OPTION PRICES AND THE UNDERLYING ASSET'S RETURN DISTRIBUTION

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April, 1991

Abstract

This work examines the relation between option prices and the true, as opposed to risk-neutral, distribution of the underlying asset. If the underlying asset follows a diffusion with an instantaneous expected return at least as large as the instantaneous risk-free rate, observed option prices can be used to place bounds on the moments of the true distribution. An illustration of the paper's results is provided by the analysis of the information concerning the mean and standard deviation of market returns contained in the prices of S&P 100 Index Options.

Although it seems natural that the prices of claims to various parts of the underlying asset's distribution should contain information about the shape of that distribution, Cox and Ross (1976) established that an option's price equals its expected payoff discounted at the risk-free rate where the expectation is taken over the 'risk-neutral', rather than the true, distribution of the underlying asset. Linking the risk-neutral distribution implicit in option prices to the true distribution remains a comparative mystery. A necessary condition for the risk-neutralized pricing methodology to be applicable is that the true and the risk neutral distribution share a common support. The only information about the true distribution that can be obtained from observed option prices *alone* is information about that support. This paper demonstrates though that observed option prices when used *in conjunction* with simple assumed restrictions on the true distribution do contain information about the non-central moments of the true distribution not directly implied by those assumed restrictions alone.[1]

The intuition for the result that option prices can be useful in placing bounds on the moments of the true distribution is straightforward. First, we generalize the results in Lo (1987) to show how the expected payoff to a call can be bounded above in terms of any chosen set of the non-central moments of the return on the underlying asset. Second, we establish restrictions on the true distribution under which the expected return on an option over its life can be bounded below by the expected return on the underlying asset. Then discounting the upper bound on the call's expected payoff at a rate known to be lower than the required return from purchasing the call, one obtains an upper bound on the call's value in terms of the non-central moments of the return on the underlying asset. All those parameter values for the chosen set of moments that imply an upper bound on the call price less than its observed price can then be ruled out.

The paper begins by showing how to use an observed European call price to place a lower bound on the range of the underlying asset's value at the option's maturity. The more valuable an option, the larger is the minimum feasible range for the underlying asset. As an example of the logic subsequently applied in the remainder of the paper, section II examines the familiar world of geometric brownian motion and non-stochastic interest rates in which option prices are determined by the Black Scholes model. The geometric brownian motion assumption places such a strong restriction on the true distribution that once one calculates the volatility parameter implied from observed option prices, the coefficient of variation of the true return distribution is known. All mean-standard deviation pairs implying some other value for the coefficient of variation are ruled out. The paper then examines what can be learned from option prices about the true distribution if it is known that the underlying asset follows a diffusion with an instantaneous expected return that is always at least the instantaneous risk-free rate. In developing the analysis two interim results are established. First, sufficient conditions are established for the elasticity of a call to be at least one. Equivalently, a bound is placed on the degree of concavity in the relation between the call price and the asset's value. Second, if the elasticity is at least one and the drift is at least the risk-free rate, the expected return on a call over its life is shown to be at least that of the underlying asset.

Section III shows how to place an upper bound on an option's expected payoff in terms of the non-central moments of the underlying asset's return distribution, and re-derives the Lo (1987) bound given knowledge of the mean and variance. Section IV examines conditions under which the required return on an option over its life is at least as large as that on the underlying asset. Section V combines the results of sections III and IV to show conditions under which observed option prices can constrain the feasible set of mean-standard deviation pairs for the underlying asset. The paper's results are illustrated in section VI using observed prices of S&P 100 Index options. Conclusions and possible extensions of this line of research are contained in section VII.

I. Call Prices and the Range of the Underlying Asset

If agents strictly prefer more to less, the risk-neutral density associated with a particular outcome for the underlying asset's value at maturity is zero if and only if the true density is also zero. One familiar implication is that if a call is valueless, there must be no chance of it finishing in-the-money. Similarly, if a call trades for exactly the difference between the asset's price and the present value of the exercise price, there must be no chance of it finishing out-of-the-money. A less familiar implication is that whenever a call is trading for more than its intrinsic value, one can determine a minimum range for the underlying asset's distribution.

Assume that the underlying asset, worth S_t, pays no dividends prior to the time T maturity of the call. Today, at time t, the call is worth $c(S_t, t, T, K)$. Let R denote the one plus risk-free rate over the period t through T. Let L and U denote the respective lower and upper bounds on S_T.

Proposition 1: *If* $c(S_t, t, T, K) > \max[0, S_t - R^{-1}K]$, *then* $L < R(S_t - c(S_t, t, T, K))$ *and* $U > KS_t/(S_t - c(S_t, t, T, K))$.

Proof: The validity of Proposition 1 is based on the absence of arbitrage opportunities. Since the call trades for more than its intrinsic value, we know immediately that $L < K < U$. If $L < U \leq K$, one could make an arbitrage profit simply by writing calls. If $K \leq L < U$, one can make an arbitrage profit by selling the call and buying the replicating portfolio by going long one share and borrowing the $R^{-1}K$.

Compare the time T payoffs of three alternate strategies: (i) Buy the call, (ii) Buy the asset and borrow $R^{-1}L$, and (iii) Buy $1 - K/U$ shares of the asset. The payoffs are shown in Figure 1. Since $Pr(S_T < L) = 0$, the payoff from strategy (ii) dominates the payoff from the call and hence the relation between their current prices must be

$$c(S_t, t, T, K) < S_t - R^{-1}L.$$

$$L < R\big(S_t - c(S_t, t, T, K)\big).$$

Since $Pr(S_T > U) = 0$, the payoff from strategy (iii) dominates the payoff from the call and hence the relation between their current prices must be

$$c(S_t, t, T, K) < S_t(1 - K/U).$$

$$U > \frac{KS_t}{S_t - c(S_t, t, T, K)}.$$

Q.E.D.

Figure 1 About Here

The upper limit on L is strictly decreasing in the call price. The lower limit on U is strictly increasing in the call price. Thus we have the intuitive result that, ceteris paribus, the more valuable a call, the larger the minimum feasible range for the distribution of returns on the underlying asset. Section IV will subsequently establish sufficient conditions under which one can make an analogous claim concerning the standard deviation of returns on the underlying asset: holding the expected return constant, the more valuable the call the larger the minimum possible standard deviation of the underlying asset's returns.

II. The True and Risk-Neutral Distributions Given Geometric Brownian Motion

Suppose the underlying asset makes no distributions prior to the time T maturity of a call on the asset and its value, S_t, follows the diffusion process

$$dS_t = \alpha(\cdot)S_t dt + \sigma(S_t, t)S_t dz_t. \tag{1}$$

The moments of the true distribution depend on the functional forms of both the drift and volatility parameters, $\alpha(\cdot)$ and $\sigma(\cdot,\cdot)$. The mean and variance of the gross return on the underlying asset over the option's life can be expressed as

$$\mu \equiv E\{\frac{S_T}{S_t}\} = \mathcal{A}\big(\alpha(\cdot), \sigma(\cdot,\cdot)\big), \tag{2}$$

$$V \equiv \sigma^2(\frac{S_T}{S_t}) = \mathcal{B}\big(\alpha(\cdot), \sigma(\cdot,\cdot)\big). \tag{3}$$

When interest rates are non-stochastic the value of a call on the asset is given by

$$\begin{aligned} c(S_t, t, T, K) &= \mathcal{C}\big(r(\cdot), \sigma(\cdot,\cdot)\big) \\ &= e^{-\int_t^T r(u)du} E\{\max[0, \widehat{S}_T - K]\}, \end{aligned} \tag{4}$$

where the expectation is taken over the risk neutral distribution implied by the diffusion

$$d\widehat{S}_t = r(t)\widehat{S}_t dt + \sigma(\widehat{S}_t, t)\widehat{S}_t dz_t$$

with $r(t)$ equal to the instantaneous risk-free rate. The three equations (2), (3) and (4) are linked by their common dependence on $\sigma(\cdot,\cdot)$. Thus, as will be shown, it can be the case that the call price gives information about $\sigma(\cdot,\cdot)$ which in conjunction with assumptions about $\alpha(\cdot)$ places restrictions on the feasible space of μ, V pairs.

In order to motivate the paper, this section provides a simple example of such a restriction when the drift and volatility parameters are constants.

$$dS_t = \alpha S_t dt + \sigma S_t dz_t.$$

$$\mu = e^{\alpha(T-t)}.$$

$$V = e^{2\alpha(T-t)}\big(e^{\sigma^2(T-t)} - 1\big).$$

$$\sqrt{V} = \mu\sqrt{e^{\sigma^2(T-t)} - 1}.$$

Using the Black Scholes model one can determine σ from an observed option price. Given knowledge of σ the feasible mean-standard deviation space is reduced to a ray through the origin with a slope of $1/\sqrt{e^{\sigma^2(T-t)} - 1}$.

It should not be concluded from the geometric brownian motion example that whenever one knows the functional form of the drift and volatility parameters one can always use observed option prices to place restrictions on the feasible mean-standard space of returns on the underlying asset. For example, suppose the diffusion in (1) takes the form

$$dS_t = [\theta - \lambda \ln(S_t)]S_t dt + \sigma S_t dz_t.$$

The expiration date value of the underlying asset, S_T, is then lognormally distributed.

$$\ln\Big(\frac{S_T}{S_t}\Big) \sim N\Big(\ln(S_t)\big(e^{-\lambda(T-t)} - 1\big) + \frac{\theta - \frac{1}{2}\sigma^2}{\lambda}\big(1 - e^{-\lambda(T-t)}\big), \frac{\sigma^2}{2\lambda}\big(1 - e^{-2\lambda(T-t)}\big)\Big).$$

The standard deviation and the mean gross return on the underlying asset are linked by

$$\sqrt{V} = \mu\sqrt{e^{\frac{\sigma^2}{2\lambda}(1-e^{-2\lambda(T-t)})} - 1}.$$

If interest rates are non-stochastic then options will be priced by Black Scholes and σ can be determined from an observed option price. But knowledge of σ alone places no restriction on the feasible mean-standard deviation space. To do so would require information about λ; i.e., information which could not be determined from option prices. This example in which the log of the underlying asset's price follows an Ornstein-Uhlenback process provides a simple illustration of the distinction between true and risk-neutral distributions. Even when returns are lognormally distributed, the true variance of the continuously compounded return on the asset over the option's life is not necessarily equal to the risk-neutral variance. Only when $\lambda = 0$ are the two variances equal. Thus even in a world where returns are lognormally distributed and options are priced by Black Scholes, it is not necessarily the case that if otherwise identical options are written on two different assets each worth the same amount, then the option on the asset with the higher true variance is more valuable.[2]

The remainder of the paper is devoted to the general case where the volatility parameter is an unspecified function of the asset price and time and the instantaneous expected return on the underlying asset is restricted to be at least as large as the instantaneous risk-free rate. When the underlying asset is a market index, this type of restriction on the drift parameter seems unobjectionable. The following section shows how knowledge of the non-central moments of the return distribution place bounds on an option's expected payoff.

