QUANTITY-ADJUSTING OPTIONS AND FORWARD CONTRACTS

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ABSTRACT

Quantity-adjusting option and forward contracts deliver a payoff on a variable quantity of underlying. This paper explains the use, pricing and hedging of such contracts and extends them to sequential investment options. These are options which guarantee optimal asset selection at switching points among a fixed set of traded assets.

Applications include domestic equity derivates held by a foreign investor and hedged into that investor's home currency. Similar hedges on foreign equities for domestic investors can be constructed using symmetry considerations if the investors are running their own hedge. If, however, the hedge is run by the same investment bank, the problems are not symmetric. Such hedges are not attainable using a buy-and-hold strategy with standard options and currency contracts.

Finally, in the presence of a forward contract on an inflation index, the real value implied by the forward contract of equity derivatives may be hedged.

Over the past four years, a spate of new investment contracts has been developed and marketed that allows investors to hold portfolios or options on portfolios of equities denominated in non-domestic currencies, but without foreign currency (hereafter Fx) risk.[1,2] The problem with using standard forward contracts or Fx options to manage the currency risk is that the quantity of domestic currency (Dx) coverage provided does not adjust to the Fx value of the underlying securities. Therefore, an investor hedging with standard instruments an investment in foreign equities would be exposed to Fx risk to the extent of any unanticipated changes in the value of foreign equities. To avoid this exposure, a new technology was developed that produced "quantity-adjusting" forward contracts (QAF) and quantity-adjusting Fx options (QAO).

For example, dollar-denominated currency-hedged put warrants are available at the American Stock Exchange on the Nikkei 225. These put warrants expire in three years, and are exercisable at any time. The warrants have a payoff that depends upon the difference between the Nikkei 225 average (in Japanese yen) on the expiration or exercise date and the strike price (in yen). If this difference is positive, there is no payoff. If the difference is negative, the investor receives the (absolute value of the) difference multiplied by the number of units specified in the contract, and translated into dollars at an exchange rate that was fixed from the outset. This put warrant differs from the standard put on the Nikkei 225 in that any gain realized on the standard warrant would generally need to be translated into dollars at the spot exchange rate prevailing at the time of the gain. (If the warrant were hedged with a standard option or forward contract, the unanticipated gain still would be subject to the Fx risk.)

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[1]The authors would like to thank Abba Krieger for his comments.

[2]Goldman, Sacks & Co. developed currency hedged options in 1986, followed by Salomon Brothers in 1987, and then a number of other firms. These securities had modest sales until 1989 and 1990, at which time sales soared. During the first quarter of 1990, the volume of trading in one such quantity-adjusting option, the Nikkei 225 put warrants, accounted for over fifty percent of all transaction volume on the American Stock Exchange.

In this paper we will not elaborate a theory of hedging, nor attempt to justify the demand for such quantity-adjusting financial contracts.[3] Rather, we take as given that there is a demand for these products, and that this demand is being supplied, to some extent. The finance literature, however, is silent on how a supplier of these instruments could offer them without assuming undue risk. This paper is an effort to fill that void.[4] We find these quantity-adjusting contracts to be special cases of a more general class of contracts (dubbed GOs here). A valuation model for GOs is set forth and numerical illustrations of the specialized cases of QAFs and QAOs are given. We also explore other applications in the family of GOs.

In Section I of this paper, the relationship between our model and the previous, more specialized models of Fischer [1978], Margrabe [1978] and Stulz [1982] is explained. In Section II we explore eight applications within the family of GOs. These applications further clarify the differences between the GOs and the options described by Fischer, Margrabe and Stulz. In Section III we present a methodology whereby an issuer of GOs could properly price and hedge these securities. The GO pricing function solves the P.D.E. which is derived in this section. Section IV derives the solution. To illustrate this methodology, in Section V we apply it to the two special cases mentioned at the outset: QAFs and QAOs. Section VI presents sequential switching or "guru" options as another example of the GO pricing methodology. Section VII concludes.

I. Relationships Between Generalized Options (GOs) and Other Options

A useful way of understanding GOs is as a generalization of extant option pricing models. There are two procedures that one might use to demonstrate that one pricing model is a special case of another. Procedure One constrains a variable or a parameter to restrict the model to a special case. Consider the case where Model A allows x to

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[3]The theory of hedging has some intriguing dimensions, and has already received some attention. For example, see Stulz [1982] and Smith and Stulz [1983].

[4]Our focus is on European style options. We leave to a second paper numerical techniques for American style option pricing and hedging.

vary, but Model B fixes x as a parameter, and the formulas to A simplify to those of B when x is constrained to be constant. Procedure One would call Model B a special case of A. Procedure Two includes procedure one but allows for an additional test which is sufficient for B to be a special case of A. By Procedure Two, one asks whether there is a portfolio which can be constructed using a buy-and-hold strategy on securities priced using Model A to create a security priced by Model B. Procedure One, for example, would not view the model for a call option spread (in a Black-Scholes economy) as a special case of the Black-Scholes option pricing model. Procedure Two would. Both procedures would view a stochastic volatility option pricing model which simplified to the Black-Scholes equation where σ was held constant as a general case of the Black-Scholes model.

With these procedures in mind, we now set forth the structure of a GO. The payoff of a GO, Z(T), at exercise is

$$Z(T) = \text{ext}\{X, Y\}\ \text{ext}\{U, V\}\ , \tag{1.1}$$

where

$$\text{ext} = \min \text{ or } \max\ .$$

It shall be assumed that

$$\frac{du}{u} = \alpha_u dt + \sigma_u d\tilde{W}_u\ , \tag{1.2}$$

u = X, Y, U, V.

The GO can be viewed as a general case of an option to exchange one asset for another as described in Margrabe [2], a general case of an option with an uncertain exercise price as in Fischer [1], or a general case of an option on the minimum or maximum of two risky assets as in Stulz [4]. Formally, a Stulz option pays

$$\text{ext}\{X, Y\}\ . \tag{1.3}$$

One can think of a GO as an option on the minimum or maximum of two risky assets where the quantity is adjusted by a multiple of the payoff to another option on the minimum or maximum of two other risky assets.

When the uncertain exercise price in Fisher's model is the price of a traded asset, the Fischer model becomes the same as Stulz' model. Margrabe's model for an option to exchange one asset for another is an option that has the payoff:

$$M(T) = \max\{X, Y\} - \min\{X, Y\} . \tag{1.4}$$

Thus, if at expiration $X > Y$, then

$$M(T) = X - Y. \tag{1.5}$$

To make the exchange and start with Y to end up with X requires owning Y. Thus, a Margrabe option plus Y gives X:

$$M(T) + Y = (X - Y) + Y = X. \tag{1.6}$$

By Procedure Two, Margrabe options are special cases of Stulz options, and Stulz options are special cases of GOs.

II. Applications of Generalized Option Technology

The generalized option pricing technology paves the way for several new kinds of option-like contracts. We present six potential applications below.

Case (i). An application of the most general specification of a GO would be an option sold to a firm, which does its accounting in U.S. dollars, on the maximum of two Japanese stocks (U and V) converted into either British pounds or French francs at pre-specified pound-yen and franc-yen rates, respectively, and valued in U.S. dollars at the future spot rate.

Case (ii). An application of GOs is to provide insurance on successive investment switching decisions, "switching options". These decisions can be timing decisions on when to switch from one project to the other one in a capital budgeting problem or when to switch between two alternative investments for a portfolio. (While the example here is kept to two periods and two alternatives, the technique used allows for n periods.) The investor makes his initial investment, $\$I_0$, at time 0, decides to switch or not to switch at time 1, and receives the payoff to his investment at time 2. For each \$1 invested at time 0, let the payoff on the investor's best decision be $\$I_1$. For each \$1 invested at time 1, let the payoff on the investor's best decision be $\$I_2$. The investor's final payoff is:

$$I(T) = I_0 \cdot I_1 \cdot I_2 \tag{2.1}$$

or

$$I(T) = I_0 \max\{X_1, Y_1\} \cdot \max\{X_2, Y_2\} \tag{2.2}$$

$$= I_0 \cdot Z(T), \tag{2.3}$$

where Z(T) is one of the four cases characterized in equation (1). The price of the GO at time 0, Z(0), is the value of perfect foresight at time 0 on the ranking of investments X and Y over the partition [0,1], [1,2]. If a fortune teller or "financial guru" came along who could predict the market, his information would be worth the value of insurance which guarantees this timing decision, Z(0). If he offered a partial deal for period 2 only, the value would be that of Stulz option.

Note that the GO model allows for nonzero serial correlation and nonzero cross section correlation. One can also compare the value of perfect information on rankings over different partitions or different sets of projects or investments.

Also the incremental value of perfect foresight over a particular information set can be compared. The information set is a variance-covariance matrix for the $n \times m$ Weiner

Processes known at the date on which one could buy a GO for the remaining investment horizon.

By comparing the cost of the GO for different information sets and partitions, one can rank them by level of information. Because the cost of the GO is the cost of insurance against bad timing on investment, the higher the cost of the GO, the lower the value of initial information.

