OPTIMAL STATE-CONTINGENT CAPITAL TAXATION: WHEN IS THERE AN INDETERMINACY?

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ABSTRACT

Several recent papers on dynamically optimal taxation have derived an indeterminacy result regarding state-contingent capital taxation in stochastic models with state-contingent government liabilities. The indeterminacy arises because the government has N degrees of freedom to set tax rates on capital income in N states of nature, only subject to a single constraint that assures an optimal level of capital investment. The paper shows that this indeterminacy result is a consequence of the assumption that the economy has only a single production technology. If there are many technologies, there will be additional constraints, because differences in capital income tax rates in different states of nature will create incentives to invest in those technologies that have high payoffs in states with relatively low tax rates. If there are a large number of technologies, both the structure of capital tax rates and the structure of government debt are tied down in many dimensions.

1. Introduction

Several recent papers on dynamically optimal taxation have derived an indeterminacy result regarding state-contingent capital taxation in stochastic models with state-contingent government liabilities; see Chari, Christiano, and Kehoe (1991a, 1991b), Zhu (1990), King (1990). Essentially, the government has N degrees of freedom to set tax rates on capital income in N states of nature, but it is only subject to a single constraint that assures an optimal (second-best) level of private capital investment. Assuming that state-contingent government debt has been issued in a way that the government is insured against all relevant "shocks" to the budget, this leaves N-1 degrees of freedom for capital taxation. Alternatively, one can view state-contingent capital taxation as the shock absorber and interpret the indeterminacy result as indeterminacy of the optimal debt structure.

This paper will show that these indeterminacy results are a consequence of the assumption that the economy has only a single production technology. With one technology, the requirement of optimal investment imposes one constraint on capital taxation. But if there are many technologies that have different patterns of state-contingent payoffs, there will generally be one additional constraint per technology. The reason is that differences in capital income tax rates in different states of nature will create incentives to invest in those technologies that have high payoffs in states with relatively low tax rates. Optimal taxation has to take these incentives into account.

In practice, the problem how to tax a multitude of technologies in a way that does not invite tax-avoidance strategies is indeed one of the key issues in capital taxation.[1] This suggests that in practice there is not much scope for shifting the

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[1] See, e.g., the Symposium on Tax Reform in the Journal of Economic Perspectives, Summer 1987.

burden of capital taxes across states of nature. Attempts to do so would likely create severe distortions in the choice of investment projects.

In general, if there are M technologies (where $1 \leq M \leq N$), the Ramsey problem has N-M degrees of freedom. Even for low values of M (but ≥ 2), both the structure of state-contingent returns on debt and the structure of state-contingent capital tax rates are non-trivial.[2] Optimal policy requires optimal tax rates on capital income and an optimal solution of a portfolio problem with at least M securities. Indeterminacy arises only if the government issues more securities than it has to.

The paper is organized as follows. Section 2 sets up a two period version of the Chari, Christiano, and Kehoe (1991b) model (referred to as CCK) with $M \geq 1$ technologies. Sections 3 and 4 examine the implications of different numbers of technologies and securities. Section 5 concludes.

2. The Model

The model is a two period version of CCK. Having more than two periods would merely complicate the conceptual points. The model has individuals (a representative) and a government. The economy starts in period t=0 and ends in period t=1. There are N states of nature $s \in S$ in period 1.

In period t=0, individuals consume c_0, supply labor inputs l_0, purchase government bonds b, and invest in M technologies indexed by m=1,..., M. In state s of period 1, individuals consume c(s) and supply labor inputs l(s). The government taxes wages at rates τ_0 and $\tau(s)$ in periods 0 and 1, respectively. Capital income in period 1 is taxed at rates $\theta(s)$. As in CCK, capital taxation at t=0 must restricted, because it would be a lump-sum tax. For simplicity, a linear technology without

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[2] For M=1, one of them can be set constant without welfare loss.

