GENERALIZED PUT-CALL PARITY

by

David F. Babbel
Laurence K. Eisenberg

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RODNEY L. WHITE CENTER FOR FINANCIAL RESEARCH
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6367

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David F. Babbel
Wharton School
University of Pennsylvania
Philadelphia, PA 19104
(215) 898-7770

Laurence K. Eisenberg
Federal Reserve Bank of Atlanta
104 Marietta Street N.W.
Atlanta, GA 30303
(404) 521-8057

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ABSTRACT

The standard put-call parity result does not include equalities based on buy-and-hold strategies for options on the minimum or maximum of two risky assets and for quantity-adjusting options. This paper generalizes put-call parity to these contracts. International put-call parity relations and the pricing of a new forward contract, absolute-value spread forwards is derived from the put-call parity generalization to options on the minimum or maximum of two risky assets. Finally an inequality comparing the price of quantity adjusting options to portfolios of standard options is presented, showing that the QAO contract, in the absence of arbitrage opportunities contracts is cheaper than the portfolio of standard currency and equity contracts which might be used to hedge the domestic value of a foreign portfolio.

Put-call parity has been a standard result in the finance literature for over twenty years (Stoll [1969]). Curiously, since Stoll's result no generalizations of put-call parity have appeared. We say "curiously" because many types of options contracts have been subject to study in the last twenty years, yet straightforward generalizations of put-call parity to some of these contracts have not appeared in this literature. Two such option contracts are options on the minimum or maximum of two risky assets, (Stulz [1982]), and quantity-adjusting options (Babbel and Eisenberg [1991]). This paper seeks to fill this gap and generalizes the now-standard-result of Stoll to these contracts.[1]

In Section I, put-call parity is generalized to options on the minimum or maximum of two assets. Section II uses the results of Section I to extend put-call parity to quantity-adjusting options. Section III presents international put-call parity relationships. Section IV applies the put-call parity results on options on the minimum or maximum if two risky assets to a new contract called an absolute value spread forward. Section V presents an inequality relating quantity-adjusting options to non-quantity adjusting options. Section VI presents more general inequalities relating generalized options (GOs), quantity-adjusting options (QAOs) and quantity-adjusting forward contracts (QAFs). Section VII concludes.

I. Put-Call Parity for Options on the Minimum or Maximum of Two Risky Assets

The standard put-call parity result states that for European options expiring on the same date T with the same exercise price, K, and on the same underlying security, and no dividends or coupon payments,

$$C(t) - P(t) + PV(t, K) = S(t) , \tag{1.1}$$

[1]The authors would like to thank the participants of the Georgia Institute of Technology Seminar for their comments, especially Nick Valerio and Jayant Kale.

where C(t) is the value of the call at time t, P(t) is the value of the put, PV(t, K) is the present value of the strike price, and S(t) is the price of the underlying security. The purpose of this section is to generalize (1.1) for options on the minimum or maximum of two risky assets. Note that (1.1) can also be written at time T as

$$\max\{0, S - K\} - \max\{0, K - S\} + K = S. \tag{1.2}$$

More generally

$$\max\{x, y\} - \max\{-x, -y\} = x + y \tag{1.3}$$

where (1.3) specializes to (1.2) if

$$x = 0; \;\; y = S - K. \tag{1.4}$$

Equation (1.3) holds because

$$\text{(i)} \quad x > y: \max\{x, y\} = x \; ; \tag{1.5}$$
$$-\max\{-x, -y\} = y$$

$$\text{(ii)} \quad x < y: \max\{x, y\} = y \; ; \tag{1.6}$$
$$-\max\{-x, -y\} = x$$

$$\text{(iii)} \quad x = y: \max\{x, y\} = x = y \; ; \tag{1.7}$$
$$-\max\{-x, -y\} = x = y \; .$$

We will proceed in this section using

$$m(x, y) \equiv \min\{x, y\} \; ; \tag{1.8}$$

$$M(x, y) \equiv \max\{x, y\} \; . \tag{1.9}$$

In this notation put-call parity would be

$$M(x, y) - M(-x, -y) = x + y \tag{1.10}$$

where x and y are given by (1.4). By reasoning similar to that which justifies (1.3) and summarizing:

$$M(x, y) + m(-x, -y) = 0 ; \tag{1.11}$$

$$m(x, y) + M(-x, -y) = 0 ; \tag{1.12}$$

$$M(x, y) + m(x, y) = x + y ; \tag{1.13}$$

$$M(x, y) - m(x, y) = |x - y| ; \tag{1.14}$$

$$m(x, y) - m(-x, -y) = x + y ; \tag{1.15}$$

$$M(x, y) - M(-x, -y) = x + y . \tag{1.16}$$

$$M(x, y) + M(-x, -y) = |x - y| .[2] \tag{1.17}$$

With the exception of (1.14) and (1.17), the right-hand side of these equations represents taking a position in the risky assets x and y. (1.14) and (1.17) require the existence of a forward contract which pays $|x-y|$ on the delivery date. In Section IV, however, we show that this contract is also spanned the contract which pays M(x, y) and the underlying securities x and y.