Case (iii). Another application is an equity option which is protected against inflation, given the existence of a futures or forward market indexed to inflation. Consider a put option on a stock portfolio. The payoff in nominal units, denoted in capital letters, is

$$P(T) = \max\{K - S, 0\} . \tag{2.4}$$

The payoff in real units, denoted in small letters, is obtained by dividing the nominal payoff by the inflation index, I(T) (dollars/real units) and is given by p(T) where[5],

$$p(t) = \frac{P(T)}{I(T)} . \tag{2.5}$$

The inflation hedged put pays off in dollars,

$$Q(T) = I(T)P(T). \tag{2.6}$$

In real units its payoff is denoted by q(T) where,

$$q(T) = P(T) . \tag{2.7}$$

That is, the payoff to the inflation-hedged put measured in real units is hedged against

[5]In the absence of complete markets where a security trades whose price is a deterministic function of the inflation rate, I(T) is the implied inflation index.

inflation. Hence the payoff in real units is independent of I(T).

Case (iv). Similar to an inflation-hedged equity put is an inflation-hedged contract which has a payoff equal to the nominal price of the stock adjusted for inflation. Its payoff is denoted by Fd(T) for forward, and in nominal units

$$\mathrm{Fd(T) = I(T)S(T)}. \tag{2.8}$$

The payoff in real units, denoted fd(T), is given by

$$\mathrm{fd(T) = S(T)}\ . \tag{2.9}$$

Case (v). If X is the spot \$/¥ exchange rate and Y is set to a constant, Y^*, ($\sigma_Y = 0$), the contract is an option which allows the investor to convert, at the better of an exchange rate written into the contract or the spot rate on exercise, the yen payoff of an option on the maximum of two risky assets denominated in yen. That payoff is:

$$\max\{X, Y^*\} \max\{U, V\}. \tag{2.10}$$

Case (vi). If we set X equal to 0, keep Y as is and set V equal to a constant ($\sigma_V = 0$), we get a QAF, a quantity-adjusting forward contract, selling yen and buying dollars adjusted to the maximum of the Japanese yen price of the Japanese stock and a zero coupon yen bond maturing on the option expiration date. This gives a payoff of:

$$Y^* \max\{U, Y^*\}\ . \tag{2.11}$$

Case (vii). To get a QAO similar to that traded on the AMEX, but European style and generalized for optional currency conversion, one has a contract with the expiration payoff:

$$C(T) = Y^* \max\{U - V, 0\}. \tag{2.12}$$

Following Procedure Two, this is a special case of a GO. Buy a GO where X is restricted to 0, and Y is set to Y^*. The expiration payoff to this security is:

$$C_1(T) = \max\{0, Y^*\} \max\{U, V\} \tag{2.13}$$

$$= Y^* \max\{U, V\}\ . \tag{2.14}$$

Sell a GO where U is further restricted to 0. The payoff to this short position is:

$$C_2(T) = -\max\{0, Y^*\} \max\{0, V\} \tag{2.15}$$

$$= -Y^*V\ . \tag{2.16}$$

Note that this is the payoff to a QAF where the adjustable quantity is the yen price of V. Adding (2.13) and (2.15) we get:

$$C_1(T) + C_2(T) = Y^* \max\{U, V\} - Y^* V \tag{2.17}$$

$$= Y^* \max\{U - V, 0\} \tag{2.18}$$

$$= C(T). \tag{2.19}$$

Note also that by restricting X and Y to equal 1 ($\sigma_X = \sigma_Y = 0$), the QAO becomes an option on the maximum of U and V, or the kind of option modeled by Stulz [4] .

Because QAOs represent a specialization of GAOs which help motivate the GAO model, it is of interest to note that Stulz options are a special case of QAOs and QAOs are not a special case of Stulz options. To see that Stulz options are a proper subset of

QAOs recall the QAO payoff as characterized in (1.3). Multiplying through by Y^* the QAO payoff is

$$C(T) = \max\{Y^*U - Y^*V, 0\} . \tag{2.20}$$

This payoff looks like that of a Fischer option, but it is not. Y^*V is just the value of a portfolio made up of a zero-coupon yen bond with a face value of V yen plus a forward contract to sell V yen for delivery into U.S. dollars on date T at Y^* dollars per yen. Y^*V, however, is a QAF and can be neither priced nor hedged using Fischer or Stulz options.

Case (viii). Similar to (vi), but with optionality on the currency instead of on the stock, consider a quantity-adjusting option to sell yen into dollars. The option is a QAO because the amount of yen to be sold is equal to the price of a Japanese stock. Using the notation of (1), the payoff is:

$$\max\,\{X, Y^*\} \cdot U. \tag{2.21}$$

Here X is the U.S. Dollar price of a Japanese yen. Y^* is the U.S. dollar strike price of a Japanese yen. U is the yen price of the Japanese stock.

III. GO Model Development – Deriving the PDE

First obtain the risk premium to the GO, $\alpha_z - r$. Second apply Ito's Lemma to obtain dZ and hence E[dZ]. Set

$$\alpha_Z - r = \frac{E\{dZ\}}{Z} - r \tag{3.1}$$

to obtain the P.D.E.

For the purpose of derivation we will use X_i, $i = 1,\ldots,4$ to denote the four underlying: X, Y, U, V. Let

$$dX_i = \alpha_i X_i dt + \sigma_i X_i d\tilde{W}_i \tag{3.2}$$

be the return process for each of the underlying securities, and

$$dZ = \alpha_z Z dt + Z(\sum_{i=1}^{4} \sigma_{z_i} d\tilde{W}_i) \tag{3.3}$$

characterize the return process for the GO. Then the risk premium for the GO, the expected return minus the risk-free rate, is

$$\alpha_z - r = \sum_{i=1}^{4} \frac{\sigma_{z_i}}{\sigma_i} (\alpha_i - r) \tag{3.4}$$

From this Lemma

$$dZ = Z_t dt + \sum_{i=1}^{4} Z_i dX_i + \sum_{i}^{4} \sum_{j}^{4} Z_{ij} dX_i dX_j \; ;$$

$$Z_{ij} \equiv \frac{\partial Z}{\partial X_i \partial X_j} \; ; \tag{3.5}$$

$$Z_i \equiv \frac{\partial Z}{\partial X_i} \, .$$

For the left-hand side of (3.4), expand and take the expectation of $\frac{dZ}{Z}$,

$$\alpha_z = \frac{Z_t}{Z} + \frac{1}{Z} \sum_{i=1}^{4} Z_i X_i \alpha_i + \frac{1}{2Z} \sum_{i=1}^{4} \sum_{j=1}^{4} Z_{ij} \sigma_i \sigma_j \rho_{ij} \, . \tag{3.6}$$

For the right-hand side of (3.4)

$$\sigma_{z_i} = \frac{Z_i X_i \sigma_i}{Z} \quad . \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.4), the P.D. E. is

$$\frac{1}{Z}\left[\frac{1}{2}\left(\sum_{i=1}^{4} \sum_{j=1}^{4} Z_{ij}\, \sigma_i \sigma_j \rho_{ij}\right) + Z_t + \sum_{i=1}^{4} Z_i X_i \alpha_i\right] - r = \sum_{i=1}^{4} \frac{Z_i X_i}{Z} (\alpha_i - r). \tag{3.8}$$

Multiply by Z and cancel redundant terms,

$$\frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} Z_{ij}\sigma_{ij} + Z_t + \sum_{i=1}^{4} Z_i X_i r - rZ = 0 . \qquad (3.9)$$

Note, however, that when X_i is foreign currency one substitutes

$$Z_i X_i (r - r_i) \qquad (3.10)$$

for

$$Z_i X_i r , \qquad (3.11)$$

where r_i is the interest rate of the currency in which asset i is denominated.

IV. Solution to the GO P.D.E.

A. The Truncated Expectation of A Multivariate Normal

In Part A, the solution to the P.D.E. is represented using a multivariate normal. Part B decouples the multivariate normal subject to the bounds of integration. Integrating the decoupled distribution, using the bounds of integration, is left for the Appendix.

The P.D.E. is solved by calculating the expected contract payoff, discounted at the risk-free rate, to obtain Z(t), the current no-arbitrage price of a GO:

$$Z(t) = e^{-r\tau} E[Z(T)] ; \qquad (4.1)$$

$$Z(T) = \max\{X_1(T), X_2(T)\} \max\{X_3(T), X_4(T)\} . \qquad (4.2)$$

Given the assumptions on the return generating processes of the underlying securities (Equation 1.2), the solution to the P.D.E. is expressed in terms of a multivariate normal. Let

$$x_i \equiv \ln X_i \qquad (4.3)$$

then

$$\begin{aligned} E[Z(T)] = \\ E[e^{(x_1 + x_3)} \mid x_1 \geq x_2, x_3 \geq x_4] \cdot \text{Prob}(x_1 \geq x_2, x_3 \geq x_4) \\ + E[e^{(x_1 + x_4)} \mid x_1 \geq x_2, x_4 \geq x_3] \cdot \text{Prob}(x_1 \geq x_2, x_3 \leq x_4) \\ + E[e^{(x_2 + x_3)} \mid x_2 \geq x_1, x_3 \geq x_4] \cdot \text{Prob}(x_2 \geq x_1, x_3 \geq x_4) \\ + E[e^{(x_2 + x_4)} \mid x_2 \geq x_1, x_4 \geq x_3] \cdot \text{Prob}(x_2 \geq x_1, x_3 \leq x_4) . \end{aligned} \tag{4.4}$$

Let $g(\mathbf{x})$ be the multivariate normal distribution with variance covariance matrix

$$\Sigma = [\sigma_{ij}] , \tag{4.5}$$

and $\mathbf{x} = (x_1, x_2, x_3, x_4)$.

