THE SUSTAINABILITY OF BUDGET DEFICITS IN A STOCHASTIC ECONOMY

by

Henning Bohn

17-91
(revision of 6-90)

RODNEY L. WHITE CENTER FOR FINANCIAL RESEARCH
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6367

The Sustainability of Budget Deficits in a Stochastic Economy

by Henning Bohn*

Department of Finance
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6367

First Draft: March 1990

Current Version: July 1991

---

* I would like to thank Andy Abel, Joe Haubrich, Jeremy Siegel, Stephen Zeldes, and the seminar participants at Princeton University, at the Board of Governors of the Federal Reserve System, and at the University of Pennsylvania for valuable comments and suggestions. Financial support from the University of Pennsylvania Research Foundation is gratefully acknowledged.

ABSTRACT

The paper studies the sustainability of government budget deficits in a stochastic economy. The general equilibrium setting permits a rigorous derivation of the relevant transversality conditions and intertemporal budget constraints. The transversality condition on government debt requires a zero limit of discounted future government debt, where the discount rate depends on the probability distribution of future debt over states of nature. The government's intertemporal budget constraint requires that the discounted present value of primary surpluses matches initial debt, where the discount rate on future government spending and taxes depends again on the probability distribution of these variables over states of nature.

Contrary to assertions made in the literature, the discount rates on future government debt, spending, and taxes are generally not related to the rates of return on government debt. Most importantly, future government debt in the transversality condition cannot be discounted at the safe interest rate, not even if government debt is safe. Debt discounted at the safe interest rate may well diverge to infinity under sustainable policies. This result raises questions about some recent empirical papers testing the sustainability of U.S. fiscal policy.

In addition, the paper shows that the average level of primary deficits provides little evidence on sustainability. Policies with permanent expected primary deficits can be sustainable, in particular when the safe interest rate is below the average growth rate of the economy.

1. Introduction

The high federal budget deficits of the 1980s have raised concerns about the long-run sustainability of U.S. fiscal policy. A number of recent empirical studies have examined whether the U.S. fiscal policy satisfies an intertemporal budget constraint; see Hamilton and Flavin (1986), Hakkio and Rush (1986), Kremers (1989), Trehan and Walsh (1988, 1991), and Wilcox (1989). The results are mixed. Both Kremers and Wilcox conclude that recent fiscal policy does not appear to be sustainable in this sense; Hamilton and Flavin, Hakkio and Rush, and Trehan and Walsh disagree.

This paper reexamines the theoretical foundations of the sustainability question by studying government policies in an explicitly stochastic general equilibrium model. A stochastic model seems appropriate not just for added realism, but because interest rates on "safe" U.S. government bonds have historically been below the average rate of economic growth. In a deterministic steady state, such low safe interest rates would indicate dynamic inefficiency. But in a stochastic model, dynamic efficiency depends on the relation between growth rate and the rate of return on "risky" capital (Abel et.al. 1989, Zilcha 1991). Based on an efficiency criterion valid for stochastic economies, Abel et.al. (1989) provide strong evidence that the U.S. economy is in fact dynamically efficient.

The combination of dynamic efficiency with a low safe interest rate raises interesting questions about the constraints on government policy. In expectation, the ratio of government debt to aggregate income would fall even if the government simply rolled over all debt with interest. On the other hand, dynamic efficiency suggests that the government cannot run a Ponzi-scheme. To examine the issue, this paper studies a simple Lucas (1978) style endowment economy with government. The Lucas economy is clearly a very simple setting, but it allows for uncertainty and it is simple enough that the constraints on deficit financing and the implications of alternative policies can be derived explicitly. Dynamic efficiency and complete markets are important maintained assumptions.[1] The model admits a safe interest rate below the average growth rate but does not require it.

The central result of the paper is that the government has to satisfy an intertemporal budget constraint and an associated transversality condition regardless of the level of the safe interest rate. Policies that satisfy these constraints will be called sustainable. In contrast to the literature, the discount factors on future government spending and on future government revenues are determined by the distribution of revenues and spending across states of nature and they are generally not related to the interest rate on government bonds.[2] Similarly, the transversality condition requires a zero limit of future government debt discounted at a rate that depends on the probability distribution of future debt (and not on the government bond rate). These results have significant implications for the empirical sustainability literature, since future fiscal variables have typically been discounted by either "the safe interest rate" (estimated as average return on some Treasury securities) or by the actual return on government debt. An example will illustrate that discounting at the safe interest rate can easily lead to erroneous rejections of sustainability.

The precise form of the intertemporal budget constraint is also important for an assessment of policy plans that show substantial budget deficits in future periods.

---

[1] The dynamic efficiency assumption seems justified by the empirical results in Abel et al. (1989). Without complete markets, it would be difficult to derive general constraints on government policy (see Blanchard and Weil 1990). The strategy here is to use a stochastic setting in which one might suspect a priori that constraints on government borrowing similar to the standard deterministic transversality constraint can be obtained. The focus is on the comparison of stochastic and deterministic constraints. Complications of the kind studied by Blanchard and Weil are therefore ruled out.

[2] This result is important because the contrary seems widely believed. For example, Sargent (1987, p.118) asserts that the intertemporal budget constraint can be written in terms of safe (multi-period) interest rates, whenever government debt is "perfectly safe." The paper shows that the assertion is only true if primary surpluses are uncorrelated with marginal rates of substitution.

The current concerns about the high U.S. budget deficits appear to be based at least in part on a belief that high budget deficits cannot be sustained for ever. Indeed, a government which starts out with an initial debt must run a primary surplus (i.e., excluding interest payments) at least in some periods and in some states of nature.[3] However, the stochastic model imposes almost no restrictions on the level of average or expected primary deficits, because the government can trade-off primary deficits in some states of nature against surpluses in other states.

Both the importance of correct discounting and the feasibility of average primary deficits will be demonstrated in a simple example, which will then be generalized. The example considers a tax rule that maintains a constant ratio of end-of-period debt to aggregate income in a setting where the safe interest rate is below the average growth rate. If the government issues safe debt, the ratio of debt to income at the beginning of the next period will be below target in expectation. The government will then—i.e., usually—run a primary deficit. A primary surplus is only needed in those states of nature in which income growth falls below the interest rate on government bonds. Ex ante, tax revenues will be below non-interest spending in expectation. Ex post, realizations of primary deficits in this economy will be positive on average.

This example is also used to illustrate the importance of correct discounting. The policy of the example is clearly sustainable. However, if one used the safe interest rate to discount future government debt in the transversality condition, one would erroneously conclude that the policy is not sustainable. The reason is that debt grows asymptotically at the same rate as income, which is above the safe interest rate.

The paper is organized as follows. Section 2 sets up the model and develops the intertemporal budget constraint under uncertainty and the associated

---

[3] This result is analogous to a result about deterministic models in McCallum (1984).

transversality conditions. In Section 3, I provide an example of a sustainable policy that would be judged non-sustainable by the traditional tests. Then I examine more generally why future government debt, spending, and taxes must usually be discounted at rates that are unrelated to the interest rates on government debt. Some empirical data on U.S. growth rates and interest rates are also provided. Section 4 concludes.

2. The Model

In this section, I will set up the government budget equation, embed the government sector in a simple general equilibrium setting, and then derive the relevant transversality and intertemporal budget constraints.

2.1. The Government

Consider a government that finances a stochastic path of government spending G_t by levying taxes T_t and by borrowing on financial markets. Taxes and government spending are stochastic processes (unrestricted for now) and they are measured in units of a consumption good.

Given the value of government liabilities at the start of period t, D_t, current government spending and taxes determine the government's total financing needs at the end of the period. The government has to issue securities with market value $\tilde{D}_t = D_t + G_t - T_t$. In a stochastic setting, spending and taxes do not provide a complete specification of government policy, because the government still has to decide what kind of securities should be issued. In general, the government may issue a portfolio of securities that promise a total payoff at the start of the next period which may depend on the state of nature at that time (see Lucas and Stokey 1983). For each state, the promised payoffs for this state constitutes the new level of government liabilities, D_{t+1}.

To obtain a relatively simple yet precise notation for the budget constraints, assume that markets are complete and that the probability distribution of states is discrete.[4] At each date t, the states of nature are denoted by s_t, where s_t takes values in a set S_t. The history of the economy up to time t is denoted by $h_t = (s_t, s_{t-1}, \ldots, s_0)$, where h_t takes values in a set H_t. Denote the probability of a history h_t by $\pi(h_t)$ and denote the period-t prices of securities that pay 1 unit of the consumption good in period t+1 in state s_{t+1} by $p(s_{t+1} \mid h_t)$.