III. Semi-Parametric Bounds on a Call's Expected Payoff

If one knew all the moments characterizing the underlying asset's return distribution, one could determine exactly the expected payoff from an option. When one knows a subset of the moments one can place bounds on the expected payoff. Lo (1987) derives the upper bound on a call's expected payoff when the expectation is taken over all return distributions for the underlying asset with a given mean and variance. This section first considers the general case where one knows the n'th moment of the gross return distribution, $E\{(S_T/S_t)^n\} = \Psi$, and then discusses the Lo bound. Throughout the paper the term 'moments of a distribution' always refers to the *non-central* moments.

A. Bounds Given Knowledge Of The n'th Moment

Given knowledge of $E\{S_T^n\} = S_t^n \Psi$ with $n \geq 1$ one can place an upper bound on the call's expected payoff. Let $\overline{B}(n, \Psi, K)$ denote this upper bound. For $0 < n \leq 1$ one can place a lower bound on the call's expected payoff.

Proposition 2: *For $n \geq 1$ consider $\overline{B}(n, \Psi, K)$, the upper bound on $E\{\max[0, S_T - K]\}$ where the expectation is taken over all gross return distributions with an n'th moment of Ψ. If $K \leq S_t \frac{n-1}{n} \Psi^{\frac{1}{n}}$, the maximizing distribution is degenerate with all the mass at $\Psi^{\frac{1}{n}}$. If $K > S_t \frac{n-1}{n} \Psi^{\frac{1}{n}}$, the maximizing distribution is a two-point distribution with outcomes*

$\frac{K}{S_t}\frac{n}{n-1}$ with probability $\frac{\Psi}{\left(\frac{K}{S_t}\frac{n}{n-1}\right)^n}$, and 0 otherwise.

$$\overline{B}(n, \Psi, K) = \begin{cases} \Psi^{\frac{1}{n}} S_t - K, & K \leq S_t \frac{n-1}{n} \Psi^{\frac{1}{n}}; \\ \Psi \frac{S_t}{n} \left(\frac{S_t}{K} \frac{n-1}{n} \right)^{n-1}, & K > S_t \frac{n-1}{n} \Psi^{\frac{1}{n}}. \end{cases}$$

Proof:

$$E\{\max[0, S_T - K]\} = \Pr(S_T > K) \times (E\{S_T | S_T > K\} - K).$$

The distribution described in Proposition 2 involves a positive probability of finishing-in-the-money. Hence the maximizing distribution cannot place a zero probability on values of $S_T > K$. Consider any non-two-point candidate distribution with the desired n'th moment and a positive probability of finishing in-the-money. Consider also an alternate two-point distribution with gross returns of u and d with probabilities p and $1 - p$ respectively. The expected return from the call is determined completely by the probability of finishing in-the-money, $Pr(S_T > K)$, and the expected gross return conditional upon finishing in-the-money, $E\{S_T | S_T > K\}/S_t$. If

$$p = Pr(S_T > K)$$
$$u = \frac{E\{S_T | S_T > K\}}{S_t}$$

and

$$d = \frac{E\{S_T | S_T \leq K\}}{S_t},$$

then the expected payoff from the call is the same under both the two-point and non-two-point distributions, the expected payoff from the asset is the same under both distributions, and under the two-point distribution $E\{S_T^n\} \leq S_t^n \Psi$ (since S_T^n is convex for $n \geq 1$). The option's expected payoff could be increased if the two-point distribution were altered by increasing u until the moment constraint became binding. Thus the distribution which maximizes the call's expected payoff has mass at no more than two points.

Setting $d = 0$ allows a further increase in u, and hence an increase in the call's expected payoff, without violating the constraint on the n'th moment. The maximizing distribution is therefore given by the values for u and p that solve:

$$\max_{p,u} \quad p(uS_t - K)$$
$$s.t. \qquad p\,u^n = \Psi$$
$$0 \leq p \leq 1.$$

Q.E.D.

Since $K \geq 0$, the call's expected payoff must be less than or equal to that of the underlying asset. If the known moment happens to be the mean, $\overline{\mathcal{B}}(n, \Psi, K)$ achieves its maximal value of $\mathcal{B}(1, \mu, K) = E\{S_T\}$. This can be seen by taking the limit as $n \to 1$. In this case, the maximizing distribution has the intuitive property that the variance is infinite.

As an illustration of Proposition 2 take $n = 2$.

$$E\{S_T^2\} = S_t^2(V + \mu^2).$$

$$\overline{\mathcal{B}}(2, V + \mu^2, K) = \begin{cases} \sqrt{V + \mu^2}S_t - K, & K \leq S_t\frac{\sqrt{V+\mu^2}}{2}; \\ \frac{1}{4}(V + \mu^2)\frac{S_t^2}{K}, & K > S_t\frac{\sqrt{V+\mu^2}}{2}. \end{cases}$$

For completeness, we also consider the lower bound on a call's expected payoff given knowledge of the n'th moment of the underlying asset's gross return distribution. When $n > 1$, knowledge of the n'th moment does not provide a strictly positive lower bound on the call's expected payoff. For $K \geq \Psi^{1/n}S_t$, the n'th moment constraint can be satisfied without assigning any mass to values of $S_T > K$. For $K < \Psi^{1/n}S_t$, the constraint is satisfied by all two-point distributions with probability $1 - p$ of the outcome K and probability p of the outcome $\widehat{S}_T$ with $\widehat{S}_T > S_t\Psi^{\frac{1}{n}}$ and $p = \frac{S_t^n\Psi - K^n}{\widehat{S}_T^n - K^n}$. The expected payoff to the call is $p(\widehat{S}_T - K)$. As $\widehat{S}_T \rightarrow \infty$, both p and $p(\widehat{S}_T - K) \rightarrow 0$.

When $0 < n \leq 1$, knowledge of the n'th moment can provide a strictly positive lower bound on the call's expected payoff.

Proposition 3: *For $0 < n \leq 1$, consider the lower bound on $E\{\max[0, S_T - K]\}$ where the expectation is taken over all return distributions with an n'th moment of Ψ. The lower bound is equal to $\max[0, S_t\Psi^{\frac{1}{n}} - K]$ and is attained by a distribution with a mass point at $S_t\Psi^{\frac{1}{n}}$.*

B. The Bound Conditional On μ And V.

Knowledge of the mean gross return, μ, provides the trivial upper bound that the call's expected payoff cannot exceed μS_t. Knowledge of the second *central* moment, the variance V, does not by itself place any upper bound on the call's expected payoff. (The mean return on the underlying asset could be infinite.) But together μ and V provide knowledge of the second moment of the gross return distribution, $V + \mu^2$. The upper bound on a call's expected payoff conditional on knowing both the first and the second non-central moment is tighter than the bound provided by knowledge of either one alone. Let $\overline{B}(\mu, V, K)$ denote the upper bound on a call's expected payoff given knowledge of the mean and variance of the underlying asset's gross return distribution.

Proposition 4: *(Lo (1987)) Consider the upper bound on $E\{\max[0, S_T - K]\}$ where the expectation is taken over all distributions with $E\{S_T\} = \mu S_t$ and $E\{S_T^2\} = S_t^2(V+\mu^2)$. The maximizing distribution is a two-point distribution: u with probability p, and d otherwise.*

For $K \leq S_t \frac{V+\mu^2}{2\mu}$: $u = \frac{V+\mu^2}{\mu}, p = \frac{\mu^2}{V+\mu^2}$ and $d = 0$.

For $K > S_t \frac{V+\mu^2}{2\mu}$: $u = \frac{K + \sqrt{[K-\mu S_t]^2 + S_t^2 V}}{S_t}$,

$p = \frac{S_t^2 V}{2([K-\mu S_t]^2 + [K-\mu S_t]\sqrt{[K-\mu S_t]^2 + S_t^2 V} + S_t^2 V)}$ and

$$d = \frac{2\mu([K-\mu S_t]^2 + [K-\mu S_t]\sqrt{[K-\mu S_t]^2 + S_t^2 V} + S_t^2 V) - (K + \sqrt{[K-\mu S_t]^2 + S_t^2 V})S_t V}{S_t^2 V + 2([K-\mu S_t]^2 + [K-\mu S_t]\sqrt{[K-\mu S_t]^2 + S_t^2 V})}.$$

$$\overline{B}(\mu, V, K) = \begin{cases} \mu S_t - K\frac{\mu^2}{V+\mu^2}, & K \leq S_t \frac{V+\mu^2}{2\mu}; \\ \frac{1}{2}\left[\mu S_t - K + \sqrt{[K-\mu S_t]^2 + S_t^2 V}\right], & K > S_t \frac{V+\mu^2}{2\mu}. \end{cases}$$

Proof: The proof follows that of Proposition 2. For any candidate non-two-point distribution with the desired mean and variance of returns there exists an alternate distribution with the same mean return, the same expected payoff from the option and a smaller variance of returns. Applying a mean-preserving spread to that alternate two-distribution allows an increase in the option's expected payoff and satisfaction of the variance restriction. Thus the maximizing distribution is given by the values for u, d and p that solve:

$$\begin{aligned} &\max_{u,d,p} \quad p(uS_t - K) \\ s.t. \quad & pu + (1-p)d = \mu \\ & pu^2 + (1-p)d^2 = V + \mu^2 \\ & 0 \leq p \leq 1 \text{ and } d \geq 0. \end{aligned}$$

Q.E.D.

Relative to the case with a single constraint on the n'th moment, the twin constraints on the first and second moments change the problem in two straightforward ways. First, provided the variance is positive the two-point distribution cannot be degenerate.

Second, the optimizing two-point distribution may not involve $d = 0$. For $K > S_t \frac{V+\mu^2}{2\mu}$, the restriction that $d \geq 0$ will not be binding at the optimum.