To see how one of (1.11-1.17) looks in comparison to standard put-call parity, note that just as (1.15) is the standard result when x and y are given by (1.4), (1.14) using those values gives

$$\max\{0, S-K\} - \min\{0, S-K\} = |S-K|. \tag{1.18}$$

Note that, by the same reasoning, the original put-call parity results (1.11 - 1.17) hold

[2]The authors would like to thank David Nachman for his comment that (1.11-1.17) is an example of a Banach lattice over the reals. See Schaefer [1974], pp. 46-56.

for not only the expiration date but for any date prior for European style options. Hence, for example, M(x, y) can be interpreted not only as a payoff on the expiration date but more generally as a contract price both prior to and at expiration of a contract whose payoff at expiration is M(x, y).

II. Put-Call Parity for Quantity-Adjusting Options

A. *Two and Three-Contract Spreads*

When x and y are given by (1.4), the equations (1.11-1.17) are arbitrage results with buy and hold strategies for non-quantity-adjusting options. The next step is to extend (1.11-1.17) to GOs (the general case of quantity-adjusting options) which were presented in (Babbel and Eisenberg [1991].[3] First, the extension is made for two contract spreads where the contracts are GOs. Analogous to (1.12-1.17), there are seven equations, (2.1-2.7). In any specific equation, E(u, v) is equal to either M(u, v) or m(u, v) but not both. The two-contract-GO equalities are:

$$M(x, y)\ E(u, v) + m(-x, -y)\ E(u, v) = 0\ ; \qquad (2.1)$$

$$m(x, y)\ E(u, v) + M(-x, -y)\ E(u, v) = 0\ ; \qquad (2.2)$$

$$M(x, y)\ E(u, v) + m(x, y)\ E(u,v) = (x + y)\ E(u, v); \qquad (2.3)$$

---

[3]GOs (generalized options) are options whose payoff on expiration is the product of the payoffs of two options, where each of the two options is an option on the minimum or maximum of two risky assets. If M(x,y) and M(u,v) are the payoffs to options which are respectively the maximum of x and y and the maximum of u and v, then

$$Z \equiv M(x,y)\ M(u,v)$$

defines the payoff to a GO. Examples of GOs include sequential investment options and (as a special case) quantity-adjusting options. An example of a sequential investment option is an option on the maximum payoff over a 2-period investment for two assets with the payoff tied to the sequence of investments which has the highest payoff. An example of a quantity-adjusting option would be an option on the Nikkei 225 stock index with payoff converted from Japanese yen into U.S. dollars at a conversion rate fixed at the outset. In the equation for Z above, let S = Nikkei 225 (¥), x = 0, y = (S − K)(¥), u = 0, v = V*($/¥). Then Z gives the stochastic payoff to a "vanilla" put option on the Nikkei 225 converted into dollars at a fixed exchange rate.

$$M(x, y)\ E(u, v)\ -\ m(x, y)\ E(u, v) = |x - y|\ E(u, v); \tag{2.4}$$

$$m(x, y)\ E(u, v)\ -\ m(-x, -y)\ E(u, v) = (x + y)\ E(u, v)\ ; \tag{2.5}$$

$$M(x, y)\ E(u, v)\ -\ M(-x, -y)\ E(u, v) = (x + y)\ E(u, v)\ ; \tag{2.6}$$

$$M(x, y)\ E(u, v) + M(-x, -y)\ E(u, v) = |x - y|\ E(u, v)\ . \tag{2.7}$$

As with previous comments regarding put-call parity for options on the minimum or maximum of two risky assets for (1.14) and (1.17), here (2.4) and (2.7) require the existence of a quantity-adjusting option whose payoff is $|x - y|\ E\ (u, v)$.