capital is assumed at t=0. Taxes are ultimately used to finance exogenous government spending g_0 and g(s). The technological constraints are

$$c_0 + g_0 + \sum_{m=1}^{M} k_m = l_0, \tag{1}$$

$$c(s) + g(s) = \sum_{m=1}^{M} F^m(k_m, l_m(s), s) \ \ \forall s \in S, \tag{2}$$

$$\sum_{m=1}^{M} l_m(s) = l(s) \ \ \forall s \in S, \tag{3}$$

where k_m is the investment in technology m, $l_m(s)$ is the amount of labor used in technology m in state s, and $F^m(\cdot)$ is a linearly-homogeneous production function describing technology m.[3]

For the budget constraint, the following notation (from CCK) is used. Wage rates in period 1 are denoted by w(s),[4] purchases of government bonds with state-contingent returns $R_b(s)$ are denoted by b, pre-tax returns on capital of type m are denoted by $r_m(s)$, and after-tax returns are denoted by $R_m(s)$. The individual budget constraints are then

$$c_0 + b + \sum_{m=1}^{M} k_m = (1-\tau_0) \cdot l_0 \tag{4}$$

$$c(s) = w(s) \cdot (1-\tau(s)) \cdot l(s) + R_b(s) \cdot b + \sum_{m=1}^{M} R_m(s) \cdot k_m \ \ \forall s, \tag{5}$$

where

$$R_m(s) = 1 + [1 - \theta(s)] \cdot [r_m(s) - 1] = r_m(s) - \theta(s) \cdot [r_m(s) - 1]. \tag{6}$$

Individuals maximize expected utility

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[3] For notational simplicity, full depreciation of capital is assumed. As in CCK, capital income taxes are levied on capital income net of depreciation. Also for simplicity, units are defined such that period-0 labor productivity is one. In the absence of capital, pre-tax income is then l_0.

[4] The period-0 wage rate equals the period-0 labor productivity, which is normalized to one.

$$U(c_0,l_0) + \beta \cdot \sum_{s \in S} \mu(s)\, U(c(s),l(s)) \qquad (7)$$

subject to these constraints, where $\mu(s)$ is the probability of state s.

CCK show that the optimal solution for taxation (in the sense of Ramsey, 1927) and the associated allocation can be obtained as the result of the following social planning problem: The government maximizes individual preferences (7) subject to the technological constraints (1)-(3) and subject to a set of constraints that assure that the government respects the individual first order conditions. (CCK prove the case M=1, but the case M>1 is analogous.)

The focus here will not be on the optimal allocation itself but on the possibility that the same allocation can be supported by several different tax policies. Assume therefore that an optimal solution to the Ramsey problem has been found. It will be taken as given from now on. Denote the resulting allocation for consumers by $\{c^*_0, l^*_0, b^*, k^*_m, c^*(s), l^*(s), l_m{}^*(s)\}$, denote the policy supporting this allocation by $\pi^* = \{\tau^*_0, \tau^*(s), \theta^*(s), R_b{}^*(s)\}$, and denote the supporting price vector by $\{w^*(s), r_m{}^*(s)\}$. Under the usual concavity assumptions, the consumer allocation and the price vector are unique. However, the policy π^* is generally not uniquely determined. This is the indeterminacy noted by CCK, King, and Zhu.

3. When is there an Indeterminacy?

The source of the policy indeterminacy is easily understood if one simply compares the number of policy instruments and the number of constraints on government policy in the Ramsey problem. The government has 3·N+1 policy instruments (if one counts state-contingent bonds returns as choice variables). The N+1 tax rates on labor income are uniquely determined by the first order conditions for labor supply

$$U_c(c_0^*,l_0^*)\cdot(1-\tau_0^*)+U_l(c_0^*,l_0^*)=0 \qquad (8)$$

$$U_c(c^*(s),l^*(s))\cdot w^*(s)\cdot(1-\tau^*(s))+U_l(c^*(s),l^*(s))=0 \qquad (9)$$

at the optimal allocation (*). To examine the constraints on bond returns and on capital tax rates, define $q^*(s)$ as the period-zero price of an (untaxed) Arrow-Debreu security that pays one unit of consumption in state s, i.e.,

$$q^*(s) = \mu(s) \cdot \beta \cdot \frac{U_c(c^*(s), l^*(s))}{U_c(c_0^*, l_0^*)}. \tag{10}$$