Stock return distributions typically display positive skewness. The two-point distribution of Proposition 4 is negatively skewed for some parameter combinations. It is interesting to examine the effect of assuming positive skewness as well as fixing μ and V. Imposing positive skewness does not tighten the upper bound on the call's expected payoff beyond $\overline{B}(\mu, V, K)$. If the maximizing two-point distribution is negatively skewed consider the alternate trinomial distribution for the gross return: d, u and x with d and u as in Proposition 4. One can choose the probability associated with the outcome x, $\pi(x)$, such that as $x \to \infty$, $\pi(x)x^2 \to 0$ while $\pi(x)x^3$ remains finite. Allowing x to become arbitrarily large produces a positively skewed distribution with mean and variance arbitrarily close to μ and V and an expected payoff from the call arbitrarily close to $\overline{B}(\mu, V, K)$.

Having determined $\overline{B}(\mu, V, K)$, one can obtain an upper bound on the value of a call in terms of the mean and variance of returns provided one can obtain a lower bound on the call's expected return. Section IV examines conditions under which the expected return on the call is bounded below by the expected return on the underlying asset.

IV. A Lower Bound on a Call's Expected Return

It is well known that the payoff from a call can be replicated by the following buy-and-hold strategy: purchase the asset and issue pure discount debt secured over the asset with a promised payoff of K at time T. One natural view is that since a call is in effect a levered claim on the underlying asset, whenever the asset is expected to earn a risk premium the call will be expected to earn an even larger risk-premium. This is certainty true if the call can be replicated by a buy-and-hold strategy of acquiring the asset and issuing riskless debt. In such a case the buy-and-hold replicating strategy involves borrowing at the riskless rate and investing at a higher expected rate. But when the debt issued as part of the buy-and-hold replication is risky, it can be the case that the borrowing rate can exceed the expected return on the underlying asset. The expected return on the call can be not only less than μ but less than R. In the following binomial option pricing example the underlying asset has an expected return in excess of the risk-free rate over each interval, yet the expected return on a call on that asset is less than the risk-free rate.

Assume the risk-free rate is a constant 10% per period. The option matures in two periods. In each period the return on the underlying asset has a two-point distribution. The tree of prices and associated probabilities is set out in Figure 2(a). The expected return on the underlying asset over the (two period) life of the option is 53.28125%, which exceeds the 21% risk-free rate over the two periods. At each node the expected single period return on the asset exceeds 10%: The expected return over the first period is 32.5%, and conditional on a price increase (decline) during the first period, the asset's second period expected return is 12.5% (31.25%).

Figure 2(a) About Here

Now consider a two period call option with a exercise price of $25. Given the two-point process for the underlying asset and the non-stochastic nature of interest rates, the no-arbitrage tree of call prices follows immediately and is given in Figure 2(b).

Figure 2(b) About Here

The expected return on the call over its life is only 8.28125%. If one thinks of the call as analogous to a long position in the asset and a short position in *two-period risky* debt secured over that asset, that debt is so risky that its required return exceeds μ. If one thinks of the call as equivalent to a dynamic strategy involving a position in the asset and *riskless one-period* debt, the replicating strategy is initially short the asset and long bonds. The call and the asset are perfectly *negatively* correlated during the first period. The expected return on the call over this first period is in fact -17.5%.

If at each trading date the underlying asset's expected return always exceeds the risk-free rate and the dynamic strategy that replicates the call is always long the asset, then the expected return on the call over its life will exceed R. If the dynamic strategy is not only always long the asset but also always involves riskless borrowing, then the expected return on the option over its life will exceed μ.

Proposition 5: *(a) If interest rates are non-stochastic and the underlying asset's price dynamics are given by* (1), *then the call price is an increasing function of the value of the underlying asset. (b) If, in addition,* $\alpha(\cdot) \geq r(t)\ \forall\ t$, *then the expected return on the call over its life is at least* R.

Proof: Let s' and s'' denote two values for S_t with $s' > s''$. The risk-neutral distribution of S_T given $S_t = s'$ first-order stochastically dominates the risk-neutral distribution of S_T given $S_t = s''$. See Proposition 2.18 of Karatzas and Shreve (1987). Intuitively, consider simultaneously starting the process at s' and at s'' with the two evolutions driven by the same Weiner process. If the processes ever meet they become identical. The process that starts at s'' finishes at or below the process that starts at s'. Thus we have

$$e^{-\int_t^T r(u)du} E\{\max[0, \widehat{S}_T - K] | S_t = s'\} > e^{-\int_t^T r(u)du} E\{\max[0, \widehat{S}_T - K] | S_t = s''\},$$

and part *(a)* is established. Now consider the true and risk-neutral distributions corresponding to the diffusions

$$dS_t = \alpha(\cdot) S_t dt + \sigma(S_t, t) S_t dz_t.$$

$$d\widehat{S}_t = r(t)\widehat{S}_t dt + \sigma(\widehat{S}_t, t)\widehat{S}_t dz_t.$$

When $\alpha(\cdot) \geq r(t)\ \forall t$, it follows from a further application of Karatzas and Shreve's Proposition 2.18 that the distribution of S_T first-order stochastically dominates the distribution of $\widehat{S}_T$. The expected return on a call over its life is then

$$\begin{aligned} \frac{E\{\max[0, S_T - K]\}}{c(S_t, t, T, K)} &= \frac{E\{\max[0, S_T - K]\}}{e^{-\int_t^T r(u)du} E\{\max[0, \widehat{S}_T - K]\}} \\ &\geq e^{\int_t^T r(u)du} = R. \end{aligned}$$

Q.E.D.

Intuitively, the result in part *(b)* follows immediately from the properties of the replicating portfolio. The option can be replicating by continually adjusting a portfolio consisting of $\frac{\partial c}{\partial S}$ units of the underlying asset and $c(S_t, t, T, K) - \frac{\partial c}{\partial S} S_t$ worth of riskless debt. Since $\frac{\partial c}{\partial S} > 0$ (part *(a)*), replication of the call's payoff always involves a long position in the underlying asset. The diffusion describing changes in the value of the call is given by

$$dc(S_t, t, T, K) = \Big(r(t) + \Omega(S_t, t)\big(\alpha(\cdot) - r(t)\big)\Big) c(S_t, t, T, K)dt + \frac{\partial c}{\partial S} \sigma(S_t, t) S_t dz_t,$$
$$\text{where} \quad \Omega(S_t, t) \equiv \frac{\partial c}{\partial S} \frac{S_t}{c(S_t, t, T, K)} > 0.$$

Provided $\alpha(\cdot) \geq r(t)\ \forall\, t$, the instantaneous expected return on the call, $r(t) + \Omega(S_t, t)\big(\alpha(\cdot) - r(t)\big)$, is always at least as large as the instantaneous risk-free rate. The expected return on the option over its life is then at least R.

We now turn to the restriction necessary to ensure that the replicating strategy is not only long the underlying asset, but is short the riskless bond.

Proposition 6: *If interest rates are non-stochastic and the underlying asset's price dynamics are given by* (1) *with* $\alpha(\cdot) \geq r(t)\ \forall\, t$ *and* $\frac{\partial c}{\partial t} \leq 0$, *then* (a) *the elasticity of the call price with respect to the value of the underlying asset always exceeds one and (b) the expected return on a call over its life is at least* μ.

Proof: The partial differential equation whose solution, subject to the appropriate boundary conditions, gives $c(S_t, t, T, K)$ is

$$\frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2(S_t, t)S_t^2 + r(t)c(S_t, t, T, K)(\Omega(S_t, t) - 1) + \frac{\partial c}{\partial t} = 0.$$

Suppose that for asset values in the region of zero, the call price is strictly concave in S_t. For values of S_t in this range, $\Omega(S_t, t) < 1$ and all three terms on the left-hand side of the p.d.e are negative. But this would imply a violation of the p.d.e. and hence the option price must be a convex function of S_t for S_t in the region of zero. Throughout this convex region, $\Omega(S_t, t) > 1$.

Figure 3 About Here

Now suppose that for some value of S_t, say s' as in Figure 3, $\Omega(s', t) \leq 1$. Then there must exist a value $s'' \leq s'$ such that the call price is a concave function of the asset's value in the region of s'' and $\Omega(s'', t) = 1$. But this implies that the p.d.e. is violated for $S_t = s''$. Thus $\Omega(S_t, t) > 1 \ \forall \ S_t$ and t, and part *(a)* is established.

Part *(a)* has established that $\Omega(S_t, t) > 1 \ \forall \ S_t$ and t. Provided $\alpha(\cdot) \geq r(t) \ \forall \ S_t$ and t, the instantaneous expected return on the call, $r(t) + \Omega(S_t, t)\big(\alpha(\cdot) - r(t)\big)$, is always at least as large as the instantaneous expected return on the asset, $\alpha(\cdot)$. Consider the gross returns on the asset and on the option over the time interval of length τ immediately prior to maturity, $\tau \leq T - t$. Using iterated expectations, the time t conditional expectation of the gross return on the asset over this interval is

$$E\left\{\frac{S_T}{S_{T-\tau}}\middle| S_t\right\} = 1 + E\left\{\frac{E\{\int_{T-\tau}^{T} \alpha(\cdot)S_u du | S_{T-\tau}\}}{S_{T-\tau}}\middle| S_t\right\}.$$

The corresponding expectation of the gross return on the option is

$$1 + E\left\{\frac{E\{\int_{T-\tau}^{T}\Big(r(u) + \Omega(S_u, u)\big(\alpha(\cdot) - r(u)\big)\Big)c(S_u, u, T, K)du | S_{T-\tau}\}}{c(S_{T-\tau}, T-\tau, T, K)}\middle| S_t\right\}.$$

For an interval of length zero both expectations equal one. As the length of the interval increases the call's conditional expected return over the interval grows at least as fast as the asset's.

$$\frac{\partial E\{\frac{\max[0,S_T-K]}{c(S_{T-\tau},T-\tau,K)}|S_t\}}{\partial\tau} = E\{r(T-\tau)+\Omega(S_{T-\tau},T-\tau)\big(\alpha(\cdot)-r(T-\tau)\big)|S_t\}$$
$$\geq E\{\alpha(\cdot)|S_t\} = \frac{\partial E\{\frac{S_T}{S_{T-\tau}}|S_t\}}{\partial\tau}.$$

For $\tau = T-t$ the expectations are just the mean gross returns on the asset and on the call over the option's life. Thus the expected return on the call over its life must be at least as large as the expected return on the underlying asset, μ. Q.E.D.