B. *Four and Five-Contract Spreads*

The next step is to extend, GO put-call parity results for five contract GO spreads. These equations are the "cross products" of (1.11 - 1.17). Each equation of the 49 cross-product equations is made by writing each of (2.1-2.7) in (x, y) and multiplying by each of (2.1-2.7) written in (u, v). The equations are listed in Appendix A. To illustrate, consider (1.14) crossed with (1.16)

$$\begin{aligned} &\left[M(x, y) - m(x, y)\right]\left[M(u, v) - M(-u, -v)\right] \\ &= M(x, y)\ M(u, v) - m(x, y)\ M(u\ , v) \\ &- M(x, y)\ M(u, v) + m(x, y)\ M(-u, -v)\ ; \\ &= |x - y|\ (u + v)\ . \end{aligned} \tag{2.8}$$

To compare (2.8) with the familiar put-call parity result let:

$$\begin{aligned} x &= 0\ ; \\ y &= S - K\ ; \\ u &= 0\ ; \\ v &= X\ - K_X\ ; \end{aligned} \tag{2.9}$$

where S is the Japanese stock price in yen, K is the strike price in yen, X is the spot dollar-yen exchange rate and K_X is the strike exchange rate. Equation (2.8) becomes

$$
\begin{aligned}
&C(S, K)\, C(X, K_X) \; - \min\{S - K, 0\} C(X, K_X) \\
&- \min\{0, S - K\}\, C(X, K_X) + \min\{0, S - K\}\, 4P(X, K_X) \\
&= |K - S|\, (X - K_X)\,.
\end{aligned} \tag{2.10}
$$

The appendix has all of the equations for five contract spreads.

III. International Put-Call Parity

Babbel and Eisenberg [1991] presented a model for QAOs. Such contracts include a European call option on the Nikkei 225, cash settled in Japanese yen, and converted into U.S. dollars at an exchange rate fixed at the date of issue of the warrant. From (2.6) set v to zero and u to a constant exchange rate. Also set y to zero and x to the yen price of the Nikkei 225 minus the yen strike price:

$$
\begin{aligned}
v &= 0\,; \\
u &= u^*(\tfrac{\$}{¥})\,; \\
y &= 0\,; \\
x &= n - k(¥)\,.
\end{aligned} \tag{3.1}
$$

This gives

$$
M(n - k, 0)\, u^* - M(k - n, 0)\, u^* = (n - k)u^* \tag{3.2}
$$

which is international put-call parity applied to QAOs. Similarly, if instead of setting v to zero, v is set to the spot dollar/yen exchange rate on expiration then one obtains

$$
\begin{aligned}
&M(n - k, 0)\, M(v, u^*) - M(k - n, 0)\, M(v, u^*) \\
&= (n - k)\, M(v, u^*),
\end{aligned} \tag{3.3}
$$

which is the international put-call parity relation for QAOs with an optional exchange rate.

In fact, any generalization of put-call parity in (A.18-A.59) where (u, v) have dimensions of exchange rates can be interpreted as an instance of international put-call parity. For example consider a GO contract which allows U.S. dollar-based investors to convert the Japanese yen payoff to an option on the maximum of two Japanese stock indices, x and y, into Deutschmarks or Swiss Francs at fixed exchange rates $v^*(\frac{SFr}{¥})$ or $u^*(\frac{DM}{¥})$, which is the maximum of these quantities valued in U.S. dollars at the spot rate on expiration. Let $a(\frac{\$}{SFr})$ be the U.S. dollar Swiss Franc spot rate and $\tilde{b}(\frac{1}{DM})$ be the dollar Deutschmark spot rate. Then international put-call parity can be applied to a contract which pays M(x, y) M(av*, bu*). This contract with other suitable contracts can be used in (A.18 - A.59). All of these can be interpreted as instances of international put-call parity.

IV. Absolute-Value Spread Forwards

Note that a portfolio of an option on the minimum or maximum of shares of two risky assets, E(x, y), and short a half share of each asset is an absolute-value spread forward,

$$M(x, y) - \frac{x + y}{2} = M(\frac{x-y}{2}, \frac{y-x}{2})$$

$$= \frac{1}{2} | x - y| , \tag{4.1}$$

and

$$m(x, y) - \frac{x + y}{2} = - \frac{1}{2} | x - y| . \tag{4.2}$$

Thus, an absolute-value spread forward is duplicated with a buy-and-hold strategy is long two maximum options on a share each of two risky assets and short the risky assets. Hence to carry out the arbitrage in (1.14), where, say, the market price permits arbitrage because the left hand side is priced higher than the right hand side, means that

$$M_t(x, y) - m_t(s, y) > 2M_t(x, y) \; - \; (x_t + y_t) \; . \qquad (4.3)$$

Hence $M_t(\cdot)$ and $m_t(\cdot)$ are taken to denote the option prices at time t, and X_t and Y_t denote the prices of the underlying, also at t. From (4.3),

$$M_t(x, y) + m_t(x, y) < x_t + y_t \; . \qquad (4.4)$$

Note that in the absence of a forward contract which pays $|x - y|$ (1.13) and (1.14) represent the same arbitrage.