The government budget equation in period t is then

$$D_t + G_t - T_t = \sum_{s_{t+1} \in S_{t+1}} p(s_{t+1} \mid h_t) \cdot D(s_{t+1} \mid h_t), \tag{1}$$

where the sum on the right hand side is the market value of newly issued state-contingent claims.[5] To be precise, the variables on the left are conditional on the particular history h_t, but the conditioning will often be omitted for notational convenience when some generic state of nature is considered.

The key question in the sustainability context is whether the path of government debt has to satisfy a set of transversality conditions. Since government borrowing is limited by the government's ability to find lenders, the answer will

---

[4] In case of non-discrete probability distributions, expectations and sums over state contingent claims prices in the equations below would have to be replaced by the corresponding integrals. The substantive results would remain unchanged. Incomplete markets could create significant problems that would be beyond the scope of this paper (see Blanchard and Weil 1990).

[5] The state-contingent claims notation is similar to Lucas and Stokey (1983), Chari et al. (1991), and King (1990). This formulation does not preclude long-term debt, because one can think of outstanding long term debt as being repurchased and re-issued at market value. The budget equation could also be written in terms of state-contingent rates of return on debt as

$$\tilde{D}_t = G_t - T_t + D_t = G_t - T_t + [1+R(s_t \mid h_{t-1})] \cdot \tilde{D}_{t-1},$$

where the distribution of returns $R(s_t \mid h_{t-1})$ across states of nature is implicitly defined by $D_t(s_t \mid h_{t-1}) = [1+R(s_t \mid h_{t-1})] \cdot \tilde{D}(h_{t-1})$. This formulation is inconvenient here because the sustainability issue is closely related to the question of how government debt is valued. That is, the rate of return formulation is only valid if one one can prove that the government has the ability to sell debt valued at $\tilde{D}_{t-1}$ by promising returns with the distribution $R(s_t \mid h_{t-1})$. But that *is* the sustainability question.

depend on the assumptions about the rest of the economy. In this sense, sustainability is inherently a general equilibrium issue.[6]

2.2. The Economy

Here I will consider a Lucas (1978) exchange economy with infinitely-lived agents. The Lucas model is used, because it appears to be the most simple general equilibrium setting in which the government sector can be embedded. Many assumptions about the economic environment can be relaxed easily without changing the policy conclusions, as indicated below.

There are I consumers indexed by $i=1,\ldots,I$. Each consumer is endowed with one unit of a "fruit tree," which yields an exogenous stochastic stream of dividends Y^i_t.[7] Dividends are goods that can be used for private consumption C^i_t or be sold to the government. The stochastic process of per capita government spending G_t is assumed to satisfy the technical feasibility constraint $G_t \leq (1/I)\cdot\sum_i Y^i_t$. (All government variables are interpreted as per capita quantities.) Taxes are lump-sum, which means that individual tax payments T^i_t can take any value on the real line.[8] Government tax revenues are $T_t=(1/I)\cdot\sum_i T^i_t$.

Each individual has preferences over consumption,

---

[6] In contrast, most empirical studies simply assert that the government is subject to certain transversality and intertemporal budget constraints, e.g., see Hamilton and Flavin (1986), Wilcox (1989), Trehan and Walsh (1988, 1991). As Section 3 will show, such ad hoc constraints may be difficult to justify.

[7] The applications below will assume a representative agent. But an assumption of identical endowments and preferences would not simplify the analysis at this point, because one cannot exclude the possibility that the government may treat (tax) individuals differently. Given that one has to allow for asymmetric allocations, one may as well permit individual differences in endowments and preferences.

[8] The implications of limited taxation are studied in Bohn (1991b). The lump-sum tax assumption may be important for thinking about situations when the government "announces" a non-sustainable policy (because it implies that a tax increase is feasible from any starting position). However, the question here is to how to determine whether a *given* government policy is sustainable or not. Arguments about how the economy might evolve over time if the government tried to implement a non-sustainable policy are not the subject of this paper.

$$\sum_{t=0}^{\infty} E_0[\beta^t \cdot U_i(C^i_t)] = \sum_{t=0}^{\infty} [\sum_{h_t \in H_t} \pi(h_t) \cdot \beta^t \cdot U_i(C^i(h_t))], \tag{2}$$

where $U_i(\cdot)$ is strictly increasing and concave and $\beta>0$. Denoting net holdings of financial assets at the start of period t by A^i_t and net purchases of state-contingent claims on period t+1 consumption by $A^i(s_{t+1} | h_t)$, the individual budget equation at time t is

$$A^i_t + Y^i_t - T^i_t = C^i_t + \sum_{s_{t+1} \in S_{t+1}} p(s_{t+1} | h_t) \cdot A^i(s_{t+1} | h_t). \tag{3}$$

The standard period-by-period and state-by-state first order conditions for optimality are

$$p(s_{t+1} | h_t) = \pi(s_{t+1} | h_t) \cdot \beta \cdot \frac{U_i'(C^i_{t+1}(h_{t+1}))}{U_i'(C^i_t(h_t))} \tag{4}$$

for all $h_{t+1} \in H_{t+1}$, where $h_{t+1} = (s_{t+1}, h_t)$ is the history at time t+1 that consists of s_{t+1} and h_t. For later reference, let $P(h_{t+N} | h_t)$ be the period-t price of a contingent claim that will pay a unit of consumption in period t+N if and only if history h_{t+N} has been realized at that time. Individual optimization implies that

$$P(h_{t+N} | h_t) = \prod_{n=1}^{N} p(s_{t+n} | h_{t+n-1}), \tag{5}$$

for all histories $h_{t+N} \in H_{t+N}$ that include the t-period history h_t; $P(h_{t+N} | h_t)=0$ applies for all other histories.[9]

For the sustainability of government debt policy, the key question is to what extent consumers are willing to lend to the government, i.e., what transversality conditions apply. This issue is examined in the next section.

---

[9] To be precise, the states s_{t+n} and histories h_{t+n-1} on the right hand side of (5) are the components of h_{t+N}, i.e., $h_{t+N} = (s_{t+N}, \ldots, s_{t+n}, h_{t+n-1})$.

2.3. *Transversality Conditions and Intertemporal Budget Constraints*

The main purpose of a transversality condition is to rule out Ponzi-schemes, i.e., financial trading strategies that involve rolling-over an initial loan with interest for ever. The objective of this section is to examine whether the government is subject to such a condition and what form it takes.

In deterministic economies, the "standard" transversality condition is the requirement that the present value of net assets in the far future converges to zero. In deterministic economies without government, such a condition can be obtained by combining two considerations (O'Connell and Zeldes 1988). First, one has to show that individuals will not find it optimal to be on the lending side of a Ponzi-scheme. Second, one has to argue that each individual's borrowing is constrained by a No-Ponzi-Game (NPG) condition that formalizes the limited willingness of all others to lend.

Uncertainty and the existence of a government complicate these arguments. The main problem is that the government is an agent whose actions may not be the result of an optimization problem. The exogenously given tax and spending policies may be such that the government accumulates financial claims issued by individuals at a rate that permits (some or all) individuals to run Ponzi-schemes against the government. Additional complications are that some limits may not exist (i.e., liminf $\neq$ limsup) and that, to rule out Ponzi-schemes in a stochastic setting, one needs a set of constraints—one for each possible history h_t—instead of a single NPG condition.

These complications cannot be ruled out a priori, but they are distracting here, because the main objective is to address policy concerns raised by excessive government borrowing (as opposed to government lending).[10] The constraints on

---

[10] See, e.g., Kane (1989) for a discussion of policy problems on the lending side.

government policy will therefore be stated in two different ways. Proposition 1 establishes that the government is *always* subject to a set of NPG-conditions. Proposition 2 adds regularity conditions that exclude technically complicated and relatively uninteresting cases.