These prices are also uniquely determined at the optimal allocation (*). The N bond returns and the N after-tax returns on capital then have to satisfy

$$\sum_{s \in S} q^*(s) \cdot R_b(s) = 1 \tag{11}$$

$$\sum_{s \in S} q^*(s) \cdot R_m(s) = 1 \tag{12}$$

Since the pre-tax returns on capital are given by the marginal products of capital, $r_m^*(s) = F^m{}_k(k_m^*, l_m^*(s), s)$,[5] equations (6) and (12) imply that

$$\sum_{s \in S} q^*(s) \cdot F^m{}_k(k_m^*, l_m^*(s), s) + \sum_{s \in S} \theta(s) \cdot q^*(s) \cdot [F^m{}_k(k_m^*, l_m^*(s), s) - 1] = 1 \tag{13}$$

has to hold for each technology m. In addition, tax revenues must be sufficient to finance government spending in each state of nature, i.e.,

$$g^*(s) = \theta(s) \cdot \sum_{m=1}^{M} [F^m{}_k(k_m^*, l_m^*(s), s) - 1] \cdot k_m^* - b^* \cdot R_b(s) + \tau^*(s) \cdot w^*(s) \cdot l^*(s) \tag{14}$$

must hold for all s.

Overall, equations (13) and (14) are a system of N+M linear constraints on $\theta(s)$ and $R_b(s)$.[6] Any set of policy choices $\{\theta(s), R_b(s)\}$ that satisfies these conditions is consistent with the Ramsey solution. Thus, one has a system of (at most) N+M

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[5] The subscript k denotes the partial derivative.

[6] Recall that there are N states of nature. Constraint (11) does not have to be imposed separately, because it is implied by (13) and (14) if one takes the sum $\sum_{s \in S} q^*(s) \cdot g^*(s)$.

linearly independent equations for 2·N policy variables.[7] If M<N, there is an indeterminacy of dimension N-M, i.e., N-M of the policy variables can be set arbitrarily.

The literature—CCK, Zhu, and King—has studied the case M=1. In the case of M=1, policy analysis simplifies nicely because either all capital tax rates $\{\theta(s)\}$ or all bond returns $\{R_b(s)\}$ can be set equal to a constant.[8] For the substantive analysis—e.g., for a characterization of the optimal policy—one may focus either on capital taxation issues and assume safe debt, or focus on debt management and assume a state-independent tax rate on capital.

In the real world, however, firms typically have a choice between many different technologies. Moreover, these different technologies seem to have very different patterns of payoffs across state of nature, e.g., if one compares investment projects in different industries, projects in different geographical areas, and projects that are more or less risky. The case M=1 should therefore be viewed as a rather extreme simplification.

The general analysis with M≥1 provides several interesting insights. First, it shows that the degree of indeterminacy depends critically on the number of technologies. The case M=1 is the polar case with the largest degree (dimension) of indeterminacy. In the opposite polar case of M=N, there is a unique optimal policy. All N capital income tax rates are then determined by the need to assure optimal

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[7] The generic case is the linear independence of the N+M equations. This will be assumed from now on. Conditions for linear independence and degenerate cases are discussed in the appendix.

[8] The constant is determined by (13) or (11), respectively. More generally, one of the two distributions can be set arbitrarily up to a scale factor. That is, one may either impose an exogenous pattern of payoffs on debt $R_b(s)/R_b(s_1)$ for states $s \neq s_1$ relative to the payoff some state s_1, or an exogenous pattern of capital tax rates $\theta(s)/\theta(s_1)$ relative to the rate some state s_1 (see Section 4).

investment levels in the N technologies, i.e. by (13). The N returns are determined by (14).[9]

Second, a look at equations (13) and (14) reveals that for all values of M above 1 (i.e. $M \geq 2$) both the structure of capital tax rates and the structure of debt returns are non-trivial. That is, one cannot set one of them constant and one cannot assume that one of them has a distribution across states that is exogenous up to a scale factor. The case M=1 is very special in this sense. If M is a reasonably large number, both the structure of capital tax rates and the structure of debt returns are tied down in many dimensions (at least M).[10]

The fact that both M=1 and M=N are special raises the next question: What can one say about debt management and about capital taxation for cases with M strictly between 1 and N? This is the third issue for which the general analysis is useful; it will be examined in the next section.

