as a function of the parameters underlying the inflation process as well as β. This variance would be sensitive to measurement error unless we deliberately modeled the process for this error. Such an approach would have more potential for misspecification (it would lead to a joint test of the CIR model and the assumed inflation process), and it is not clear to us that it would offer any real advantages over the moment restrictions chosen here. Rather than test models of inflation, we wanted to follow a path that would allow us to investigate models for real bond returns.

Finally, there is nothing in the procedure that requires us to examine monthly (say) holding period returns on bonds of various maturities. In constructing the moment conditions there are very few restrictions on the holding period of the gross real returns on the nominal discount bonds. We could, for example, choose the holding period to correspond to the maturity of each bond. Selecting a holding period in this way leads to an examination of the data on the *ex post* real yields-to-maturity of these bills. However, in this case the estimation must take into account the fact that there is substantial overlap (for example, with monthly data and yields on twelve month Bills, there are eleven months of overlap); therefore, we must choose higher values for m in the Newey-West procedure. We will examine the sensitivity of our empirical results when we use real yields instead of real returns.

5 Data Description and Summary Statistics

The empirical results reported in this paper are based on monthly data from 1964 through 1989. Data on U.S. Treasury Bills were obtained from the Government Bond Files of the Center for Research in Security Prices (CRSP) at the University of Chicago. The Bureau

[20] Even if the measurement error had serial correlation induced by a moving average process of low enough order, the computation in equations (31) through (33) remains valid for lags that are sufficiently long to remove the dependency induced by the measurement error.

of Labor Statistics' series on the Consumers' Price Index, corrected for the home ownership interest component, were kindly provided to us by John Huizinga. From these two sources, gross holding period real returns ${}_tR_{t+u}(\tau)$ were constructed for each month t, for maturities (τ) of one, three, six, and twelve months.

In principle, the CIR model, in providing the real price of a default free pure discount bond, enables one to value coupon-bearing bonds as a linear combination of pure discount instruments. Therefore, the test of the model could be conducted with a rich structure of maturities by employing coupon-bearing instruments. We chose to avoid this for three reasons. First, the nonlinear structure of the pricing formula for coupon bonds would place a heavy burden on the algorithm which searches for the parameter values satisfying the moment conditions. Second, the differential taxation of the income and capital gains component would undoubtedly affect the pricing of coupon bonds. The effects of differential taxation would be exacerbated by the restrictions placed on the Treasury during the sample period as to the maximum coupons payable. Any call features on these instruments would preclude a simple approach to the valuation of coupon-bearing bonds. It should also be noted that although prices of stripped, single-payment certificates derived from coupon-bearing Treasuries are now available, we lack a sufficient history of these for our purposes. Third, the problems from overlapping observations (which becomes an issue when we employ data on yields till maturity) become progressively worse as we use bonds with long maturities. In addition to these three considerations, we are not persuaded that the benefits from including long bonds are great. As Dunn and Singleton (1984) argue, the variation in long term bond returns is large, so it may be more difficult to estimate parameters from long bond data with great precision.

Although CRSP reports Treasury Bill prices for several maturities at each month's end, this study restricts the maturities to one, three, six, and twelve month bills. The last three of these are the most heavily traded "on-the-run" bills; therefore, their month-end prices from CRSP are most likely to be current and simultaneous quotations. Treasury Bills for other maturities are usually not as heavily traded, and the potential for nonsynchronous prices and measurement error is greater with these. It should be recognized that the CPI series that is employed has consumption goods' prices that are usually sampled during a month, and therefore there is error induced in taking the CPI as a month-end price level for the purpose of computing the real return. This error is unavoidable, whereas the measurement errors in nominal bill prices may be reduced, as is argued here, by employing the bills on-the-run.

The estimation procedures employed in this study build on the moments of the real, monthly holding period returns. At each month's end, it is generally not possible to find

Treasury Bills with maturities of exactly one, three, six, and twelve months; however, there are bills with maturities surrounding these. We constructed the prices of these Treasury Bills by linearly interpolating between the annualized yields of the two bills that immediately surrounded the desired maturity.[21]

Figures 1A through 1D plot these data series, which are used for the bulk of the tests reported in Section 6. Panel A of Table 1 provides summary statistics of the real return series for the four maturities. The statistics reported in Panel A indicate that the mean and standard deviation of the real monthly returns increases as the maturity increases; the correlations between bills of adjacent maturity are high relative to the others. The autocorrelations at lag 1 are highest for the one month bills. While the autocorrelations decay quickly at higher lags for six month and twelve month bills, they are slow to die out for the one month and three month bills.

The Consumers' Price Index, which we use in computing the real returns may induce autocorrelations in the series. This will occur if the reported index values are computed from prices that are sampled for different goods in sequence, and for an individual good periodically. We are careful to employ comoment conditions which only use nonoverlapping returns; we discuss this in Section 6.2 where we report on some diagnostics.

Panel B provides the statistics on the real yields (to maturity) of the same bills. Because the monthly data for real yields-to-maturity involve overlapping intervals, the autocorrelations should at least reflect that degree of overlap. For instance, the autocorrelations for the yields on three month bills should be affected by the overlap for at least two months; the autocorrelations beyond lag 2 should reflect the autocorrelation in the structural model underlying the nonoverlapping returns. For the one month yield series there is no overlap, and the significant autocorrelations are, in the absence of measurement error, an indication of the structural model underlying the returns. For the other series, the autocorrelations are significant for lags well beyond their degree of overlap, and this also provides information on the process that generates the data. The contemporaneous correlation coefficients also display the effects of overlapping intervals across the observations; its greatest impact is, as expected, in the computed correlation between six and twelve month yields, which is 0.95.

Figure 1 and Table 1 indicate that the average *ex post* real return was large over this sample period by historical standards. For example, Fama (1975) reports the average real

[21] In the actual estimation we denoted time in units of years and treated a one month Bill as if it matured in 30/365 years, a three month Bill as if it matured in 90/365 years, a six month Bill as if it matured in 180/365 years, and a twelve month Bill as if it matured in 345/365 years. Since we rely on end of month prices of Treasury Bills, we found the average maturity of the longest bill was approximately 345 days.

returns on one month Treasury Bills was seven basis points per month, from 1953 to 1971; our higher value of 13 basis points reflects recent experience. Despite the high average return, Figure 1 shows that the *ex post* returns were negative over some periods, especially in the late seventies.[22]

The variation in the *ex post* real return series (the standard deviation of 0.273% per month for Bills of one month maturity) is substantially greater than that reported in Fama and Gibbons (1982, 1984) for the sample period 1953-1977. Again, the sample period in our work includes relatively volatile periods.

6 Results

In Section 6.1 we report the empirical results using the moments given in equations (30) and (33). To test these with the data, we limit our attention to a few discount bonds and some specific moments. As mentioned in Section 5, we rely on monthly returns on Treasury Bills maturing in one, three, six, and twelve months. Section 6.2 investigates the sensitivity of our findings from alternative econometric specifications. Using the CIR model Section 6.3 focuses on a particular implication about the dynamics for real returns on bonds; here we report that the CIR framework is inconsistent with a time series feature in the data.