PROPOSITION 1:

In the economy considered here, the government cannot run a Ponzi-scheme. That is, government policy must satisfy

$$\liminf_{N\to\infty} \sum_{h_{t+N}\in H_{t+N}} P(h_{t+N} \mid h_t) \cdot D(h_{t+N}) \leq 0 \tag{6}$$

for $h_t \in H_t$ and for all t. ♦

PROPOSITION 2:

Provided the government does not permit individuals to run Ponzi-schemes against the government and provided all relevant limits exist, government policy must satisfy the transversality conditions

$$\lim_{N\to\infty} \sum_{h_{t+N}\in H_{t+N}} P(h_{t+N} \mid h_t) \cdot D(h_{t+N}) = 0, \tag{7}$$

and the intertemporal budget constraints

$$D_t = \sum_{n\geq 0} \Big[\sum_{h_{t+n}\in H_{t+n}} P(h_{t+n} \mid h_t) \cdot T(h_{t+n})] - \sum_{h_{t+n}\in H_{t+n}} P(h_{t+n} \mid h_t) \cdot G(h_{t+n})\Big] \tag{8}$$

for all $h_t \in H_t$ and for all t. Analogous transversality conditions and budget constraints apply to all individuals. ♦

PROOFS: See the Appendix. ♦

Both proofs use the same idea as the proof of the NPG-conditions in deterministic economies without government (see Cass 1972, O'Connell and Zeldes

1988):[11] On can show that an optimizing individual will never buy financial claims from an entity that intends to roll them over indefinitely. This line of argument explains why the government cannot run a Ponzi-scheme (Proposition 1). Proposition 2 focuses on policies for which (6) holds as equality and for which the limit is a proper limit. Then private Ponzi-schemess are also ruled out and equation (6) reduces to (7). Equation (8) follows directly from (7) and (1).

From now on, the paper will consider policies that satisfy the assumption of Proposition 2, so that constraints (7) and (8) apply.[12]

One implication of the intertemporal budget constraint (8) is immediate: All government debt has to be backed by future primary surpluses (defined as differences T-G) of equal present value. A government that starts out with positive debt must run a primary surplus in some state of nature in some subsequent period. This generalizes McCallum's (1984) result about the impossibility of permanent primary deficits. It is an important difference to models with dynamic inefficiency, in which permanent primary deficits are possible (Diamond 1965). Intuitively, the impossibility of permanent primary deficits follows directly from the NPG-conditions, because a policy a with initial debt and no subsequent primary surpluses would be a Ponzi-scheme.

---

[11] Unfortunately, the proofs are substantially more lengthy and technically more involved, because one has to specify how individuals will behave if the government allows some of them to run Ponzi-schemes in some states of nature (i.e., if the strict inequality in (6) applies for some h_t), and because one has to condition on different states of nature. The proofs are therefore placed in an appendix.

[12] The main purpose of Proposition 1 is to remove any suspicion that Proposition 2 might be an artifact of the regularity conditions. (Another purpose is to point out that one does need regularity conditions to obtain an intertemporal budget constraint.) Problems with non-existing limits arise only when taxes alternate between positive and negative values with increasing amplitude, so that the expression in (6) has liminf≠limsup. Note that an assumption of non-negative government debt ($D_t \geq 0$) would be sufficient for an equality sign in (6). Given an equality in (6), a sufficient condition for a proper limit is the assumption of non-negative taxes ($T^i_t \geq 0$). Both assumptions seem satisfied in many countries, suggesting that the assumptions of Proposition 2 are not very stringent.

Another interesting feature of the constraints (7) and (8) is that these equations involve sums of fiscal variables weighted by state-contingent claims prices. In general, they cannot be written in terms of discounted expected values of these variables. This feature distinguishes the stochastic model with risk-aversion from deterministic and certainty-equivalance models (see, e.g., Blanchard-Fischer 1989).

To highlight this difference, it is instructive to write the constraints in terms of conditional expectations and to simplify slightly. To simplify, I will abstract from heterogeneity and assume that all individuals are equal (so that the index i can be omitted). Then all individuals have equilibrium consumption $C_t = Y_t - G_t$. The prices of state contingent claims are therefore

$$P(h_{t+N} \mid h_t) = \pi(h_{t+N} \mid h_t) \cdot \beta^N \cdot \frac{U'(Y_{t+N}-G_{t+N})}{U'(Y_t-G_t)}$$

where $\pi(h_{t+N} \mid h_t)$ denotes the conditional probability of h_{t+N} given h_t. Since the state-contingent claims prices are proportional to the relevant conditional probabilities, constraints (7) and (8) can be rewritten in terms of expectations as

$$\lim_{N\to\infty} E_t[u_{t,N} \cdot D_{t+N}] = 0, \tag{9}$$

$$D_t = \sum_{n\geq 0} E_t[u_{t,n} \cdot (T_{t+n} - G_{t+n})] \tag{10}$$

where $E_t[\cdot]$ denotes the expectation conditional on some state h_t and where

$$u_{t,n} = \beta^n \cdot \frac{U'(Y_{t+n}-G_{t+n})}{U'(Y_t-G_t)} \tag{11}$$

denotes the equilibrium value of the marginal rate of substitution between period-t and period-(t+n) consumption.

These expressions look similar to the discounted expected value constraints familiar from the literature, except that debt, taxes, and spending are not multiplied by some fixed discount factor but by a marginal rate of substitution that may vary over time and across states of nature. Since the fiscal variables may be correlated with the marginal rates of substitution, it is not generally possible to write these constraints in terms of fixed discount factors. The objective of the following section will be to show that such correlations are indeed important for the interpretation of fiscal policies.

Before moving on, it is worth noting that a number of simplifying assumptions could be relaxed rather easily. The assumption of time-additive utility is for notational convenience only. It could be replaced by any other preference specification, provided utility is strictly increasing in consumption in all periods and in all states of nature (so that individuals are not voluntarily lending into a Ponzi-scheme). For example, habit formation or preferences of the Kreps-Porteus type could be accommodated.[13] The assumption of exogenous endowments could be replaced by a production technology with capital and labor, since individual optimization would imply a transversality constraint on capital. The assumption of lump-sum-taxation could be replaced by distortionary taxation (say, due to collection cost in an endowment economy or due to incentive problems in a production economy). Distortionary taxation could introduce a complication in that tax revenues might have an upper limit rather than being any number on the real line. But even with limited taxation or other additional policy constraints, equations (9)

---

[13] This point may be important in the context of Mehra and Prescott's (1985) work, which suggests that the basic Lucas model has difficulties in explaining the observed low level of safe interest rates. In light of the ongoing controversy on this issue, I will proceed with the Lucas model for expositional simplicity, but I want to emphasize that the results are not specific to this model. Readers who are comfortable with the arguments of Kocherlakota (1990), Benninga-Protopapagakis (1990), and Kandel-Stambaugh (1990, 1991) may want to take the model literally for the empirical interpretation, including the implicit assumption of a rather high degree of relative risk aversion. Others may interpret the use of the Lucas model as an expositional simplification. Those readers may replace the expression for state-contingent claims prices in (4) by whatever expression comes out of their favorite "solution" to the Mehra-Prescott puzzle, e.g., habit formation as in Constantinides (1990), or non-expected utility as in Weil (1989).

and (10) would still be necessary conditions for sustainability.[14] Overall, the main result of this section is that, depending on the specific assumptions, constraints for the form (6), (7)–(8), or (9)–(10) are the correct constraints on government policy in a large class of stochastic, dynamically efficient economies.

3. Sustainability Conditions in a Stochastic World

This section will show that constraints (9)–(10) differ significantly and in practically relevant ways from the more specialized versions obtained under certainty or risk-neutrality. The differences will first be demonstrated in a simple example, which will then be generalized. In particular, the analysis will question the apparently widespread belief that the government budget constraint and the transversality condition can be expressed in terms of the rate of return on government debt—or at least in terms of the safe interest rate if government debt is safe. As this section will show, such beliefs are generally unwarranted.[15]

3.1. An Example

To gain some intuition about the implications of the transversality and intertemporal budget constraints in a stochastic economy, this section will examine a fiscal policy with safe debt in a very simple stochastic setting. The example is motivated by the fact that many governments issue essentially safe debt in a world in

---

[14] It is straightforward to show that (9) and (10) are also sufficient in the Lucas model, i.e., that the government can implement any policy that satisfies these constraints for some set of market-clearing prices.

[15] In his textbook, Sargent (1987, p.118) make such an assertion in the context of the Lucas model, the same model used here. He apparently omits a covariance term (see below). Wilcox (1989) asserts without proof that realized rates of returns on government debt can be used for discounting in the intertemporal budget constraint. Hamilton and Flavin (1986) recognize that discounting by the average realized return on government debt involves an approximation, but they still assert (also without proof) that the government must satisfy a constraint written in terms of the average realized return on government debt. Similar constraints are invoked by Hakkio and Rush (1986) and Trehan and Walsh (1988, 1991).

which the safe interest rate has often been below the average rate of economic growth. The example and the subsequent generalizations will demonstrate that some popular certainty-based sustainability criteria can be highly misleading under these conditions.