$E[Z(T)]$ expressed in integral form is

$$E[Z(T)] = \int_{\Omega_{12,34}} e^{(x_1 + x_3)} g(\mathbf{x})\, d\mathbf{x} + \int_{\Omega_{21,43}} e^{(x_2+x_4)} g(\mathbf{x})\, d\mathbf{x} \tag{4.6}$$

$$+ \int_{\Omega_{21,34}} e^{(x_2 + x_3)} g(\mathbf{x})\, d\mathbf{x} + \int_{\Omega_{12,43}} e^{(x_1 + x_4)} g(\mathbf{x})\, d\mathbf{x} ;$$

$$g(\mathbf{x}) = Ke^{-\frac{1}{2}\mathbf{x}'-\mu\Sigma^{-1}(\mathbf{x}-\mu)} ; \tag{4.7}$$

$$K = \frac{\prod_{i=1}^{p} d_i^{\frac{1}{2}}}{(2\pi)^{p/2}} ; \quad p = 4 ; \tag{4.8}$$

$$\Omega_{ij,kl} = \{\mathbf{x}: x_i \geq x_j,\; x_k \geq x_l\} \tag{4.9}$$

for i, j, k and l between 1 and 4 inclusive. d_i , i = 1,...,4 are the characteristic roots of Σ. In (4.6) the integral sign represents a multiple integral in 4-space. For example, with the bounds of integration in the first term in (4.6),

$$\int_{\Omega_{12,34}} f(\mathbf{x})d\mathbf{x} = \int_{x_4=-\infty}^{\infty} \int_{x_2=-\infty}^{\infty} \int_{x_3=x_4}^{\infty} \int_{x_1=x_2}^{\infty} f(\mathbf{x})d\mathbf{x} \qquad (4.10)$$

for any $f(\mathbf{x})$. Note that generalizations to switching options or basket QAOs and QAFs, for example, could be for N greater than 4.

Following standard practice, subtract off the means from x, to get

$$\mathbf{y} \equiv \mathbf{x} - \boldsymbol{\mu} \ . \qquad (4.11)$$

The Jacobian of the transformation is unity. Obtain for each original region of integration $\Omega_{ij,kl}$ with $x_i(y_i) = y_i + \mu_i$ for i = 1, . . . , 4,

$$E[Z(T)] = \int_{\Omega_{12,34}(\mathbf{x}(\mathbf{y}))} e^{[x_1(y_1) + x_3(y_3)]} K e^{-\frac{1}{2}\mathbf{y}'\Sigma^{-1}\mathbf{y}} d\mathbf{y} \qquad (4.12)$$

$$+ \int_{\Omega_{12,43}(\mathbf{x}(\mathbf{y}))} e^{[x_1(y_1) + x_4(y_4)]} K e^{-\frac{1}{2}\mathbf{y}'\Sigma^{-1}\mathbf{y}} d\mathbf{y}$$

$$+ \int_{\Omega_{21,34}(\mathbf{x}(\mathbf{y}))} e^{[x_2(y_2) + x_3(y_3)]} K e^{-\frac{1}{2}\mathbf{y}'\Sigma^{-1}\mathbf{y}} d\mathbf{y}$$

$$+ \int_{\Omega_{21,43}(\mathbf{x}(\mathbf{y}))} e^{[x_2(y_2) + x_4(y_4)]} K e^{-\frac{1}{2}\mathbf{y}'\Sigma^{-1}\mathbf{y}} d\mathbf{y} \ .$$

Thus we have obtained a solution to the P.D.E. in terms of 4 regions of integration to a 4-vector multivariate normal.

B. Diagonalizing the Variance-Covariance Matrix

The next step is to simplify (4.12). We do this by diagonalizing the variance-covariance matrix. The bounds of integration do not allow the integral to be represented as the product of 4 cumulative normals but we are able to simplify the expression. Since Σ is positive definite, by a standard theorem on quadratic forms there exists an orthogonal matrix P such that

$$\mathrm{P}' \, \Sigma^{-1} \, \mathrm{P} = \mathrm{D} \ , \tag{4.13}$$

where D is diagonal with the characteristic roots d_i of Σ^{-1} displayed on the diagonal. $g(\cdot)$ is to be integrated over each of the four regions given in (4.6). Decouple the multivariate normal subject to the constraints from the integration bounds.

For the most general derivation of the GO, intuition might suggest that the 4-vector multivariate normal decouples into the product of two bivariates. Because of the bounds of integration, however, even with the variance-covariance matrix diagonalized, we have not obtained this result. For the more specialized applications of the GOs, namely QAFs and QAOs decoupling does occur.

The GO Model applied to the QAF case uses a multivariate normal decoupled into the product of four univariate normals. One obtains the result that QAF prices are a function of the expectation of a normal and the integrals drop out altogether. For the case of a QAO one has again a univariate normal, but because the bounds of integration are not the whole line the price formula has a cumulative normal.

Transform $\mathbf{y}$ to $\mathbf{t}$ by the transformation

$$\mathbf{y} = \mathbf{P}\mathbf{t} \, . \tag{4.14}$$

The determinant of an orthogonal matrix is 1. Substituting for $\mathbf{y}$,

$$\mathbf{y}'\Sigma^{-1}\,\mathbf{y} = \mathbf{t}'\mathbf{P}'\Sigma^{-1}\,\mathbf{Pt} \;; \tag{4.15}$$
$$= \mathbf{t}'\mathbf{D}\,\mathbf{t}.$$

For the bounds of integration,

$$\Omega_{ij,kl}(\mathbf{x}(\mathbf{y})) \tag{4.16}$$

becomes

$$\Omega_{ij,kl}\big(\mathbf{x}(\mathbf{y}(\mathbf{t})\big) = \Omega_{ij,kl}(\mathbf{x}(\mathbf{Pt}))\;.$$

The expected value of Z(T) becomes

$$E(Z(T)] = \tag{4.17}$$

$$\int\limits_{\Omega_{12,34}(\mathbf{x}(\mathbf{Pt}))} e^{[x_1(\mathbf{Pt})+x_3(\mathbf{Pt})]}\,Ke^{-\frac{1}{2}\mathbf{t}'\mathbf{Dt}}\,d\mathbf{t} + \int\limits_{\Omega_{12,43}(\mathbf{x}(\mathbf{Pt}))} e^{[x_1(\mathbf{Pt})\,+\,x_4(\mathbf{Pt})]}\,Ke^{-\frac{1}{2}\mathbf{t}'\mathbf{Dt}}\,d\mathbf{t}$$

$$+ \int\limits_{\Omega_{21,34}(\mathbf{x}(\mathbf{Pt}))} e^{[x_2(\mathbf{Pt})\,+\,x_3(\mathbf{Pt})]}\,Ke^{-\frac{1}{2}\mathbf{t}'\mathbf{Dt}}\,d\mathbf{t} + \int\limits_{\Omega_{21,43}(\mathbf{x}(\mathbf{Pt}))} e^{[x_2(\mathbf{Pt})\,+\,x_4(\mathbf{Pt})]}\,Ke^{-\frac{1}{2}\mathbf{t}'\mathbf{Dt}}\,d\mathbf{t}\,.$$

Because **D** is a diagonal matrix,

$$E[Z(T)] = \int\limits_{\Omega_{12,34}(\mathbf{x}(\mathbf{Pt}))} e^{[x_1(\mathbf{Pt})\,+\,x_3(\mathbf{Pt})]}\,K\exp^{\left(\frac{1}{2}\sum_{i=1}^{4} d_i t_i^2\right)}\,d\mathbf{t} \tag{4.18}$$

$$+ \int\limits_{\Omega_{12,43}(\mathbf{x}(\mathbf{Pt}))} e^{[x_1(\mathbf{Pt})\,+\,x_4(\mathbf{Pt})]}\,K\exp^{\left(\frac{1}{2}\sum_{i=1}^{4} d_i t_i^2\right)}\,d\mathbf{t}$$

$$+ \int\limits_{\Omega_{21,34}(\mathbf{x}(\mathbf{Pt}))} e^{[x_2(\mathbf{Pt})\,+\,x_3(\mathbf{Pt})]}\,K\exp^{\left(\frac{1}{2}\sum_{i=1}^{4} d_i t_i^2\right)}\,d\mathbf{t}$$

$$+ \int_{\Omega_{21,43}(\mathbf{x}(\mathbf{Pt}))} e^{[x_2(\mathbf{Pt}) + x_4(\mathbf{Pt})]} \, K \exp^{\left(-\frac{1}{2}\sum_{i=1}^{4} d_i t_i^2\right)} \mathbf{dt} \ .$$

Because x_i is an affine transformation of the components of $\mathbf{t}$, $(\mathbf{x} = \mathbf{Pt} + \boldsymbol{\mu})$, the power of e for each of the four integrals is a quadratic in the components of $\mathbf{t}$. For the last region of integration in the above equation, which we denote as I_4, for example,

$$I_4 = \int_{\Omega_{21,43}(\mathbf{x}(\mathbf{Pt}))} K \exp^{\left(-\frac{1}{2}\sum_{i=1}^{4} d_i t_i^2 + \sum_{j=1}^{4} a_j t_j + b\right)} \mathbf{dt} \ ; \tag{4.19}$$

$$a_j \equiv \mathbf{P}_{2j} + \mathbf{P}_{4j} \ ; \quad b = \mu_2 + \mu_4 \ .$$

By "completing the square," and multiplying by e taken to the power of a suitable constant, the exponent can be represented as a quadratic form with a diagonal matrix. Subject to the bounds of integration, each 4-integral becomes the integral of the product of four non-standard normals. These are easily transformed into cumulative standard normals.