6.1 Empirical Tests with 14 Moments

To estimate the parameters and test the implications of the CIR model, we need to summarize the historical data using some sample moments and then compare these sample moments with values implied by the theory. For a given set of parameter values, equation (30) provides the theoretical prediction about the first moment while equation (33) supplies similar restrictions regarding second moments. We must first decide which first and second moments to use.

Obviously, we would like to use sample moments that provide a good summary of the historical data and capture the important stylized facts about this time period. We would also like to use enough moments to generate overidentifying restrictions to confront the theory. However, we recognize that the GMM approach has only asymptotic justification, so we want to avoid the use of too many moments, especially if the information in one moment

[22]This is not inconsistent with the CIR model, which predicts that the *ex ante* real rate, over all holding periods, is nonnegative. This feature of their model is an outcome of their decision to model the process on the single state variable as they did; there are no *a priori* reasons to expect a nonnegative *ex ante* real rate, except if the consumption good could be stored costlessly.

may be redundant with the other moments.

We selected the first moment for all bond maturities, which seems like an obvious choice. Equation (30) generates four first moment conditions since we have four maturities.

The choice of the sample comoments to be used in equation (33) is not as straightforward. Even with a small number of bond maturities, the restrictions implied by equation (33) permit an unmanageably large number of comoment conditions, obtained by varying the lag structure. Since the sampling characteristics of the comoments are probably superior at small lags, we focused on short lags.

We rely on the comoments to capture the dynamic characteristics of the historical data. The degree of mean reversion in the sample is an obvious summary of the temporal behavior. Thus, we examined one serial comoment for each maturity, providing four additional moment conditions. The serial comoment, even though it is not a central moment, is closely related to the first order autocovariance.

We also wanted a measure of the correlation among bond returns of differing maturities. Because we do not specify a process for inflation, we are unable to compute a theoretical value for the contemporaneous correlation among bonds with different maturities. As a substitute for the correlation, we rely on serial cross-comoments which provide some information about the association among bond returns — as well as some information about the dynamics of bond returns. We correlated a lagged return on a one month bill with the returns on the other three maturities, and we correlated a lagged return on a twelve month bill with the returns on the other three maturities. These comoments correspond to using the shortest maturity bill to predict the subsequent returns on the other bills and to using the longest bill to predict the subsequent returns on the other bills. This information represents six cross comoments for fitting the CIR model.

To summarize, we have four first moments (one for each maturity), four serial comoments (one for each maturity), and six cross comoments. The second column in panel B of Table 2 provides a list of the specific moments.[23]

[23]So far, we have chosen to interpret the expectations in equations (30) and (33) as noncentral moments of returns. In panel B of Table 2, we have scaled these moments up by 100,000. Now columns 3 and 5 can be given another interpretation. Column 3 represents the historical average of the real wealth one obtained by investing \$100,000 in bonds. Column 5 indicates the expected real wealth from the same investment strategy assuming the CIR model is correct with particular parameter values. In the case of the four first moments, the holding period of the investment is one month. In the case of the ten second moments, the holding period is two months. Each investment strategy differs by the maturity of the bond(s) purchased during the holding period.

In applying GMM we sought a set of parameter estimates for β that fixed the 14 population moments as close as possible to the sample moments given in the third column of panel B of Table 2. The system is overidentified, so it is not possible to perfectly match the sample moments in the third column. We minimized the quadratic form given in equation (38) to determine the optimal set of estimates.[24] Table 2 provides a test statistic for the overidentifying restrictions. Essentially, this test measures whether deviations from the sample moments in Table 2 are small, as would be expected if the theory is true. The deviations are measured by using the quadratic form in equation (38), which is distributed χ^2_{10} under the null hypothesis. Since the fitted moments implied by the parameter estimates in the fifth column in Panel B of Table 2 seem very close to the actual sample moments in the third column, it is not surprising to find that the CIR model cannot be rejected at traditional levels of significance. Having failed to reject the overidentifying restrictions, we now turn our attention to the parameter estimates which generated column 5 of Table 2.

The parameter estimates along with standard errors based on asymptotic theory are given in the second column in Panel A of Table 2. All the estimates are more than two standard errors away from zero. The point estimate for θ is large in magnitude (154 basis points per annum) relative to a similar parameter estimated by Fama (1975). This probably reflects our use of more recent history where the real return on Treasury Bills has been high by historical standards. Indeed, Section 6.3 confirms the importance of the particular time period selected when results by subperiods are presented. For the reader's convenience, the last column in Panel A of Table 2 also reports the implied parameter estimates for the parametrization using the original CIR notation.

The degree of mean reversion as measured by ρ is quick relative to the random walk model in Fama and Gibbons (1982), which implies that ρ should be close to one. This parameter provides some guidance for the speed of adjustment of the intercept of the "real" yield curve in a CIR world. Equation (19) in CIR (1985, page 392) provides a formula for the conditional expectation of r. Restating their equation using a parametrization based on ρ gives[25]

$$E\left[r(s)|r(t)\right] = \theta + \rho^{(s-t)}[r(t) - \theta]\,. \tag{44}$$

Thus, employing the estimates from Table 2, if the current intercept of the yield curve is 2.77% (which is one standard deviation above the mean of the steady state distribution), we expect it to be 1.98% in one month hence and almost equal to the long-run mean of 1.54%

[24]The optimization was done using conjugate gradient methods as implemented in *Mathematica*. We also confirmed our results with the numerical minimization routine in *Gauss*, which uses an algorithm based on the Broyden-Fletcher-Goldfarb-Shanno positive-definite secant update method.

[25]In equation (44), $(s-t)$ is measured in units of months, not years. Recall that we defined $\rho \equiv \exp(-\kappa/12)$; we omitted the 12 in equation (44).

by six months. This adjustment seems quick given recent behavior in the bond market when the short-term real rate was high by historical standards and remained at that high level for sustained periods. This rapid speed of adjustment is the focus of Section 6.3 below, so we postpone discussion of this point till that subsection.

The estimate of σ_v provides a measure of the unconditional standard deviation of r; for our sample period we estimate σ_v to be 1.23%. Consistent with recent experience when real rates were high, the estimates in Table 2 do imply that the conditional variation in the instantaneous rate is sensitive to the level of the rate. For example, using equation (19) in CIR (1985, page 392) the standard deviation of $r(s)$ conditional on $r(t) = 2.77\%$ is 1.36% and 1.23% for $s - t$ equal to one and twelve months, respectively. The sensitivity of the conditional variance of $r(s)$ to the current level of $r(t)$ is based on just the 14 sample moments in Table 2. None of these 14 moments are the sample variances nor do any of the 14 moments relate directly to the predictability of the variance based on current interest rates. Section 6.2 will examine an extension to Table 2 where we build in some information about the sample correlation between the conditional variance and predictors of this variance.