For the example, consider a Lucas economy where endowments Y_t grows at a rate $y_t = Y_t/Y_{t-1}-1$, which is i.i.d. with mean y. Government spending G_t is a fixed fraction $g = G_t/Y_t$ of income, which implies that consumption is also a fixed fraction of income, $C_t = (1-g) \cdot Y_t$. Utility is CRRA with risk aversion parameter α. Income growth is assumed to be low enough that utility is finite.

As tax policy in this economy, consider the policy that keeps the ratio of end-of-period debt to income constant at some level $d = \tilde{D}_t/Y_t$ at all times, i.e., that sets

$$T_t = G_t + D_t - d \cdot Y_t. \tag{12}$$

This policy is chosen as a simple representative of policies that stabilize the debt-income ratio at some time horizon (see Section 3.3. below). An example with stable debt-income ratio is also convenient because some authors have argued that polices with unbounded debt-income ratios would not be practically feasible (e.g., Blanchard 1984, Kremers 1989). One cannot dismiss this example with reference to such considerations.[16]

By assumption, the government issues safe debt. The safe interest rate r is determined by the Euler condition $\beta \cdot E_{t-1}[(1+r) \cdot (1+y_t)^{-\alpha}]=1$. Given debt $\tilde{D}_{t-1} = d \cdot Y_{t-1}$ at the end of period t-1, debt at the start of period t will be $D_t = (1+r) \cdot d \cdot Y_{t-1}$. The ratio of debt D_t to current income Y_t will below the end-of-period target d whenever the growth rate y_t has been above r, and vice versa. To bring the debt-income ratio back

---

[16] I want to note this without taking a position on the empirical issue of what is practically feasible. Bohn (1991b) extends the model of this paper to economic environments with limited taxation in which a bounded debt-income ratio is necessary for feasibility. Policy (12) satisfies such stronger feasibility conditions, provided one picks a reasonably low value for d.

to the target value, relatively high taxes must be imposed whenever income growth is low.[17]

It should be intuitively clear that this policy can be sustained for ever without problems. More formally, the path of debt is such that

$$\lim_{N\to\infty} E_t[u_{t,N} \cdot D_{t+N}] = d \cdot Y_t \cdot \lim_{N\to\infty} \frac{1}{(1+v)^{N-1}} = 0, \tag{13}$$

where $v>0$ is defined by $\beta \cdot E[(1+y_t)^{1-\alpha}] = 1/(1+v)$. The first equality in (13) holds because debt D_{t+N} is proportional to income Y_{t+N-1} for all states of nature h_{t+N-1}. The second equality holds because finite expected utility requires $\beta \cdot E[(1+y_t)^{1-\alpha}]<1$, which implies $v>0$.[18] The key point in this derivation is that the value of debt in the far future must be discounted at roughly the same rate as future income.[19] Government debt consists of safe securities, but the level of debt evolves stochastically over time.

This point is important, because the level of the safe interest rate depends on risk-aversion and on the variance of income growth. If individuals are sufficiently risk averse and if the variance of income growth is sufficiently large, the safe interest rate (r) may well be negative and/or far below the average growth rate of the economy (y). For the following arguments, suppose $r<y$ holds. (The empirical relevance is discussed below.) Then the example has two interesting features. First,

---

[17] For the interpretation, one should think of a period as a very long time interval, say 10 years or even 100 years. A negative relation between income growth and taxes rates would be counterfactual in the short run, but since low income growth increases the debt-income ratio (unless debt has an income-indexed rate of return), a negative relation in the long run is almost inevitable for tax policies that stabilize the debt-income ratio. In this sense, the example provides an indication of the sustainability properties of policies that stabilize the debt-income ratio in the very long run (see Section 3.3 for generalizations). The example abstracts from cyclical and other short run fluctuations.

[18] Since $\sum_{n\geq 1} E_t[u_{t,n} \cdot Y_{t+n}] = 1/v \cdot Y_t$, the constant v may be interpreted as the dividend yield on the fruit tree.

[19] Discounting D_{t+N} involves $(1+v)^{N-1}$ and not the N-th power because debt accumulates deterministically on a period by period basis (from t+N-1 to t+N). For large N, this distinction becomes less and less important.

the expected value of future debt discounted at the safe interest rate diverges to infinity,

$$\lim_{N\to\infty} E_t[\frac{1}{(1+r)^N}\cdot D_{t+N}] = d\cdot Y_t \cdot \lim_{N\to\infty} \frac{(1+y)^{N-1}}{(1+r)^{N-1}} = \infty. \tag{14}$$

Second, expected tax revenues as share of income are always less than expected government spending as share of income,

$$E_{t-1}[T_t/Y_t - G_t/Y_t] = (r-y)\cdot d < 0$$

If r is so low that $E[1/(1+y_t)]<1/(1+r)$ holds (a slightly stronger condition because of Jensen's inequality), then expected tax revenues are also below expected spending in levels at all times, i.e.,

$$E_{t-1}[T_t - G_t] = E[(1+r)/(1+y_t)-1]\cdot d\cdot Y_{t-1} < 0.$$

The first feature is interesting, because several papers in the sustainability literature (e.g., Hamilton and Flavin 1986, Wilcox 1989) have asserted that a zero limit in (14) is a necessary condition for the sustainability of government policy. Hamilton and Flavin discount by the average interest rate on government debt and Wilcox by the realized return, but here both are equal to the safe rate r. The example shows, however, that a zero limit in (14) is <u>not</u> a necessary condition for sustainability in a stochastic economy.[20] The expression in (14) will diverge whenever the growth rate of government debt exceeds the safe interest rate. But in a stochastic environment where the interest rate is below the growth rate of income, debt growth at a rate above the safe interest rate does not necessarily pose a sustainability problem.

---

[20] The cointegration literature (Hakkio and Rush 1986, Trehan and Walsh 1988, 1991) does not test (14) directly, but it uses the same (or similar) transversality condition in deriving the cointegration restrictions. Thus, the problem is not limited to the "discounting" tests. One might wonder, however, whether cointegration tests could be written in a form that they represent a test of the correct transversality condition (9). This question is left for future research.

The second feature is interesting, because policy plans with persistent primary deficits are often viewed as problematic in the political discussion. In this example, point estimates of future government revenues and spending—e.g., CBO or administration projections—would show primary deficits at all times and at all horizons. Still, the policy is clearly sustainable.

The fact that an example with these features exists suggests that one should examine more generally under what conditions the path of discounted debt—the expression in (14)—provides evidence about sustainability and in what way the sustainability conditions (9) and (10) restrict the government's ability to run primary deficits.

3.2. *Discounting Fiscal Variables and the Rate of Return on Government Debt*

The key distinction between the transversality and budget constraints of this paper and the similar-looking constraints in the literature is the fact that (9) and (10) are expressed in terms of marginal rates of substitution rather than interest rates. In general, marginal rates of substitution, state-contingent claims prices (in (7) and (8)), and distributions of assets returns are linked by the Euler equations

$$E_t[u_{t,1} \cdot (1+R_{t+1})] = \sum_{s_{t+1} \in S_{t+1}} p(s_{t+1} \mid h_t) \cdot (1+R(s_{t+1} \mid h_t)) = 1, \tag{15}$$

where $R_{t+1} = R(s_{t+1} \mid h_t)$ denotes the return on a financial asset in period t+1 when state s_{t+1} is realized.

However, the product $u_{t,1} \cdot (1+R_{t+1})$ does not have to equal one in every state of nature. Therefore, the marginal rates of substitution in (9) and (10) can generally not be replaced by present value factors. Specifically, they cannot be replaced by discount factors involving the realized return on government debt (cf. Wilcox 1989), they cannot be replaced by the average return on government debt (cf. Hamilton and Flavin 1986), and they cannot be replaced by safe interest rates, even if all government debt is perfectly safe (cf. Sargent 1987, p.118). All three points were demonstrated in the example, which did involve safe debt.

To see why safe government debt is not a sufficient condition for using rates of return in the constraints, note that the marginal rate of substitution is related to the n-period return $r_t(n)$ on a default-free discount bond by $E_t[u_{t,n}] = \sum_{h_{t+n} \in H_{t+n}} P(h_{t+n} | h_t) = (1+r_t(n))^{-n}$. The intertemporal budget constraint (10) can therefore be written as

$$D_t = \sum_{n \geq 0} \{(1+r_t(n))^{-n} \cdot E_t[T_{t+n} - G_{t+n}] + Cov_t[u_{t,n}, T_{t+n} - G_{t+n}]\} \qquad (16)$$

The covariance terms will drop out, if the economy evolves deterministically or if individuals are risk-neutral. With risk aversion, however, the covariances will only vanish if future primary surpluses are uncorrelated with future marginal utility (cf. Sargent 1987, p.118). In practice, such uncorrelatedness will probably be rare, since it is difficult to imagine a tax and spending policy that is uncorrelated with government spending (itself) and with aggregate income, which are the variables determining the marginal utility of consumption (see (11)). In the example, the covariance terms were large enough that the intertemporal budget constraint was satisfied even though the expected primary surplus $E_t[T_{t+n} - G_{t+n}]$ was negative for all $n>0$.