Denote the exponent to e for the last term in (4.19) as Γ with

$$\frac{b}{4}\sum_{i=1}^{4} \gamma_i \equiv \Gamma \ ; \ \ \gamma_i \equiv \ -\tfrac{1}{2}\, d_i t_i^2 \ + a_i t_i \ . \tag{4.20}$$

$$\gamma_i = \ -\frac{1}{2}\, d_i \left(t_i - \frac{a_i}{d_i}\right)^2 + \frac{a_i^2}{2d_i} \ . \tag{4.21}$$

$$e^{\gamma_i} = e^{\frac{a_i^2}{2d_i}} \ e^{-\frac{1}{2}\left(\frac{t_i - \mu_i'}{\sigma_i'}\right)^2} \ ; \ \ \mu_i' = \frac{a_i}{d_i} \ \ ; \ \sigma_i' = \frac{1}{\sqrt{d_i}} \tag{4.22}$$

$$I_4 = K' \int_{\Omega_{21,43}(\mathbf{x}(\mathbf{Pt}))} \prod_{i=1}^{4} e^{-\frac{1}{2}\left(\frac{t_i - \mu_i'}{\sigma_i'}\right)^2} dt_i \; ; \tag{4.23}$$

$$K' = \exp\left\{ \sum_{i=1}^{4} \frac{\mu_i'^2}{2\sigma_i'} + b \right\} \prod_{i=1}^{4} \frac{1}{\sqrt{2\pi}} \sigma_i' \; . \tag{4.24}$$

$$I_4 = \exp\left\{ \sum_{i=1}^{4} \frac{\mu_i'^2}{\sigma_i'} + b \right\} \int_{\Omega_{21,43}} \prod_{i=1}^{4} n(t_i;\, \mu_i',\, \sigma_i')\, dt_i \, , \tag{4.25}$$

where $n(x_i,\, \mu_i';\, \sigma_i')$ is a normal with mean μ_i' and standard deviation σ_i'. With μ_i', σ_i' and b suitably defined, similar equations hold for I_i, i = 1,...,3. Conditions which simplify (4.25) as applied to $I_1, \ldots, I_4$ are given in the Appendix, "Bounds of Integration and Decoupling the Multivariate Distribution".

V. QAFs and QAOs

To illustrate an application of the GO pricing model, consider an investor who wishes to take a long position in non-domestic equities (denominated in a currency other than his own), but who desires to avoid any direct foreign exchange risk.[6] Such a position could be achieved by purchasing some foreign equities directly, and by purchasing either a quantity-adjusting forward contract (QAF) to repurchase domestic currency (Fx) at or above a given Fx strike price. The quantity of Fx coverage would adjust over time to whatever level would be required to accommodate the full Fx value of equities. The QAF would lock in a given exchange rate, whereas the QAO would give the investor the opportunity to benefit from any appreciation in the value of Fx.

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[6]It is clear that at some level, an investor who hedges away the direct Fx risk is only hedging away a part of Fx risk. For example, the foreign business market may be affected by the international business climate, or there may be component parts of the products being produced abroad that are acquired through trade, and subject to Fx risk.

A distinction is best made at this point between the purchaser of the QAO or QAF and the vendor of this security. An investor who purchases this derivative as a currency-hedged investment would naturally buy such a derivative on non-domestic securities only. The vendor, presumably a bank or trading firm, might sell such securities to foreign investors as well as domestic investors. This presents the bank with the need to synthesize and value QAOs and QAFs on both domestic securities with cash delivery in foreign currency and non-domestic securities with cash delivery in domestic currency.

Let us call the former derivative an "incoming" QAO or QAF and the latter an "outgoing" QAO or QAF. It is immediately apparent that in the absence of arbitrage opportunities, there can be only one price for a contract. For example, suppose that a Japanese broker were to sell an outgoing QAO (to an American pension fund) on Japanese stock with delivery in U.S. dollars. In the absence of arbitrage opportunities, the price charged by Japanese broker would equal the price charged by an American broker for an incoming QAO on the Japanese stock. Nonetheless the hedges would not be identical. This results from the lack of symmetry to the two problems. In fact, the securities available to trade for an outgoing QAO are not those available to trade for an incoming contract. When the Japanese broker trades Japanese stock to hedge the outgoing QAO, he trades a security denominated in Japanese yen, the domestic currency. The same security, when traded by the American bank to hedge an incoming QAO, is a security whose U.S. dollar value is the product of the Japanese yen price of the Japanese stock and the spot U.S. dollar/Japanese yen exchange rate.

On the other hand, if the American bank were to sell an outgoing QAO on an American stock to a Japanese investor, the hedging formulae would be identical to that used by the Japanese broker in the previous problem, even though the parameter values are different.

A. Outgoing QAOs

The outgoing QAO price is given by:

$$C(\tau) = \frac{X(\tau)}{X} S(\tau) \exp(\tau(r - r_f + \rho\psi\sigma)) N(q) - \frac{X(\tau)}{X} K \exp(-\tau r_f) N(q - \sigma\sqrt{\tau});$$

$$q = \frac{1}{\sigma\sqrt{\tau}} [(r + \psi\sigma\rho)\tau + \ln(S/K)] + \frac{\sigma\sqrt{\tau}}{2}; \quad (5.1)$$

$S \equiv$ stock price in domestic currency ;

$r \equiv$ domestic interest rate ;

$r_f \equiv$ foreign interest rate ;

$K \equiv$ strike price in domestic currency ;

$\sigma \equiv$ standard deviation of returns to domestic stock ;

$\psi \equiv$ standard deviation of returns to the domestic price of a unit of foreign currency;

$\rho \equiv$ correlation between returns to *domestic* stock and returns of a unit of foreign currency;

$X \equiv$ U.S. dollar price of one unit of Japanese yen written into the contract ;

$X(\tau) \equiv$ same for spot rate τ time units before expiration ;

$X_f(\tau) \equiv$ forward exchange rate for delivery τ time units in the future (at T).

Note that if we want to write the option price function using forward, instead of spot rates, then one has no change in h_1 or h_2, with the new equation given by

$$C(\tau) = \frac{X_f(\tau)}{X} S(\tau) \exp(\tau\rho\psi\sigma) N(q) - \frac{X_f(\tau)}{X} K \exp(\tau r) N(q - \sigma\sqrt{\tau}) ;$$

$$q = \frac{1}{\sigma\sqrt{\tau}} [(r + \psi\sigma\rho)\tau + \ln(S/K)] + \frac{\sigma\sqrt{\tau}}{2} . \quad (5.2)$$

B. Outgoing QAFs

The outgoing QAF price is obtained from taking the QAO price formula and setting K = 0. This gives N(q) = 1. Equation (5.5) becomes:

$$F(\tau) = \frac{X(\tau)}{X} e^{-\tau r_f} \left[S(\tau)\exp(\tau(r + \rho\psi\sigma)) - K\right] . \quad (5.3)$$

For a zero-delivery price K = 0, we have currency-hedged stock:

$$F(\tau) = \frac{X(\tau)}{X} S(\tau) \exp(\tau(r - r_f + \rho\psi\sigma)) \quad (5.4)$$

Again if we want to write the forward price function using forward, instead of spot rates, the new equation with a zero delivery price is given by:

$$F(\tau) = \frac{X_f(\tau)}{X} S(\tau) \exp(\tau\rho\psi\sigma)) . \quad (5.5)$$

C. Incoming QAOs

The Incoming QAO price is given by:

$$C(\tau) = \frac{X(\tau)}{X} Z(\tau) \exp(\tau(r_f - r - \rho\psi\sigma)) N(q) - \frac{X(\tau)}{X} K \exp(\tau r) N(q - \sigma\sqrt{\tau}) ;$$

$$q = \frac{1}{\sigma\sqrt{\tau}} [(r_f - \psi\sigma\rho)\tau + \ln(S/K)] + \frac{\sigma\sqrt{\tau}}{2} . \quad (5.6)$$

Z $\equiv$ foreign stock price in domestic currency;

$K \equiv$ foreign strike price in foreign currency ;

$\sigma \equiv$ standard deviation of returns to foreign stock in foreign currency;

$\rho \equiv$ correlation between returns of *foreign* stock and returns of a unit of foreign currency;

Formulas for Incoming QAFs are obtained from (5.6) as formulas for Outgoing QAFs (5.3-5.5) were obtained from (5.1) and (5.2).

VI. Sequential Switching or "Guru" Options

A. Description

The second example cited as an application of the GO model in Section II was that of a contract which guarantees its owner optimal switching between two different investments (which are traded assets) over two time periods. Sequential switching options need not be restricted to the 2×2 case, but may be generalized to n investment opportunities over m periods which need not be of equal length. Using the solution technique described in Section IV requires integrating a $(m \times n)-$dimensional multivariate normal. We will restrict ourselves to the 4-dimensional case.