Consistent with traditional theories of term premiums, the value for y_∞ is positive in Table 2. Thus, the steady state yield curve is positively sloped with a spread between the asymptote and the intercept of the yield curve of $3.01\% - 1.54\% = 1.47\%$. Theories of the term structure also make predictions about the expected returns on short versus long bonds. The last column of Panel A of Table 2 reports a value for λ that is negative. Equation (22) indicates that the instantaneous expected return on any pure discount indexed bond is equal to $r - \lambda B(\tau)r$, and $B(\tau) > 0$ for all τ. This implies the instantaneous expected real return on a bond is positively related to its interest sensitivity (and its maturity). To provide an alternative perspective, Panel A of Figure 2 provides a plot of the unconditional expected returns over a discrete holding period of one month for bonds of various maturities. Based on the parameter estimates in Table 2, the curve for unconditional expected returns asymptotes at 2.98% for maturities in excess of one year. In fact, the unconditional expected return is 2.90% even for bonds with six months to maturity. Fama (1984) reports that the unconditional sample average returns on T-Bills are not monotonically increasing after six months till maturity. Our parameter estimates for the CIR model suggest that expected returns may effectively asymptote around six months till maturity.

To summarize, we have failed to reject some overidentifying restrictions implied by the CIR model, and we have parameter estimates which we view as plausible. Panel B of Figure 2 attempts to summarize the results in another way by providing a plot of the term structure of real yields on indexed bonds for different values of the current instantaneous rate (using

the parameter estimates in the second column of Table 2). All yield curves in Figure 2B asymptote to y_∞, which is 3.01%. For cases where the r is such that it is less than the long-run yield, the term structure is uniformly increasing. None of the yield curves in Figure 2B display a "hump". In principle the CIR model can generate a hump in the yield curve; however, our particular parameter estimates preclude such a shape for any value of r.

6.2 Sensitivity Analysis

While our econometric framework is conceptually straightforward, the optimization requires a solution to a difficult nonlinear problem. With many of our initial runs, we had a difficult time finding the proper set of starting values in order to achieve convergence. Furthermore, we experienced some situations where the minimum was apparently found; yet our numerical calculation of the hessian suggested that it was not positive definite. We have attempted to do a thorough search over the parameter space to confirm our estimates reported in Table 2.[26] Figure 3 provides some graphical evidence on the shape of the objective function. In these graphs we held three of the four parameter estimates fixed at the values given in Table 2, then we graphed the objective against the fourth parameter along the horizontal axis. In all cases, the graphs suggest a (locally) unimodal objective function around the optimal estimate of that fourth parameter.

We also examined the robustness of our results to various changes in the econometric specification. Table 3 summarizes the results for the sensitivity analysis. The first row of Table 3 repeats the results in Table 2 to allow for easier comparison between the initial result and alternative specifications. The sensitivity checks can be classified into four groups. These are:

1. Lags used in the Newey-West weighting matrix (reported in row 2 of Table 3);
2. Lag structure in the second moment conditions (reported in row 3 of Table 3);
3. Additional overidentifying restrictions based on third moments (reported in row 4 of Table 3); and
4. Moment conditions using yields to maturity (reported in row 5 of Table 3).

[26] Since the value of the χ^2 statistic is small, this is also a good indication that the algorithm has successfully found a global minimum. Of course, the converse does not necessarily follow. A large value for this statistic need not imply a local minimum has been found, for a large value is also consistent with the case that the theoretical model is misspecified.

Each of the above categories will be discussed in turn.

1.) LAGS USED IN THE NEWEY-WEST WEIGHTING MATRIX: The choice of the number of lags (i.e., m) to use in the Newey-West weighting matrix [see equation (39)] is somewhat arbitrary. In Table 2 we set $m = 4$ to account for our inability to observe and hence condition on r, which has a first-order autoregressive structure. We varied m between 0 and 8; the second row of Table 3 illustrates the effect of increasing m to 8. Fortunately, our results in Table 2 were not significantly affected by alternative choices of m. The point estimates and the standard errors in the first two rows of Table 3 are similar.

2.) LAG STRUCTURE IN THE SECOND MOMENT CONDITIONS: Two considerations motivated the lag structure used in Table 2. First, analyzing too many lags may lead to small sample problems with GMM. Second, using short lags are probably superior to long lags because the statistical precision is greater for the former. However, we did experiment with alternative lag lengths, but we found little change from the results in Table 2. The parameter estimate with the greatest sensitivity to the lag structure is σ_U. The third row of Table 3 illustrates the impact of using comoments at a two month lag rather than at the one month used in Table 2.

While we examined other lag structures as well,[27] the two month lag is perhaps the most interesting. The advantage of the two month lag over the one month lag is threefold. First, we are more confident that the actual dating of the returns does not overlap (either due to nonsynchronous trading or the measurement problems with inflation). Second, extending the lag length in the comoments minimizes problems associated with autocorrelated measurement error as long as the measurement error follows a moving average process of small order. Finally, a slightly longer lag length guarantees that the information that we viewed as known in deriving equation (33) is in fact known by the market when it sets the prices of bonds. The results in the third row are reassuring because they suggest that our findings in the first row are robust to such measurement errors.

3.) ADDITIONAL OVERIDENTIFYING RESTRICTIONS BASED ON THIRD MOMENTS: Not only is the selection of the lag length somewhat arbitrary, but it is also unclear why we should limit our analysis to just the first and second moments. In Section 4, we computed the first and second moments for real returns on nominal bonds under the CIR specification; in the appendix we generalize these results to higher order moments.

[27]For example, when we used a twelve month lag instead of the one month lag in Table 2 we found the standard error on σ_U increased, and we could no longer reject the hypothesis that σ_U was equal to zero—even though the point estimate did not change substantially.

Originally we relied on just the first and second moments because of the difficulty in estimating higher order moments precisely. We extended our investigation to include a noncentral third moment, for it is closely related to the conditional standard deviation of r, which is not constant in the CIR world. Clearly, the conditional variance is related to third moments since it reflects the expectation of the square of the random variable times a lagged value of the random variable. We estimated the model keeping the original 14 moments reported in Table 2, while adding eight more sample moments. These eight additional moments had the form:

$$E[{}_t\tilde{R}_{t+1}(\tau_1) \times {}_{t+1}\tilde{R}_{t+2}(\tau_2) \times {}_{t+2}\tilde{R}_{t+3}(\tau_2)] \,, \tag{45}$$

where τ_1 was set equal to one month or twelve months and τ_2 was set equal to each of the four maturities. That is, we correlated the lagged value of the one month (or twelve month) Bill return with the serial comoments of all four maturities.

As in Table 2, the p-value of the χ^2_{18} statistic (reported in row 4 of Table 3) exceeded .05 despite the additional constraints placed on the model. Except for σ_U (and to a lesser extent y_∞), the estimates in Table 2 were not significantly affected by the additional eight moment conditions. As reported in the fourth row of Table 3, the estimate for σ_U was reduced to 0.86, but its standard error decreased as well. The reduction in the standard error provides some evidence that the third moment has information for the conditional variance of r and hence for σ_U.

