TRADING MECHANISMS IN SECURITIES MARKETS

by

Ananth N. Madhavan

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RODNEY L. WHITE CENTER FOR FINANCIAL RESEARCH
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6367

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Ananth Madhavan[1]

The Wharton School
University of Pennsylvania
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Abstract

This paper analyzes and contrasts the process of price formation under two alternative trading mechanisms: a continuous *quote-driven* system where dealers post prices before order submission and an *order-driven* system where traders submit their orders before prices are determined. The order-driven system can operate either as a *continuous auction*, providing immediate order execution, or as a *periodic auction*, where orders are batched for execution at a pre-determined time. Throughout, trading is modeled as a game where order quantities and beliefs are determined endogenously and agents act strategically. We show that prices in the continuous dealer system are more efficient and less variable than prices in the continuous auction system. The auction mechanism is more robust in the sense that it can operate where the dealer mechanism cannot, but the reverse is not true. The two mechanisms are equivalent only in a 'large' market.

We demonstrate that a periodic mechanism, by pooling orders for simultaneous execution, can overcome the problems of information asymmetry that cause failure in a continuous mechanism where trading takes place sequentially. When both mechanisms are viable, a periodic system offers greater price efficiency but requires traders to sacrifice continuity. The results provide a partial explanation for differences in market organization across assets and markets.

1 Introduction

There is an remarkable diversity in the method by which trading is accomplished around the world and across assets. For example, a trader on the International Stock Exchange (London) can obtain firm price quotations before trading and order execution at these prices is assured. By contrast, some stocks on smaller European exchanges can be traded only once a day and orders must irrevocably submitted before prices are determined. Yet, we know little about the relative performance of different trading designs. This paper develops a model of price formation under different forms of market organization when information is imperfect and traders act strategically.

Understanding the relative merits of different trading designs is important for theoretical and applied reasons. From a theoretical viewpoint, seemingly subtle variations in trading protocols can lead to very different equilibrium outcomes. Further, models of the trading process that incorporate institutional detail and strategic behavior may possess new equilibria not found in frictionless Walrasian models. Recent empirical research suggests that market structure has important effects on properties of asset prices. For example, Amihud and Mendelson (1987) find significant differences in the distribution of New York Stock Exchange (NYSE) stock returns from open-to-open, where trading is organized as an auction market, and close-to-close, where prices are determined in a specialist market.[1] Stoll and Whaley (1990) confirm these findings, attributing them to NYSE opening practices. Finally, the securities industry is in the process of rapid structural changes generated by inter-market competition, innovations in communications technology, and the proliferation of new financial instruments. Understanding the relationship between market structure and performance is necessary to evaluate the impact of these changes and to guide public policy.

A trading mechanism transforms the latent demands of investors into realized transac-

[1]See Stoll (1985) for a description and analysis of the NYSE's specialist system.

tions. The key to this transformation is *price discovery*, the process of finding market clearing prices. It this function, not the routine services of order routing and settlement, that differentiates market organizations. No two trading mechanisms are alike in the performance of price discovery; they differ in the technology of order submission, the times at which trading can occur, the amount of market information conveyed to investors at the time of order submission, and the reliance upon intermediaries to balance supply and demand. A useful taxonomic distinction, which helps to focus attention on the process of price formation, is between *quote-driven* and *order-driven* mechanisms.

In a posted-price or quote-driven system, such as NASDAQ or the International Stock Exchange (London), investors can obtain firm price quotations from intermediaries who buy or sell on demand.[2] This type of mechanism is *continuous* because an investor need not wait for his or her order to be executed but trades immediately with the dealer. A continuous dealer system is characterized by a sequence of bilateral transactions at (possibly) different prices.

By contrast, in an order-driven system, investors submit their orders for execution through an auction process. There are two types of order-driven systems. In the first type, known as a *periodic auction* (or *batch market*), the orders of investors are accumulated (batched) for simultaneous execution at a single price which clears the market. Market clearing is accomplished through a set of multilateral transactions at one price.[3] The second type of order-driven system, known as a *continuous auction*, is increasingly prevalent. In a continuous auction, investors submit orders for immediate execution by dealers on an exchange floor. The system is continuous, since orders are executed upon arrival, but also functions

[2]These intermediaries, known as dealers or market makers, buy at the bid price and sell at the ask price, and their quotations typically depend on the size of the order. In London, for example, dealers quote price schedules.

[3]Periodic systems are used to open many continuous markets such as the New York Stock Exchange and Tokyo Stock Exchange. In addition, many European stock markets operate as periodic auctions. See Pagano and Roell (1990) for a detailed description and analysis of the operation of stock markets in Europe.

as an auction because the price is determined by 'crowd trading' or bidding by competitive floor traders.[4] Most trading mechanisms are complex hybrids of these three systems. For example, the NYSE opens with an auction or batch market, and then switches to a dealer market for most stocks. Highly active NYSE stocks trade in a continuous double auction. These compound mechanisms can be understood only through the analysis of their simpler component structures.

Previous studies of market mechanisms (e.g., Garbade and Silber (1979), Zabel (1981), Pithyachariyakul (1986), and Mendelson (1987)) have stressed the differences between continuous and periodic trading when traders have fixed reservation prices. This paper differs from previous research in its emphasis on price discovery and strategic behavior under imperfect information. Specifically, the trading mechanisms we compare differ in the sequencing of trade. In the order-driven system, investors must move first, while in the market maker system, dealers move first, providing an option to traders who can pick off their quotes. The trading sequence also matters because it affects the quality of information provided by the mechanism to traders. We show that the quoted bid and ask prices in the market maker system convey information because they are sufficient statistics for the history of trading. In a periodic auction investors condition their beliefs on the single clearing price. The auction price also aggregates information, but does so noisily, unlike the dealer's quotes. Finally, the strategic response of traders differs across trading mechanisms. In a continuous dealer system, transaction prices are determined bilaterally whereas a trader's trading strategy in a periodic auction depends on the strategies of all other traders. The impact of these factors on the properties of prices and the viability of trade is the subject of this paper.

We argue that there is a trade-off between the robustness of the auction system to

[4]Examples of continuous auction systems include the *à la criée* system for active stocks on the Paris Bourse, the Swiss Option and Financial Futures Exchange (SOFFEX), the Frankfurt Stock Exchange, the Toronto Stock Exchange's Computer Assisted Trading System (CATS), and the Tokyo Stock Exchange's Computer Assisted Routing and Execution System (CORES).

problems of asymmetric information and the efficiency of the market maker system. The intuition is straightforward. Equilibrium may not exist in a continuous dealer system because dealers have a 'first-mover' disadvantage. Dealers must supply firm quotes to traders who then determine their order. If the quotes reveal information, then all traders are at least as well informed as the dealer, and a certain minimum amount of non-information trading is required to preclude market failure. If equilibrium exists, competition among dealers to provide price quotations leads to prices that are semi-strong form efficient. By contrast, both the continuous and periodic auction mechanisms are more robust than the dealer system because orders are pooled for execution before prices are determined, mitigating the adverse selection problems. The trade-off comes in the form of inefficient prices in the continuous auction and loss of continuity in the periodic auction.