The condition $\frac{\partial c}{\partial t} \leq 0$ does not rule out, but does limit the degree of, concavity in the relation between $c(S_t,t,T,K)$ and S_t. Note that convexity of the call price in S_t is sufficient, but not necessary, for $\Omega(S_t,t) > 1\ \forall\ S_t,t$. Sufficient conditions for $\frac{\partial c}{\partial t} \leq 0$ are that either $\sigma(S_t,t) = \sigma(S_t)$ and $r(t) = r\ \forall\ t$, or that $\sigma(S_t,t) = \sigma(t)\ \forall\ t$. If $\sigma(S_t,t) = \sigma(S_t)$ and $r(t) = r\ \forall\ t$, the call price depends only on the time difference $T-t$, and $\frac{\partial c}{\partial t} = -\frac{\partial c}{\partial T}$ which is nonpositive since an option with a longer time to maturity must be at least as valuable as a shorter maturity option. If $\sigma(S_t,t) = \sigma(t)\ \forall\ t$, the option can be priced by Black Scholes and the call price is strictly convex in S_t. Strict convexity implies $\Omega(S_t,t) > 1$ and the p.d.e. implies that $\frac{\partial c}{\partial t} < 0$.

Perrakis and Ryan (1984) develop an alternate sufficient condition for the expected return on the option to be at least that on the underlying asset. Let $g(\cdot)$ and $f(\cdot)$ denote the respective risk-neutral and true densities associated with the distribution of S_T. For $g(s)/f(s)$ non-increasing in s the following relation is satisfied.

$$R \leq \mu \leq \frac{E\{\max[0,S_T-K]\}}{c(S_t,t,T,K)} \leq R + \frac{S_t}{c(S_t,t,T,K)}(\mu - R). \tag{5}$$

For $g(s)/f(s)$ non-decreasing in s,

$$R \geq \mu \geq \frac{E\{\max[0,S_T-K]\}}{c(S_t,t,T,K)} \geq R + \frac{S_t}{c(S_t,t,T,K)}(\mu - R). \tag{6}$$

In the binomial option pricing example beginning this section, the expected return on the underlying asset exceeds the risk-free rate yet the expected return on the call is less than the risk-free rate. Neither (5) nor (6) are satisfied. In that example the ratio of the risk-neutral to the true probability is non-monotonic in S_T.

The Proposition 6 condition that $\alpha(\cdot) \geq r(t)\ \forall\ t$ does not imply the Perrakis-Ryan condition that the ratio of the risk-neutral to the true density, $g(\cdot)/f(\cdot)$, is non-increasing in S_T. But when both the drift and volatility parameters are non-stochastic we have

$$\frac{g(S_T)}{f(S_T)} = \exp\Big(\frac{\ln(S_T/S_t)\int_t^T(r(u)-\alpha(u))du + \frac{1}{2}\big(\big(\int_t^T r(u)du\big)^2 - \big(\int_t^T \alpha(u)du\big)^2\big)}{\int_t^T \sigma^2(u)du}\Big)$$

and if $\alpha(t) \geq r(t)\ \forall\ t$, then $g(\cdot)/f(\cdot)$ is non-increasing in S_T.[3]

$$\frac{\partial \frac{g(S_T)}{f(S_T)}}{\partial S_T} = \frac{\int_t^T(r(u)-\alpha(u))du}{S_T\int_t^T \sigma^2(u)du}\frac{g(S_T)}{f(S_T)} \leq 0.$$

Section III has developed the upper bound on a call's expected payoff given the mean and variance of the return on the underlying asset. This section has examined an intuitive condition under which the expected return on the call is bounded below by the expected return on the underlying asset. We turn next to the implications of combining these two results.

V. The Feasible Set Of μ, V Combinations Given An Observed Call Price

That the call price equals its expected payoff at maturity discounted at the expected return on the call over its life is an identity. Knowledge of the non-central moments (with $n > 1$) of the return distribution of the underlying asset places an upper bound on the call's expected payoff. One can then obtain an upper bound on the value of the call provided one has a lower bound on the discount rate. The tighter the lower bound on the discount rate the tighter the upper bound on the call.

$$c \equiv \frac{\text{Expected Payoff}}{1 + \text{Expected Return}} \leq \frac{\overline{B}(n, \Psi, K)}{1 + \text{Expected Return}}$$
$$\leq \frac{\overline{B}(n, \Psi, K)}{\text{Lower Bound on } (1 + \text{Expected Return})}.$$

Proposition 5 has established intuitive conditions under which the risk-free rate serves as a lower bound on the expected return on the call. Under these conditions, one can use Proposition 2 and an observed call price to place lower bounds on each of the non-central moments of the underlying assets' return distribution with $n > 1$.

A. Bounds On μ And V.

Proposition 6 established conditions under which the mean return on the asset provides a lower bound on the expected return on the call.

Proposition 7: *If the expected return on a call is at least as large as the expected return on the underlying asset and the call is trading at c, all feasible μ, V pairs satisfy*

$$c \leq \frac{\overline{B}(\mu, V, K)}{\mu} = \begin{cases} S - K\frac{\mu}{V+\mu^2}, & K \leq S\frac{V+\mu^2}{2\mu}; \\ \frac{1}{2}\left[S - \frac{K}{\mu} + \sqrt{[\frac{K}{\mu} - S]^2 + \frac{S^2 V}{\mu^2}}\right], & K > S\frac{V+\mu^2}{2\mu}. \end{cases}$$

Equivalently, given μ and V, the call price is bounded above by $\frac{\overline{B}(\mu,V,K)}{\mu}$.

The upper bound on the call price has the following intuitive properties: As $V \to \infty$, the upper bound approaches S_t. As $V \to 0$ and $\mu \to R$, the upper bound approaches $\max[0, S_t - K/R]$. The maximum call price for which the pair μ, V is consistent with an expected return on the call at least as large as that on the asset is a decreasing, convex, differentiable function of K. Figure 4 depicts the relation between $\frac{\overline{B}(\mu,V,K)}{\mu}$ and K. The figure is drawn assuming that the return on the underlying asset over the life of the option has $\mu - 1 = 20\%$ and a standard deviation of 50%. At an exercise price of $K = S_t\frac{V+\mu^2}{2\mu}$, the upper bound on the call price is $\frac{1}{2}S_t$ and the relation switches from linearity to strict convexity.

Figure 4 About Here

Figure 5 portrays lower boundaries of the feasible sets of mean-standard deviation pairs corresponding to observed call prices of $0.3S_t$, $0.5S_t$ and $0.65S_t$ for an at-the-money call; i.e., call value contours in mean-standard deviation space. For an at-the-money call to be worth $0.3S_t$, the mean and standard deviation of the underlying asset would have to plot to the north-east of the contour portrayed with a solid line.

Figure 5 About Here

First consider varying V while holding μ constant. Suppose that $\mu - 1 = 20\%$. If the standard deviation of the underlying asset's returns over the option's life is only 30%, the call will sell for less than $0.3S_t$. For the call to be worth $0.3S_t$, the standard deviation of returns would have to be at least 48%. Similarly, given $\mu - 1 = 20\%$ a call price greater than $0.65S_t$ requires a standard deviation of returns in excess of 141%. For a given mean, the upper bound on the call's price is increasing in the standard deviation. This is logical since the expected payoff to the call is increasing in V. Proposition 1 established that the minimum feasible range for the underlying asset's return distribution is an increasing function of the call price. Proposition 7 establishes an analogous result: for a given μ, the minimum feasible standard deviation of returns consistent with an expected return on the call of at least μ is an increasing function of the call price.

Now consider varying μ while holding the V constant. An increase in the mean will both increase the upper bound on the call's expected payoff and increase the lower bound on the discount rate. For sufficiently high values of V relative to μ the increase in the numerator of $\frac{\overline{B}(\mu,V,K)}{\mu}$ is outweighed by the increase in the denominator, and the upper bound on the call's value actually declines with an increase in μ. This effect can be seen in the dashed contour in Figure 5. With a standard deviation of 141% and $\mu - 1 = 20\%$, a call price of $0.65S_t$ is feasible. Given the same standard deviation and $\mu - 1 = 40\%$, the maximum feasible call price decreases slightly to $0.6454S_t$.

B. Consistency With Geometric Brownian Motion and Black Scholes Option Pricing

Suppose the diffusion describing changes in the asset's value is given by (1) with $\sigma(S_t, t) = \sigma(t)\ \forall S_t, t$. Option prices will be determined by Black Scholes. Now suppose in addition that the drift parameter in (1) is non-stochastic. When the drift and volatility parameters are non-stochastic

$$\sqrt{V} = \mu\sqrt{e^{\int_t^T \sigma^2(u)du} - 1}.$$

Suppose the price of an at-the-money call is $0.3S_t$. The variance of the *continuously compounded* return on the asset over the option's life, $\int_t^T \sigma^2(u)du$, implied from the call price and the Black Scholes model (given $R = 1.2$) is 0.32796. Thus $\sqrt{V} = \mu \times 0.623$ and all feasible mean-standard deviation pairs plot along the ray (through the origin) in Figure 5.

Now add the further restriction that $\alpha(t) > r(t)\ \forall\ t$ and let the risk-free rate over the option's life be 20%. Points below the horizontal line in Figure 5 are infeasible: they involve $\mu - 1 < 20\%$. Thus the feasible set becomes those mean-standard deviation pairs on the ray with values for $\mu - 1$ of at least 20%. Naturally this segment of the ray is a subset of the set of mean-standard deviation combinations lying to the north-east of the contour portrayed by the solid-line.

C. A Ratio Of Bounds Or A Bound On The Ratio

The ratio $\frac{\overline{B}(\mu,V,K)}{\mu}$ is just the ratio of an upper bound on the call's expected payoff to a lower bound on its expected gross return. The natural question to ask is whether there exists a distribution with mean gross return μ and variance V such that both bounds are simultaneously satisfied and the call price attains its upper bound. The answer is yes (almost). When the exercise price is sufficiently low there does exist a two-point distribution with the desired mean and variance such that the call is worth $\frac{\overline{B}(\mu,V,K)}{\mu}$. Otherwise, there exists a trinomial distribution with mean and variance arbitrarily close to μ and V such that $\frac{\overline{B}(\mu,V,K)}{\mu}$ is a feasible call price.

First, consider the case $K \leq S_t \frac{V+\mu^2}{2\mu}$. The expected payoff to the call over all distributions with a mean and variance of μ and V is maximized by the two-point gross return distribution: $\frac{V+\mu^2}{\mu}$ with probability $\frac{\mu^2}{V+\mu^2}$, 0 otherwise (Proposition 4). For this two-point distribution, ownership of a call on the asset is equivalent to ownership of the fraction $1 - \frac{K}{S_t} \frac{\mu}{(V+\mu^2)}$ of the asset itself and the required return on the call equals the required return on the asset. To prevent arbitrage, the call must sell for $S_t - K \frac{\mu}{V+\mu^2}$ which is just $\frac{\overline{B}(\mu,V,K)}{\mu}$ for $K \leq S_t \frac{V+\mu^2}{\mu}$.