4.) MOMENT CONDITIONS USING YIELDS TO MATURITY: Our final sensitivity check examines the measurement error associated with on-the-run bills versus off-the-run bills. Table 2 is based on returns where the holding period is one month. For example, a six month bill is purchased and then sold when its maturity is five months. When this bill is sold, it is no longer on-the-run, and the measurement error in the market price of the security is somewhat greater due to decreased trading activity in the market. We can circumvent this problem by computing real returns from holding an on-the-run bill until it matures. Such a procedure for measuring real returns also provides the *ex post* real yield on the bond. We adopt the terminology of real yields to distinguish this analysis from the cases involving one month holding periods. Since the Treasury does not auction one month bills, we also exclude this maturity in this sensitivity check in an attempt to provide a clean set of prices for measuring real yields based only on on-the-run instruments.

The fifth row in Table 3 summarizes the results when real yields are used instead of one month holding period returns. At first glance it seems that the measurement error may be important, for the point estimates of θ, ρ, and y_∞ are smaller than those in the earlier

rows in this table. However, the standard errors are now much larger, so the discrepancies are less significant than they first appear. The increase in the standard errors is to be expected because the real yields generates a large amount of overlap in the observations. For example, the real yield on a twelve month bill has eleven months of overlap with the adjacent observation of the same series. This overlap requires us to extend the Newey-West lag structure of the weighting matrix; in Table 3 we report results where m is equal to 22.[28]

While the point estimates in the fifth row are bothersome, we found the χ^2_6 statistic for the overidentifying restrictions more troubling, for now we are rejecting the CIR model at the usual levels of significance. One could attribute all the results in the fifth row to small sample properties of the econometric procedure due to the presence of extreme amounts of serial dependence induced by overlapping observations. A second explanation may claim that the measurement error in off-the-run bill prices is substantial and biases against rejection. A third possibility may be the presence of one month bills in the first four rows of Table 3 and the exclusion of this maturity in the fifth row.

To investigate this last possibility, we extended the results in Table 2 to a setting where returns with a one month holding period are used but where we excluded the one month bill. This exclusion decreases the number of moment conditions to ten and reduces the number of overidentifying restrictions. The results of this case are reported in the sixth row of Table 3. As expected the precision of the estimates in the final row is greater than that reported in the fifth row since the one month holding period returns eliminate the overlap in the data. Now the point estimate for y_∞ is comparable to that in the earlier rows. However, the estimates for θ and ρ are different from the earlier rows. The inclusion of the one month maturity provides a more reasonable estimate for θ which may be expected since θ represents the expected return on a very short-term maturity bond. Measurement error in the off-the-run prices for bills is not an adequate explanation of the discrepancy between the fifth row and the earlier rows, for the χ^2_6 continues to reject the over-identifying restrictions in the sixth row where off-the-run prices are used in computing the real returns. (Again, we defer our discussion of ρ till the next subsection.)

The sixth row of Table 3 suggests that the CIR model fits the very short-end of the term structure better than it fits the intermediate range. Row 6 of Table 3 provides an empirical discrepancy between the CIR model and the data. The next subsection discusses another deficiency of the CIR model.

[28]We tried alternative values for m which did not reduce the degree of discrepancy between the row for yields and the other rows in Table 3.

6.3 A Deficiency of the Model

Based on the previous subsections, one might conclude that the CIR model provides a reasonable characterization of the real returns on nominal bills — at least for maturities twelve months or less. For the most part the estimates are reasonable, the implied shape of the term structure for indexed bonds is plausible, and the results are not very sensitive to the exact specification of the econometric model.

The estimate of the autocorrelation, ρ, is the most troubling. As was noted in Section 6.1 above, the yield on indexed bonds reverts rather quickly to θ given the parameters estimates in Table 2. We also found in the last row of Table 3 some evidence suggesting higher estimates for ρ for alternative econometric specifications. Figure 2A illustrates the impact on unconditional expected returns when ρ is increased from 0.35 to 0.95. As ρ increases, the graph for unconditional expected returns displays curvature even for long maturities.

Other implications from the low value of ρ show up in Table 4. Using the fitted values for the first moment and for the serial comoment in Table 2, we backed out the implied value for the autocovariance. Then we divided these theoretical autocovariances by the *sample* variances to compute a standardized measure of serial dependence based on autocorrelations; these numbers are reported in Table 4. (Since we have not specified a process for inflation, the theory does not make a prediction about variances, and we must rely on sample variances. We divided by the sample variance in order to produce numbers that are a little easier to interpret.) In all rows of Table 4, we are reporting implied autocorrelations for real returns on bonds with a one month holding period.

The results in Table 4 are striking in that the autocorrelations are quite close to zero, die off quickly, and for most maturities the autocorrelations are negative. In contrast, Panel A of Table 1 suggests that the sample autocorrelations are substantially different from zero, die off slowly, and are always positive. It is difficult to reconcile the results in Tables 1 and 4, where the model performs poorly, with Table 2 where the model seemed to fit well. Recall that in Section 6.2 we reported that the fit of the model was not affected by the use of an alternative lag structure. The very low value of the implied autocorrelation in Table 4 for even twelve month lags does not seem to be a problem in fitting the model despite the high sample autocorrelation reported in Table 1.

One possible explanation is that the parameter estimates in Table 2 are not based on moments reflecting the sample variance. As noted earlier, the CIR model has no implications for the variance without specifying the inflation process. The predicted autocorrelations in

Table 4 would be much higher if the variances used in transforming autocovariances into autocorrelations were lower. However, even if the standard deviations were lower, the signs in Tables 1 and 4 are troublesome. How can the model fit so well in Table 2 and yet the sign of the implied autocorrelation be wrong? The answer to this question may lie in the fact that we used noncentral second moments. As a result little penalty is attached to situations where the central second moments (after adjusting for the first moment) have the wrong sign relative to the sample central second moments.

Casual empiricism is also troubling for this aspect of the CIR model. During the middle 1980's expected real returns were high (by the standards of the fitted CIR model) and stayed high (i.e., mean reversion was slow). Given we estimate θ to be about 1.54%, how can we have a time period like the 1980's where the "expected" real returns on short-term bonds are substantially greater than implied by this value of θ? When nominal interest rates were high during the middle 1980's, it is hard to imagine that anyone forecasted inflation to be a comparably high number — especially for short-term bond maturities. If the mean reversion is quick (or ρ is only 0.35), it would seem that the "expected" real return should have been smaller.[29]

One obvious solution is to increase the number of parameters in the CIR model and allow for a time-dependent value of θ — an extension that is discussed in Cox, Ingersoll, and Ross (1985b). Because we do not have a good model for θ, we have not pursued this line of inquiry; however, we do report parameter estimates from two subperiods in the last two rows of Table 3. Not surprisingly, θ and y_∞ are much lower in the first subperiod (1964–1976) than in the second subperiod. In line with Fama (1975), $\hat{\theta}$ is 85 basis points. We also find $\hat{\sigma}_U$ is essentially zero in the first subperiod[30] and much higher in the second. Again, Fama (1975) concludes that one cannot reject the hypothesis that the real return is constant during a time period that largely overlaps with our first subperiod. The higher value of σ_U is to be expected in a time period which includes the change in Fed monetary policy. While $\hat{\rho}$ is higher in the first subperiod than in the second, it is not close to unity which is implied by a view that the real rate follows a random walk. In both subperiods, the overidentifying restrictions are not rejected.