A similar argument applies to the transversality condition (9). The expression $E_t[u_{t,n} \cdot D_{t+n}]$ can only be written as $E_t[D_{t+n}]/(1+r_t(n))^n$, if the path of debt D_{t+n} is uncorrelated with the marginal rate of substitution. Even if all debt is safe, this will only rarely be satisfied in a stochastic economy with risk aversion, because the amount of new debt issued in future periods will often be correlated with income and government spending (hence with $u_{t,n}$, see (11)).

Concerning primary deficits, the Euler equation (15) suggests that sustainability imposes very few restrictions on the average level of deficits in a stochastic economy with risk aversion. Even if the safe interest were relatively high, assets with return distributions that have a sufficiently high negative covariance with the marginal rate of substitution would have expected returns below the average growth rate. To design a policy with positive expected primary deficits, the government could use the same tax policy as in the example, provided it issues state-contingent debt with the appropriate correlation properties instead of safe debt. Different average levels of the primary deficit could be generated by varying the covariance between the returns on government debt and the marginal rate of substitution.

To summarize, the analysis has shown the traditional practice of discounting fiscal variables at a fixed interest rate appears questionable from a theoretical perspective. In the example, it led to very misleading conclusions about the sustainability of government policy. This raises the question how important correct discounting is under practically relevant conditions.

3.3. Generalizations and the Empirical Relation between Growth and Interest Rates

This section will show that many assumptions of the example can be relaxed without changing the conclusions. I will argue that a safe interest rate below the average growth rate of the economy is the assumption that is primarily responsible for the most striking conclusions of the example, and that this inequality between growth and interest rates is highly relevant for the United States.

Table 1 displays average U.S. growth rates and interest rates for various sample periods. Several different interest rate series are provided, because we are interested in rates of return on government debt as well as the safe interest rate. Since there is no strictly safe (inflation-indexed) asset, average real returns on "almost safe" assets must be used as proxies.[21] Panel A shows the raw data, Panel B shows the differences between average growth rates (y) and average returns (r). As one can see, the difference y-r was generally negative during the 19th century and for the 1980s, but positive for most of the 20th century.[22] The 20th century values are so large that the long 1800-1988 average is also positive. This suggests that the inequality $r<y$ applies for most empirical studies that use 20th century U.S. data. Indeed, Hamilton-Flavin (1986), Wilcox (1989), and Trehan-Walsh (1991) all study the sample period 1960-84, over which U.S. real interest rates were below the U.S. growth rate.[23]

Additional evidence for the importance of low real interest rates is provided in Table 2, which shows average values of the U.S. primary surplus. On average, the U.S. government has run primary deficits over all 20th-century sample periods and also over the full 189 year sample.[24] It seems difficult to reconcile the traditional

---

[21] Average Treasury bill and bond returns are ex post measures of the return on government debt. The average real yield and the average interest charge (the ratio of net interest payments to initial debt) contain some ex ante information. None of the series is a perfect proxy for the safe return, but it is reassuring that all series show similar averages.

[22] If one took after-tax interest rates, the differences would be even larger. An explanation for the low real interest rates is beyond the scope of the paper. (See the literature on the Mehra-Prescott puzzle cited above.) Since extreme assumptions about risk aversion might be needed to reconcile low real interest rates with time-additive utility, it should be emphasized that time-additive utility is only assumed to it simplify the exposition. As noted above, constraints (6)–(10) could also be written in terms of more general utility functions.

[23] The debt-GNP ratio fell on average over this period (using either conventional or the Hamilton-Flavin debt series), so that the growth rate of government debt was even below the rate of economic growth. Since Hamilton-Flavin and Wilcox use interest rates in their tests, one can verify directly that the y-r differences were positive (+2.02% and +1.04%, respectively). Trehan and Walsh (1991) discuss interest rate issues, but they still test transversality conditions that are difficult to justify in a setting with low real interest rates.

[24] The largest deficits were of course run during wars, but not only: The primary deficit was positive in 32 out of the 60 years between 1929 and 1988. In any case, traditional deterministic or certainty-equivalence models would require that peacetime surpluses dominate wartime deficits on average. This has not happened. Interestingly, the sign of the primary deficit and the sign of the growth-interest difference y-r match for all sample periods in Tables 1 and 2 except for the 1980s. In the 19th century, y-r was negative and the government ran primary surpluses on average. For most of the 20th century, y-r was negative and the government ran

expected present value budget constraint with a fiscal policy that ran primary deficits on average over such long periods. In contrast, average primary deficits are easily explained in the model used here. In view of equation (16), the presence of average primary deficits in the empirical data suggest that the covariance term cannot be ignored in practice.

Concerning the generality of the example, I will show that the problem of misleading inferences is relevant for a rather large class of government policies. First, consider the class of policies that yield a bounded debt-income ratio. About the economic environment, assume only that aggregate income has a finite present value.[25] If $\bar{d}$ denotes the upper bound on the debt-income ratio, one has

$$\lim_{N\to\infty} E_t[u_{t,N} \cdot D_{t+N}] \leq \bar{d} \cdot \lim_{N\to\infty} E_t[u_{t,N} \cdot Y_{t+N}] = 0. \tag{17}$$

Thus, a bounded debt income-ratio combined with a finite present value of income is a sufficient condition for sustainability.

Instead of examining the limiting behavior of $E_t[u_{t,N} \cdot D_{t+N}]$, traditional sustainability tests have examined the limiting behavior of expressions of the form $E_t[D_{t+N}/(1+R_{t,N})^N]$, where $R_{t,N}$ is an interest rate. In the context of Hamilton and Flavin (1986), $R_{t,N}$ would be interpreted as the safe real rate. In the context of Wilcox (1989), $R_{t,N}$ would be interpreted as the N-period actual return on government debt. To assess such tests, consider the class of policies with a (small) positive lower bound on the debt-income ratio $\underline{d}>0$ and consider an economic environment where the average interest rate is below the average growth rate of the economy in the sense that $E_t[(1+R_{t,n})^{-n} \cdot Y_{t+N}/Y_t] \geq 1$ for large n. Then

---

primary deficits on average. But in the 1980s, the U.S. government has continued to run primary deficits even though real interest rates were much higher than before.

[25] The formal assumption is $\sum_{n\geq 1} E_t[u_{t,n} \cdot Y_{t+n}] < \infty$; this implies the zero limit in (17). No additional assumptions about the utility function, about the stochastic processes of income and government spending, or about interest rates are needed.

$$\lim_{N\to\infty} E_t[D_{t+N}/(1+R_{t,N})^N] \geq \underline{d} \cdot E_t[Y_{t+N}/(1+R_{t,N})^N] \geq \underline{d} \cdot Y_t > 0, \tag{18}$$

which means that the traditional criterion identifies all such policies as non-sustainable.

Note that this inference is made even if the debt-income ratio has an upper bound, i.e., even if (17) applies. Thus, whenever the average interest rate is below the average growth rate of the economy, the traditional tests yield misleading results for all policies that imply a finite upper bound and a positive lower bound on the debt-income ratio. Given that the upper bound may be arbitrarily high (but fixed), this statement covers a rather large class of policies. In addition, these are merely sufficient conditions. Misleading inferences may also arise when these conditions are not met, but that would have to be examined on a case-by-case basis.[26] The important point is that one cannot rely on the traditional sustainability criterium. In this sense, the example of Section 3.1 is far from being a special case.

4. Conclusions

The paper has derived the intertemporal constraints on government policy in a stochastic, dynamically efficient economy. A key result is that the discounting of future government debt, taxes, and spending in the intertemporal budget constraint and the transversality conditions generally depends on the probability distribution of these variables across states of nature. Except for special cases, these constraints cannot be written in terms of expected fiscal variables discounted at a fixed interest rate.

---

[26] Note that there is some empirical evidence on long run mean-reversion in the debt-income ratio and in related variables (see, e.g., Kremers 1989, Bohn 1991c). For policies with upper bound on the debt-income ratio but without strictly positive lower bound and for economies with high real interest rates, the traditional tests may yield the correct conclusion, but not always. For policies without upper bound on the debt-income ratio, more details about the policy and about the economy would be needed to apply either test.