The rest of the paper proceeds as follows: In section 2, we set up the basic framework and in section 3, we analyze a continuous market where dealers post prices before trading occurs. In section 4, we model an order-driven mechanism where investors submit orders for execution by floor traders. This mechanism can operate as a continuous auction or as a periodic mechanism. Section 5 compares the operation of these mechanisms, and section 6 summarizes the paper. All proofs are in the appendix.

2 The Model

There are two assets, cash and a single risky asset with a stochastic liquidation value, denoted by $\tilde{v}$, which is realized after time 1. There are two types of agents in this model. The first type, termed 'traders' (indexed by i) enter the market according to an exogenous stochastic process at calendar times $\{t_i\}_{i=1}^{\infty}$ in the interval $(0,1)$. The second type of agent, termed 'dealers' or'market makers' provide liquidity by trading with the first group of traders. We examine two different forms of market organization: a continuous system (which operates either as a continuous dealer or continuous auction system) and a periodic auction mechanism.

In the continuous dealer market, each trader transacts with a market maker upon arrival. For simplicity, we assume that each trader trades only once.[5] Trading in the dealer market is characterized by a sequence of bilateral transactions at different prices. By contrast, in the auction market, trader's and dealer's orders are batched for simultaneous execution at a pre-determined point in time and the market clears in a set of multilateral transactions at a single price. The special case where a trade is executed upon arrival by dealers on the exchange floor corresponds to the continuous auction.

We first describe the objectives and information of traders and then turn to the dealers. Let q_i represent the order quantity of trader i with the convention that $q > 0$ denotes a trader purchase and $q < 0$ a trader sale. Denote by p_i the security's price at time t_i. Traders maximize the expected utility of final period wealth. Each trader is assumed to have a negative exponential utility function:

$$u(W_{1i}) = -e^{-\rho W_{1i}}$$

where W_{1i} is the final period wealth of trader i, and $\rho > 0$ is the coefficient of absolute risk aversion. Trader i's initial endowment is described by the vector (x_i, c_{0i}), where x_i is the number of risky securities in the initial portfolio and c_{0i} represents cash holdings. Endowments of the risky asset are distributed normally across traders with mean 0 and precision (the reciprocal of the variance) ψ.[6] Since traders are risk averse, variation in asset endowments generates portfolio hedging which is not information motivated. The amount of 'noise' or 'liquidity' trading is inversely related to the value of ψ.[7]

[5]Equivalently, the probability of repeat trading is very low. The assumption could be motivated by the prisoner's dilemma nature of competition among traders with private information. Since all traders possess private information, a trader who fails to fully exploit his information when he gets a chance to trade runs the risk that his information will already be impounded in prices when he next obtains an opportunity to trade.

[6]The assumption of a zero mean is without loss of generality. We also assume the security is 'widely' held so that information on x_i does not convey information about x_j, for $i \neq j$.

[7]Liquidity trading in this model arises endogenously, in contrast to most models which assume an exogenous noise shock to generate non-information based trade.

For trader i, final period wealth, $\widetilde{W}_{1i}$, is a random variable given by:

$$\widetilde{W}_{1i} = (q_i + x_i)\tilde{v} + c_{0i} - p_i q_i \ . \tag{1}$$

Let Φ_i represent the information set of trader i.[8] If $\widetilde{W}_{1i}$ is normally distributed conditional upon Φ_i, then maximizing expected utility is equivalent to maximizing (see, e.g., Grossman (1976)):

$$E[\widetilde{W}_{1i}|\Phi_i] - \left(\frac{\rho}{2}\right) \text{Var}[\widetilde{W}_{1i}|\Phi_i] \tag{2}$$

where $E[\,\cdot\,|\Phi_i]$ and $\text{Var}[\,\cdot\,|\Phi_i]$ represent the conditional expectation and variance operators relative to Φ_i. We turn now to the determinants of traders' information sets. At time 0, the risky security is known to be distributed normally with mean μ and precision τ. This is public information and is known to both dealers and traders. In addition to public information, trader i ($i = 1, \ldots, N$) observes the realization of a random variable, $\tilde{y}_i = v + \tilde{\epsilon}_i$ where $\tilde{\epsilon}_i$ is white noise and v is the time 1 value of the risky asset. We assume that $\tilde{\epsilon}_i$ is normally distributed with mean 0 and precision θ and $E[\tilde{\epsilon}_i\tilde{\epsilon}_j] = 0$ if $i \neq j$. The realized value of the random variable $\tilde{y}_i$ is denoted y_i. The private information of trader i is the pair (x_i, y_i). Then, trader i's prior distribution of $\tilde{v}$ is normal with mean y_i and precision θ. The signal can be interpreted as an unbiased estimate of the security return based upon access to different information sets, as described by Working (1958):

> The amount of pertinent information potentially available to traders in most modern markets is far beyond what any one trader can both acquire and use to good effect. Circumstance and inclination lead different traders to seek out and use different sorts of information. In short, traders are forced to engage in a sort of informal division of labor in their use of available information. Using different information, different traders must find themselves often of different opinions, one buying at the same time that another sells, even though all may stand at an equal level of intelligence, steadiness in judgement, and quantity of information at their command.

Alternatively, idiosyncratic differences in valuations can be regarded as arising from diverse

[8]Formally, the information set is the σ-algebra generated by public and private information.

opinions or beliefs. The assumption of independent signals can be relaxed to allow correlation among conditional expectations.

Turning now to the dealers, we assume that there are $M > 2$ competitive dealers. These dealers are assumed to be risk-neutral and their objective is to maximize expected profits, subject to competitive constraints. Unlike traders, dealers do not receive private information signals; they know the prior distribution of $\tilde{v}$ which is public information, and, over time, make inferences about the security's value from the order flow. Since the signals dealers receive from order flow will depend on the way in which trading is accomplished, we must first discuss the specifics of the trading arrangements. Before doing so, however, it is useful to define a measure of information quality at the beginning of the trading day. Define Υ as:

$$\Upsilon \equiv \frac{\theta^2}{\tau} + \theta. \tag{3}$$

The measure Υ has an important role in our analysis. It can be thought of as a proxy for the degree of information asymmetry in the market. A security with pronounced information asymmetries has (relatively) large values of θ and low values of τ, implying Υ is large. Conversely, for securities where information asymmetries are inconsequential, τ is large and θ is small, implying Υ is small. Intuition suggests that the measure is also negatively correlated with market value and the level of trading activity. With this framework in place, we now address the specifics of market organization.

3 The Continuous Dealer Mechanism

We consider first a continuous dealer mechanism. The dealer market model is based on the single-period model of Glosten (1989), who contrasts a monopolistic market maker system with a competitive market maker system. Trading is accomplished through M competitive, risk-neutral market makers who take the opposite side of all transactions. The critical feature of this mechanism is that it is quote-driven; market makers provide bid-ask quotations to traders on demand, and can revise their quotes only after a transaction is complete. Strict price priority prevails, so that the market maker with the lowest ask price or the highest bid price is matched with the trader. Since market makers have access only to public information, their initial prior distribution of $\tilde{v}$ is normal with mean μ and precision τ.