Now consider the case $K > S_t \frac{V+\mu^2}{2\mu}$ and the corresponding values for u, p and d specified in Proposition 4. For the moment think of μ and V as parameters used in defining u, d and p. Suppose S_T has a trinomial distribution taking on the values uS_t, dS_t and 0 with probabilities p, $(1-p)(1-\epsilon)$ and $(1-p)\epsilon$ respectively, with ϵ some arbitrarily small positive value. Assume that the associated risk-neutral probabilities are $\frac{Rp}{\mu}$, $\frac{R(1-p)}{\mu}$ and $1 - \frac{R}{\mu}$ respectively.[4]

Provided $K > S_t \frac{V+\mu^2}{2\mu}$, the value of the call is equal to $\frac{\overline{B}(\mu,V,K)}{\mu}$. The mean gross return on the call is μ. As $\epsilon \to 0$, $E\{S_T\} \to \mu S_t$ and $\sigma^2(S_T) \to S_t^2 V$.

Section V has examined the implications for μ and V of an observation on the price of a single call option. Section VI shows how to optimally combine the information implicit in a set of observed option prices.

VI. Feasible μ, V Combinations Given A Set Of Call Prices

Suppose one observed the prices of n options on the same underlying asset with the options sharing a common maturity but differing in their exercise prices, $K_1, \cdots, K_n$. Suppose also that the conditions of Proposition 6 are satisfied. The n option prices restrict the feasible set of μ, V combinations in two ways. First, and most obviously, the feasible set contains only μ, V pairs which are consistent with *all* of the observed options having an expected return at least equal to μ. Second, the n option prices place bounds on the unobserved prices of options with exercise prices $\notin \{K_1, \cdots, K_n\}$. The feasible set of μ, V pairs contains only those parameter values which, for *all* exercise prices, are consistent with the ratio of each call's maximum possible payoff to its minimum possible price being at least μ.

To illustrate these restrictions we examine the prices of four options on the S&P 100 Index. On September 7'th 1989 the index closed at \$324.62. Call options on the index with exercise prices of \$290, \$320, \$340 and \$345 and a common November 17'th maturity closed at \$41, \$14, $\$4\frac{1}{4}$ and $\$2\frac{15}{16}$ respectively. The annualized yield on t-bills maturing November 16'th was 8.02%, equivalent to a 1.512% yield over the option's life. The stock comprising the index are assumed to be pay a continuous dividend, d, at an annual rate of 3.4%. These four options are assumed to be priced as if they are effectively European in nature. It seems not unreasonable to assume that the expected return on these index options exceeds the expected return on the underlying index. Proposition 7 is then applicable with $S_t e^{-d(T-t)}$ replacing S_t.[5]

A. *The Information In The Observed Option Prices*

Proposition 1 sets out the relation between call prices and the minimum feasible range for the underlying asset. Proposition 1 extends naturally to the case where one observes a number of options with different exercise prices. One can conclude from the observed S&P option prices that the minimum feasible range for the index on November 17'th was \$240.65 to \$356.19; i.e., over the period September 7 through November 17 there was some chance of a capital loss on the market of 25.9% or more and some chance of a capital gain of 9.7% or more.

If the expected return on each of the individual options is to be at least μ, then feasible μ, V combinations must satisfy

$$\frac{\overline{B}(\mu, V, K)}{c(S_t, t, T, K)} \geq \mu \; \forall \; K \in \{K_1, \cdots, K_n\}.$$

For the four options on the S&P 100 Index, the feasible mean-standard deviation space is portrayed in Figure 6.[6] The mean and standard deviation of returns refer to a September 7 through November 17 holding period.[7] Mean-standard deviation combinations in the intersection of the four "feasible" sets determined from each option in isolation are consistent with all four options having expected gross returns at least equal to the market's. If, say, the expected return on the market from September 9'th through November 17'th were 1.88%, then given the observed option prices the standard deviation of returns would have to have been at least 7.2%. With instead a much higher expected return on the market of 7%, the standard deviation of returns need only have been 1.93%.

Figure 6 About Here

B. *The Information In The Lower Bound On The Prices Of The Unobserved Options*

The more observations on call prices one has, the more one can potentially restrict the feasible μ, V space. Given n observed option prices, the property that $c(S_t, t, T, K)$ is a convex function of K implies that relatively tight bounds can be placed on the prices of all other options. Figure 7 depicts the situation with four observed prices. Let $\underline{\underline{c}}(S_t, t, T, K)$

denote the minimum possible price of an option with exercise price K given the observed prices of options with exercise prices $K_1, \cdots, K_n$ and the absence of arbitrage.[8] Feasible μ, V combinations must satisfy

$$\frac{\overline{B}(\mu, V, K)}{\underline{c}(S_t, t, T, K)} \geq \mu \ \forall \ K. \tag{7}$$

Figure 7 About Here

If when the expected payoff attains its upper bound given μ and V and simultaneously the option sells for its minimum possible price consistent with the absence of arbitrage, the expected gross return on the option is still less than μ, then that particular μ, V pair can be ruled out.

Figure 8 illustrates such an occurrence. The upper bound on call prices depicted by the curved line, $\frac{\overline{B}(\mu,V,K)}{\mu}$, is constructed for $\mu - 1 = 20\%$ and a 50% standard deviation. Prices of two options with exercise prices K_1 and K_2 (marked with a $\square$) are observed. Although individually each observed price is consistent with $\mu = 1.2$ and $V = 0.25$, the price of the call with the lower exercise price is too low relative to the call with the higher exercise price for that μ, V pair to be feasible. The minimum possible prices consistent with the absence of arbitrage of options with exercise prices in the range K_2 to K_3 exceed the upper bound given by $\frac{\overline{B}(\mu,V,K)}{\mu}$. Thus the candidate μ, V pair is ruled out.

Figure 8 About Here

The dashed-dotted line in Figure 6 shows the lower boundary of the feasible mean-standard deviation space once the four observed prices are used to determine the minimum possible prices of all unobserved options. The dashed-dotted line overlies the contours determined from the four observed option prices for $\mu - 1$ in the range 1.6% to 5.6%. But for $\mu - 1$ less than 1.6% or greater than 5.6%, consideration of the minimum possible values of all unobserved options does further restrict the feasible mean-standard deviation space.

VII. Concluding Remarks

This work explores a potential link between the moments of an asset's return distribution, determined by the diffusion $dS_t = \alpha(\cdot)S_t dt + \sigma(S_t, t)S_t dz_t$, and the value of call options on the asset, determined from the corresponding risk-neutral diffusion $d\widehat{S}_t = r(t)\widehat{S}_t dt + \sigma(\widehat{S}_t, t)\widehat{S}_t dz_t$. Provided the instantaneous expected return on the asset is always at least as large as the instantaneous risk-free rate, observing a call price places a bound on the feasible space of mean-standard deviation pairs. Conversely, knowledge of the mean and standard deviation of the returns on the underlying asset over the option's life places an upper bound on the call price. An illustration of the paper's results is provided by the analysis of the information concerning the mean and standard deviation of market returns contained in the prices of S&P 100 Index options.

Although the bounds derived here are weak, so too is the restriction $\alpha(\cdot) \geq r(t) \; \forall \, t$. Two avenues for extending this work appear promising. The first involves imposing more a priori restrictions on the return distributions; e.g., for some assets it may be reasonable to assume that the return distribution is positively skewed with kurtosis bounded below some level. The second involves extending the results to debt options. The no-arbitrage bounds on debt options are interesting: Par value provides a natural upper bound on the value of the underlying pure discount bond at the option's expiration, and the price of a debt option must be at least as large as the price of an otherwise equivalent option written on a later maturing bond. The difficulty in extending the results to debt options lies in the determination of a natural bound on a debt option's expected return. Debt option pricing inherently involves consideration of the equilibrium price of interest-rate risk. For a given distribution of the expiration date value of the underlying bond, it will be interesting to explore the bounds on debt option prices implied by various specifications of the sign and functional form of the price of interest-rate risk.

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Footnotes

* The Wharton School of the University of Pennsylvania. I am grateful for the comments of Avi Bick, Michael Brennan, Chi-fu Huang, Ravi Jagannathan, Peter Kempthorne, Andy Lo, Francis Longstaff, Robert Whitelaw and seminar participants at the Australian GSM, Melbourne GSM, MIT, Wharton, UBC and UCLA.

[1] Other work exploring the link between option prices and the underlying equilibrium includes Bailey and Stultz (1989), Bates (1991a) and (1991b), Bick (1985) and (1987), Breeden and Litzenberger (1978), Brennan (1979), Jagannathan (1984), Perrakis and Ryan (1984) and Rubinstein (1976).

[2] The Ornstein-Uhlenback example should be thought of as an illustration of the point first made in Jagannathan (1983, p.16). Jagannathan (1984) shows that the more "risky" is the risk-neutral distribution of the underlying asset in the Rothschild-Stiglitz sense, the more valuable an option on the asset and that this relation need not apply to the true distribution.

[3] For $\alpha(t) = r(t) \; \forall \; t$, the true and risk-neutral densitys are identical. For $\alpha(t) \leq r(t) \; \forall \; t$, the ratio of the risk-neutral to the true density is non-decreasing in S_T.

[4] For the outcomes 0, dS_t and uS_t, the ratio of the risk-neutral to the true probability equals $\frac{\mu-R}{\mu(1-p)\epsilon}$, $\frac{R}{\mu(1-\epsilon)}$ and $\frac{R}{\mu}$. For ϵ sufficiently small, the ratio is decreasing in S_T.