[29]Here we use "expected" to mean the forecasts of interest rates and inflation held by investors in the market — not necessarily the conditional expectation calculated by a formal theoretical model.

[30]Based on our experience with the estimation, the low standard errors reported for the first subperiod is a by-product of the low value of σ_U.

7 Conclusion

We have presented a test of the model of the term structure developed in Cox, Ingersoll, and Ross (1985b) and estimates of its parameters. Our tests and estimates indicate that the model, which is at once quite complicated in its structure and rather simplistic in its dependence on a single state variable, performs reasonably well when confronted with data on short-term Treasury Bills. The parameter estimates indicate that term premiums are positive, and that the term structure of indexed bonds can assume a number of shapes.

Much work remains to be done. We hope to exploit our estimates in a procedure which extracts a time series for the unobservable economic variable (i.e., $r(t)$); to extend the tests to a class of models which explicitly incorporate inflation and find the nominal price of a nominally riskless bond; and to conduct empirical tests of models which find the nominal prices of derivative securities. We hope that future research will extend the theory to a general equilibrium setting wherein money has a useful economic function. Such an extension would provide an endogenous process for inflation and for interest rates.

Appendix

In this Appendix we compute the restrictions on the first and second moments of ${}_t\tilde{R}_{t+u}(\tau)$, denoted $\Phi_1(u,\tau;\beta)$ and $\Phi_2(u,v,w,\tau_1,\tau_2;\beta)$ as defined in equations (30) and (33), respectively, in the text. These computations permit the application of the GMM procedure.

Consider first the computation of $\Phi_1(u,\tau;\beta)$ in (30):

$$\Phi_1(u,\tau;\beta) \equiv \frac{A(\tau-u)}{A(\tau)} E\left\{\exp\left(-B(\tau-u)\tilde{r}(t+u)+B(\tau)\tilde{r}(t)\right)\right\}. \tag{46}$$

The expected value in the RHS of (46) needs to be computed. Now defining $p \equiv B(\tau-u)$ and $q \equiv B(\tau)$ we can write

$$E\left\{\exp\left(-p\,\tilde{r}(t+u)+q\,\tilde{r}(t)\right)\right\} = E\left\{\exp\left(q\,\tilde{r}(t)\right) E_t\left[\exp\left(-p\,\tilde{r}(t+u)\right)\right]\right\} \tag{47}$$

$$E_t\left\{\exp\left(-p\,\tilde{r}(t+u)\right)\right\} = C_1 \exp\left[C_2 r(t)\right] \tag{48}$$

$$\text{where} \quad C_1 \equiv \left[1-\frac{p\sigma^2\left(1-\exp\{-\kappa u\}\right)}{2\kappa}\right]^{-2\kappa\theta/\sigma^2} \tag{49}$$

$$\text{and} \quad C_2 \equiv \left[\frac{2\kappa p}{2\kappa - p\sigma^2\left[1-\exp\{-\kappa u\}\right]}\right]. \tag{50}$$

Equation (48) relies on the fact that the distribution of $\tilde{r}(t+u)$ given $\tilde{r}(t)$ is noncentral χ^2 (see CIR(1985b)), whose properties are given in Johnson and Kotz (1970, Chapter 28). Use

(48) in (47)

$$E\left\{\exp\left(-p\,\tilde{r}(t+u)+q\,\tilde{r}(t)\right)\right\}=\mathcal{C}_1 E\left\{\exp\left((q+\mathcal{C}_2)\tilde{r}(t)\right)\right\}. \tag{51}$$

The distribution to be used in computing the RHS of (51) is the unconditional or steady state distribution of $r(t)$, which is a gamma. From CIR(1985b) and Johnson and Kotz (1970, Chapter 17)

$$E\left[\exp\left(\phi\tilde{r}(t)\right)\right] = \left[1-\frac{\phi}{\omega}\right]^{-\nu} \quad \text{for} \quad \phi<\omega, \tag{52}$$

$$\text{where} \quad \omega \equiv \frac{2\kappa}{\sigma^2}, \tag{53}$$

$$\text{and} \quad \nu \equiv \frac{2\kappa\theta}{\sigma^2}. \tag{54}$$

The function $\Phi_1(u,\tau;\beta)$ is found from substituting (52) in (51), and the result in (46).

The condition on the second moment of ${}_t\tilde{R}_{t+u}(\tau)$ follows from relation (33). This involves the computation of the unconditional expectation involving $r(t), r(t+u), r(t+v)$, and $r(t+w)$ of the following form:

$$E\left\{\exp\left[a\,\tilde{r}(t)+b\,\tilde{r}(t+u)+c\,\tilde{r}(t+v)+d\,\tilde{r}(t+w)\right]\right\}, \tag{55}$$

where a, b, c, and d are of the form $B(\tau_i)$ and hence a function of the parameters and the holding periods. This expectation can be rewritten as

$$E\left\{\exp\left[a\,\tilde{r}(t)\right]\cdot E_t\left\{\exp\left[b\,\tilde{r}(t+u)\right]\cdot E_{t+u}\left\{\exp\left[c\,\tilde{r}(t+v)\right]\cdot E_{t+v}\left\{\exp\left[d\,\tilde{r}(t+w)\right]\right\}\right\}\right\}\right\}. \tag{56}$$

By repeated use of the arguments in (47) through (52), this expectation can be expressed as a function of a, b, c and d (which are defined in terms of CIR parameters), the holding periods (t to $t+u$ and $t+v$ to $t+w$) and the times to maturity. This defines $\Phi_2(u,v,w,\tau_1,\tau_2;\beta)$ which is the unconditional expectation defined in the comoment (33). The restriction in the computation of this comoment is that the holding periods should not overlap.

This procedure can be extended to compute comoments of higher order; for example, in computing the third comoment the expectation would involve six distinct values for r at different times t. Section 6.2 summarizes our empirical results using an additional eight third moments.