4. The Relation between Technology Choices and Debt Management

The main point of this section is to show that the number of technologies determines the minimum degree of "complexity" of the government's debt management policy. Specifically, I will show that in an economy with M

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[9] It is plausible that states of nature are differentiated in more dimensions than just by technologies, suggesting that the case M=N is also unrealistic. Overall, values for M that are large but below N should be most relevant for policy analysis.

[10] Further constraints on capital taxation may arise in the context of technical innovation. If the government uses capital income taxes to generate revenue in some states of nature, firms will have an incentive to find new investment projects that have relatively high payoffs in states of nature with low taxes. Then the government cannot take the set of M technologies as exogenous in its optimization problem. Alternatively, one might try to interpret M as the number of current and potential future technologies. In any case, state-contingent capital income tax rates will presumably be constrained rather tightly by incentive conditions.

technologies, the government's optimal portfolio has to consist of at least J=M different securities. Each additional security (J>M) buys one degree of freedom in setting tax rates on capital income. If M is large, debt management will have to be relatively sophisticated in this sense.

In the optimal taxation literature, the government's debt management problem is often formalized as the problem of choosing the distribution of returns across states of nature (see CCK, Zhu, King, and the model above; cf. Bohn 1990). In finance, the analogous problem of an individual investor is usually formulated as a problem of choosing a portfolio of exogenously defined securities. Here the finance perspective is useful to link technology choices and optimal debt management.

In terms of the state-contingent claims framework of the previous section, a security is defined by its promised payoffs in different states of nature. Suppose one has defined a number of securities indexed by j=1,...,J. The payoffs of security j in period 1 are denoted by $V_j(s)$. The price of security j in period zero, P_j, is given by the Euler equation

$$P_j = \sum_{s \in S} q^*(s) \cdot V_j(s) \tag{15}$$

where the state-contingent claims prices $q^*(s)$ were defined above. The gross rate of return on security j is simply $R_j(s) = V_j(s)/P_j$. Thus, the pattern of returns across states of nature (up to the scale factor $1/P_j$) is determined by the characteristics of the security. The level of returns—the scale factor P_j—is determined by the market.

The total value of government liabilities b can then be interpreted as a portfolio of securities,

$$b = \sum_j P_j \cdot d_j, \tag{16}$$

where d_j is the number of securities of type j that is being issued by the government.[11] The promised total payoff on the government's portfolio in state s is

$$R_b(s)\cdot b = \sum_j V_j(s)\cdot d_j. \tag{17}$$

Now one can replace b and $R_b(s)\cdot b$ in all previous equations by the expressions in equations (16) and (17), respectively. If there are J=N securities with linearly independent payoffs (i.e., complete markets), the choice of a return distribution $\{R_b(s)\}$ examined in Sections 2 and 3 and the choice of a security portfolio $\{d_j\}_{j=1,...,J}$ are equivalent policy problems. However, the formulation with securities provides a new characterization of the indeterminacy result and it allows an extension to an incomplete markets environment.

The government's choice problem is now defined over quantities d_j. If the government issues J different securities, where $M \leq J \leq N$, the Ramsey solution puts N+M linear restrictions on the choices of the N+J variables $\{d_j\}$ and $\{\theta(s)\}$. Thus, there is an indeterminacy, if and only if the number of securities issued by the government exceeds the number of technologies (J>M).[12]

This result has several implications for a substantive analysis of Ramsey-optimal government policies. If one solves the Ramsey problem for a given number of technologies M, one may assume without loss of generality that the government issues exactly J=M securities. This may be a convenient assumption for theoretical analysis, because the optimal net supply of these securities will be uniquely

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[11] If a security has a natural private sector issuer, "issuing" should be interpreted as selling short. Negative values of d_j should be interpreted as security holdings; they are not restricted.

[12] The N+M constraints are equations (13) and equations (14) with $R_b(s)\cdot b$ replaced by (17). It should be reemphasized that degenerate cases have been excluded. (They are discussed in the appendix.) Note that (16) does not have to be imposed separately, because this constraint is implied by (13), (14), and (17) if one takes the sum $\sum_{s \in S} q^*(s)\cdot g^*(s)$.

determined.[13] The "type" of securities is arbitrary, provided they have linearly independent returns. Alternatively, one may assume that net supplies of J-M securities (J-M≥0) are set arbitrarily and that the remaining M are supplied optimally. Or one may optimize over more than M securities to gain degrees of freedom in capital taxation.