We model the dealer mechanism as a two-stage game. In the first stage, the dealers determine their quoted prices and in the second stage the trader chooses his or her order given the quoted prices. Bertrand competition (pure price competition) forces the expected profits of market makers on each trade to zero. Rational dealers set prices so that the price for a given order size is an unbiased estimate of the asset's value given prior information and the information provided by the size of the order, i.e., that quoted prices are 'regret-free' or *ex post* rational.[9] Since order size is variable, this discussion implies that the quoted prices are contingent on order size. Formally, at time t_i, market makers determine a quotation schedule $p_i(\cdot)$, and then observe q_i, the order placed by trader i, after which market makers can revise their quotation schedules for trader $i+1$, i.e., choose $p_{i+1}(\cdot)$. In setting $p_i(\cdot)$, market makers use the information contained in the trading history, $h_i = \{(p_1, q_1), \ldots, (p_{i-1}, q_{i-1})\}$, and their prior information (μ, τ). Let Φ_d^i represent the pre-trade information set of market makers at time t_i.[10] The dealer mechanism in each period is described as a game $\Gamma_c = (\{p_i(\cdot), q_i(y_i, x_i)\})$ characterized by the players and their strategies.

Definition 1 *At time t_i, equilibrium for the mechanism Γ_c is a price quotation function, $p_i : \Re \rightarrow \Re_+$, and a corresponding demand, q_i, such that:*

(a) Each market maker ($j = 1, \ldots, M$) quotes a price schedule that ensures non-negative expected profits on each transaction

$$(E[\tilde{v}|q_i \wedge \Phi_d^i] - p_i(q_i))(-q_i) \geq 0 \ .$$

[9]Glosten and Milgrom (1985) use this condition to demonstrate the existence of an information-based bid-ask spread. In their model, order size is fixed.

[10]Formally, this set is the σ-algebra generated by the history h_i and public information.

(b) *Strict price priority prevails, i.e., there does not exist another function,* p_i^o, *satisfying condition (a) such that:*

$$\begin{aligned} qp_i^o(q) &\le qp_i(q) \quad \textit{for all} \quad q \in \Re \\ \textit{and} \quad qp_i^o(q) &< qp_i(q) \quad \textit{for some} \quad q \in \Re \, . \end{aligned}$$

(c) *Trader* i *maximizes expected utility given* $p_i(\cdot)$*:*

$$q_i \in \text{argmax}_{\{q_i\}}\{E[u(\widetilde{W}_{1i}(q_i))|\Phi_i \wedge p_i\} \, .$$

Condition (a) is the requirement that market makers earn non-negative expected profits given post-trade information, while condition (b) requires that p_i not be dominated by another quotation function, p_i^o, which has lower ask prices and/or higher bid prices. These two conditions capture the sequential nature of the game Γ_c, where dealers must move first to quote prices. Together, they imply that market makers earn zero expected profits on every trade. Finally, condition (c) states that each trader maximizes the expected utility of final period wealth given the price schedule posted by dealers. From definition 1, equilibrium corresponds to a fixed point in the space of continuous functions.

The bid-ask spread is implicit in the quotation schedule defined above. For an order quantity q, the dealer's ask price is $p(|q|)$ and bid price is $p(-|q|)$. The definition of the bid-ask spread follows.

Definition 2 *At time* t_i*, the effective bid-ask spread for an order* q *is a function* $s_i : \Re \rightarrow \Re_+$ *where:*

$$s_i(q) = p_i(|q|) - p_i(-|q|) \, .$$

The bid-ask quotation for one round lot is the pair $(p_i^a, p_i^b) = (p_i(1), p_i(-1))$.

3.1 Dealer Market Equilibrium

Our first objective is to characterize equilibrium in the continuous dealer market. Proposition 1 shows that a necessary condition for the existence of a unique equilibrium is that the information parameter Υ be bounded above.

Proposition 1 (Dealer System) : *If Υ is bounded above, equilibrium exists at time t_i $(i = 1, 2, \ldots)$. At time t_i equilibrium is linear:*

(a) The demand function is:

$$q_i = \frac{E[\tilde{v}|\Phi_i] - p_{i-1} - \alpha_i x_i}{\alpha_i + 2\lambda_i}.$$

(b) The price quotation schedule is:

$$p_i(q_i) = p_{i-1} + \lambda_i q_i$$

where λ_i and α_i are constants defined in the appendix and $p_0 = \mu$.

The sequence of the game, where dealers 'move first,' means the trader faces a quotation schedule and not a single price. Given this schedule, the trader has no incentive other than to choose the utility maximizing order quantity. The trader selects a point on this schedule, taking into account the effect of order quantity at the margin on the price of the entire trade. Only if the information asymmetry parameter $\Upsilon = (\theta^2/\tau) + \theta$ is bounded, does equilibrium exist in every period. The upper bound is related to the motives for non-information trading, and we discuss it in greater detail in section 5. The proposition implies that if information asymmetries are sufficiently extreme, market makers may be unable to open markets without suffering expected losses. If markets cannot open in the first period, they cannot open in any subsequent periods, and there is no way for information to be aggregated over time. We refer to this phenomenon as market failure. Note that market makers cannot 'experiment' at cost to induce the revelation of information that permits them to open markets in the future because competition generates a free-rider problem.

3.2 Price Dynamics

Define the Bayesian predictive (forecast) error, e_i, as $e_i = (v - p_i)$. The absolute predictive error $|e_i|$, is a measure of price efficiency at time t_i. Proposition 2 shows that prices are efficient with respect to public information, but that this result is consistent with autocorrelation in the Bayesian predictive errors.

Proposition 2 (Price Dynamics): *If equilibrium exists:*

(a) Transaction prices follow a martingale, i.e., $E[\tilde{p}_{i+1}|p_i] = p_i$*, and prices are semi-strong form efficient, i.e.,* $E[\tilde{v}|\Phi_d^{i+1}] = p_i$*.*

(b) The Bayesian predictive errors are positively correlated:

$$E[\tilde{e}_{i+1}|e_i] = \eta_i e_i$$

where $0 < \eta_i < 1$ *for all* i*.*

(c) The effective spread is an increasing function of order size, $s(q_i) = 2\lambda_i|q_i|$*. Further, the quoted bid-ask spread decreases over the day:*

$$(p_i^a - p_i^b) < (p_{i-1}^a - p_{i-1}^b) \; .$$

The martingale property of prices is present in models of competitive dealer markets such as Glosten and Milgrom (1985) and Easley and O'Hara (1987). Part (b) shows that although prices follow a martingale (relative to post-trade public information Φ_d^{i+1}), the Bayesian predictive errors are still positively correlated, so that mispricing in any period is carried into the next period. Observing autocorrelation in transaction prices or trends is not inconsistent with part (a). The parameter η_i measures the informational efficiency of the market, i.e., the rate of convergence of transaction prices to the full-information price. Higher values of η_i are associated with less rapid convergence, since errors persist longer. Since $\eta_i < 1$ for all i, the effect of mispricing at time t_i exerts diminishing influence on future prices.