[5] If the drift and/or volatility parameters in the diffusion describing changes in S_t are stochastic then $E\{S_T\} \neq S_t e^{-d(t,T)}\mu$, with μ defined as the mean gross return over

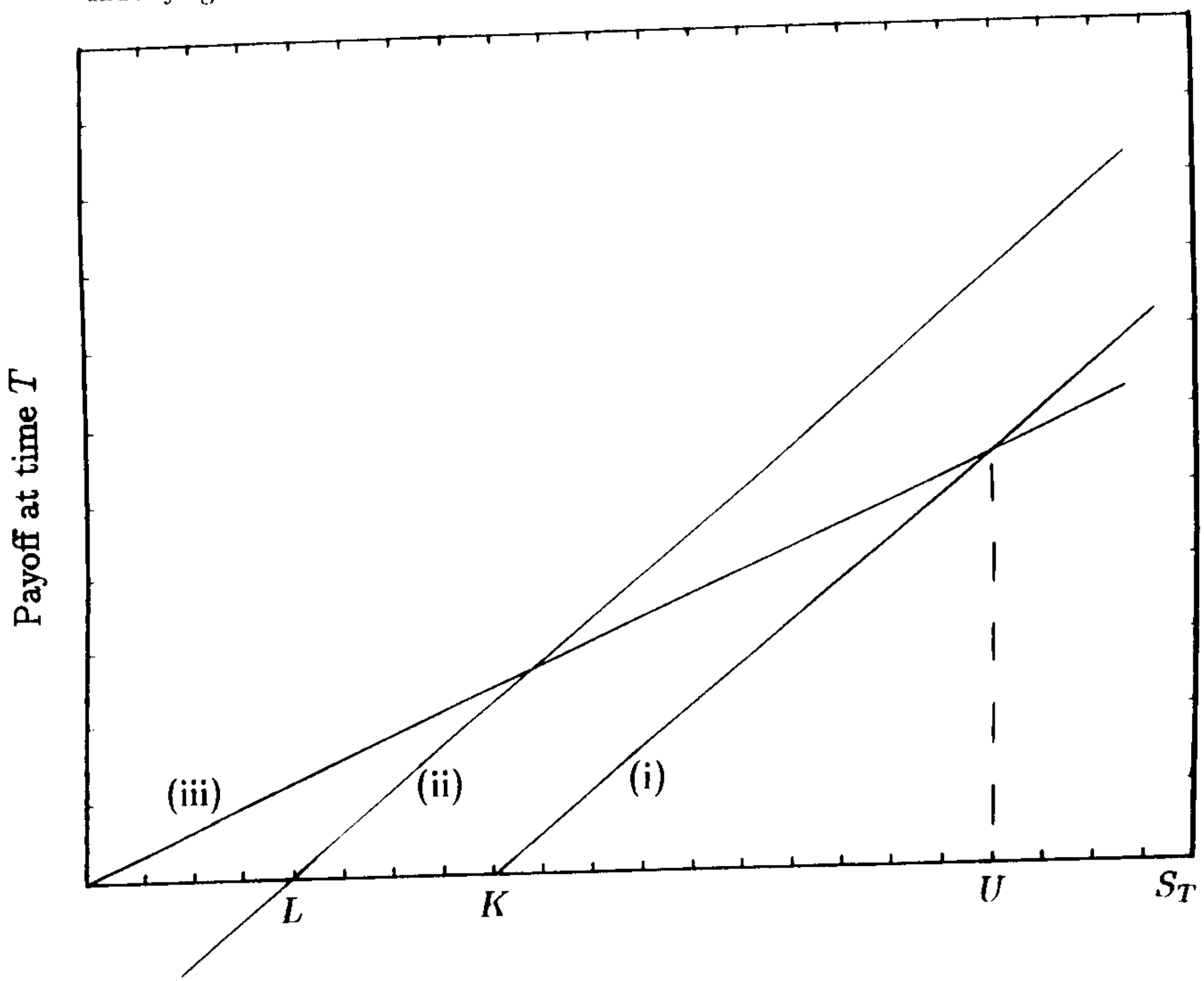
the option's life from a long position in the stock with dividend re-investment. With only 71 days to maturity, the error involved in using this approximation is presumably trivial. Note $e^{-0.034\frac{71}{365}} = 0.9934$.

[6] It bears pointing out that all the issues associated with non-simultaneous observations and bid-ask spreads arise here.

[7] Although annualized data are a more familiar metric, annualization would require that one know the functional forms of the drift and volatility parameters of the diffusion describing changes in the index.

[8] With changes in S_t described by (1), non-stochastic interest rates and a perfect capital market, the introduction of additional options will not change the prices of the existing options.

Figure 1. Time T Payoffs from three alternate investment strategies. The underlying asset is worth S_T at time T. Strategy (i): Long a call with an exercise price of K. Strategy (ii): Long the underlying asset plus borrowing of $R^{-1}L$. The one plus risk-free rate over the investment horizon is R. Strategy (iii): Long $1 - K/U$ shares of the underlying asset.



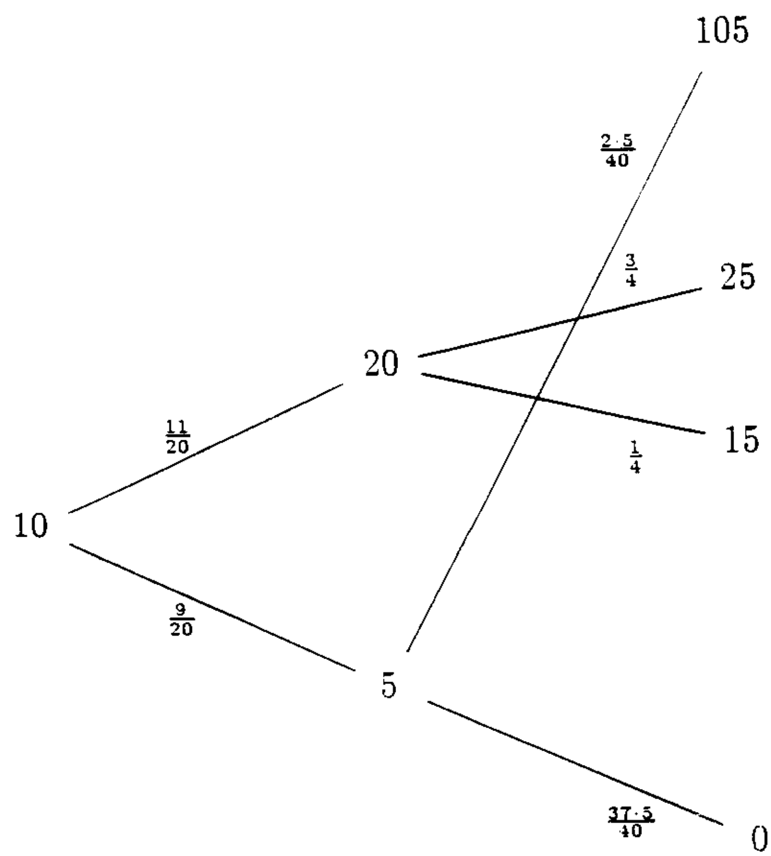


Figure 2(a). Price and probability tree for an underlying asset. The underlying asset is a firm which switches to holding relatively low risk investments if it initially does well and switches to investing in deep-out-of-the-money options if it initially does poorly.

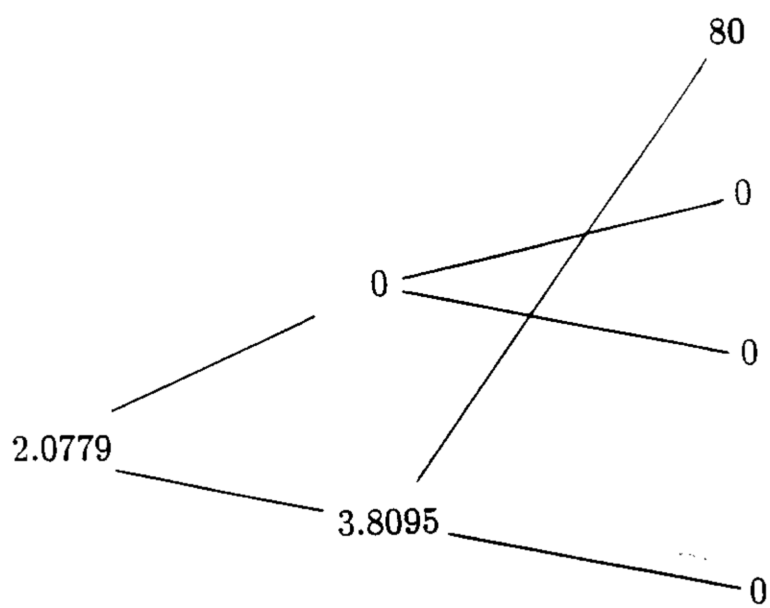


Figure 2(b). Tree of call prices. Prices of a two-period call with an exercise price of $25 written on the underlying asset portrayed in Figure 2(a). The risk-free rate is 10% per period.

Figure 3. **A relation between $C(S_t, t, T, K)$ and V_t that would imply the existence of arbitrage opportunities if the option is a wasting asset.** For $S_t = s''$, $\Omega(s'', t) = 1$ and $\frac{\partial^2 c}{\partial S^2}\big|_{S_t = s''} < 0$.