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Table 1

Summary Statistics

(Means, Std Deviations, Autocorrelations to 12 Lags)
Data: Monthly Observations, 1964/1–1989/12, on
US Treasury Bills of 1, 3, 6 and 12 Month Maturities

Panel A: Real Monthly Holding Period Returns (% per month)

Maturity	Mean (%)	S.D. (%)	Correlations				Autocorrelations											
			1 Mo.	3 Mo.	6 Mo.	12 Mo.	$\rho(1)$	$\rho(2)$	$\rho(3)$	$\rho(4)$	$\rho(5)$	$\rho(6)$	$\rho(7)$	$\rho(8)$	$\rho(9)$	$\rho(10)$	$\rho(11)$	$\rho(12)$
1 Mo.	0.130	0.273	1.000	0.934	0.773	0.583	0.561	0.446	0.369	0.387	0.365	0.361	0.377	0.371	0.396	0.370	0.361	0.414
3 Mo.	0.193	0.322	0.934	1.000	0.925	0.774	0.529	0.379	0.344	0.368	0.418	0.362	0.379	0.345	0.367	0.318	0.350	0.359
6 Mo.	0.221	0.434	0.773	0.925	1.000	0.937	0.375	0.172	0.159	0.175	0.289	0.207	0.275	0.160	0.237	0.194	0.245	0.166
12 Mo.	0.213	0.699	0.583	0.774	0.937	1.000	0.274	0.001	0.010	0.041	0.189	0.086	0.224	−0.006	0.123	0.067	0.150	0.077

Panel B: Real Yields to Maturity: Overlapping Data

Maturity	Mean (%)	S.D. (%)	Correlations				Autocorrelations											
			1 Mo.	3 Mo.	6 Mo.	12 Mo.	$\rho(1)$	$\rho(2)$	$\rho(3)$	$\rho(4)$	$\rho(5)$	$\rho(6)$	$\rho(7)$	$\rho(8)$	$\rho(9)$	$\rho(10)$	$\rho(11)$	$\rho(12)$
1 Mo.	0.130	0.273	1.000	0.811	0.704	0.671	0.561	0.446	0.369	0.387	0.365	0.361	0.377	0.371	0.396	0.370	0.361	0.414
3 Mo.	0.466	0.720	0.811	1.000	0.908	0.858	0.888	0.734	0.608	0.581	0.580	0.574	0.564	0.563	0.555	0.556	0.566	0.545
6 Mo.	1.063	1.395	0.704	0.908	1.000	0.950	0.947	0.875	0.806	0.745	0.695	0.657	0.637	0.652	0.658	0.659	0.652	0.613
12 Mo.	2.179	2.886	0.671	0.858	0.950	1.000	0.968	0.926	0.891	0.861	0.836	0.805	0.689	0.776	0.755	0.733	0.713	0.661

Table 2
Test of the Cox, Ingersoll & Ross Model of the Term Structure Using the Generalised Method of Moments

Data : Real, monthly holding period returns on 1, 3, 6, and 12 month U S Treasury Bills: 1964-89.
Moment Conditions : 4 first moments, 4 autocovariances and 6 serial cross-covariances (lagged 1 month).
Newey-West Lag : $m = 4$ for weighting matrix.
$_t\tilde{R}_{t+u}(\tau)$: Gross real return on τ-month T Bill held from t to $t+u$.

Panel A: Parameter Estimates[1]

Yield Curve Based Parameters	Estimate	Standard Error	Correlations θ	ρ	y_∞	σ_U	CIR Model Based Parameters
$\theta(\%)$	1.54	0.26	1.000	0.436	0.893	0.295	$\theta = 1.54\%$
ρ	0.35	0.06	0.436	1.000	0.422	0.050	$\kappa = 12.43$
$y_\infty(\%)$	3.01	0.30	0.893	0.422	1.000	0.225	$\lambda = -6.08$
$\sigma_U(\%)$	1.23	0.44	0.295	0.050	0.225	1.000	$\sigma = 0.49$

Test Statistic, χ^2_{10} : 13.39
p-value : 0.203

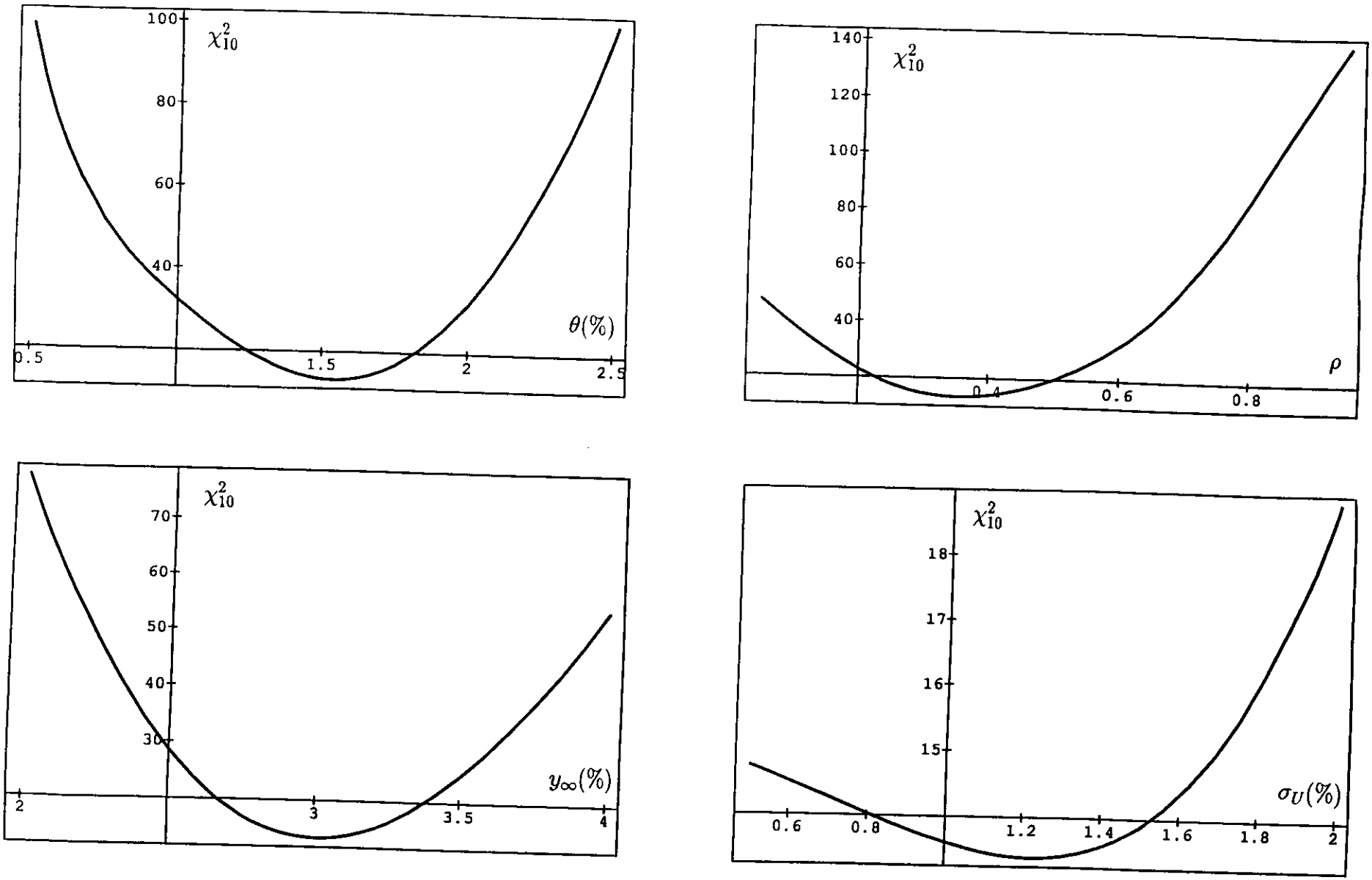
Panel B: Sample Moments and Fitted Moments ($\times 100,000$)[2]