The model provides some insights into the nature of the bid-ask spread implicit in the price-quotation schedule. First, observe that the size and placement of the spread are informative, and form a sufficient statistic for the entire history of trading. The mid-quote,

$(p_i^a + p_i^b)/2$, is simply $E[\tilde{v}|\Phi_d^i]$, a point estimate of the asset's value given pre-trade public information, and the precision of this estimate can be inferred from the spread since τ_i is a monotonically decreasing in $(p_i^a - p_i^b)$. Second, the linearity of the quotation schedule allows the entire quotation schedule can be inferred from a single bid-ask quotation. For a given order, the effective bid-ask spread, $s_i(q)$, is strictly increasing in q, and the quoted bid-ask spread, $(p_i^a - p_i^b)$ is strictly decreasing with the number of trades.[11] The result suggests that the components of volume (order size and the frequency of trading) have opposite effects on the bid-ask spread. A security whose trading volume is composed of a few large trades will have a wider effective spread than a security with identical volume composed of many small-sized trades. This point has been ignored in previous empirical studies of the cross-sectional determinant of bid-ask spreads which use total transaction or dollar volume as an independent variable. Third, spreads narrow over the day because each trade reduces the information asymmetry between dealers and traders.[12]

4 The Order-Driven Mechanism

An alternative to the quote-driven continuous dealer system is an order-driven system where traders submit orders to an exchange for execution by floor traders or dealers. This type of system includes *continuous auctions*, where orders are executed immediately, as well as *batch systems* or *periodic auctions* where orders are accumulated for simultaneous execution at a single market clearing price.[13]

[11] A similar prediction for order quantity arises from the model of Ho and Stoll (1983) because of the dynamic inventory control policies pursued by market makers. Easley and O'Hara (1987) consider a model with two order sizes and show that the spread is strictly higher for the larger order size.

[12] Evidence for a decline in spreads over the day is provided by McInish and Wood (1988) who analyze intraday bid-ask spreads for NYSE stocks for five months of 1987. McInish and Wood group the sample into quintiles by trading frequency and find that bid-ask spreads decline rapidly after opening within each quintile. The decline is most evident in the first 15 minutes of trading, after which spreads remain stable, possibly because the discreteness of prices generates a minimum spread.

[13] Batch markets have been examined by Ho, Schwartz, and Whitcomb (1985), Mendelson (1982), and Mendelson (1987), but in these models traders do not condition on prices or act strategically.

Unlike a posted-price system, the transaction price in an auction mechanism is not known at the time of order submission. We argue, however, that traders can effectively condition their beliefs on the price. Rational investors know that the equilibrium price reveals information, so that the demand schedule they submit is the set of price-quantity combinations such that the quantity demanded at each price is the desired order quantity given that price clears the market.[14] The resulting allocation resembles the model of Kyle (1989). Kyle demonstrates the existence of a rational expectations equilibrium with finite numbers of traders who act strategically, and shows that this equilibrium is distinct the rational expectations equilibrium that obtains when traders act competitively. This paper differs from Kyle (1989) because liquidity trading is endogenous here.[15] Consider an auction with N traders and M dealers. We denote the vector of trader demand functions by $\vec{q} = (q_1(\cdot), \ldots, q_N(\cdot))$, and the vector of dealer demands by $\vec{d} = (d_1(\cdot), \ldots, d_M(\cdot))$.[16] Let $\vec{q}_{-i} = (q_1, .., q_{i-1}, q_{i+1}, .., q_N)$, and similarly define $\vec{d}_{-j}$. We model the order-driven process as a game $\Gamma_a = (\{q_i\}_{i=1}^N, \{d_j\}_{j=1}^M)$ indexed by the players and their strategy functions. Then, we define a Bayes-Nash equilibrium for the game Γ_a as follows:

Definition 3 *A Bayes-Nash equilibrium for the mechanism Γ_a consists of a vector of strategy functions, $\vec{d}$, for dealers $j = 1, \ldots, M$, a vector of strategy functions, $\vec{q}$, for traders $i = 1, \ldots, N$, and a price, p^*, such that:*

(a) $\sum_j^M d_j(p^*) + \sum_i^N q_i(p^*; y_i, x_i) = 0$

(b) $q_i(p^*; y_i, x_i) \in \text{argmax } E_i[u(\widetilde{W}_{1i}|p^* \wedge (\vec{q}_{-i}, \vec{d})]$

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[14]This is analogous to our construction of the dealer quotation schedule using a sequential rationality argument.

[15]In Kyle's model, there is exogenous noise trading in addition to trading by investors and speculators. In our paper, liquidity trading takes the form of portfolio adjustment for hedging purposes. The endogeneity of liquidity trading is critical to our argument about the viability of trading systems. Kyle shows, for example, that even arbitrarily small amounts of *exogenous* noise is sufficient for markets to be open, although trading volume is infinitesimal.

[16]By convention $d_j > 0$ denotes a purchase and $d_j < 0$ a sale for dealers $j = 1, \ldots, M$.

(c) $d_j(\cdot) \in \text{argmax}_{\{d_j\}}\{E_j[(\tilde{v} - p)d_j(p)|p^* \wedge (\vec{q}, \vec{d}_{-j})]\}$, *subject to non-negativity.*

Condition (a) is requires that the market clear in equilibrium while condition (b) requires that the strategy of trader i maximize expected utility given the equilibrium price and the strategy functions of other agents. Finally, condition (c) requires that the strategy (demand schedule submitted) of a dealer be a best-response to the strategies of other dealers and traders. Competition among dealers takes the form of competition in demand schedules rather than pure price competition as in a quote-driven system.

4.1 A Continuous Auction

Consider a continuous auction, where an order is executed upon arrival by the 'trading crowd' of dealers who are present on the exchange floor.[17] The continuous auction is a special case of Γ_a, with $N = 1$. Let d_j^i be the demand of dealer j in time t_i. We represent this mechanism by the game Γ_{ca}.

Proposition 3 *If Υ is bounded above, there exists an equilibrium for the mechanism Γ_{ca} at time t_i where:*

(a) Dealer j's ($j = 1, \ldots, M$) strategy function is linear:

$$d_j^i(p_i) = \gamma_i(p_{i-1} - p_i) - \gamma_i \zeta_{i-1} q_{i-1}.$$

(b) The trader's demand function is:

$$q_i(p_i) = \frac{E[\tilde{v}|\Phi_i] - p_i - \alpha_i x_i}{\alpha_i + \zeta_i}.$$

(c) The equilibrium price is:

$$p_i^* = \frac{[M\gamma_i(\alpha_i + \zeta_i) + (1 - \delta_i)](p_{i-1} - \zeta_{i-1} q_{i-1}) + \delta_i y_i - \alpha_i x_i}{M\gamma_i(\alpha_i + \zeta_i) + 1}$$

where $\alpha_i, \gamma_i, \delta_i$, and ζ_i are constants described in the appendix and $p_0 = \mu$.

[17] It is convenient to think of orders being price-contingent, but whether traders submit limit orders or market orders has no effect on the equilibrium outcome, as shown in the appendix.

Proposition 3 demonstrates the existence of equilibrium under general conditions. We require $M > 2$ for a linear equilibrium; it is possible that other equilibria exist (e.g., mixed strategy equilibria) but throughout the paper we will focus on linear equilibria to facilitate the comparisons across different mechanisms. We focus on the properties of prices (including price variability, efficiency, and the informativeness of prices) and the conditions required for the market to function. These criteria are singled out in public policy discussions, and at this stage we make no normative statements regarding the desirability of one mechanism over another. The equilibrium for Γ_{ca} differs in significant ways from the equilibrium for Γ_c, as shown below.