X_1, X_2, Y_1, Y_2 as they used in (2.1-2.3) are gross returns,

$$I_1 = \max\{1 + r_{X_1},\ 1 + r_{y_1}\} ; \tag{6.1}$$

$$I_2 = \max\{1 + r_{X_2},\ 1 + r_{y_2}\} ; \tag{6.2}$$

$$I(T) = I_0 \cdot I_1 \cdot I_2 . \tag{6.3}$$

While it is useful to think of the sequential switching case of a GO as a quantity-adjusting contract on *returns*, to be consistent with our presentation of GOs as options on traded assets, relate an option on returns to an option on traded assets by

$$Z(T) = I_0 \max\{\frac{X_1}{X_0}, \frac{Y_1}{Y_0}\} \max\{\frac{X_2}{X_1}, \frac{Y_2}{Y_1}\} . \tag{6.4}$$

In terms of traded assets, (6.4) says that a quantity of I_0 GO contracts are purchased. Each GO pays the maximum payoff of \$1 invested in X_0 or Y_0 times the maximum payoff at time 2 of \$1 invested in X_1 or Y_1 at time 1. With the time 0 prices of X and Y given by X_0 and Y_0 the first-period GO payoff is the maximum of $1/X_0$ units of X and $1/Y_0$ units of Y. This gives a payoff of X_1/X_0 and Y_1/Y_0 respectively. These are the gross returns for period one. Period two gross returns are construced similarly.

B. Hedging Forwards and European Options when the Payoff is Determined by Asset Prices at more than One Future Date

Pricing and hedging forwards and European options whose contract parameters (strike or delivery price and quantity) are set at the date of sale is standard. GOs are among the set of contracts which have parameters set at a future date.

With the exception of sequential switching options, all of the applications of GOs mentioned in Section II have only one parameter set of a future date: the quantity of currency sold at the predetermined or optional foreign or inflation rate. This quantity is determined by the "state of the world" at only one date: the expiration date. The adjustable quantity is hedged by trading all parts of the GO payoff represented by asset prices starting at the issue date and continuing to the maturity or expiration date.

With the case of sequential switching options, however, this adjustable quantity is determined by "state of the world" at more than one future date. Unlike the purely "cross-sectional" GO applications, for a sequential switching option, the asset prices which determine the quantity-adjusting payoff are *not* hedged by trading all relevant assets starting with the contract issue date.

For example, in (2.1, 2.2) where there are 3 dates (0, 1, 2), I_2 is not determined, much less does it change before date 1. Dynamic trading in the components $I_2(X_2, Y_2)$

does not start until date 1. With this observation in mind some comments about contracts with this property follow.

(i) Sequential Spread Forwards

(a) Payoff

$$Fd(T_2) = [S(T_2) - S(T_1)] \,. \tag{6.5}$$

(b) Hedge

At T_0:

(1) Buy a forward contract with a 0 delivery price on 1 share of stock for delivery at T_2 for an immediate cash outflow of $S(T_0)$.

(2) Sell (a+1) forward contracts with the same specification but for delivery on T_1 for an immediate cash inflow of $(1+a)S(T_0)$ where

$$a = 1 + r[(T_2 - T_1) - (T_2 - T_1)]. \tag{6.6}$$

At T_1:

Borrow $\$(1+a)S(T_1)$, at the risk-free rate and close short position.

At T_2:

(1) Receive $\$S(T_2)$.

(2) Payout $\$S(T_1)$.

(c) Application

Hedging options struck at-the-money at a future date.

(ii) Sequential Ratio Forwards

(a) Payoff

$$Fd(T_2) = S(T_2)/S(T_1) \,. \tag{6.7}$$

(b) Hedge

At T_0:

Buy a zero-coupon bond with \$1 face value and maturity at T_1 for a cash outflow

of \$a.

At T_1:

Buy $1/S(T_1)$ shares of $S(T_1)$ using the \$1 from the maturing bond.

At T_2:

Receive $\$S(T_2)/S(T_1)$.

(c) Application

Hedging guru options.

(iii) Sequential Spread Options

(a) Payoff

$$C(T_2) = \max\{0, Fd(T_2)] , \tag{6.8}$$

where $Fd(T_2)$ is given by (6.5).

(b) Hedge

Hedge as standard option but with the forward in (6.5) as the underlying where

$$Fd(0) = a , \tag{6.9}$$

and where a is given in (6.6).

(iv) Sequential Ratio Options

(a) Payoff

$$C(T_2) = \max\{0, Fd(T_2)\} , \tag{6.10}$$

where $Fd(T_2)$ is given by (6.7) .

(b) Hedge

Hedge as a standard option but with the forward in (6.7) as the underlying.

(c) Application

Stulz option applied to \$1 invested at a future date.

(v) Sequential Ratio Forwards as the Underlying to "Guru" Options

It is immediately apparent that a sequential ratio forward with a zero delivery price is an underlying to a guru option.

C. The Variance-Covariance Matrix for Guru Options

So far, Section VI has described "guru" options in some greater detail and demonstrated that by constructing sequential ratio forwards, markets for guru options are complete. To solve for the pricing formula and hedge ratios, the next step is to derive the variance-covariance matrix for a 2×2 guru options, Σ.

Recall that the payoff on a 2×2 guru option is

$$Z(T) \quad = \quad I_0 \max\left\{\frac{X(T_1)}{X(T_0)}, \frac{Y(T_1)}{Y(T_0)}\right\} \max\left\{\frac{X(T_2)}{X(T_1)}, \frac{Y(T_2)}{Y(T_1)}\right\} . \tag{6.11}$$

As before, Σ is calculated using the returns after making a change of variables. Taking natural logs,

$$\begin{aligned} x_i &\equiv \ln X(T_i) ; \\ y_i &\equiv \ln Y(T_i) . \end{aligned} \tag{6.12}$$

Recall from (1.2) that

$$\frac{du}{u} = \alpha_u dt = \sigma_u dW_u . \tag{6.13}$$

$$\Delta u_i = u_i - u_{i-1} = \int_{T_{i-1}}^{T_i} \frac{du}{u} . \tag{6.14}$$

$$u = x_1 y ; \quad i = 1, 2 .$$

$$\Delta u_i = \alpha u_i \Delta T_i + \sigma u_i \Delta W u_i ; \tag{6.15}$$

$u = x, y; \quad i = 1, 2$.

$$Z(T) = I_0 \max\left\{e^{\Delta x_1}, e^{\Delta y_1}\right\} \max\left\{e^{\Delta x_2}, e^{\Delta y_2}\right\} . \tag{6.17}$$

The next step is to calculate, $\sum (\Delta x_1, \Delta y_1, \Delta x_2, \Delta y_2)$.

The data used to calculate $\sum$ are the four variances of the return process σ_{x_1}, σ_{x_2}, σ_{y_1}, σ_{y_2}, and the correlations between the disturbance terms:

$$\rho_{x_i,y_j} = \frac{1}{\Delta T} E\left\{(\Delta W_{x_i})(\Delta W_{y_j})\right\} . \tag{6.18}$$

The off-diagonal elements of Σ are determined by:

$$\sigma_{x_i,y_i} \equiv \mathrm{Cov}(\Delta x_i, \Delta y_i) \tag{6.19}$$

$$= E\left\{(\sigma_{x_i}\Delta W_{x_i})(\sigma_{y_j}\Delta W_{y_j})\right\}$$

$$= \sigma_{x_i}\sigma_{y_i}\rho_{x_i,y_i}\sqrt{\Delta T_i \Delta T_j} \quad ;$$

$i, j = 1, 2$. If $x = y$, then for an off-diagonal element, $i \neq j$.

If the time partition is uniform then $\Delta T_i = \Delta T_j = \Delta T$, and

$$\sigma_{x_i,y_j} = \sigma_{x_i}\sigma_{y_j}\rho_{x_i,y_j}\Delta T. \tag{6.20}$$

Summarizing, let $u = x, y$ then

$$\mathrm{Cov}[(\sigma_{u_1}W_{u_1})(\sigma_{u_1}W_{u_1})] \tag{6.21}$$

are the entries to the upper left 2×2 submatrix. Similarly

$$\mathrm{Cov}[(\sigma_{u_2} W_{u_2})(\sigma_{u_2} W_{u_2})] \tag{6.22}$$

for u = x, y are the entries of the lower right 2×2 submatrix. For the off-diagonal 2×2 submatrices we have the serial covariance entries

$$\mathrm{Cov}[(\sigma_{u_i} \Delta W_{u_i})(\sigma_{u_j} \Delta W_{u_j})] \; . \tag{6.23}$$

D. Summary

Having shown how the underlying to guru options may be hedged, and how the P.D.E. is solved, we can price and hedge such options.

VII. Concluding Remarks

The GO model includes QAOs and QAFs as special cases. The QAO model, in turn, includes Stulz options as a special case. GAOs can also be used to model "switching options." These are options which guarantee over a finite number of decision dates and a finite number of investments, optimal sequential switching between investments to obtain the maximum return over the investment period which ends on option expiration. Examples of closed-form solutions to GO models were given for the QAO and QAF cases.

APPENDIX

Bounds of Integration and Decoupling the Multivariate Distribution

This appendix simplifies the numerical integration of a 4-vector multivariate normal as it appears in the GO price formlua, (4.18). There are four parts to this section. The bounds of integration for **x** and **t** are given respectively in parts (i) and (ii). An example is given in (iii), and two lemmas which generalize the example are given in (iv). The example shows how the integration over **dt** ϵ R^4 can be reduced to integrating over **dt** ϵ R^2. The first lemma proves that certain conditions are necessary and sufficient to reduce the dimension of the integration. The second lemma gives necessary and sufficient conditions to determine the bounds of integration once the dimensions to integrate over are reduced.