Moment Number	Definition	Sample Mean	Standard Error	Fitted Value
1	$E[_t\tilde{R}_{t+1}(1)]$	100129.09	26.18	100153.71
2	$E[_t\tilde{R}_{t+1}(3)]$	100192.28	30.05	100214.95
3	$E[_t\tilde{R}_{t+1}(6)]$	100220.20	34.78	100241.13
4	$E[_t\tilde{R}_{t+1}(12)]$	100212.81	47.96	100247.70
5	$E[_t\tilde{R}_{t+1}(1)\,_{t+1}\tilde{R}_{t+2}(1)]$	100258.63	51.85	100307.68
6	$E[_t\tilde{R}_{t+1}(3)\,_{t+1}\tilde{R}_{t+2}(3)]$	100384.85	59.54	100430.28
7	$E[_t\tilde{R}_{t+1}(6)\,_{t+1}\tilde{R}_{t+2}(6)]$	100441.32	68.51	100482.72
8	$E[_t\tilde{R}_{t+1}(12)\,_{t+1}\tilde{R}_{t+2}(12)]$	100427.41	93.51	100495.87
9	$E[_t\tilde{R}_{t+1}(1)\,_{t+1}\tilde{R}_{t+2}(3)]$	100321.96	55.42	100369.02
10	$E[_t\tilde{R}_{t+1}(1)\,_{t+1}\tilde{R}_{t+2}(6)]$	100349.96	58.89	100395.25
11	$E[_t\tilde{R}_{t+1}(1)\,_{t+1}\tilde{R}_{t+2}(12)]$	100342.67	69.33	100401.83
12	$E[_t\tilde{R}_{t+1}(12)\,_{t+1}\tilde{R}_{t+2}(1)]$	100342.74	68.52	100401.70
13	$E[_t\tilde{R}_{t+1}(12)\,_{t+1}\tilde{R}_{t+2}(3)]$	100406.24	73.64	100463.05
14	$E[_t\tilde{R}_{t+1}(12)\,_{t+1}\tilde{R}_{t+2}(6)]$	100434.40	79.90	100489.29

[1] θ is the mean of the steady state distribution of r in CIR's model, and also the intercept of the "steady state" yield curve(% per year); ρ is its autoregressive parameter at 1 month lag, where $\rho = \exp(-\kappa/12)$; y_∞ is the asymptote of the CIR yield curve (% per year); and σ_U is the unconditional value for the standard deviation of r (% per year).

[2] The standard errors in the fourth column are the square roots of the diagonal elements of the inverse of the Newey-West weighting matrix.

Figure 3
Value of the Objective Function Versus Individual Parameter Values.



Note: For all four graphs, the minimum value of the objective function (χ^2_{10}) occurs at $\theta = 1.54\%$, $\rho = 0.35$, $y_\infty = 3.01\%$, and $\sigma_U = 1.23\%$. Each of these graphs plots one of the four parameters holding the other three fixed at the optimal estimate.

Table 3

Sensitivity Analysis of Econometric Estimates for the CIR Model
Monthly Data on Real Returns and Yields, 1964-89[a]

Case Description	T Bill Maturities	No. of Moments	Newey-West Lag	CIR Parameters & Std Errors				Test Statistic (p-value)
				θ	ρ	y_∞	σ_U	
1. Returns	1,3,6,12	14	4	1.54 (0.26)	0.35 (0.06)	3.01 (0.30)	1.23 (0.45)	$\chi^2_{10} = 13.39$ (0.20)
2. Returns	1,3,6,12	14	8	1.50 (0.29)	0.36 (0.06)	3.00 (0.32)	1.16 (0.43)	$\chi^2_{10} = 13.10$ (0.22)
3. Returns[b]	1,3,6,12	14	4	1.40 (0.27)	0.36 (0.06)	3.02 (0.32)	1.81 (0.62)	$\chi^2_{10} = 9.54$ (0.48)
4. Returns (3d Moments)	1,3,6,12	22	12	1.40 (0.22)	0.32 (0.05)	2.77 (0.26)	0.86 (0.20)	$\chi^2_{18} = 19.39$ (0.37)
5. Yields	3,6,12	10	22	0.03 (10.68)	0.08 (0.72)	1.27 (0.43)	1.34 (26.08)	$\chi^2_6 = 14.04$ (0.03)
6. Returns	3,6,12	10	4	3.05 (0.39)	0.62 (1.21)	3.05 (0.35)	1.64 (5.42)	$\chi^2_6 = 50.62$ (0.00)
7. Returns (1964-76)	1,3,6,12	14	4	0.85 (0.00)	0.48 (0.00)	2.34 (0.00)	0.00 (0.00)	$\chi^2_{10} = 7.30$ (0.70)
8. Returns (1977-89)	1,3,6,12	14	4	2.64 (0.36)	0.30 (0.06)	4.53 (0.43)	1.65 (0.92)	$\chi^2_{10} = 7.57$ (0.67)

Notes: a. All returns are for monthly holding periods; all yields are for holding periods equal to the Bill's maturity.

b. In Row 3, the comoment restrictions employ a two month lag — in all other rows, the lag is one month.

Table 4

Theoretical Autocorrelations Using Covariances Implied By CIR Parameter Estimates and Sample Standard Deviations