Proposition 4 (Continuous Markets): *If $M < \infty$, then:*

(a) If an equilibrium exists for the dealer mechanism Γ_c, then it exists for the auction mechanism Γ_{ca}. Further, Γ_{ca} is viable in markets where the continuous dealer mechanism Γ_c fails.

(b) Prices in the continuous auction system do not follow a martingale (i.e., $E[\tilde{p}_{i+1}|p_i] \neq p_i$, a.s.), and prices are not semi-strong form efficient (i.e., $E[\tilde{v}|p_i \wedge \Phi_d^i] \neq p_i$, a.s.).

(c) The unconditional variability of prices in the continuous auction system is higher than in the continuous dealer system. However, the trading history is equally informative, i.e., $E[\tilde{v}|h_i]$ is the same in both mechanisms.

The proposition demonstrates that continuous trading is more robust to problems of asymmetric information if organized as a continuous auction rather than a dealer system. Conversely, if both mechanisms are viable, prices in the auction system are not efficient and are more volatile than in the dealer system. Intuitively, in the market maker system, the first stage price competition among dealers eliminates the 'wedge' between the transaction price and the expected value of the asset that is the source of dealers' expected profits. In the second stage, traders condition upon these quotes, exposing dealers to adverse selection costs. If the degree of information asymmetry is high, i.e., if Υ is high, there may be no equilibrium in a dealer market. In the auction system the clearing price is simultaneously determined. With a finite numbers of players, each player has some influence on price. In equilibrium, each player's order quantity is determined through a strategy function which is a best-response to the strategy functions of all other players. Strategic behavior distorts prices, inducing inefficiency. However, the trading history is equally informative because rational traders can 'undo' the distortion due to strategic behavior in their inference process. A trade-off arises, however, because the auction system does not suffer from the 'first-mover disadvantage' present in the market maker system, and can operate in economies where that system fails.

When are the two mechanisms are equivalent? Proposition 5 shows that as the number of dealers grows asymptotically large, the equilibria for the two mechanisms coincide and the trade-off between robustness and efficiency disappears.

Proposition 5 (Equivalence:) *For any time period t_i, as $M \to \infty$, the equilibrium price-quantity pair (p_i, q_i) for the mechanism Γ_{ca} converges to the equilibrium price-quantity pair of Γ_c.*

Having examined the nature of quote-driven and order-driven continuous systems, we turn our attention to periodic systems.

4.2 A Periodic Auction

Consider the general case for the mechanism Γ_a, where $N > 1$. To ensure a fair comparison between continuous and periodic systems, we assume the the number of dealers is large so that the continuous dealer and continuous auction systems are equivalent. The next proposition shows that if the number of traders in Γ_a is sufficiently large, the periodic mechanism provides less variable prices and is more robust than the continuous system.

Proposition 6 (Periodic Auction): *There exists N such that:*

(a) The auction mechanism Γ_a is viable in economies where the continuous mechanism Γ_c fails. The equilibrium strategy functions of traders and dealers are linear in price.

(b) In a particular periodic auction, the price is semi-strong form efficient and over a sequence of periodic auctions, the auction prices follow a martingale.

(c) The price in the auction mechanism Γ_a is less variable than each of the corresponding sequence of N prices in the continuous mechanism Γ_c.

Proposition 6 shows that a large enough auction can provide more efficient prices than a continuous market. In a periodic system, all traders observe a noisy estimate of their aggregate information, in addition to public and private information signals. The more traders participating in the auction, the more efficient the price is as a signal of asset value. Further, unlike the mechanisms Γ_c and Γ_{ca}, the auction system does not require intermediaries to function. Even if the quality of public information is so poor that no dealers are willing to absorb order flow (i.e., $M = 0$), the N traders can share risk among themselves, provided the quality of private information signals, θ, is bounded above.[18] The disadvantage of such a system is that it does not provide for continuous trading. Rather, investors must wait until pre-specified times for order execution.[19]

5 Continuous versus Periodic Trading

Propositions 1 and 5 shows that market failure can occur in a continuous market if the degree of information asymmetry Υ, exceeds a critical bound, $\Upsilon^c = \frac{\rho^2}{\psi}$. In this case, dealers cannot make non-negative expected profits on the first trade and on all subsequent trades. The

[18] The auction price is efficient and is equal to the price in a Walrasian auction. The details are provided in the appendix.

[19] See Garbade and Silber (1979) who provide a model where there exists an optimal time between market clearings that minimizes investors' liquidity risk.

upper bound Υ^c is a measure of the non-information motives for trading; it increases with risk aversion, ρ, and the variance of initial endowments of the risky asset, ψ^{-1}. For some securities there may be periods when θ is high relative to τ, so that Υ exceeds the critical bound and a continuous system may not be viable. Possible violations may be at the start of the trading day or immediately preceding the public revelation of new information.

Our analysis of the continuous system implies that trading could be restarted by providing market makers in a continuous dealer system with a public information signal, raising τ and hence lowering Υ.[20] Casual observation suggests many devices and procedures to supply dealers with information on current market conditions when market failure is likely. For example, many continuous markets open (or re-open following a trading halt) with a call auction to allow the assimilation of new information.

However, it is not always the case that continuous trading can be restarted with a public information signal of sufficiently high quality. To see this, note that as $\tau \to \infty, \Upsilon \downarrow \theta$, so that when $\Upsilon^c < \theta$, a continuous market is not viable even if the quality of public information is very high. The results suggest that proposals to reduce market stress in continuous systems with 'circuit breakers' (trading halts triggered by large price movements) can exacerbate the original problem. Once trading is halted, it can be difficult or impossible to restart the process. Proportion 6 suggests a possible solution is to switch to a periodic trading mechanism in times of market stress, since this system can operate even if dealers choose not to make markets.

The choice of market design may be affected by other factors, especially operating costs. The periodic system can impose high submission costs on traders, requiring either the physical presence of traders on the exchange floor or the costly submission of written demand schedules. Traders' information costs in a dealer market are lower because traders need

[20]Since the bid-ask quotes of market makers completely reveal their information to traders, it is not necessary to disseminate this information directly to traders.

only solicit the dealer's bid-ask quotes. These quotes are sufficient statistics for the entire trading history. By contrast, in an auction market, investors must collect and process the past trading information, which imposes private costs and social costs through duplication. Conversely, the direct costs on dealers imposed by the communications requirements of a continuous system can be quite high relative to a simple batch market. It is not possible *a priori* to identify the system with lower costs. We have focused on the measures of market quality emphasized by policy-makers, in particular, the informativeness and volatility of prices and the possibility of market failure. Without unambiguous results regarding the potential for trading, it is not possible to make unambiguous normative statements.

6 Conclusions

This paper models the process of price discovery under two alternative forms of market organization: a quote-driven system and an order-driven system. The quote-driven system relies on competitive dealers to post prices before orders are submitted while the order-driven system requires order submission before prices are determined. The quote-driven system operates as a continuous system where orders are executed upon submission. The order-driven system can operate either as a continuous system, with immediate execution on the exchange floor, or as a periodic system, where orders are stored for simultaneous execution. Trading is modeled as a game where order quantities and beliefs are determined endogenously and players act strategically. We showed that a continuous dealer system provides greater price efficiency than a continuous auction system. However, the dealer market is less robust to problems of asymmetric information than an auction market. There is a trade-off between price efficiency and stability. In the limit, with a 'large' number of dealers, the equilibria of two mechanisms coincide.