Correct discounting seems to be especially important if the safe interest rate (or the interest rate on government bonds) is on average below the rate of the economic growth. If the transversality condition were erroneously written in terms of the safe interest rate (or the interest rate on government bonds), even fiscal policies that maintain a stable debt-income ratio would be identified as non-sustainable, although they are clearly sustainable.

One reason to study the sustainability of government policy in a stochastic setting is because the case of dynamically efficient economy with safe interest rate below the average growth rate seems to be empirically relevant for the United States. Such a scenario could not be modeled in a deterministic economy, because dynamic efficiency would require a steady state interest rate above the growth rate. In the U.S., the safe interest rate has in fact been below the average growth rate for long periods. Moreover, the U.S. government has apparently exploited this inequality by running policies that on average resulted in primary deficits.

References

Abel, Andrew, Gregory Mankiw, Larry Summers, and Richard Zeckhauser, 1989, Assessing Dynamic Efficiency: Theory and Evidence, Review of Economic Studies 56, 1-20.

Benninga, Simon, and Aris Protopapagakis, 1990, Leverage, Time Preference, and the 'Equity Premium Puzzle,' Journal of Monetary Economics 25, 49-58.

Blanchard, Olivier, 1984, Current and Anticipated Deficits, Interest Rates and Economic Activity, European Economic Review 25, 7-27.

Blanchard, Olivier, and Stanley Fischer, 1989, Lectures on Macroeconomics, MIT Press, Cambridge.

Blanchard, Olivier, and Philippe Weil, 1990, Dynamic Efficiency and Debt Ponzi-Games under Uncertainty, Working Paper, Massachusetts Institute of Technology.

Blume, Marshall, and Jeremy Siegel, 1990, Long-Term Financial and Economic Returns, Working Paper, The Wharton School of the University of Pennsylvania.

Bohn, Henning, 1991a, Budget Balance through Revenue or Spending Adjustments? Some Historical Evidence for the United States, forthcoming, Journal of Monetary Economics.

Bohn, Henning, 1991b, The Sustainability of Budget Deficits with Lump-Sum and with Income-Based Taxation, forthcoming, Journal of Money, Credit, and Banking.

Bohn, Henning, 1991c, On Testing the Sustainability of Government Deficits in a Stochastic Environment, Working Paper, The Wharton School of the University of Pennsylvania.

Cass, David, 1972, On Capital Overaccumulation in the Aggregative Neoclassical Model of Economic Growth, Journal of Economic Theory 4, 200-223.

Chari, V.V., Lawrence Christiano, and Patrick Kehoe, 1991, Optimal Fiscal and Monetary Policy: Some Recent Results, forthcoming, Journal of Money, Credit, and Banking.

Constantinides, George, 1990, Habit Formation: A Resolution to the Equity Premium Puzzle, Journal of Political Economy 98, 519-543.

Diamond, Peter, 1965, National Debt in a Neoclassical Growth Model, American Economic Review 55, 1126-1150.

Hakkio, Craig, and Mark Rush, 1986, Co-Integration and the Government's Budget Deficit, Working Paper, Federal Reserve Bank of Kansas City.

Hamilton, James, and Majorie Flavin, 1986, On the Limitations of Government Borrowing: A Framework for Empirical Testing, American Economic Review 76, 808-819.

Kandel, Shmuel, and Robert Stambaugh, 1990, Expectations and Volatility of Consumption and Asset Returns, Review of Financial Studies 3, 207-232.

Kandel, Shmuel, and Robert Stambaugh, 1991, Asset Returns and Intertemporal Preferences, forthcoming Journal of Monetary Economics.

Kane, Edward, 1989, The S&L Insurance Mess: How Did It Happen? Urban Institute: Washington, D.C.

King, Robert, 1990, Observable Implications of Dynamically Optimal Taxation, Working Paper, University of Rochester.

Kocherlakota, Narayana, 1990, On the 'Discount' Factor in Growth Economies, Journal of Monetary Economics 25, 43-48.

Kremers, Jeroen, 1989, U.S. Federal Indebtedness and the Conduct of Fiscal Policy, Journal of Monetary Economics 23, 219-238.

Lucas, Robert, 1978, Asset Prices in an Exchange Economy, Econometrica 46, 1429-1445.

McCallum, Bennett, 1984, Are Bond-financed Deficits Inflationary? A Ricardian Analysis, Journal of Political Economy 92, 123-135.

Mehra, Rajnish, and Edward Prescott, 1985, The Equity-Premium: A Puzzle, Journal of Monetary Economics 15, 145-162.

O'Connell, Stephen, and Stephen Zeldes, 1988, Rational Ponzi Games, International Economic Review 29, 431-450.

Sargent, Thomas, 1987, Dynamic Macroeconomic Theory, Harvard University Press, Cambridge.

Trehan, Bharat, and Carl Walsh, 1988, Common Trends, The Government Budget Constraint, and Revenue Smoothing, Journal of Economic Dynamics and Control 12, 425-444.

Trehan, Bharat, and Carl Walsh, 1991, Testing Intertemporal Budget Constraints: Theory and Applications to U.S. Federal Budget and Current Account Deficits, Journal of Money, Credit and Banking 23, 210-223.

Weil, Philippe, 1989, The Equity Premium Puzzle and the Risk-Free Rate Puzzle, Journal of Monetary Economics 24, 401-422.

Wilcox, David, 1989, The Sustainability of Government Deficits: Implications of the Present-Value Borrowing Constraint, Journal of Money, Credit and Banking 21, 291-306.

Zilcha, Itzhak, 1991, Characterizing Efficiency in Stochastic Overlapping Generations Models, Working Paper, Tel-Aviv University.

Table 1: A Comparison of Growth Rates and Interest Rates

Panel A: Basic Data

	Growth GNP	T-Bill return	T-bond return	T-bond yield	Int. Charge	HF return	W return	Memo: Inflation
1929-88	2.97	0.23	0.96	1.72	0.45			3.33
1954-88	3.10	1.11	0.53	2.27	0.89			4.46
1980-88	2.87	3.62	6.53	5.93	3.73			4.62
1960-84	3.14	0.95	-0.51	1.88	0.44	1.12	2.11	5.36
1800-1899	4.28		6.08	5.33	5.62			-0.44
1900-1988	3.14		-0.18	1.38	0.43			3.29
1800-1988	3.74		3.08	3.45	3.18			1.30

Panel B: Differences of GNP-growth and return series

using...	T-Bill return	T-bond return	T-bond yield	Int. Charge	HF return	W return
1929-88	2.74	2.02	1.44	2.52		
1954-88	1.99	2.57	1.08	2.21		
1980-88	-0.75	-3.66	-2.70	-0.86		
1960-84	2.19	3.66	1.64	2.70	2.02	1.04
1800-1988		0.66	0.47	0.56		
1900-1988		3.32	1.89	2.71		
1800-1899		-1.80	-0.82	-1.34		

Legend: For the post-1929 data, GNP is from the Commerce Department; T-bill returns (1 month), T-bond returns, and inflation (CPI) are from the CRSP files. Returns labeled HF and W are the average real rate and the average real return of government debt as measured by Hamilton-Flavin (1986) and Wilcox (1989), respectively. The long (pre 1929) series for GNP and inflation (GNP-deflator) are taken from Bohn (1991a). The series "Int. charge" refers to interest payments on the federal debt divided by debt at the start of the fiscal year, also taken from Bohn (1991a). T-bond yields for all samples and the T-bond returns for the long sample are from Blume and Siegel (1990).

Table 2: U.S. Primary Surpluses

Sample Period	Average Primary Surplus/GNP (in percent)
1929-1988	-1.77
1954-1988	-0.28
1980-1988	-1.47
1960-1984	-0.39
1800-1988	-0.40
1900-1988	-1.34
1800-1899	0.43

Legend:

The average primary surplus/GNP is the difference of federal receipts and non-interest outlays divided by GNP. All data are from Bohn (1991a).

Appendix to:

The Sustainability of Budget Deficits in a Stochastic Economy

Proofs of Propositions 1-2 and of other Assertions

by Henning Bohn

Department of Finance
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6367

July 1991

This appendix contains the proofs of Propositions 1 and 2, some remarks on the definition of a Ponzi-scheme in a stochastic context, and the proofs of the assertions made in footnote 12.