For the special case of M=1, the result here reduces to the CCK-King-Zhu result that the debt structure is completely indeterminate: The government may issue any kind of security. For M>1, the "complexity" of optimal government debt policy increases with the number of technologies M in the sense that the number of different securities which have to be issued and be chosen optimally increases in M. The minimum degree of complexity of capital income taxation also increases with M, because even if there are many securities, optimal capital income tax rates will be tied down in at least M dimensions.

Finally, consider a scenario with incomplete markets. If markets are incomplete in the sense that only J securities (with given $V_j(s)$ and J<N) can be traded, the above result implies that the CCK solution to of the consumer allocation problem is feasible only if J≥M. The policy to support this allocation is unique, if J=M. It is indeterminate, if J>M.

In practice, both J and M are presumably rather large numbers. It is far from obvious, which one is bigger. If one takes J as given, this suggests that a determinate solution to the Ramsey problem should be viewed as a "knife-edge" case that will only rarely be practically relevant. On the other hand, one might think of J as the number of securities that the government *wants* to issue, assuming the government could design a new security whenever needed. The indeterminacy result is then equivalent to the observation that the government has the ability to create and issue

[13] Optimal tax rates on capital income are then also uniquely determined.

more securities than it has to on optimality grounds (more than M). Given that the number of available technologies M is rather large so that the government's portfolio problem for J=M is already quite complex, one may wonder how important this ability is in practice.

5. Conclusions

The paper shows that the indeterminacy results about capital taxation and debt management obtained by Chari, Christiano, and Kehoe (1991a, 1991b), Zhu (1990), and King (1990) depend critically on the menu of technologies available to private agents. If there are many production technologies with different patterns of payoffs in different states, differences in capital income tax rates in different states of nature will create incentives to invest in those technologies that have high payoffs in states with relatively low tax rates. Optimal state-contingent capital income tax rates are therefore tied down by incentive constraints in many dimensions.

The assumption of a single aggregate production function is standard in real business cycle models, but it ignores the choice of technologies. It will likely provide a misleading impression about the government's freedom to choose capital income tax rates. One should keep this limitation in mind when one uses real business cycle models for policy analysis.

The need to model production on a disaggregate level is clearly inconvenient for macroeconomic policy analysis, in particular if one wants to move towards calibration or estimation and one does not know much about the set of technologies. Fortunately, there is a simplifying assumption that should be appropriate when there are a "large" number of technologies: With many technologies, optimal state-contingent capital tax rates are tied down in so many dimensions that there should be little scope to use capital income taxation as budgetary shock absorber. For macroeconomic purposes, one may then treat the relative pattern of capital income tax rates across states of nature as exogenously given (up to a scale factor), namely as determined by micro-level incentive issues that one wants to abstract from. Budgetary considerations will then determine the optimal structure of government debt.

References

Bohn. Henning, 1990, Tax-Smoothing with Financial Instruments, American Economic Review 80, 1217-1230.

Chari, V.V., Lawrence Christiano, and Patrick Kehoe, 1991a, Optimal Fiscal and Monetary Policy: Some Recent Results, forthcoming, Journal of Money, Credit, and Banking.

Chari, V.V., Lawrence Christiano, and Patrick Kehoe, 1991b, Optimal Fiscal Policy in a Business Cycle Model, Working Paper, Federal Reserve Bank of Minneapolis.

King, Robert, 1990, Observable Implications of Dynamically Optimal Taxation, Working Paper, University of Rochester.

Ramsey, F., 1927, A Contribution to the Theory of Taxation, Economic Journal 37, 47-61.

Zhu, Xiandong, 1990, Optimal Taxation Under Uncertainty, Working Paper, University of Chicago.

Appendix: Degenerate Cases

Throughout the paper, degenerate cases were excluded. This appendix will discuss under what conditions the relevant rank conditions could be violated.