We demonstrated that a periodic trading mechanism can function where a continuous market. This occurs because pooling orders for simultaneous execution can overcome the problems of information asymmetry that cause failure in a mechanism where trading takes place sequentially. However, a periodic system cannot provide immediate order execution. Our results show that if a continuous market fails, it cannot re-open unless the degree of information asymmetry is lowered. This has important implications for public policy since it suggests that halting trading in times of market stress (e.g., circuit breakers) can actually exacerbate the problem. Our results suggest a switch to a batch market is superior to a trading halt since the periodic system can operate when dealers refuse to make markets, thereby providing information signals that allow continuous trading to re-open.

Casual observation suggests that thickly traded securities are generally traded in continuous markets, whereas thinly traded securities are traded in periodic auction systems. To the extent that information asymmetry is inversely related to the market value and trading activity, the model is consistent with this observation.

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Appendix

Proof of Proposition 1 :

The proposition is a straightforward extension of Glosten's single-period model. We provide a detailed proof here for the sake of completeness. The price at time t_i is a function of the order quantity of the trader, and we write $p_i = p_i(q)$. Using (1), we see:

$$E[\widetilde{W}_{1i}|\Phi_i] = (q_i + x_i)E[\tilde{v}|\Phi_i] + c_{0i} - p(q_i)q_i \tag{A.1}$$

and

$$\text{Var}[\widetilde{W}_{1i}|\Phi_i] = \text{Var}[\tilde{v}|\Phi_i](q_i + x_i)^2 \; . \tag{A.2}$$

To simplify the notation, define $r_i \equiv E[\tilde{v}|\Phi_i]$ and $\alpha_i \equiv \rho\text{Var}[\tilde{v}|\Phi_i]$. Expected utility maximization in equation (2) implies that q_i, solves:

$$r_i - p_i(q_i) - p_i'(q_i)q_i - \alpha_i q_i - \alpha_i x_i = 0 \; . \tag{A.3}$$

The second order condition is:

$$-p_i''(q_i)q_i - 2p_i'(q_i) - \alpha_i < 0 \; . \tag{A.4}$$

Suppose trader i places an order for q_i securities, given $p_i(q)$, the market maker's quotation function. Define $\hat{r}_i$ as follows:

$$\hat{r}_i(q_i) \equiv p_i(q_i) + p_i'(q_i)q_i + \alpha_i q_i. \tag{A.5}$$

At time t_i, suppose the prior distribution of $\tilde{v}$ given public information (including any information revealed through the quotation schedule itself) is a normal distribution with mean μ_i and precision τ_i. Applying a basic proposition from statistical decision theory (see, e.g., DeGroot (1970)), the trader's posterior mean of $\tilde{v}$ given the signal, y_i, is:

$$r_i = \frac{\theta y_i + \tau_i \mu_i}{(\theta + \tau_i)} \; . \tag{A.6}$$

The posterior precision of this estimate is $(\theta + \tau_i)$. The definition of α_i yields:

$$\alpha_i = \frac{\rho}{(\theta + \tau_i)} \; . \tag{A.7}$$

Define $z_i(q_i)$ by:

$$z_i(q_i) = \frac{\hat{r}_i(q_i)(\tau_i + \theta) - \mu_i \tau_i}{\theta} \; . \tag{A.8}$$

Using the definition of $\hat{r}(q)$, we see that $\hat{r}_i(q_i) = r_i - \alpha_i x_i$. Substituting this and equation (A.6) into equation (A.8), we obtain:

$$z_i(q_i) = \frac{\theta y_i - \alpha_i x_i(\tau_i + \theta)}{\theta} \; . \tag{A.9}$$

Using the definition of α_i in (A.7), $z_i = y_i - (\rho/\theta)x_i$. Recall $y_i = v + \epsilon_i$ so we can write:

$$z_i = v + \epsilon_i - x_i \left(\frac{\rho}{\theta}\right) \; . \tag{A.10}$$

Using equation (A.10), we see that, from a market maker's perspective, z_i is drawn from a normal distribution with mean v and precision ω, where:

$$\omega \equiv \frac{\theta^2}{\left(\theta + \frac{\rho^2}{\psi}\right)} . \tag{A.11}$$

The parameter ω is a critical constant, reappearing in our analysis of the order-driven mechanism. Note that it is independent of i. Because $\text{Cov}(z_i, \mu_i) = 0$, we see that the market maker's posterior distribution of $\tilde{v}$ has mean:

$$E[\tilde{v}|q_i \wedge \Phi_d^i] = \frac{\omega z_i + \tau_i \mu_i}{(\omega + \tau_i)} \tag{A.12}$$

and precision $(\omega + \tau_i)$. Competition among dealers in making price quotations implies zero expected profits conditional upon public information including trade size:

$$p_i(q) = E[\tilde{v}|q_i \wedge \Phi_d^i] . \tag{A.13}$$

To solve for the equilibrium quotation schedule, we express z_i in terms of q_i by substituting equation (A.5) into (A.8):

$$z_i = \frac{(\tau_i + \theta)(p_i(q) + p_i'(q)q + \alpha_i q) - \mu_i \tau_i}{\theta} . \tag{A.14}$$

Substituting (A.14) into equation (A.12), we obtain:

$$p_i(q) = \frac{\omega}{(\omega + \tau_i)} \left[\frac{(\tau_i + \theta)}{\theta} (p_i(q) + p_i'(q)q + \alpha_i q) - \frac{\tau_i \mu_i}{\theta} \right] + \frac{\tau_i \mu_i}{(\omega + \tau_i)} .$$

Some algebraic manipulation yields the following expression:

$$q p_i'(q) = \kappa_i (p_i(q) - \mu_i) - \alpha_i q \tag{A.15}$$

a first-order differential equation in q, where κ_i is a constant:

$$\kappa_i \equiv \frac{\tau_i(\theta - \omega)}{\omega(\theta + \tau_i)} . \tag{A.16}$$

The unique functional form solving (A.15) is given by:

$$p_i(q) = \mu_i + \left[\frac{\alpha_i}{\kappa_i - 1} \right] q + K \text{sign}(q) |q|^{\kappa_i} \tag{A.17}$$

where $\kappa_i \neq 1$, K is an arbitrary constant of integration. In the special case where $\kappa_i = 1$, the solution to (A.15) is:

$$p_i(q) = \mu + Kq - \alpha_i q \ln(|q|) .$$

We have derived the general functional form of the quotation function. The next step in the proof is to show that the linear function is the unique function satisfying the conditions of definition 1.

Consider first the general case, where $\kappa_i \neq 1$. It is convenient to define $\lambda_i \equiv \alpha_i / (\kappa_i - 1)$. Suppose $\kappa_i < 1$ (i.e., $\lambda_i < 0$) and $K < 0$. Then, $p(q) < p(-q)$ for all $q > 0$, an arbitrage opportunity that will be eliminated by interdealer trading. Now suppose $K > 0$. Using (A.17), the second order condition (A.4) is:

$$\kappa_i(\kappa_i - 1)K|q|^{\kappa_i - 1} + 2\lambda_i + 2\kappa_i K |q|^{\kappa_i - 1} > -\alpha_i .$$

Inspection of the second order condition shows there exist some q^* and q_*, where $q^* > 0 > q_*$, such that the second order condition (A.4) is violated if q does not lie in the interval (q^*, q_*), so that this cannot be an equilibrium.