In a setting where a potentially non-optimizing agent—the government—may allow others to run Ponzi-schemes, one has to distinguish carefully between borrowing and lending transactions in the budget equations. Let therefore $A^{i+}{}_t$ be an agent's asset holding of financial assets issued by other private agents, $D^i{}_t$ the agent's holdings of government debt, and $A^{i-}{}_t$ the agents supply of financial assets to others (including the government), where $A^{i+}{}_t \geq 0$, $D^i{}_t \geq 0$, and $A^{i-}{}_t \geq 0$ apply by definition. The net holdings of financial assets are $A^i{}_t = A^{i+}{}_t + D^i{}_t - A^{i-}{}_t$. The budget equation (3) can then be written in more detail as

$$A^{i+}{}_t + D^i{}_t - A^{i-}{}_t + Y^i{}_t - T^i{}_t = C^i{}_t + \sum_{s_{t+n} \in S_{t+1}} p(s_{t+1} \mid h_t) \cdot [A^{i+}(s_{t+1} \mid h_t) + D(s_{t+1} \mid h_t) - A^{i-}(s_{t+1} \mid h_t)]. \qquad \text{(A1)}$$

The two propositions will follow from the following lemma, which shows that an optimizing agent will (i) never lend into other agents' Ponzi-schemes, (ii) never over-accumulate government debt, and (iii) always exploit an opportunity to a run Ponzi-scheme against other agents or against the government, if they permit it. Some useful notation for limit expressions is also defined in the context of the lemma.

LEMMA A1:

Suppose an individual maximizes utility subject to the budget equation (A1) and subject to a set of limiting conditions

$$\liminf_{N \to \infty} \sum_{h_{t+N} \in H_{t+N}} P(h_{t+N} \mid h_t) \cdot A^{i-}(h_{t+N}) \equiv b^i(h_t) \leq b^{i*}(h_t) \qquad \text{(A2)}$$

for all h_t, where $b^{i*}(h_t) \geq 0$ are exogenous constants satisfying $b^{i*}(h_t) = \sum_{s_{t+1} \in S_{t+1}} p(s_{t+1} \mid h_t) \cdot b^{i*}(h_{t+1})$ for $h_{t+1} = (s_{t+1}, h_t)$. Then

(i) $$\liminf_{N \to \infty} \sum_{h_{t+N} \in H_{t+N}} P(h_{t+N} \mid h_t) \cdot A^{i+}(h_{t+N}) \equiv a^{i+}(h_t) = 0$$

(ii) $$\liminf_{N \to \infty} \sum_{h_{t+N} \in H_{t+N}} P(h_{t+N} \mid h_t) \cdot D^{i}(h_{t+N}) \equiv d^{i}(h_t) = 0$$

(iii) $$\limsup_{N \to \infty} \sum_{h_{t+N} \in H_{t+N}} P(h_{t+N} \mid h_t) \cdot A^{i-}(h_{t+N}) \equiv a^{i-}(h_t) \geq b^{i*}(h_t).$$

for all $h_t \in H_t$.

PROOF:

To start, note that the conditions relating $b^{i*}(h_t)$ to the values of $b^{i*}(h_{t+1})$ are without loss of generality, because the same equalities connect $a^{i-}(h_t)$ to the values of $a^{i-}(h_{t+1})$ and $b^{i-}(h_t)$ to the values of $b^{i-}(h_{t+1})$. (Otherwise, one would have to introduce a notation for the b^{i*}-values in (iii) that differs from (A2) and define the values in (iii) recursively from those in (A2) based on this set of equalities.) Next, note that

$$\liminf_{N \to \infty} \sum_{h_{t+N} \in H_{t+N}} P(h_{t+N} \mid h_t) \cdot A^{i}(h_{t+N}) \equiv a^{i}(h_t)$$

$$= a^{i+}(h_t) + d^{i}(h_t) - a^{i-}(h_t) \geq -b^{i*}(h_t) \qquad \text{(A3)}$$

follows from (A2) and from the non-negativity of A^{i+} and of D^{i} (and from the relation liminf(·) = -limsup[-(·)].)

The main argument is by contradiction. Consider an optimal plan (*) of consumption and of financial market activity, $\{C^{i}(h_t)=C^{*}(h_t),\ A^{i+}(h_t)=A^{+*}(h_t),\ A^{i-}(h_t)=A^{-*}(h_t),\ D^{i}(h_t)=D^{*}(h_t)$, for all $h_t\}$. Assume for contradiction that $a^{i}(h^{\dagger}_t) > -b^{i*}(h^{\dagger}_t)$ for some specific history $h^{\dagger}_T$ at some time T under the optimal plan (*). Then consider an alternative consumption plan $\{C^{i}(h_t)=C^{**}(h_t),\ A^{i+}(h_t)=A^{+**}(h_t),\ A^{i-}(h_t)=A^{-**}(h_t),\ D^{i}(h_t)=D^{**}(h_t)$, for all $h_t\}$, which is defined as follows.

Let $C^{**}(h^{\dagger}_{T}) = C^{*}(h^{\dagger}_{T}) + \Delta$, where $\Delta = [a^{i}(h^{\dagger}_{T}) + b^{i*}(h^{\dagger}_{T})]/3>0$, and $C^{**}(h_t) = C^{*}(h_t)$ for all states $h_t \neq h^{\dagger}_{T}$. Plan (**) yields higher utility than (*) because consumption is higher in state $h^{\dagger}_{T}$. To define the financial market activity under plan (**), three cases must be distinguished, depending on why $a^{i}(h^{\dagger}_{T}) > -b^{i*}(h^{\dagger}_{T})$ under plan (*). Since $a^{i}(h^{\dagger}_{T}) + b^{i*}(h^{\dagger}_{T}) = a^{i+}(h^{\dagger}_{T})+d^{i}(h^{\dagger}_{T})+ [b^{i*}(h^{\dagger}_{T})-a^{i-}(h^{\dagger}_{T})]$, plan (*) satisfies either (a) $a^{i-}(h^{\dagger}_{T}) \leq b^{i*}(h^{\dagger}_{T}) - \Delta$, or (b) $a^{i+}(h^{\dagger}_{T}) \geq \Delta$, or (c) $d^{i}(h^{\dagger}_{T}) \geq \Delta$.

In case (a), leave A^{i+} and D^{i} unchanged, i.e., set $A^{+**}(h_t) = A^{+*}(h_t)$ and $D^{**}(h_t)=D^{*}(h_t)$ for all h_t. Increase A^{i-} by

$$\xi(h_{T+n}) = [b^{i*}(h_{T+n})-a^{i-}(h_{T+n})] \cdot \frac{\Delta}{b^{i*}(h^{\dagger}_{T})-a^{i-}(h^{\dagger}_{T})} \geq 0 \tag{A4}$$

for all h_{T+n} that include the history $h^{\dagger}_{T}$. That is, set $A^{-**}(h_{T+n}) = A^{-*}(h_{T+n}) + \xi(h_{T+n})$, so that $A^{**}(h_{T+n}) = A^{*}(h_{T+n}) - \xi(h_{T+n})$. Since

$$\sum_{h_{T+N} \in H_{T+N}} P(h_{T+N} \mid h^{\dagger}_{T}) \cdot \xi(h_{T+N}) = \Delta$$

holds for all N>0, plan (**) satisfies the period-by-period budget equations (3) and (A1). Plan (**) also satisfies condition (A2) for all h_{T+n} that include the history $h^{\dagger}_{T}$. This is because $A^{i-}(h_{T+n})$ is increased by an amount less or equal than the original difference $b^{i*}(h_{T+n})-a^{i-}(h_{T+n}) \geq 0$ under (*), and because this difference is less or equal than the original difference of $b^{i*}(h_{T+n})$ and $b^{i-}(h_{T+n})$. For all other h_t, condition (A2) remains unchanged. Thus, plan (**) satisfies all constraints in case (a).

In case (b), set $A^{+**}(h_{T+n})=0$ for all h_{T+n} that include the history $h^{\dagger}_{T}$ and define $A^{-***}(h_{T+n}) = A^{-*}(h_{T+n}) - A^{+*}(h_{T+n})$. Since $a^{i+}(h^{\dagger}_{T}) \geq \Delta$ under plan (*), one has $a^{i-}(h^{\dagger}_{T}) \leq b^{i*}(h^{\dagger}_{T}) - \Delta$ if the limit $a^{i-}(h^{\dagger}_{T})$ is computed for the modified values $A^{i-}(h_{T+n}) = A^{-***}(h_{T+n})$. Therefore, case (a) can be applied, provided the values of $A^{-***}(\cdot)$ are substituted for $A^{-*}(\cdot)$ in the definition of A^{-**} and provided $a^{i-}(\cdot)$ in (A4) is evaluated under $A^{i-}(\cdot)=A^{-***}(\cdot)$.