The key system of equations for the analysis in Section 3 is the system of N+M equations {(13), (14)}. In matrix notation, it can be written as

$$(F \cdot Q) \cdot \theta = 1_M \tag{A1}$$

$$(-b^* \cdot I_N) \cdot R_b + T \cdot \theta = G \tag{A2}$$

where R_b is the 1×N vector with elements $R_b(s)$, θ is the 1×N vector with elements $\theta(s)$, F is the M×N matrix of marginal products $F^m{}_k(k_m^*, l_m^*(s), s)$ (states in columns, technologies in rows), Q the N×N diagonal matrix with prices $q^*(s)$ on the diagonal, I_N an identity matrix of dimension N×N, and T the N×N diagonal matrix with the "tax bases" for capital income taxes $T(s) = \sum_m [F^m{}_k(k_m^*, l_m^*(s), s) - 1] \cdot k_m^*$ on the diagonal (for each state s), 1_M a 1×M vector of ones, and G the 1×N vector of with elements $g^*(s) - \tau^*(s) \cdot w^*(s) \cdot l^*(s)$.

Equation (A2) shows that R_b is uniquely determined by θ. The number of constraints on θ is determined by the rank of (A1). A sufficient condition for (A1) to have rank M is that $q^*(s) \neq 0$ for all s (so that P has rank N) and that F has rank M, i.e., that the m vectors of marginal products $(F^m{}_k(k_m^*, l_m^*(s_1), s_1), \ldots, F^m{}_k(k_m^*, l_m^*(s_N), s_N))$, m=1,...,M, are linearly independent at the Ramsey solution.

In Section 4, the relevant system of constraints is

$$(F \cdot Q) \cdot \theta = 1_M \tag{A3}$$

$$-V \cdot d + T \cdot \theta = G \tag{A4}$$

where d is the 1×J vector of security supplies d_j and V is the N×J matrix of promised payoffs on security j in state s. Equation (A3) is the same as (A1) (and the same as (13)). Equation (A4) is obtained from (14) and (17).

For the case J=M, sufficient conditions for rank N+M in this system are (a) $T(s) \neq 0$ for all s, (b) $q^*(s) \neq 0$ for all s, (c) linear independence of the marginal productivities of capital (i.e., F has rank M) and (d) linear independence of the security returns (i.e., V has rank M). The proof is as follows. Because of (a), θ is uniquely determined as a function of d, $\theta = T^{-1} [G + V \cdot d]$. Inserting this in (A3), d is constrained by

$$(F \cdot Q \cdot T^{-1} \cdot V) \cdot d = 1_M - F \cdot Q \cdot T^{-1} \cdot G. \tag{A5}$$

To show that (A5) yields a unique solution for d, one has to show that the M×M matrix $F \cdot Q \cdot T^{-1} \cdot V$ has full rank M. But this is implied by assumptions (a)-(d), which assure that each of the four components has full rank. For J>M (so that $F \cdot Q \cdot T^{-1} \cdot V$ has dimensions M×J), one clearly has an indeterminacy in (A5).

Failure of the rank conditions has different implications depending on which conditions fail. For J<M and for J≥M with rank of V less than M, one will have generic non-existence of a solution. That is, the CCK-solution can generally not be supported with fewer than M securities. If F has rank f<M, only f securities will be needed (because intuitively, there are only f relevant technologies). An indeterminacy would also occur if $q^*(s)=0$ for some state s, but then state s would have $\mu(s)=0$ and should be excluded.

Finally, if $T(s)=0$ for some s, then the associated $\theta(s)$ variables drop out of (A4). If the number of states n with $T(s)=0$ is less or equal to M, here is still a unique solution in the case of J=M. The argument is (to save space, it is presented without new notation) that N-n of the $\theta(s)$-values are determined by M-n of the d_j-values in N-n of the equations in (A4), which are in turn determined by N-n equations in (A3). The remaining n equations in (A4) determine the remaining n d_j-values and the remaining n equations in (A3) determine the $\theta(s)$-values that do not appear in (A4). New problems arise if n>M, i.e., if the tax base for capital income taxes happens to be zero in more than M states (which is obviously a very degenerate case). Then variations capital tax rates cannot be used to satisfy revenue requirements in "enough" states. One needs at least n securities to satisfy (14)—with uniquely determined d_j values—and there is an indeterminacy in capital tax rates in that there are n-M degrees of freedom in setting $\theta(s)$.