Now assume $\kappa_i > 1$ (equivalently $\lambda_i > 0$) and $K < 0$. The previous argument applies directly, so that $K \geq 0$. If $K > 0$, then the second order condition (A.4) is always satisfied. However, the family of curves defined by equation (A.17) is bounded from below (above) in the positive (negative) orthant by the linear equilibrium, and using condition (b) of definition 1, $K = 0$. Hence, if $\kappa_i \neq 1$, then $K = 0$ in equilibrium, and the equilibrium is unique and linear in every period. Finally, the analysis of the case where $\kappa_i = 1$ is treated exactly like the case of $\kappa_i < 1$. The linear equilibrium is the unique equilibrium if $\kappa_i > 1$.

At time t_{i+1}, market makers' posterior distribution of v, given q_i, is normal if the initial prior is normal, because the normal distribution is closed under sampling (a conjugate family). This distribution becomes the prior distribution in determining $p_{i+1}(q)$. At time t_{i+1}, the dealer's prior distribution of $\tilde{v}$ is normal with mean μ_{i+1}. By definition, $p_i(q_i)$ is the posterior mean of $\tilde{v}$ given q_i:

$$\mu_{i+1} = p_i(q_i) \tag{A.18}$$

and the posterior precision is:

$$\tau_{i+1} = \tau_i + \omega \tag{A.19}$$

where ω is defined in (A.11). In the first period, market makers have a normal prior distribution with mean μ and precision τ. Hence, by induction, the proposition holds, if equilibrium exists in every period. It remains to show that if the equilibrium condition is met in the first period, it will be satisfied in all subsequent periods.

When $\kappa_i > 1$ (equivalently, when $\lambda_i > 0$), we see that $\tau_i(\rho^2/\psi - \theta) > \theta^2$. From (A.19), τ_i is increasing in i, so that if $\tau\rho^2/\psi > \theta(\theta + \tau)$ at time 1 then equilibrium exists in all subsequent periods. So, from equations (A.17), (A.18), and the definition of λ_i, the unique equilibrium at time t_i is linear. Then

$$p_i(q_i) = p_{i-1}(q_{i-1}) + \lambda_i q_i \tag{A.20}$$

where $p_0 = \mu$ and λ_i is given by:

$$\lambda_i = \frac{\rho\theta}{\frac{\tau_i\rho^2}{\psi} - \theta(\theta + \tau_i)}. \tag{A.21}$$

Let $\Upsilon^c = \frac{\rho^2}{\psi}$. Clearly, if $\Upsilon < \Upsilon^c$, the unique equilibrium is given by (A.20) and (A.21), proving the first part of the proposition. ∎

Proof of Proposition 2 :

(a) Note from proposition 1 that:

$$p_{i+1} = p_i + \lambda_{i+1} q_{i+1}$$

so that it is sufficient to show:

$$E[\tilde{q}_{i+1} | p_i] = 0 \ .$$

From (A.3), q_{i+1} is:

$$q_{i+1} = \frac{(r_{i+1} - p_i) - \alpha_{i+1} x_{i+1}}{(\alpha_{i+1} + 2\lambda_{i+1})}. \tag{A.22}$$

Now $E[\tilde{r}_{i+1} - p_i|p_i] = E[\frac{\tilde{y}_{i+1}\theta + p_i\tau_{i+1}}{\theta+\tau_{i+1}} - p_i|p_i] = \frac{\theta}{\theta+\tau_{i+1}}(E[\tilde{v} + \tilde{\epsilon}_{i+1}|p_i] - p_i)$. Since $p_i = E[\tilde{v}|p_i]$ and $E[\tilde{\epsilon}_{i+1}|p_i] = E[\tilde{x}|p_i] = 0$, the result follows immediately.

(b) Since $e_{i+1} = v - p_{i+1}$, we obtain using (A.20):

$$E[\tilde{e}_{i+1}|e_i] = -\lambda_{i+1}E[\tilde{q}_{i+1}|e_i] + e_i.$$

From the definition of q_i, we obtain:

$$E[\tilde{e}_{i+1}|e_i] = \eta_i e_i$$

where:

$$\eta_i = 1 - \frac{\lambda_{i+1}}{(\alpha_{i+1} + 2\lambda_{i+1})} \cdot \frac{\theta}{(\theta + \tau_{i+1})} .$$

Clearly, $0 < \eta_i < 1$. Thus the Bayesian predictive errors are positively correlated.

(c) The parameter λ_i depends on i through τ_i. Comparative statics shows that λ_i is a decreasing function of τ_i. Since τ_i is strictly increasing in i, it follows from (A.21) that $\lambda_i < \lambda_{i-1}$, and hence $s_i(1) < s_{i-1}(1)$. ∎

Proof of Proposition 3 :

The proof constructs the Bayes-Nash equilibrium for Γ_{ca} by solving for a trader's best response to the conjectured strategies adopted by other traders and then shows the conjectures are consistent. Suppose at time t_i, the prior distribution of $\tilde{v}$, based on public information, is normal with mean μ_i and precision τ_i, where $\mu_1 = \mu$ and $\tau_1 = \tau$. Suppose trader i, who arrives at time t_i, conjectures that dealers (indexed by $j = 1, \ldots, M$) adopt strategy functions of the form:

$$d_j^i(p_i) = \gamma_i(\mu_i - p_i) . \tag{A.23}$$

We will show this conjecture is correct in equilibrium, and that the conjectured demand function satisfies all the conditions of the definition 3. Using part (a) of definition 3, we see that in equilibrium:

$$M\gamma_i(\mu_i - p_i) + q_i = 0 . \tag{A.24}$$

Using (A.24) the price can be written as a function of q_i:

$$p_i = \mu_i + \zeta_i q_i \tag{A.25}$$

where $\zeta_i \equiv \frac{1}{M\gamma_i}$. We turn now to the functional form of the demand schedule submitted by trader i. From equation (A.6), it follows that:

$$E[\tilde{v}|\Phi_i] = \delta_i y_i + (1 - \delta_i)\mu_i \tag{A.26}$$

where $\delta_i = \theta/(\tau_i + \theta)$ is a constant. From the utility maximization condition (A.3), the demand schedule $q_i(p)$ given the price functional (A.25) is:

$$q_i(p) = \frac{E[\tilde{v}|\Phi_i] - p_i - \alpha_i x_i}{(\alpha_i + \zeta_i)} . \tag{A.27}$$

where we define $\alpha_i = \rho/(\tau_i + \theta)$.

Substituting (A.26) and equation (A.25) into equation (A.27) and simplifying, we can express the optimal demand of trader i as:

$$q_i = \frac{\delta(y_i - \mu_i) - \alpha_i x_i}{\alpha_i + 2\zeta_i} . \tag{A.28}$$

This equation shows that if the trader has rational expectations and correctly conjectures the price functional, he can submit a market order that is equivalent to the limit order, so that order form is irrelevant.