V. An Inequality: Hybrid Put versus Standard Puts

This section shows that in the absence of arbitrage opportunities, a hybrid put option, Z(t), on a non-domestic equity portfolio, is as cheap or cheaper than a portfolio using standard equity and foreign currency put options.

For example, suppose a U.S. dollar-based investor owns a Japanese equity portfolio and wants to protect the U.S. dollar value of that portfolio at some arbitrary U.S. dollar level denoted by G(\$). Decompose G(\$) into the product of any two arbitrary numbers with dimensions Japanese yen and U.S. dollars/Japanese yen respectively:

$$G(\$) = K(¥) \cdot K_X\left(\frac{\$}{¥}\right) . \qquad (5.1)$$

Let K(¥) be the exercise price of a standard put option on the Japanese stock portfolio, and $K_X\left(\frac{\$}{¥}\right)$ be the strike price of K put options on Japanese yen to sell 1 yen for at least X(0, T)(\$).

In the absence of arbitrage opportunities, the dollar-based investor will find the cost of this portfolio of K currency put options and one standard yen-denominated equity put option greater than or equal to the cost of a hybrid put option:[4]

$$Z(T) = \max\{\tilde{X}(T), K_X\}(\tfrac{\$}{¥}) \cdot \max\{0, \tilde{N}(T) - K\}(\tfrac{\$}{¥}), \tag{5.2}$$

where

$$T \equiv \text{ expiration date of the hybrid option;} \tag{5.3}$$

$$N(¥) = \text{price of Japanese stock portfolio in Japanese yen;}$$

$$X(T)\left(\tfrac{\$}{¥}\right) \equiv \text{ the spot dollar-yen exchange rate at T.}$$

$$Z(T) = (\max\{0, K_X - X(T)\} + X(T)) \cdot \max\{0, K - N(T)\}; \tag{5.4}$$

$$Z(T) = \max\{0, K_X - X(T)\} \cdot \max\{0, K - N(T)\} + X(T) \max\{0, K - N(T)\}. \tag{5.5}$$

Since

$$\max\{0, K - N(T)\} \leq K \tag{5.6}$$

$$Z(T) \leq \max\{0, K_X - X(T)\} \cdot K + X(T) \cdot \max\{0, K - N(T)\}. \tag{5.7}$$

Note strict inequality holds in (5.7) if $N(T) > 0$. The first term on the right hand side is the payoff at time T in U.S. dollars of a put option on K Japanese yen struck at K_X dollars per yen. The second term on the right hand side is the payoff of a standard put option on the Japanese stock portfolio struck at K Japanese yen and valued in U.S. dollars on expiration at the spot rate.

Note that the hedge with the portfolio of "vanilla" options on currency and foreign stock, respectively, does not provide an exact hedge of currency translation risk because

[4]Note that the hybrid option is a portfolio made up the security given in case (v) of Section II. By Procedure Two of Section I in that paper, the hybrid option is a special case of Case (v) in Babbel and Eisenberg [1991].

the currency options are on a fixed amount of foreign currency. This fixed amount need not be equal to the yen payoff of the Nikkei 225 put.

Note also that the dominance result in this section is *not* a version of the theorem that a portfolio of options is worth more than an option a portfolio, Merton [1973]. In the result here, instead of an option on a portfolio, we have a quantity-adjusting option, and instead of a portfolio, all with options as stock, we have a portfolio of options on stock and on foreign currency.

VI. More General Comparisons of Dominance Relations Between Different Types of Options

The previous section demonstrated that in the absence of arbitrage opportunities, a QAO put on stock, say, Japanese stock, is cheaper than a portfolio made up of an at-the-money "vanilla" put on the stock (denominated in a Japanese yen) and a vanilla currency put to sell yen into the investor's home currency, say U.S. dollars, where the currency put is on a quantity of yen equal to the current yen price of the Japanese stock. In part A of this section, we will consider a more general equation and consequent inequality that will allow us to compare not only QAOs to standard options (options on one risky asset), but also GOs to options on the minimum or maximum of two risky assets and to QAOs. In parts B, C and D, three other inequalities are given. The first compares the prices of GOs to QAOs. The second and third inequalities compare prices of QAOs to QAFs, and GOs to QAFs, respectively.