Reduction of the dimensions of integration from 4 to 2 is taken to mean that instead of integrating a multivariate normal over four dimensions, the integration is carried out in two stages as illustrated in (A.21) to obtain (A.23). First cumulative normals are calculated. Then a function of the product of the cumulative normals is integrated when this function can be expressed as a function of (t_3, t_4) ϵ R^2 instead of R^4. To illustrate, consider the number of calculations needed to integrate over 4 dimensions versus the approach which reduces to (A.23). Suppose that over each dimension the numerical integration is calculated 100 times. In 4 dimensions this gives a 4-D grid of 10^8 points at which the calculation is be carried out. Using a "lookup table" or an approximating polynomial for the cumulative univariate normals, calculations are performed over 2 dimensions instead of 4, reducing the number of points at which calculations are made from 100 million to 10,000.

(i) Bounds of Integration of x

Consider the first region of integration in (4.18). Similar reasoning applies to the other three regions:

$$\infty \geq x_4 \geq -\infty \; ; \tag{A.1}$$

$$\infty \geq x_2 \geq -\infty \; ; \tag{A.2}$$

$$x_3 \geq x_4 \; ; \tag{A.3}$$

$$x_1 \geq x_2 \, . \tag{A.4}$$

Next determine the region of integration for **t**.

(ii) Bounds of Integration of t

Recall that

$$\mathbf{x} = \mathbf{Pt} + \boldsymbol{\mu} \, . \tag{A.5}$$

Note that (A.1, A.2) establish no constraints, so we shall work only with (A.3, A.4). To illustrate, for the region of integration defined in (A.1) - (A.4), we work out I_1, the first term in (4.18), for a special case; then we generalize. From (A.3, A.4)

$$\sum_{j=1}^{4} \alpha_j t_j + \mu_3 - \mu_4 \geq 0 \; ; \quad \alpha_j \equiv P_{3j} - P_{4j} \; ; \tag{A.6}$$

$$\sum_{j=1}^{4} \beta_j t_j + \mu_1 - \mu_2 \geq 0 \; ; \quad \beta_j \equiv P_{1j} - P_{2j} \, . \tag{A.7}$$

(iii) An Example: Reducing the Dimension of Integration From 4 to 2

From (A.6) if $\alpha_1 > 0$,

$$t_1 \geq -\sum_{j=2}^{4} \frac{\alpha_j}{\alpha_1} t_j - \frac{1}{\alpha_1} (\mu_3 - \mu_4) \, . \tag{A.8}$$

Similarly from (A.7) if $\beta_2 > 0$,

$$t_2 \geq -\sum_{j=1}^{4} \frac{\beta_j}{\beta_2} t_j - \frac{1}{\beta_2} (\mu_1 - \mu_2) \, . \tag{A.9}$$

Now substitute for t_2 in (A.8) using (A.9) and contingent on $\frac{\alpha_2}{\alpha_1} < 0$ and obtain,

$$t_1 \geq -\sum_{j=3}^{4} \frac{\alpha_j}{\alpha_1} t_j - \frac{1}{\alpha_1}(\mu_3 - \mu_4) + \frac{\alpha_2}{\alpha_1}\Big(\sum_{j \neq 2}^{4} \frac{\beta_j}{\beta_2} t_j + \frac{1}{\beta_2}(\mu_1 - \mu_2)\Big). \quad \text{(A.10)}$$

$$t_1\Big(1 - \frac{\alpha_2}{\alpha_1}\frac{\beta_1}{\beta_2}\Big) \geq -\sum_{j=3}^{4}\Big(\frac{\alpha_j}{\alpha_1} - \frac{\alpha_2}{\alpha_1}\frac{\beta_j}{\beta_2}\Big) t_j - \frac{1}{\alpha_1}(\mu_3 - \mu_4) + \frac{\alpha_2}{\alpha_1\beta_2}(\mu_1 - \mu_2). \quad \text{(A.11)}$$

If $1 - \frac{\alpha_2\beta_1}{\alpha_1\beta_2} > 0$,

$$t_1 \geq -\Big(1 - \frac{\alpha_2\beta_1}{\alpha_1\beta_2}\Big)^{-1} \sum_{j=3}^{4}\Big(\frac{\alpha_j}{\alpha_1} - \frac{\alpha_2\beta_j}{\alpha_1\beta_2}\Big) t_j - \Big(1 - \frac{\alpha_2\beta_1}{\alpha_1\beta_2}\Big)^{-1} \frac{1}{\alpha_1}(\mu_3 - \mu_4) \quad \text{(A.12)}$$

$$+ \Big(1 - \frac{\alpha_2\beta_1}{\alpha_1\beta_2}\Big)^{-1} \frac{\alpha_2}{\alpha_1\beta_2}(\mu_1 - \mu_2).$$

Because

$$\Big(1 - \frac{\alpha_2\beta_1}{\alpha_1\beta_2}\Big)^{-1} = \frac{\alpha_1\beta_2}{\alpha_1\beta_2 - \alpha_2\beta_1} \quad \text{(A.13)}$$

(A.12) becomes

$$t_1 \geq \sum_{j=3}^{4} \gamma_{1j} t_j + \gamma_{10}. \quad \text{(A.14)}$$

$$\gamma_{1j} = -\frac{\alpha_j\beta_2 - \alpha_2\beta_j}{\alpha_1\beta_2 - \alpha_2\beta_1}; \quad \text{(A.15)}$$

$$\gamma_{10} = -\Big(\frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1}\Big)\Big(\frac{1}{\alpha_2}(\mu_1 - \mu_2)\Big) - \beta_2(\mu_3 - \mu_4)\Big). \quad \text{(A.16)}$$

Note that to obtain (A.14) it has been necessary to check the signs of: α_1, $\frac{\alpha_2}{\alpha_1}$ and $(1-\frac{\alpha_2\beta_1}{\alpha_1\beta_2})$.

Let us now obtain the analog to (A.14) for t_2. If $\beta_2 > 0$ and $-\frac{\beta_1}{\beta_2} > 0$ substitute (A.8) into (A.9) to obtain

$$t_2 \geq \frac{\beta_1}{\beta_2}\Big(\sum_{j=2}^{4} \frac{\alpha_j}{\alpha_1} t_j + \frac{1}{\alpha_1}(\mu_3-\mu_4)\Big) - \sum_{j=3}^{4} \frac{\beta_j}{\beta_2} t_j - \frac{1}{\beta_2}(\mu_1-\mu_2)\,. \tag{A.17}$$

This gives

$$t_2\Big(\frac{\beta_2\alpha_1-\beta_1\alpha_2}{\beta_2\alpha_1}\Big) \geq \sum_{j=3}^{4} \frac{\beta_1\alpha_j-\beta_j\alpha_1}{\beta_2\alpha_1} t_j + \frac{\beta_1}{\beta_2\alpha_1}(\mu_3-\mu_4) - \frac{1}{\beta_2}(\mu_1-\mu_2)\,. \tag{A.18}$$

Contingent on $\Big(\frac{\beta_2\alpha_1-\beta_1\alpha_2}{\beta_2\alpha_1}\Big) > 0$,

$$t_2 \geq \sum_{j=3}^{4} \gamma_{2j}\, t_j + \gamma_{20}\,; \tag{A.19}$$

$$\gamma_{2j} = \frac{\beta_1\alpha_j-\alpha_1\beta_j}{\beta_2\alpha_1-\beta_1\alpha_2}\;;$$

$$\gamma_{20} = \Big(\frac{1}{\beta_2\alpha_1-\beta_1\alpha_2}\Big)\Big(\beta_1(\mu_3-\mu_4) - \alpha_1(\mu_1-\mu_2)\Big);\quad t_0 \equiv 1\,.$$

Note that the inequality in (A.16) was contingent on the signs of: β_2, $\frac{\beta_1}{\beta_2}$ and $(1-\frac{\beta_1\alpha_2}{\beta_2\alpha_1})$. For each one of these which is negative change the direction of the inequality. The same holds for (A.14). By the arguments just made

$$\int_{\Omega_{21,43}} \prod_{j=1}^{4} g_j(t_j)\, dt_j = \tag{A.20}$$

$$\int_{-\infty}^{\infty} g_4(t_4)\left[\int_{-\infty}^{\infty} g_3(t_3)\left[\left(\int_{t_2=t_2(t_3,t_4)}^{\infty} g_2(t_2)dt_2\right)\left(\int_{t_1=t_1(t_3,t_4)}^{\infty} g_1(t_1)dt_1\right)\right] dt_3\right] dt_4 \;,$$

where $g_i \equiv n(t_i,\ \mu_i',\ \sigma_i')$. Integrating over dt_1 and dt_2 in (A.20) where

$$N_i(t_3,t_4) = \int_{-\infty}^{t_i(t_3,t_4)} g_i(t_i)\ dt_i \tag{A.21}$$

and g_i is a non-standard normal,

$$\int_{\Omega_{21,43}} \prod_{j=1}^{4} g_j\ (t_j)\ dt_j = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_4(t_4)\ g_3(t_3)(1-N_2(t_3,\ t_4))(1-N_1(t_3,t_4))dt_3dt_4 \tag{A.22}$$

$$= 1 + \int_{-\infty}^{\infty}\int_{\infty}^{\infty} g_4(t_4)\ N_2(t_3,\ t_4)\ N_1(t_3,\ t_4)\ g_3(t_3)\ dt_3\ dt_4 \tag{A.23}$$

$$- \int_{-\infty}^{\infty}\int_{\infty}^{\infty} g_4(t_4)\ N_1(t_3,\ t_4)\ g_3(t_3)\ dt_3\ dt_4\ \ - \int_{-\infty}^{\infty}\int_{\infty}^{\infty} g_4(t_4)\ N_2(t_3,\ t_4)\ g_3(t_3)\ dt_3\ dt_4\ .$$

Given that the conditions on α_1, α_2, β_1 and β_2 in this example are statistical, the integration over $\mathbf{dt} \in R^4$ in (4.18) reduces to an integration over $\mathbf{dt} \in R^2$ in (A.23) and simplifies the numerical integration.