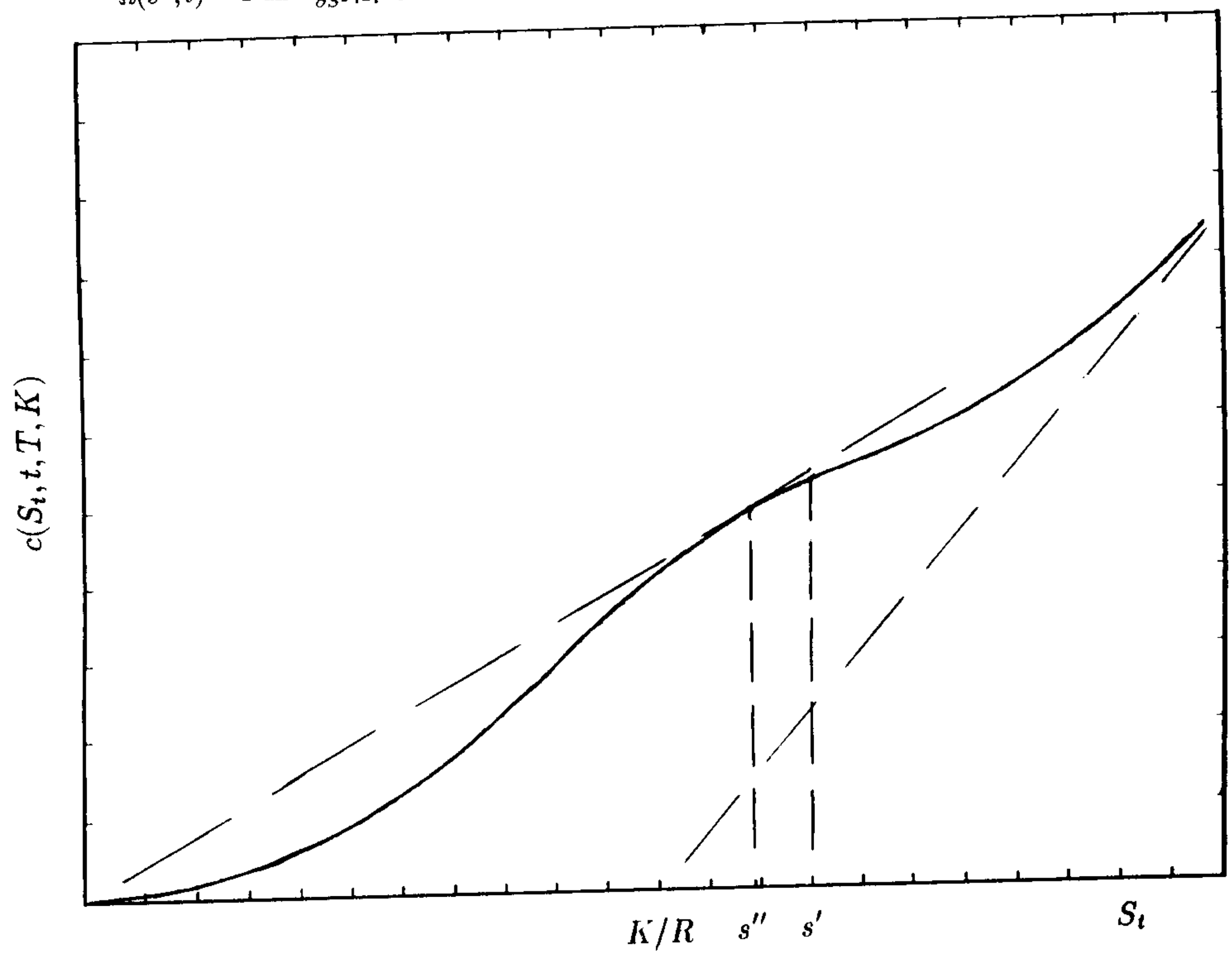


Figure 4. Upper bound on the value of a call option as a function of the exercise price. The upper bound is portrayed by the convex curve, $\frac{\overline{B}(\mu,V,K)}{\mu}$. The bound is calculated assuming that the net return on the underlying asset over the option's life has a mean, $\mu - 1$, of 20% and a standard deviation, $\sqrt{V}$, of 25%.

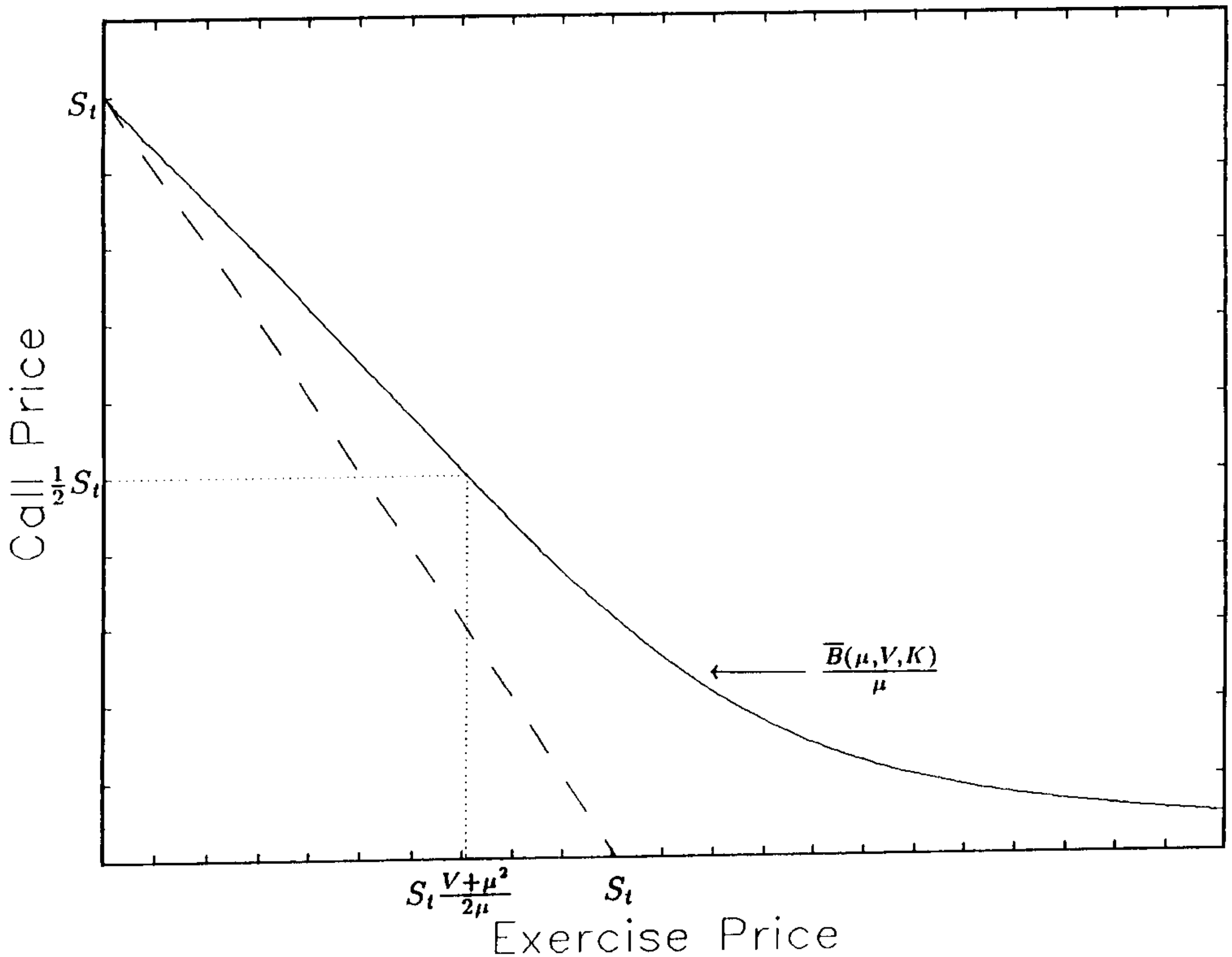


Figure 5. Feasible mean-standard deviation space given various prices of an at-the-money call. If the call is selling for $0.3S_t$, feasible mean-standard deviation pairs consistent with an expected return on the option at least as large as that on the underlying asset plot to the northeast of the solid contour. If the call is selling for $0.5S_t$ (for $0.65S_t$), the feasible space is to the northeast of the dotted (dashed) contour.

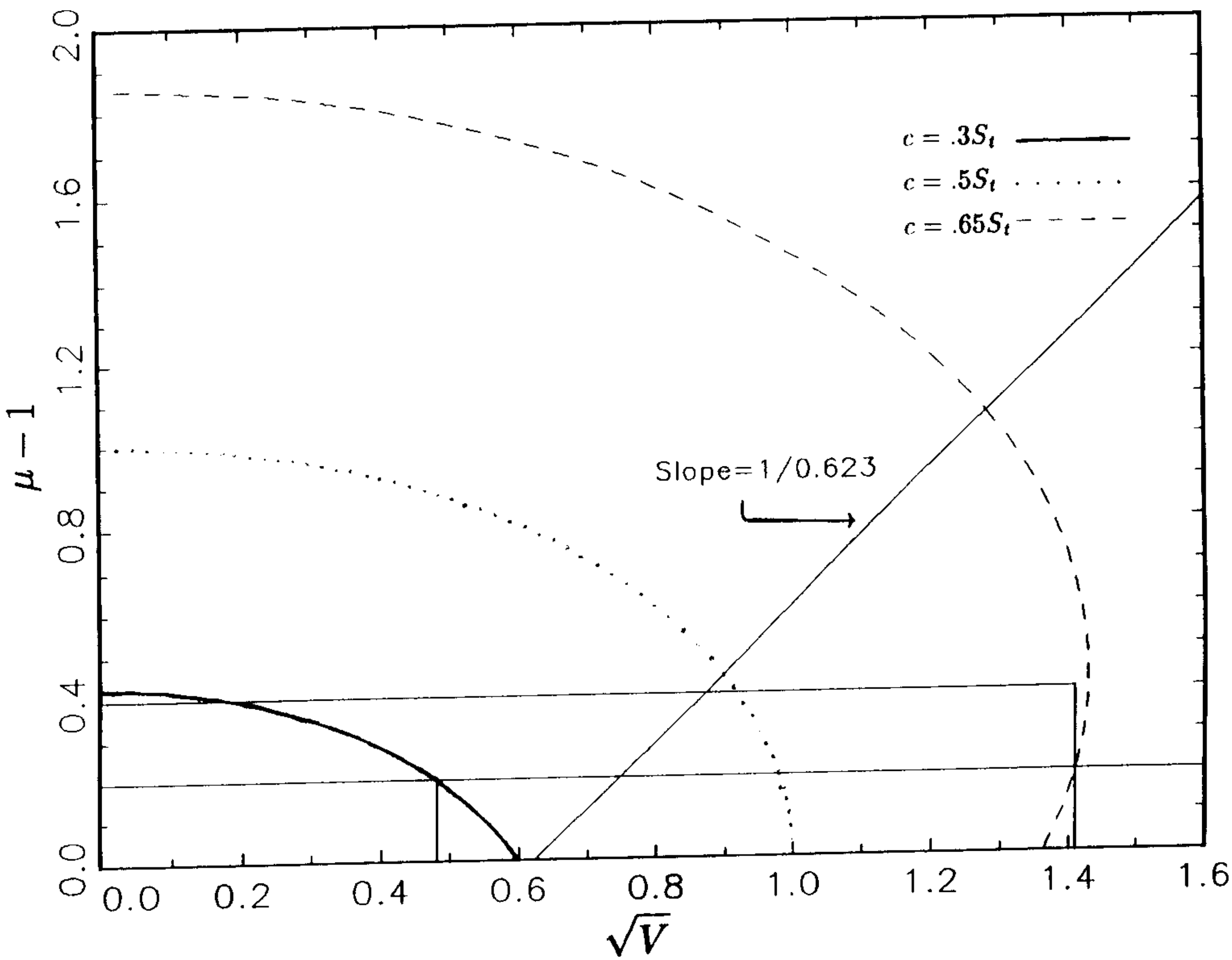


Figure 6. Feasible mean-standard deviation space given prices observed on 9/7/89 of S&P 100 Index options with a 11/17/90 maturity. The contours corresponding to the four individual observed options are portrayed either with dots exclusively or with dashes exclusively. Mean-standard deviation combinations consistent with all options having expected returns at least as large as the underlying index (relation (7) in the text) lie to the northeast of the contour portrayed with both dots and dashes.