A. *Inequalities Relating GOs, QAOs and QAFs*

From (4.4),

$$M(x, y) - \frac{x+y}{2} = \frac{1}{2}\,|x-y|\ ; \tag{6.1}$$

$$M(u, v) - \frac{u+v}{2} = \frac{1}{2}\,|\,u-v|\ .$$

Hence,

$$M(x, y)M(u, v) - \tfrac{1}{2}(x + y)M(u, v) - \tfrac{1}{2}(u + v)M(x, y)$$

$$+ \tfrac{1}{4}(x + y)(u + v) = \tfrac{1}{4}|x - y|\,|u - v| \geq \frac{\psi}{4}; \tag{6.2}$$

$$\psi = (x - y)(u - v),\ (y - x)(u - v),\ (x - y)(v - u),\ (x - y)(u - v). \tag{6.3}$$

Taking the first of the four cases for ψ in (6.3), then (6.2) becomes,

$$M(x, y)M(u, v) - \tfrac{1}{2}(x + y)M(u, v) - \tfrac{1}{2}(u + v)M(x, y)$$

$$+ \tfrac{1}{4}(xu + xv + yu + yv) \tag{6.4}$$

$$\geq \tfrac{1}{4}(xu - xv - yv + yv).$$

Using ϕ as defined below, this simplifies to

$$M(x, y)M(u, v) - \tfrac{1}{2}(x + y)M(u, v) - \tfrac{1}{2}(u + v)M(x, y)$$

$$\geq -\frac{\phi}{2} \equiv \frac{v}{2}(x + y) \tag{6.5}$$

The following (ψ, ϕ) pairs obtain:

$$\psi = (y - x)(u - v);\quad \phi = yv + xu; \tag{6.6}$$

$$\psi = (x - y)(u - v);\quad \phi = xv + yu; \tag{6.7}$$

$$\psi(x - y)(u - v);\quad \phi = xv + yv. \tag{6.8}$$

Equations (6.5) – (6.8) give dominance relations between GOs, QAOs and QAFs. Note that in (6.5) the first term is the payoff of a GO at expiration (or more generally its price prior to T). xM(u, v), yM(u, v), uM(x, y) and vM(x, y) are the same for QAOs. The term on the righthand is the sum of two QAFs.

B. An Inequality for GOs and QAOs Only

Another inequality compares the price of a GO directly to that of QAOs independent of the pricing of QAFs:

$$M(x, y)\, M(u, v) \;\geq\; \tfrac{1}{2}\,(x + y)M(u, v),\; \tfrac{1}{2}\,(u + v)\, M(x, y)\;. \tag{6.9}$$

Equation (6.9) is obtained as follows. Note that

$$M(x, y) \;\geq\; x, y \;\;; \tag{6.10}$$

$$2M(x, y) \;\geq\; x + y\;; \tag{6.11}$$

$$M(x + y) \;\geq\; \frac{x + y}{2}\;. \tag{6.12}$$

Similarly,

$$M(u, v) \;\geq\; \frac{u + v}{2}\;. \tag{6.13}$$

Multiplying (6.12) and (6.13), respectively, by M(u, v) and M(x, y) get (6.9).

C. An Inequality for QAOs and QAFs only

By (6.10) and (6.13),

$$(u + v)M(x, y), (x + y)M(u, v) \geq \frac{1}{2}(x + y)(u + v). \quad (6.14)$$

D. An Inequality for GOs and QAFs Only

By (6.9) and (6.14),

$$M(x, y)\, M(u,v) \geq \frac{1}{4}(x + y)(u + v). \quad (6.15)$$

VII. Conclusion

Buy and hold trading strategies such as those represented by put-call parity are robust to characterizations of the stochastic process of the returns to the underlying security.

In periods when traders may be unwilling to rely on a dynamic strategy dependent upon a characterization of the stochastic return processes to the underlying risky assets (such as on October 19, 1987), buy-and-hold strategies can be useful both to arbitrage existing markets or to create new markets because such strategies are robust to assumptions on return-processes.

Since put-call parity was first represented in the literature, many types of options contracts have been discussed in the literature. This paper has generalized the standard result to options on the minimum or maximum of two assets and to quantity-adjusting options with applications to international put-call parity.

Appendix - Generalized Put Call Parity Results

A. Two-Option Spread Equalities for Non-Quantity-Adjusting Options

$$M(x, y) + m(-x, -y) = 0\,; \quad (A.1)$$

$$m(x, y) + M(-x, -y) = 0\,; \quad (A.2)$$

$$M(x, y) + m(x, y) = x + y\,; \quad (A.3)$$

$$M(x,y) - m(x, y) = |x - y|\,; \quad (A.4)$$

$$m(x, y) - m(-x, -y) = x + y \; ; \tag{A.5}$$

$$M(x, y) - M(-x, -y) = x + y \; ; \tag{A.6}$$

$$M(x, y) + M(-x, -y) = |x - y| \; . \tag{A.7}$$

B. Two-Option Equalities for Non-Quantity Adjusting Options In Standard Put and Call Notation

Using the following notation,

$$x = 0 \; ; \;\; y = S - K \; ; \tag{A.8}$$

$$C = \max\{0, S(T) - K)\} \; ; \tag{A.9}$$

$$P = \max\{0, K - S(T)\} \; . \tag{A.10}$$

(A.1 - A.7) become respectively:

$$C + \min\{0, K - S\} = 0 \; ; \tag{A.11}$$

$$\min\{0, S - K\} + P = 0 \; ; \tag{A.12}$$

$$C + \min\{0, S - K\} = S - K \; ; \tag{A.13}$$

$$C - \min\{0, S - K\} = |K - S| \; ; \tag{A.14}$$

$$\min\{0, S - K\} - \min\{0, K - S\} = S - K \; ; \tag{A.15}$$

$$C - P = S - K \; ; \text{ (put-call parity)} \tag{A.16}$$

$$C + P = |K - S| \; . \tag{A.17}$$

C. Three-Option GO Spread Equalities (Includes Non-Quantity Adjusting Options)

In any of the equations in this section E(u,v) is to be interpreted for all occurrences in that equation as either m(u, v) or M(u, v). (A.1 - A.7) become respectively:

$$M(x, y)\, E(u, v) + m(-x, -y)\, E(u, v) = 0 \tag{A.18}$$

$$m(x, y)\, E(u, v) + M(-x, -y)\, E(u, v) = 0 \tag{A.19}$$

$$M(x, y)\, E(u, v) + m(x, y)\, E(u, v) = (x + y)\, E(u, v)\ ; \tag{A.20}$$

$$M(x, y)\, E(u, v)\ - m(x, y)\, E(u, v) = |x - y|\, E(u, v)\ ; \tag{A.21}$$

$$m(x, y)\, E(u, v)\ - m(\text{-}x, \text{-}y)\, E(u, v) = (x + y)\, E(u, v)\ ; \tag{A.22}$$

$$M(x, y)\, E(u, v)\ - M(\text{-}x, \text{-}y)\, E(u, v) = (x + y)\, E(u, v); \tag{A.23}$$

$$M(x, y)\, E(u, v) + M(\text{-}x, \text{-}y)\, E(u, v) = |x - y|\, E(u, v)\ . \tag{A.24}$$

D. *Four-Option GO-Spread Equalities in Standard Put and Call Notation*

Along with (A.8 - A.10) let

$$u = 0\ ;\ \ v = X - K_X\ . \tag{A.25}$$

Obtain

$$C\, E(u, v) + \min\{0, K - s\}\, E(u, v) = 0\ ; \tag{A.26}$$

$$\min\{0, S - K\}\, E(u, v) + P\, E(u, v) = 0\ ; \tag{A.27}$$

$$C\, E(u, v) + \min\{0, S - K\}\, E(u, v) = (S - K)\, E(u, v); \tag{A.28}$$

$$C\, E(u, v)\ -\ \min\{0, S - K\}\, E(u, v) = |K - S|\, E(u, v)\ ; \tag{A.29}$$

$$\min\{0, S - K\}\, E(u, v)\ - \min\{0, K - S\}\, E(u, v)$$
$$= (S - K)\, E(u, v); \tag{A.30}$$

$$C\, E(u, v) + P\, E(u, v) = |K - S|\, E(u, v)\ . \tag{A.31}$$

E. *Five-Option-GO-Spread Equalities*

Each of the following equations may be thought of as corresponding to a cell of a (7 × 7) matrix. Only the upper triangular portion of the matrix is given. Equalities corresponding to a lower triangular (i, j) may be obtained from the equation corresponding to upper triangular cell (j, i) and switching each occurrence of x with that of u and each occurrence of y with v. The figure below gives the equation numbers corresponding to the upper-triangular cells:

Four-Option GO Spread Equalities

	A.1	A.2	A.3	A.4	A.5	A.6	A.7
A.1	11	12	13	14	15	16	17
A.2		22	23	24	25	26	27
A.3			33	34	35	36	37
A.4				44	45	46	47
A.5					55	56	57
A.6						66	67
A.7							77