The implication of the failure to obtain the inequalities for t_1 and t_2 (A.14, A.19) or similar inequalities for any two of t_1, ... t_4 in this example is that (4.18) does not simplify to (A.20) and (A.23).

If, for example, (A.14), the inequality expressing t_1 in terms of t_3 and t_4 cannot be obtained, then t_1 must be represented with an inequality in terms of not only t_3 and t_4

but t_2, also. The inner brackets to (A.20) becomes

$$\int_{t_2 = t_2(t_3,\, t_4)}^{\infty} g_2(t_2) \left(\int_{t_1 = t_1(t_2,\, t_3,\, t_4)}^{\infty} g_1(t_1) dt_1 \right) dt_2 \,. \tag{A.24}$$

The right-hand side of (A.22) becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_4(t_4)\, g_3(t_3)(1 - N_2(t_3, t_4)\ (1 - N_1(t_2, t_3, t_4))dt_2 dt_3 dt_4 \tag{A.25}$$

and the third term to (A.23) becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_4(t_4)\, g_3(t_3) N_1(t_2, t_3, t_4)\ dt_2 dt_3 dt_4 \tag{A.26}$$

If no inequality for t_2 in terms if only t_3 and t_4 could be obtained either (and no such inequalities are available for any other pair of t's either), then similar to (A.26) the last term in (A.23) becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_4(t_4)\, N_2(t_1, t_3, t_4) g_3(t_3) dt_1 dt_3 dt_4 \tag{A.27}$$

(iv) General Case

LEMMA 1:

Given from (A.6) and (A.7), then (a) if $\exists\, j' \ni \alpha_{j'}\beta_{j'} < 0$ then $\forall\, j'' \ni j'' \neq j'$ an explicit inequality for $t_{j''}$ can be obtained; (b) If there also exists a $j'' \ni j'' \neq j'$ and $\alpha_{j''}, \beta_{j''} < 0$, then a second inequality for $t_{j'}$ can be obtained.

Proof:

(1) From (A.6) and (A.7),

$$\sum_{j=0}^{4} \alpha_j t_j \geq 0 \ ; \ \sum_{j=0}^{4} \beta_j t_j \geq 0 \ ; \tag{A.28}$$

(2) For $j', j'' \epsilon \{1,....,4\}$, consider $\alpha_{j'}$ and $\beta_{j'}$. Either $\alpha_{j'} > 0$, $\alpha_{j'} < 0$ or $\alpha_{j'} = 0$. If for all $j' \epsilon \{1,...,4\}$, $a_{j'} = 0$, then from (A.6) and (A.3), $x_3 \geq x_4$ reduces to $\mu_3 \geq \mu_4$ so that $x_3 - \mu_3 = x_4 - \mu_4$ for all (x_3, x_4), or $(x_3, x_4) = (\mu_3, \mu_4)$ and (x_3, x_4) is constant. In this the GO contract, defined by its payoff in (1.1) reduces to C Max$\{x_1, x_2\}$ where C is a known constant and C = max$\{x_3, x_4\}$. The problem then reduces to solving for the value of C Stulz options. By similar reasoning if for all $j' \epsilon \{1,...,4\}$, $\beta_{j''} = 0$, then x_1 and x_2 are constants and again the GO problem reduces to a Stulz problem. If both $\alpha_{j'}$ and $\beta_{j''}$ are zero for all j's and j''s, then all uncertainity is removed from the problem. The same can be said of $\beta_{j''}$. We shall make a table checking each of the four nonzero cases:

	$\alpha_{j'} < 0$	$\alpha_{j'} > 0$
$\beta_{j''} < 0$	(i)	(ii)
$\beta_{j''} > 0$	(iii)	(iv)

Table 1

Consider $a_{j'} > 0$, then

$$\sum_{\substack{j=0 \\ j \neq j'}}^{4} \frac{\alpha_j}{\alpha_{j'}} t_j \geq -t_{j'} ; \quad -\sum_{\substack{j=0 \\ j \neq j'}}^{4} \frac{\alpha_j}{\alpha_{j'}} t_j \leq t_{j'} . \tag{A.29}$$

(3) Second consider $\alpha_{j'} < 0$, then

$$\sum_{\substack{j=0 \\ j \neq j'}}^{4} \frac{\alpha_j}{\alpha_{j'}} t_j \leq -t_{j'} , \text{or} \quad -\sum_{\substack{j=0 \\ j \neq j'}}^{4} \frac{\alpha_j}{\alpha_{j'}} t_j \geq t_{j'} . \tag{A.30}$$

(4) Consider $\beta_{j''}$. Let $\beta_{j''} > 0$, then

$$-\sum_{\substack{j=0 \\ j \neq j''}}^{4} \frac{\beta_j}{\beta_{j''}} t_j \leq t_{j''} . \qquad \text{(A.31)}$$

(5) Let $\beta_{j''} < 0$, then (A.32)

$$-\sum_{\substack{j=0 \\ j \neq j''}}^{4} \frac{\beta_j}{\beta_{j'}} t_j \leq t_{j''} . \qquad \text{(A.33)}$$

The object is to determine if an unambiguous inequality exists for $t_{j''}$ or $t_{j'}$. The idea is to solve for $t_{j'}$ by substituting for $t_{j''}$ in (A.6) or to solve for $t_{j''}$ by substituting for $t_{j'}$ in (A.7)

(6) Case (i): Consider substituting for $t_{j''}$ in the α-equation first. Either $\alpha_{j''} > 0$, $\alpha_{j''} = 0$. We have already assumed in case (i) that $\alpha_{j'} < 0$. If $\alpha_{j''} = 0$, then an inequality in $t_{j'}$ is immediate. In the two other cases $-\alpha_{j''}/\alpha_{j'} > 0$ and $-\alpha_{j''}/\alpha_{j'} < 0$ respectively.

$$-\sum_{\substack{j=0 \\ j \neq j' j''}}^{4} \frac{\alpha_j}{\alpha_{j'}} t_j + \frac{\alpha_{j''}}{a_{j'}} \left(\sum_{\substack{j=0 \\ j \neq j''}}^{4} \frac{\beta_j}{\beta_{j''}} t_j \right) t_{j''} > t_{j'} \quad . \qquad \text{(A.34)}$$

If $-\alpha_{j''}/\alpha_{j'} > 0$.

(8) If $-\alpha_{j''}/\alpha_{j'} < 0$, then multiplying (A.33) by a negative number changes the direction of the inequality in (A.33). The result is that substituting for $t_{j''}$ in (A.26) makes the left hand side of (A.26) smaller and the direction of the resulting inequality is ambiguous so that an inequality in $t_{j'}$ not using $t_{j''}$ is unavailable.

(9) Note that for case (i) an inequality for $t_{j'}$ is obtained when $\alpha_{j''} > 0$.

(10) By symmetry one cannot substitute for $t_{j'}$ in the β-equation using the α-equation if $-\beta_{j'}/\beta_{j''} < 0$, but one can if $-\beta_{j'}/\beta_{j''} > 0$. (Symmetry here means switch α with β and j′ with j″.)

(11) Again as in (9) $-\beta_{j'}/\beta_{j} > 0$ implies $\beta_{j'} > 0$ in case (i) it is assumed that $\beta_{j'} < 0$. Thus given the assumptions of case (i), an explicit solution is obtained if and only if $\beta_{j'} > 0$.

(12) Summarizing case (i):

$$\alpha_{j'},\ \beta_{j''} < 0 < \alpha_{j''} \ \Rightarrow\ t_{j'} \tag{A.35}$$

$$\alpha_{j'},\ \beta_{j''} < 0 < \beta_{j'} \ \Rightarrow\ t_{j''} \tag{A.36}$$

Table 2

In Table 2, $t_{j'}$ on the right hand side of the arrows means that there exists an affine function of t_i, $i \neq j'$, $f(t_i\colon i \neq j')$ such that (with similar results for $\epsilon_{j''}$).

$$f(t_i\colon i \neq j') = \sum_{\substack{i=0 \\ i \neq j'}}^{4} \gamma_{j'}\, t_i \ \geq\ t_{j'}\ . \tag{A.37}$$

B. Case (ii): $\alpha_{j'} > 0 > \beta_{j''}$.

(13) In this case (A.25) and (A.29) hold. As in (6) for case (i) consider substituting for $t_{j''}$ in (A.25) using (A.32). If $-\alpha_{j''}/\alpha_{j'} < 0$, then we obtain an unambiguous inequality.

$$-\sum_{\substack{j \neq 0 \\ j \neq j', j''}}^{4} \frac{\alpha_j}{\alpha_{j'}}\, t_j + \frac{\alpha_{j'}}{\alpha_{j''}} \left(\sum_{\substack{j=0 \\ j \neq j'}}^{4} \frac{\beta_j}{\beta_{j'}}\, t_j \right) \leq t_{j'}\ . \tag{A.38}$$

If $-\ \alpha_{j''}/\alpha_{j'} > 0$, no unambiguous inequality results. Note that $-\alpha_{j''}/\alpha_{j'} \Rightarrow \alpha_{j''} > 0$, and an inequality for $t_{j'}$ is obtainable.