In case (c), set $D^{**}(h_{T+n})=0$ for all h_{T+n} that include the history h^{+}_{T} and set $A^{-***}(h_{T+n}) = A^{-*}(h_{T+n}) - D^{**}(h_{T+n})$. Again, case (a) can be applied, provided $A^{-***}(\cdot)$ is used instead of $A^{-*}(\cdot)$.

Overall, plan (**) satisfies all constraints on the consumer problem in all cases and it provides higher utility than (*), which proves the contradiction. Thus, an optimal plan must satisfy $a^{i}(h_t) = -b^{i*}(h_t)$ for all h_t. Given $a^{i-}(h_t) \leq b^{i*}(h_t)$ and given the non-negativity of $d^{i}(\cdot)$ and $a^{i+}(\cdot)$, this requires (i), (ii), and (iii). Q.E.D. ♦

Proposition 1 follows from the lemma, when one recognizes that government debt is limited by agents' demands for financial assets. To clarify the government's financial activities, denote the government's holdings of financial assets issued by individual agents by A^{G}_{t} and gross government debt by D^{+}_{t}, where $A^{G}_{t} \geq 0$ and $D^{+}_{t} \geq 0$. Net debt is $D_t = D^{+}_{t} - A^{G}_{t}$. All three variables are defined on a per capita basis.

PROOF OF PROPOSITION 1:

In a market equilibrium, gross government debt per capita equals agents' average lending to the government, i.e. $D^{+}_{t} = (1/I)\cdot\sum_{i} D^{i}_{t}$. Since this holds for all states of nature, one can take the lower limit

$$\liminf_{N\to\infty} \sum_{h_{t+N}\in H_{t+N}} P(h_{t+N} \mid h_t)\cdot D^{+}(h_{t+N})$$

$$= (1/I)\cdot \sum_{i=1}^{I} \Big[\liminf_{N\to\infty} \sum_{h_{t+N}\in H_{t+N}} P(h_{t+N} \mid h_t)\cdot D^{i}(h_{t+N})\Big] = 0$$

where the last inequality follows from Lemma A1, condition (ii). Equation (6) follows because of $D_t \leq D^{+}_{t}$. Q.E.D. ♦

The main idea for the proof of Proposition 2 that if the government is not willing to lend into a Ponzi-scheme, then nobody is, which means that all agents have to maximize subject to $b^{i*}(h_t)=0$.

PROOF OF PROPOSITION 2:

Given the non-negativity of D^+, the assumed existence of the limit in equation (6) implies

$$\lim_{N\to\infty} \sum_{h_{t+N}\in H_{t+N}} P(h_{t+N} | h_t) \cdot D^+(h_{t+N}) = 0. \tag{A5}$$

By assumption, the government does not over-accumulate financial assets purchased by individual agents, i.e.,

$$\liminf_{N\to\infty} \sum_{h_{t+N}\in H_{t+N}} P(h_{t+N} | h_t) \cdot A^G(h_{t+N}) \leq 0. \tag{A6}$$

Non-negativity and the assumed existence of the limit implies

$$\lim_{N\to\infty} \sum_{h_{t+N}\in H_{t+N}} P(h_{t+N} | h_t) \cdot A^G(h_{t+N}) = 0. \tag{A7}$$

Equations (A5) and (A7) imply (7). If one recursively eliminates future debt, equation (1) implies

$$\begin{aligned} D_t = \sum_{n=0}^{N-1} [\sum_{h_{t+n}\in H_{t+n}} P(h_{t+n} | h_t) \cdot T(h_{t+n})] - \sum_{n=0}^{N-1} [\sum_{h_{t+n}\in H_{t+n}} P(h_{t+n} | h_t) \cdot G(h_{t+n})] \\ + \sum_{h_{t+N}\in H_{t+N}} P(h_{t+N} | h_t) \cdot D(h_{t+N}) \end{aligned} \tag{A8}$$

Since the term on the second line converges to zero due to (7), (A8) implies (8).

To prove the analogous constraints for individuals, note that equilibrium on the market for privately issued securities requires that borrowing equals lending at all times, i.e.,

$$\sum_{i=1}^{I} A^{i-}(h_t) = I \cdot A^G(h_t) + \sum_{i=1}^{I} A^{i+}(h_t).$$

Taking the limit (where the proper limits exists by assumption), this equality together with (A7) and result (i) in Lemma A1 implies $\sum_i b^i(h_t) = \sum_i a^{i+}(h_t) \leq 0$. Given non-negativity, this requires $b^i(h_t) = 0$ for all i. That is, $b^{i*}(h_t)=0$ applies as limiting condition on individual optimization for all individuals and all states h_t. Existence of limits and condition (iii) in Lemma A1 then implies the transversality condition

$$\lim_{N\to\infty} \sum_{h_{t+N}\in H_{t+N}} P(h_{t+N}|h_t) \cdot A^i(h_{t+N}) = 0$$

for all h_t and all i. Together with (4), this implies the intertemporal budget constraints

$$A^i{}_t = \sum_{n\geq 0} \Big[\sum_{h_{t+n}\in H_{t+n}} P(h_{t+n}|h_t) \cdot \{Y^i(h_{t+n})-C^i(h_{t+n})-T^i(h_{t+n})\}\Big].$$

Q.E.D.♦

Finally, a comment on the definition of a Ponzi-scheme and a proof of the assertions in footnote 12 should be provided. The term "Ponzi-scheme" refers to a trading strategy on financial markets that yields a path of net assets that satisfies

$$\limsup_{N\to\infty} \sum_{h_{t+N}\in H_{t+N}} P(h_{t+N}|h_t) \cdot A^i(h_{t+N}) <0,$$

for some state h_t, where A^i would be replaced by (-D) in case of a government. Most importantly, the definition means that one can say that the government runs a Ponzi-scheme, if and only if (6) is false.

The definition seems appropriate, since a sequence of net asset positions with this property would enable the agent to increase consumption at time t in state h_t without reducing it later on (as one can show by an argument similar to the one used in the proof of Lemma A1). O'Connell and Zeldes (1988; p.434) use an analogous definition in a deterministic setting. My definition would reduce to theirs if there were no uncertainty. In any case, the definition is just a label for a particular sequence of asset positions; no substantive results the paper depend on whether one finds the definition appropriate.

The argument why $T^i_t \geq 0$ implies the existence of a proper limit in (7) is a follows. Define total individual wealth

$$W^i_t = A^i_t + Y^i_t + \sum_{n\geq 0} \sum_{h_{t+n}\in H_{t+n}} P(h_{t+n} \mid h_t) \cdot Y^i(h_{t+n}),$$

where the infinite sum exists because the income stream has a finite value by assumption. Define the sequence

$$w^i_n(h_t) = \sum_{h_{t+n}\in H_{t+n}} P(h_{t+n} \mid h_t) \cdot W^i(h_{t+n})).$$

Since the budget equation (4) implies $W^i_t = w^i_0(h_t) = C^i_t + T^i_t + w^i_1(h_t)$, and since

$$w^i_n(h_t) = \sum_{h_{t+n}\in H_{t+n}} P(h_{t+n} \mid h_t) \cdot [C^i(h_{t+n})+T^i(h_{t+n})] + w^i_{n+1}(h_t)$$

the sequence $w^i_n(h_t)$ is monotonously declining in n, provided $C^i_t \geq 0$ and $T^i_t \geq 0$. This is important, because monotonous sequences have a proper limit (including $-\infty$). Since the income stream has a finite value at all times, this implies a proper limit for

$$\lim_{N\to\infty} \sum_{h_{t+N}\in H_{t+N}} P(h_{t+N} \mid h_t) \cdot A^i(h_{t+N}).$$

Since $D(h_{t+n}) = (1/I) \cdot \sum_i A^i(h_{t+n})$ at all times, the limit

$$\lim_{N\to\infty} \sum_{h_{t+N}\in H_{t+N}} P(h_{t+N} \mid h_t) \cdot D(h_{t+N}),$$

must then exists. It must be zero, because the liminf is non-positive and the limsup is non-negative. Q.E.D. Turning to the assertion that $D_t \geq 0$ rules out private Ponzi-schemes, note that the assertion is explained by the fact that $D_t \geq 0$ implies $0 \leq A^G_t \leq D^+_t = (1/I) \cdot \sum_i D^i_t$, which in turn implies (A6) because of condition (ii) in Lemma A1.