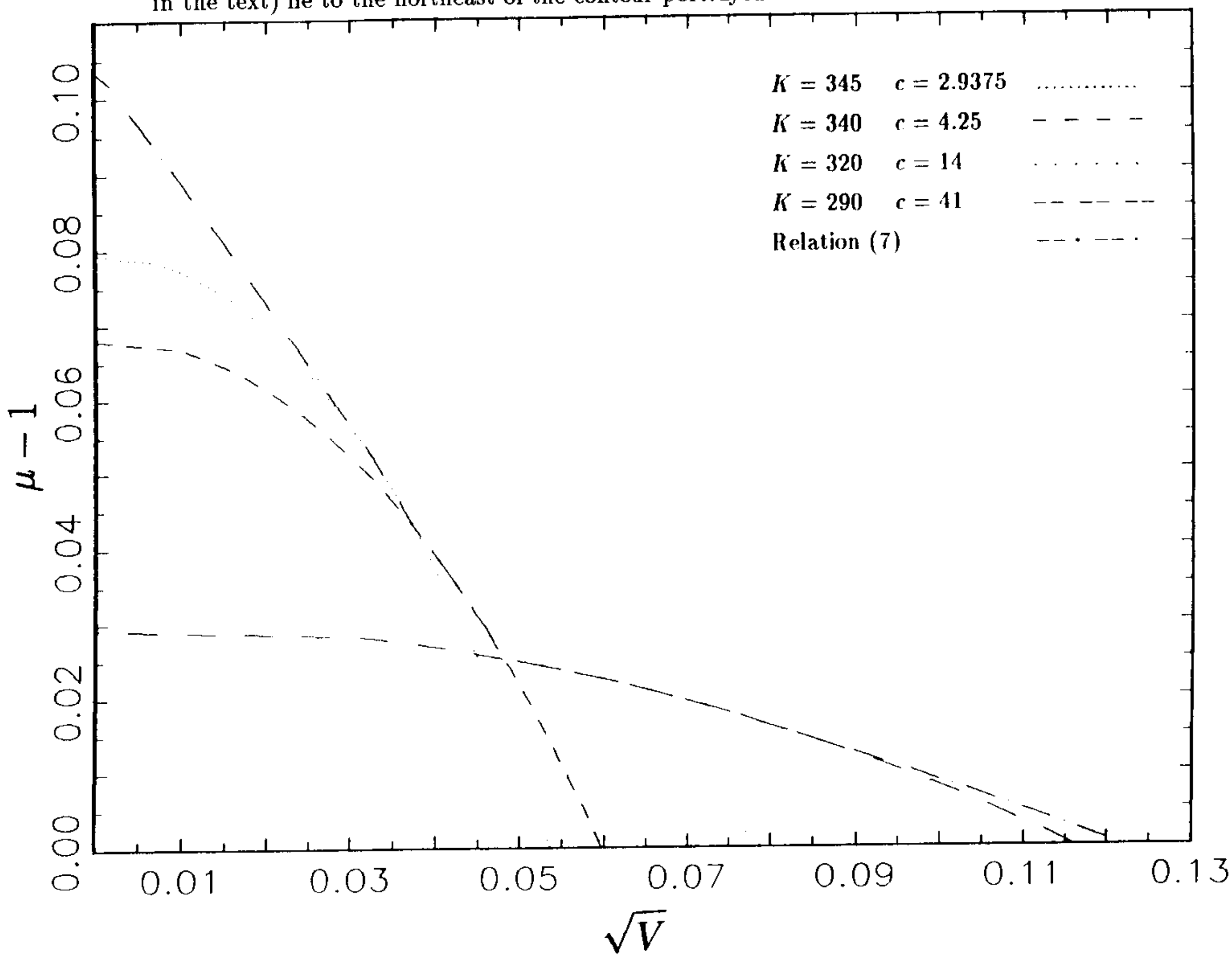
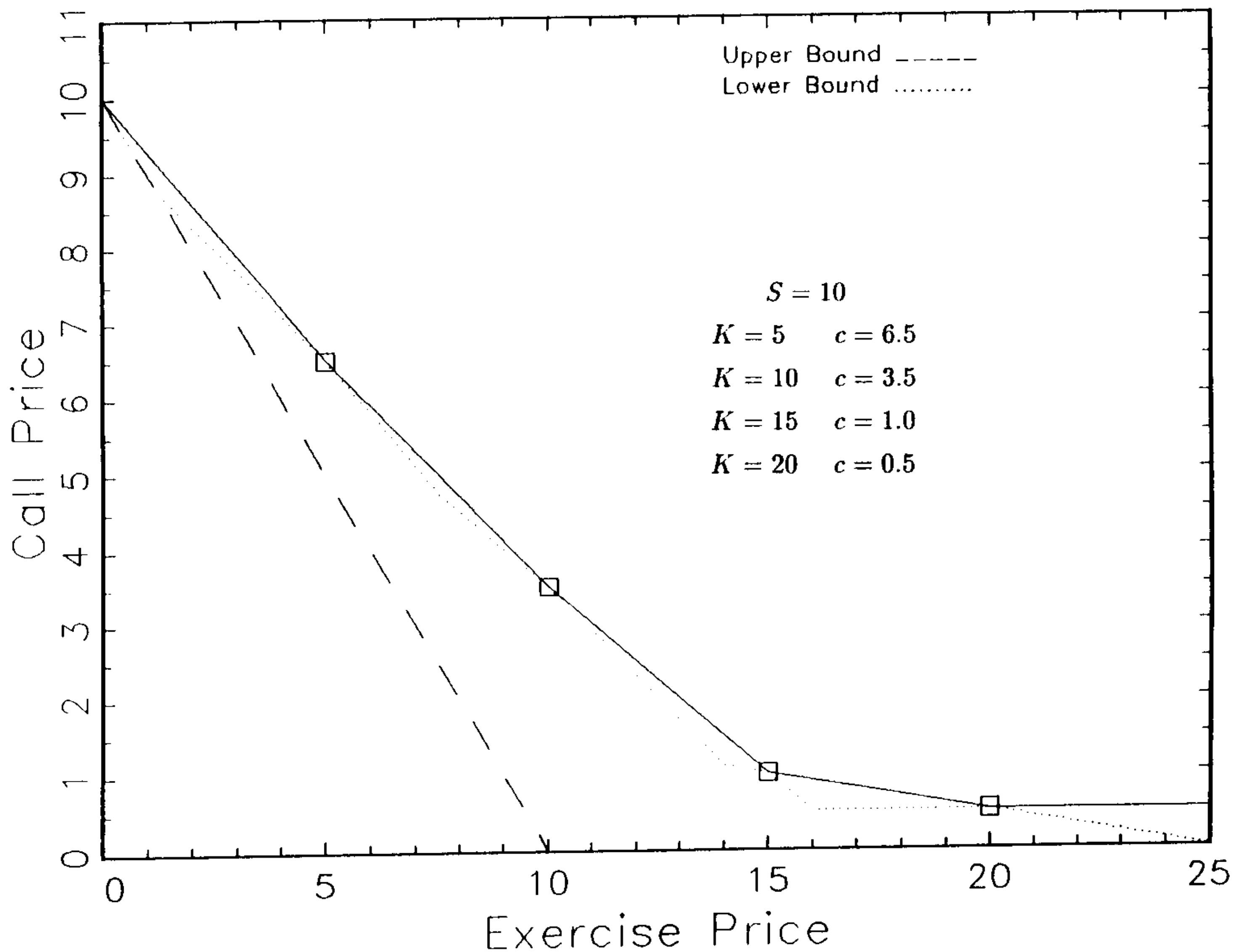


Figure 7. Upper and lower bounds on option prices implied from a set of four observed prices. The symbol □ is used to plot the assumed prices of options with exercise prices of $5, $10, $15 and $20.



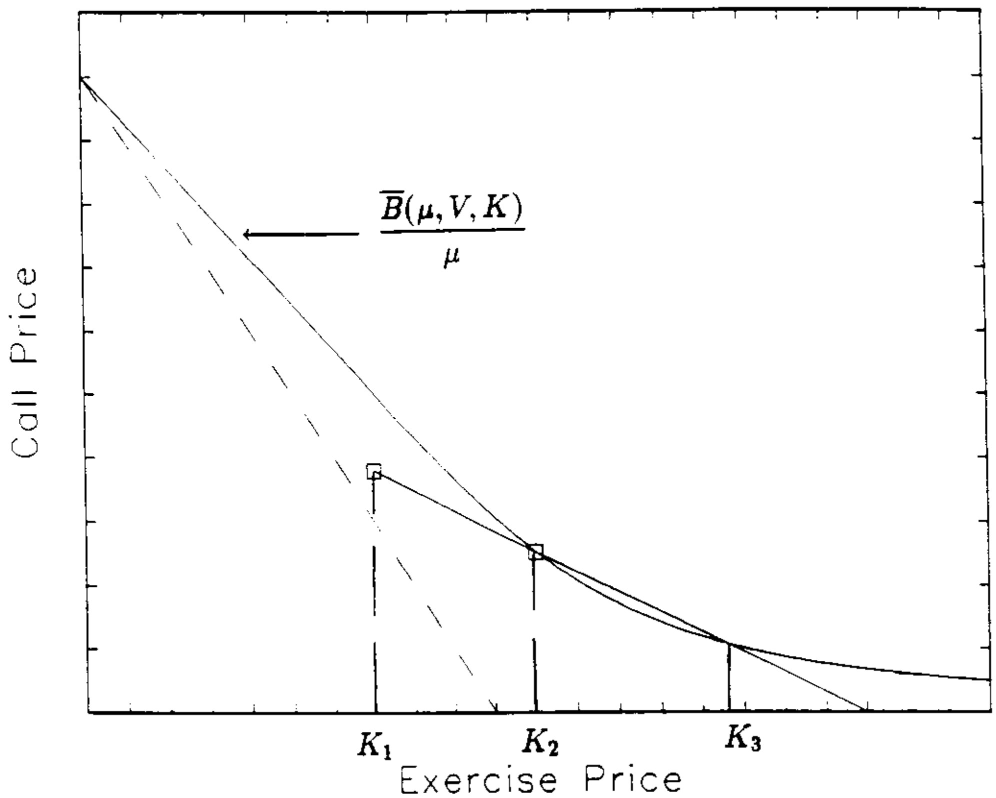


Figure 8. Example where a particular mean-standard deviation pair is inconsistent with all options having an expected return at least that of the underlying asset given the observed prices of two options. The assumed prices of options with exercise prices of K_1 and K_2 are plotted with the symbol □. Given the mean-standard pair used to calculate $\frac{\overline{B}(\mu,V,K)}{\mu}$, the upper bound on the value of options with $K_2 < K < K_3$ is less than the lower bound on their value consistent with the assumed values for $c(S_t, K_1)$ and $c(S_t, K_2)$ and the absence of arbitrage.