(14) Consider substituting for $t_{j'}$ in (A.29) using (A.25). If $-\beta_{j'}/\beta_{j''} < 0$, an unambiguous inequality for t_j is obtained. Since $0 > \beta_j$, obtain $\beta_{j''} < 0$. Summarizing case (ii)

$$\alpha_{j''},\ \alpha_{j'} > 0 > \beta_{j''} \ \Rightarrow\ t_{j'} \tag{A.39}$$

$$\alpha_{j'} > 0 > \beta_{j''}\ ,\ \beta_{j'} \Rightarrow\ t_{j''} \tag{A.40}$$

Table 3

C. Case (iii): $\beta_{j''} > 0 > \alpha_{j'}$.

(15) This case is identical to case (ii) with β and α switched and j′ and j″ switched, so it is immediate that we get

$$\beta_{j'},\ \beta_{j''} > 0 > \alpha_{j''} \ \Rightarrow\ t_{j''} \tag{A.41}$$

$$\beta_{j''} > 0 > \alpha_{j''}\ ,\ \alpha_{j''} \Rightarrow t_{j''} \tag{A.42}$$

Table 4

D. Case (iv): $\alpha_{j'}\ ,\ \beta_{j''} > 0$.

(16) In this case (A.25) and (A.27) hold. Consider substituting for $t_{j''}$ in (A.25) using (A.30). If $-\alpha_{j''}/\alpha_{j'} > 0$ an unambiguous inequality holds for $t_{j'}$. Note that $\alpha_{j'} - \alpha_{j''}/\alpha_{j'} > 0\ \Rightarrow \alpha_{j''} < 0$.

(17) Similarly an inequality for $t_{j''}$ can be obtained if $-\beta_{j'}/\beta_{j''} > 0$. Note that $\beta_{j''}$, $-\beta_{j'}/\beta_{j''} > 0 \Rightarrow \beta_{j''} < 0$. Summarizing case (iv) we get ,

$\alpha_{j'}, \beta_{j''} > 0 > \alpha_{j''} \Rightarrow t_{j'}$	(A.43)
$\alpha_{j'}, \beta_{j''} > 0 > \beta_{j'} \Rightarrow t_{j''}$.	(A.44)

Table 5

(18) Summarizing over cases (i) - (iv) obtain

$\alpha_{j'}, \beta_{j''} < 0 < \alpha_{j''} \Rightarrow t_{j'}$	(A.45)
$\alpha_{j'}, \beta_{j''} < 0 < \beta_{j'} \Rightarrow t_{j''}$	(A.46)
$\alpha_{j'} > 0 > \beta_{j''}, \beta_{j'} \Rightarrow t_{j''}$	(A.47)
$\alpha_{j''}, \alpha_{j'} > 0 > \beta_{j''} \Rightarrow t_{j'}$	(A.48)
$\beta_{j'}, \beta_{j''} > 0 > \alpha_{j'} \Rightarrow t_{j''}$	(A.49)
$\beta_{j''} > 0 > \alpha_{j'}, \alpha_{j''} \Rightarrow t_{j'}$	(A.50)
$\alpha_{j'}, \beta_{j''} > 0 > \alpha_{j''} \Rightarrow t_{j'}$	(A.51)
$\alpha_{j'}, \beta_{j''} > \beta_{j'} \Rightarrow t_{j''}$	(A.52)

Table 6

One can see that if $\alpha_{j'}$ and $\beta_{j'}$ are of opposite sign, and hence $\alpha_{j'}\ \beta_{j''} < 0$, then the inequality for $t_{j''}$ can be obtained. Similarly, if $\alpha_{j''}\beta_{j''} < 0$, an inequality for $t_{j'}$ is obtained QED.

The next step is to summarize the direction of the inequalities.

For each case (i) - (iv) in Lemma 1, Lemma 2 derives the direction of the inequalities for $t_{j'}$ and $t_{j''}$. These inequalities are used to determine the bounds of integration (whether the ∞'s occur at the lower or the upper bound in the simplified form of (4.18)).

LEMMA 2.

Define $\psi \equiv \dfrac{\alpha_{j''}\beta_j}{\alpha_{j'}\beta_{j''}}$, then in Table 7 if $\psi < 1$, then on each line, given the first two inequalities, the last inequality is to be read that $t_j < 1$ (or $>$) $F(t_j)$. Thus in the first line of Table 7, for $j = j', j''$, $t_j < F(t_j)$. If $\psi > 1$, then the inequality is reversed.

$\alpha_{j'}, \beta_{j''} < 0 < \alpha_{j''} < F(t_j)$
$\alpha_{j'}, \beta_{j''} < 0 < \beta_{j'} > F(t_j)$
$\alpha_{j'} > 0 > \beta_{j''}, \beta_{j'} < F(t_j)$
$\alpha_{j''}, \alpha_{j'} > 0 > \beta_{j''} > F(t_j)$
$\beta_{j'}, \beta_{j''} > 0 > \alpha_{j'} > F(t_j)$
$\beta_{j''} > 0 > \alpha_{j'}, \alpha_{j''} < F(t_j)$
$\alpha_{j'}, \beta_{j''} > 0 > \alpha_{j''} < F(t_j)$
$\alpha_{j'}, \beta_{j''} > 0 > \beta_{j'} > F(t_j)$

Table 7

Proof:

A. *Case (ia) :* $\alpha_{j'}, \beta_{j''} < 0 < \alpha_{j''}$.

(1) For $t_{j'}$ obtain from (A.30),

$$\sum_{\substack{j=0 \\ j \neq j', j''}}^{4} \frac{\alpha_j}{\alpha_{j'}} t_j + \frac{\alpha_{j''}}{\alpha_{j'}} \left(\sum_{\substack{j=0 \\ j \neq j''}}^{4} \frac{\beta_j}{\beta_{j''}} t_j \right) \geq t_{j'} \; ; \tag{A.53}$$

$$-\sum_{\substack{j=0\\ j\neq j',j''}}^{4} \frac{\alpha_j}{\alpha_{j'}}\, t_j + \frac{\alpha_{j''}}{\alpha_{j'}} \left(\sum_{\substack{j=0\\ j\neq j''}}^{4} \frac{\beta_j}{\beta_{j''}}\, t_j \right) \geq t_{j'} \left(1 - \frac{\alpha_{j''}}{\alpha_{j'}}\, \frac{\beta_{j'}}{\beta_{j''}}\right) \tag{A.54}$$

(2) Define

$$\psi' \equiv \frac{\alpha_{j''}\beta_{j'}}{\alpha_{j'}\beta_{j''}} < 1\ , \tag{A.55}$$

obtain,

$$t_{j'} \leq \sum_{\substack{i=0\\ i\neq j,j''}}^{4} \gamma_{j'}\, t_i\ ,\quad t_0 \equiv 1\ ; \tag{A.56}$$

$$\gamma_{ij'} = \left(1 - \frac{\alpha_{j''}\beta_{j'}}{\alpha_{j'}\beta_{j''}}\right)^{-1} \left(-\frac{\alpha_j}{\alpha_{j'}} + \frac{\alpha_{j''}\beta_j}{\alpha_{j'}\beta_{j''}}\right). \tag{A.57}$$

For $\psi' > 1$, the inequality in (A.51) changes direction.

Case (ib): $\alpha_{j'}, \beta_{j''} < 0 < \beta_{j'}$:

(3) For $t_{j''}$ by symmetry under the assumptions of case (i) to obtain the inequality for $t_{j''}$ (which is possible only if (A.35) holds) rewrite (A.35) switching " $'$ " with " $''$ " and α with β. Obtain

$$\psi = \psi'' \equiv \frac{\beta_{j'}\,\alpha_{j''}}{\beta_{j''}\,\alpha_{j'}} = \psi'\ . \tag{A.58}$$

Thus both inequalities when they exist have the same direction in case (i).

B. *Case (iia):* $\alpha_{j''}, \alpha_{j'} > 0 > \beta_{j'}$.

(5) Then (A.25) and (A.29) hold. Subsituting for $t_{j''}$ in (A.25) from (A.29) get (A.48) with the inequality reversed.

(6) Define ψ' as before. If $\psi' < 1$, then get the opposite result as in case (i).

Case (iib): $\alpha_{j'} > 0 > \beta_{j''}, \ \beta_{j'}$.

(7) By reasoning identical to *(ib)*, we get the same result as *(ib)*.

C. *Case (iiia):* $\beta_{j''}, \ \beta_{j'} > 0 > \alpha_{j'}$.

The result is identical to *(ia)*.

Case (iiib): $\beta_{j'}, \beta_{j''} > 0 > \alpha_{j'}$.

The result is the opposite to *(ib)*.

D. *Case (iva):* $\alpha_{j'}, \beta_{j''} > 0 > \alpha_{j''}$.

Get the opposite result to *(ia)*.

Case (ivb): $\alpha_{j'}, \ \beta_{j''} > 0 > \beta_{j'}$.

Get the opposite result to *(iib)*.

(b) If $\psi' > 1$, then the inequalities in Table 7 are reversed.

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