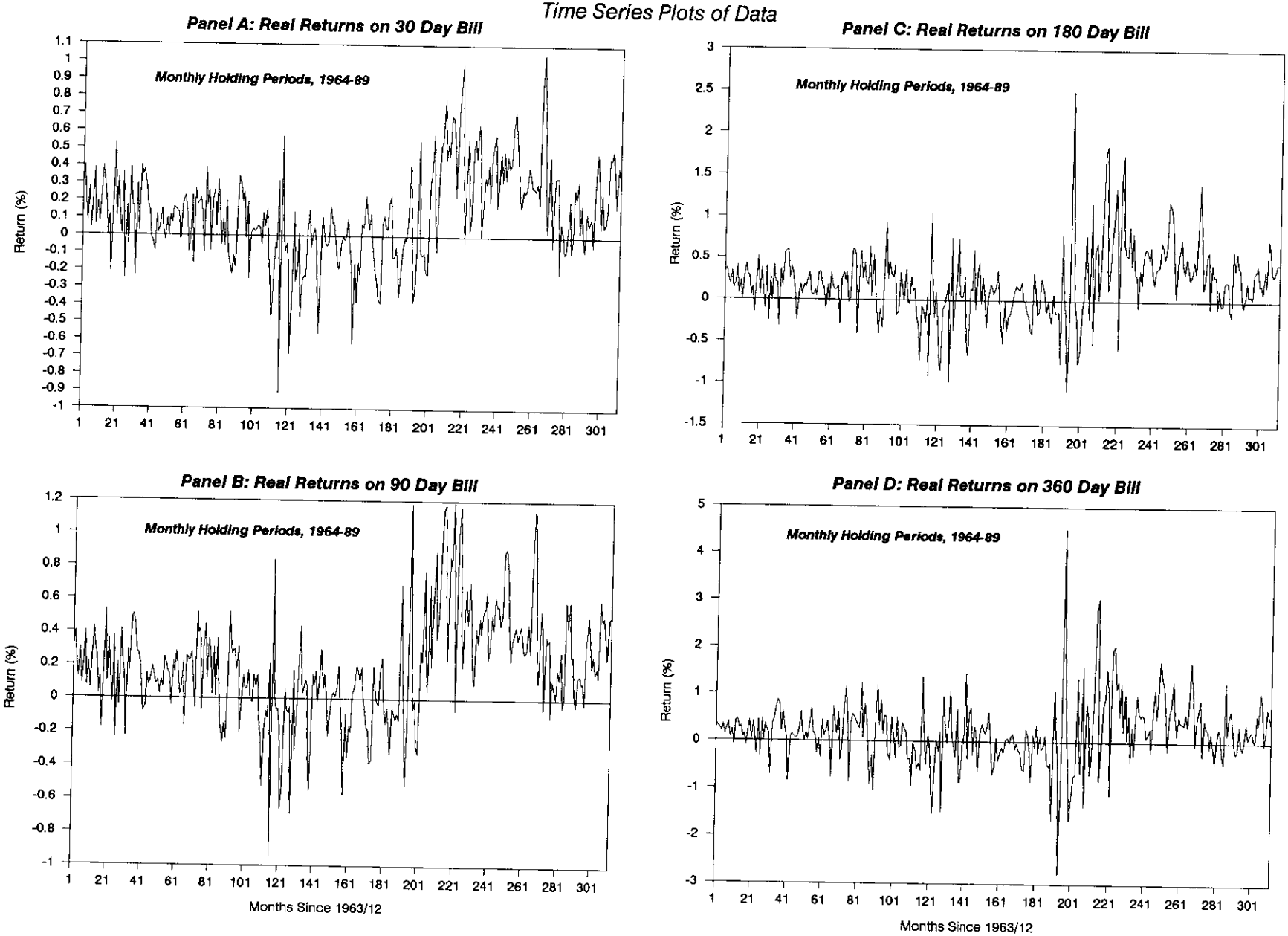
Monthly Returns on 1, 3, 6 and 12 Month Bills, Using CIR Formula Prices
CIR Parameter Estimates Used: $\theta = 0.0154, \kappa = 12.43$
$\lambda = -6.08$ and $\sigma = 0.49$

Sample Standard Deviations, from monthly data, 1964-89:
0.273%, 0.322%, 0.434% and 0.699% for
1, 3, 6, and 12 month Bills respectively.

Bill Maturity	*Autocorrelations at Lag*											
	1	2	3	4	5	6	7	8	9	10	11	12
1 Mo.	0.030	0.011	0.004	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3 Mo.	-0.074	-0.023	-0.009	-0.003	-0.001	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
6 Mo.	-0.073	-0.026	-0.009	-0.003	-0.001	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
12 Mo.	-0.031	-0.011	-0.004	-0.001	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000

Figure 1

Time Series Plots of Data



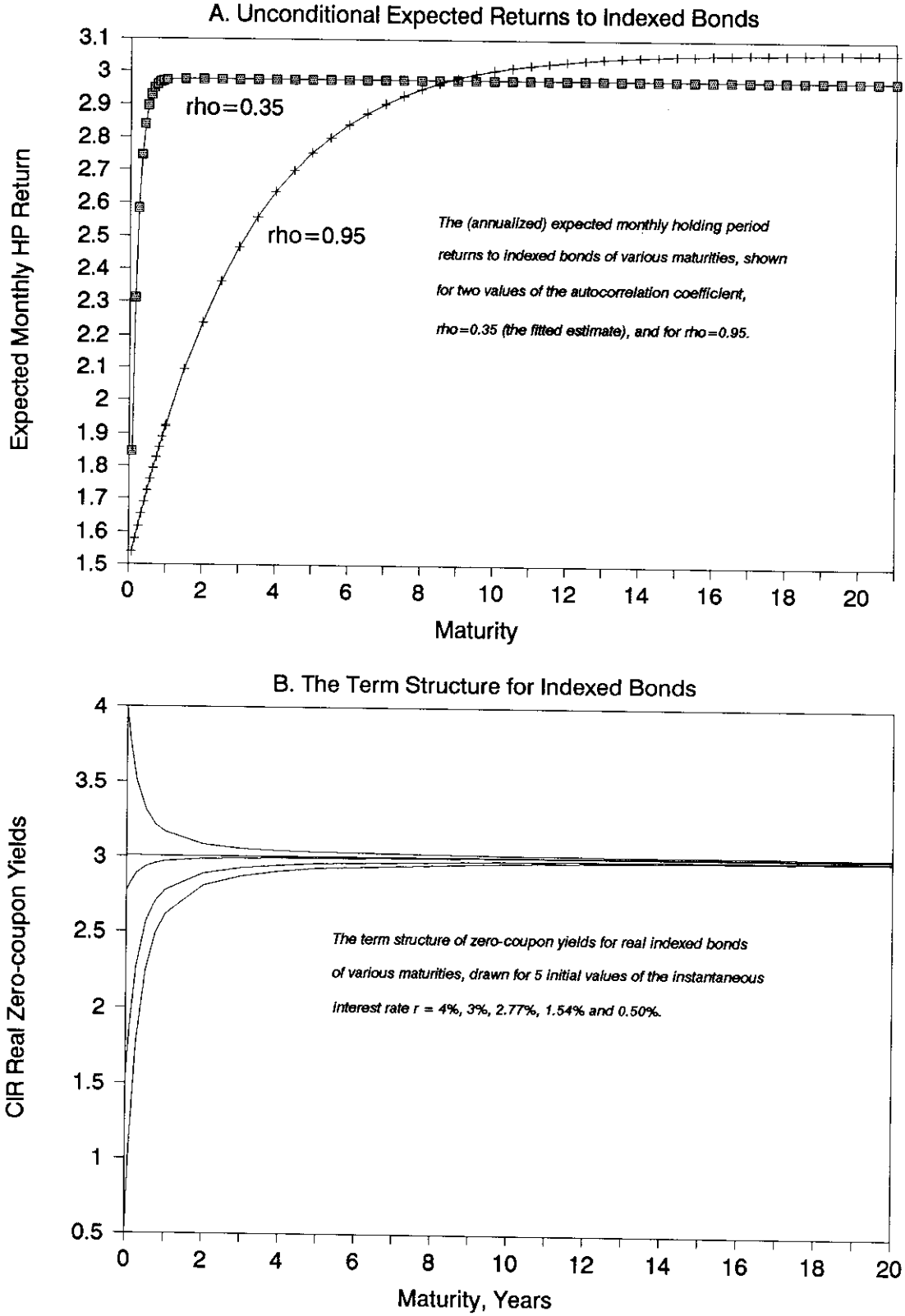


Figure 2. Bond Returns and Yields-to-Maturity for the CIR Model, Using GMM Parameter Estimates, 1964-89