(i) A.1 Equalities

Going from 1 to 7 in the matrix:

$$[M(x, y) + m(-x, -y)][M(u, v) + m(-u, -v)] \tag{A.32}$$
$$= M(x, y)M(u, v) + m(-x, -y)M(u, v)$$
$$+ M(x, y)m(-u, -v) + m(-x, -y)m(-u, -v) ;$$
$$= 0 .$$

$$[M(x, y)m(-x, -y)][m(u, v) + M(-u, -v)] \tag{A.33}$$
$$= M(x, y)m(u, v) + m(-x, -y)m(u, v)$$
$$+ M(x, y)M(-u, -v) + m(-x, -y)M(-u, -v) ;$$
$$= 0 .$$

$$[M(x, y) + m(-x, -y)][M(u, v) + m(u, v)] \tag{A.34}$$
$$= M(x, y)M(u, v) + m(-x, -y)M(u, v)$$
$$+ M(x, y)m(u, v) + m(-x, -y)m(u, v) ;$$
$$= 0 .$$

$$[M(x, y)m(-x, -y)][M(u, v) - m(u, v)] \tag{A.35}$$
$$= M(x, y)M(u, v) + m(-x, -y)m(u, v)$$
$$+ m(-x, -y)M(u, v) - m(-x, -y)m(u, v) ;$$
$$= 0 .$$

$$[M(x, y) + m(-x, -y)][m(u, v) - m(-u, -v)] \tag{A.36}$$
$$= M(x, y)m(u, v) + m(-x, -y)m(u, v)$$
$$+ m(-x, -y)m(u, v) - m(-x, -y)m(-u, -v) ;$$
$$= 0 .$$

$$[M(x, y)m(-x, -y)][M(u, v) - M(-u, -v)] \tag{A.37}$$
$$= M(x, y)M(u, v) + m(-x, -y)M(u, v)$$
$$- M(x, y)M(-u, -v) - m(-x, -y)M(-u, -v) ;$$
$$= 0 .$$

$$[M(x, y) + m(-x, -y)][M(u, v) + M(-u, -v)] \tag{A.38}$$
$$= M(x, y)M(u, v) + m(-x, -y)M(u, v)$$
$$+ M(x, y)M(-u, -v) + m(-x, -y)M(-u, -v) ;$$
$$= 0 .$$

(ii) A.2 Equalities

Going from cell 22 to cell 27 gives six equalities.

$$[m(x, y) + M(-x, -y)][m(u, v) + M(-u, -v)] \tag{A.39}$$
$$= m(x, y)m(u, v) + M(-x, -y)m(u, v) ;$$
$$= 0 .$$

$$[m(x, y) + M(-x, -y)][M(u, v) + m(-u, -v)] \tag{A.40}$$
$$= m(x, y)M(u, v) + M(-x, -y)M(u, v) ;$$
$$+ m(x, y)m(u, v) + M(-x, -y) m(u, v);$$
$$= 0 .$$

$$[m(x, y) + M(-x, -y)][M(u, v) - m(u, v)] \tag{A.41}$$
$$= m(x, y)\ M(u, v) + M(-x, -y)\ M(u, v)\ ;$$
$$- m(x, y)m(u, v) - M(-x, -y)\ m(u, v);$$
$$= 0\ .$$

$$[m(x, y) + M(-x, -y)][m(u, v) - m(-u, -v)] \tag{A.42}$$
$$= m(x, y)M(u, v) + M(-x, -y)M(u, v)\ ;$$
$$+ m(-x, -y)m(u, v) - M(-x, -y)m(-u, -v)\ ;$$
$$= 0\ .$$

$$[m(x, y) + M(-x, -y)][M(u, v) - M(-u, -v)] \tag{A.43}$$
$$= m(x, y)M(u, v) - m(x, y)M(-u, -v)\ ;$$
$$= M(-x, -y)M(u, v) - M(-x, -y)M(-u, -v)\ ;$$
$$= 0\ .$$

$$[m(x, y) + M(-x, -y)][M(u, v) + M(-u, -v)] \tag{A.44}$$
$$= m(x, y)M(u, v) + M(-x, -y)M(u, v)\ ;$$
$$+ m(x, y)M(-u, -v) + M(-x, -y)M(-u, -v)\ ;$$
$$= 0\ .$$

(iii) <u>A.3</u> <u>Equalities</u>

Starting with cell 33 and continuing to cell 37 gives five equalities.

$$[M(x, y) + m(x, y)][M(u, v) + m(u, v)] \tag{A.45}$$
$$= M(x, y)M(u, v) + m(x, y)m(u, v)\ ;$$
$$+ M(x, y)m(u, v) + m(x, y)m(u, v)$$
$$= (x + y)(u + v)\ .$$

$$[M(x, y) + m(x, y)][M(u, v) - m(u, v)] \tag{A.46}$$
$$= M(x, y)M(u, v) + m(x, y)M(u,v)\ ;$$
$$- M(x, y)m(u, v) - m(x, y)m(u, v)\ ;$$
$$= (x + y)\ |u - v|\ .$$

$$[M(x, y) + m(x, y)][m(u, v) - m(-u, -v)] \tag{A.47}$$

$$= M(x, y)\, m(u, v) + m(x, y)\, m(u, v)\ ;$$

$$- M(x, y) m(-u, -v) - m(x, y)\, m(-u, -v);$$

$$= (x + y)(u + v)\ .$$

$$[M(x, y) + m(x, y)][M(u, v) - M(-u, -v)] \tag{A.48}$$

$$= M(x, y)M(u, v) + m(x, y)M(u, v)\ ;$$

$$- M(x, y)\, M(-u, -v) - m(x, y)m(-u, -v)\ ;$$

$$= (x + y)(u + v)\ .$$

$$[M(x, y) + m(x, y)][M(u, v) + M(-u, -v)] \tag{A.49}$$

$$= M(x, y)M(u, v) + m(x, y)M(u, v)$$

$$+ M(x, y)M(-u, -v) + m(x, y)M(-u, -v)\ ;$$

$$= (x + y)\ |u - v|\ .$$

(iv) A.4 Equalities

Starting with cell 44 and continuing to cell 47 gives three equalities

$$[M(x, y) - m(x, y)][M(u, v) - m(u, v)] \tag{A.50}$$

$$= M(x, y)M(u, v) - m(x, y)M(u, v)\ ;$$

$$- M(x, y)m(u, v) + m(x, y)m(u, v)\ ;$$

$$= |x - y|\ |u - v|\ .$$

$$[M(x, y) - m(x, y)][m(u, v) - m(-u, -v)] \tag{A.51}$$

$$= M(x, y)m(u, v) - m(x, y)m(u, v)\ ;$$

$$- M(x, y)m(-u, -v) + m(x, y)m(-u, -v)\ ;$$

$$= |x + y|\ (u + v)\ .$$

$$[M(x, y) - m(x, y)][M(u, v) - M(-u, -v)] \tag{A.52}$$

$$= M(x, y)M(u, v) - m(x, y)M(u, v)\ ;$$

$$- M(x, y)M(-u, -v) + m(x, y)M(-u, -v)\ ;$$

$$= |x - y|\ (u - v)\ .$$

$$[M(x, y) - m(x, y)][M(u, v) + M(-u, -v)] \tag{A.53}$$

$$= M(x, y)\, M(u, v) - m(x, y)\, M(u, v) ;$$

$$+ M(x, y)\, M(-u, -v) - m(x, y)\, M(-u, -v);$$

$$= |x + y|\, |u + v| .$$

(v) <u>A.5</u> <u>Equalities</u>

Cell 55 through 57 gives three equations.

$$[m(x, y) - m(-x, -y)][m(u, v) - m(-u, -v)] \tag{A.54}$$

$$= m(x, y)m(u, v) - m(-x, -y)\, m(u, v) ;$$

$$= (x + y)(u + v) .$$

$$[m(x, y) - m(-x, -y)][M(u, v) - M(-u, -v)] \tag{A.55}$$

$$= m(x, y)M(-u, -v) - m(-x, -y)M(u, v) ;$$

$$- m(x, y)\, M(-u, -v) + m(-x, -y)\, M(-u, -v) ;$$

$$= (x + y)\, (u - v) .$$

$$[m(x, y) - m(-x, -y)][M(u, v) - M(-u, -v)] \tag{A.56}$$

$$= m(x, y)M(u, v) - m(-x, -y)M(u, v) ;$$

$$+ m(x, y)\, M(-u, -v) - m(-x, -y)\, M(-u, -v) ;$$

$$= (x + y)\, |u - v| .$$

(vi) <u>A.6</u> <u>Equalities</u>

$$[M(x, y) - M(-x, -y)][M(u, v) - M(-u, -v)] \tag{A.57}$$

$$= M(x, y)M(u, v) - M(-x, -y)M(u, v) ;$$

$$- M(x, y)\, M(-u, -v) + M(-x, -y)\, M(-u, -v) ;$$

$$= (x + y)\, (u + v) .$$

$$[M(x, y) - M(-x, -y)][M(u, v) - M(-u, -v)] \tag{A.58}$$

$$= M(x, y)M(u, v) - M(-x, -y)M(u, v) ;$$

$$+ M(x, y)\, M(-u, -v) \; - \; M(-x, -y)\, M(-u, -v)\ ;$$

$$= (x + y)\ |u - v|\ .$$

(vii) A.7 Equality

With only the diagonal element left, there is one equation.

$$[M(x, y) \; - \; M(-x, -y)][M(u, v) + M(-u, -v)] \tag{A.59}$$

$$= M(x, y)M(u, v) + M(-x, -y)M(u, v)\ ;$$

$$+ M(x, y)\, M(-u, -v) + M(-x, -y)\, M(-u, -v)\ ;$$

$$= |x + y|\ |u - v|\ .$$

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