Consider now the strategic decision of a dealer, say dealer m. From (A.24) and the market clearing condition, we obtain:

$$(M-1)\gamma_i(\mu_i - p_i) + q_i + d^i_m(p_i) = 0 . \tag{A.29}$$

It follows that for dealer m the price functional is given by:

$$p_i = (\mu_i + \beta_i q_i) + \beta_i d_m \tag{A.30}$$

where we define $\beta_i \equiv \frac{1}{(M-1)\gamma_i}$. The expected profits of dealer m at time t_i are given by:

$$\pi_i = (E[\tilde{v}|p_i] - p_i)d^i_m(p_i) \tag{A.31}$$

where the conditional expectation reflects rational expectations on the part of the dealer. Substituting (A.30) into (A.31), and solving for the optimal demand, we obtain:

$$d^i_m(p) = \frac{E[\tilde{v}|p_i] - \mu_i - \beta_i q_i}{2\beta_i} . \tag{A.32}$$

Now consider the conditional expectation of the dealer. The dealer has rational expectations, and learns from the market clearing price and order submitted. Note that a dealer who knows his own order quantity and the price knows q_i since $q_i = -Md_m$; under the conjectured strategies the trader's order is absorbed equally by the M dealers. From equation (A.28), observing q_i is equivalent to observing:

$$\hat{y}(q_i) = \mu_i + \frac{\alpha_i + 2\zeta_i}{\delta_i} q_i. \tag{A.33}$$

From equation (A.28), we can express the signal as:

$$\hat{y}(q_i) = y_i - \left(\frac{\alpha_i}{\delta_i}\right) x_i. \tag{A.34}$$

So, $\hat{y}$ is the minimum variance unbiased estimator of the private information of trader i a dealer can make given the order quantity q_i. Since $\alpha_i/\delta_i = \rho/\theta$, the signal is distributed normally with mean v and precision ω, where ω is given by (A.11). Using Bayes' rule, we obtain:

$$E[\tilde{v}|p_i] = \chi_i \hat{y}(q_i) + (1 - \chi_i)\mu_i \tag{A.35}$$

where $\chi_i = \frac{\omega}{\omega + \tau_i}$.

Using (A.30) in (A.32), we can write the desired order quantity of dealer m as:

$$d^i_m(p) = \frac{E[\tilde{v}|p] - p}{\beta_i} . \tag{A.36}$$

Using (A.35) and (A.33), (A.36) can be written as:

$$d^i_m(p) = \frac{\mu_i - p + C_i q_i}{\beta_i} \tag{A.37}$$

where $C_i \equiv \frac{\chi_i(\alpha_i + 2\zeta_i)}{\delta_i}$ is a constant. Since $q_i = -M d^i_m$, we obtain:

$$d^i_m(p) = \frac{\mu_i - p}{MC_i + \beta_i} \tag{A.38}$$

which has the conjectured form, with $\gamma_i = (MC_i + \beta_i)^{-1}$. Since both β_i and C_i depend on γ_i, we must verify that γ_i is well-defined and satisfies the second order conditions for a maximum. This requires that $\gamma_i > 0$. Only then is the proposed strategies an equilibrium. Substituting in the definitions of C_i and β_i, we see that γ_i satisfies:

$$\frac{1}{\gamma_i} = M\chi_i \left(\frac{\rho}{\theta} + \frac{2}{M\delta_i\gamma_i} \right) + \frac{1}{\gamma_i(M-1)} . \tag{A.39}$$

The solution has:

$$\gamma_i = \frac{\theta(\omega + \tau_i)}{M\omega\rho} \left[1 - \frac{2\omega(\theta + \tau_i)}{\theta(\omega + \tau_i)} - \frac{1}{M-1} \right] . \tag{A.40}$$

Given that $M > 2$, for $\gamma_i > 0$ we require (using equation (A.40)) that:

$$\frac{M-2}{M-1} > \frac{2\omega(\theta + \tau_i)}{\theta(\omega + \tau_i)} . \tag{A.41}$$

Let $k(M) = \frac{M-2}{M-1}$. Substituting the expression for ω, we obtain:

$$\frac{k(M)\tau\rho^2}{\psi} > (2 - k(M))\theta(\theta + \tau_i). \tag{A.42}$$

If the inequality in (A.42) is satisfied, $\gamma_i > 0$ and equilibrium is well-defined. Note that if γ_1 exists, then equilibrium exists in all subsequent periods since $\tau_i = \tau + (i-1)\omega$. The conjugate property of the normal distribution ensures that the prior distribution is in fact normal. The prior is given by $\mu_i = p_{i-1} - \zeta_{i-1} q_{i-1}$, so the strategies and price functional can be easily expressed in the form stated in the proposition. ∎

Proof of Proposition 4 :
(a) From equation (A.42), that equilibrium for the auction system exists only if γ_1 exists. This requires that:

$$\frac{\theta(\theta + \tau)}{\tau} < \frac{k(M)\rho^2}{(2 - k(M))\psi}. \tag{A.43}$$

Define $\Upsilon^a \equiv \frac{k(M)\rho^2}{(2-k(M))\psi}$. Since $k(M) = \frac{M-2}{M-1} > 1$, it follows immediately that $\Upsilon^c < \Upsilon^a$. This shows the mechanism Γ_{ca} is viable in economies where Γ_c does not possess an equilibrium.

(b) To show that prices are not efficient (in a semi-strong form sense), suppose to the contrary that $p_i = E[\tilde{v}|p_i]$. Using (A.36), this implies that $d_j^i = 0$ for all $j = 1, \dots, M$. This implies that $q_i = 0$ for all $i = 1, \dots, N$, or equivalently that:

$$\delta_i(\tilde{y}_i - \tilde{\mu}_i) - \alpha_i \tilde{x}_i = 0 \ .$$

Since the probability of this event is (almost surely) zero, we obtain a contradiction. From equation (A.36) we obtain:

$$p_i = E[\tilde{v}|\Phi_d^i \wedge q_i] + \zeta_i q_i \tag{A.44}$$

which shows that prices are not efficient.

(c) To distinguish the prices in the two mechanisms, at time t_i, let p_i^{ca} denote the price in Γ_{ca} and denote by p_i^c the price in Γ_c. Now, using equation (A.35) and the definition of χ, we see that the first term in equation (A.44) is the price that prevails in time t_i in the continuous dealer mechanism, i.e., $E[\tilde{v}|\Phi_d^i \wedge q_i] = p_i^c$. Accordingly, we can write the continuous auction price as:

$$p_i^{ca} = p_i^c + \zeta_i q_i \ . \tag{A.45}$$

Note that the quantity q_i differs from the quantity traded in the continuous dealer market because of differences in market liquidity as measured by ζ_i. Equation (A.45) shows that the trading history under a continuous auction contains the information necessary to recover the unbiased dealer market prices; both histories are equally informative. Taking the unconditional variance of p_i^{ca} in (A.45), we obtain:

$$\text{Var}[\tilde{p}_i] = \text{Var}[\tilde{p}_i^c] + 2\zeta_i \text{Cov}(\tilde{p}_i^c, \tilde{q}_i) + \zeta_i^2 \text{Var}[\tilde{q}_i] \ .$$

Since the covariance term is positive, the variance of price at times $\{t_i\}$ in the continuous auction is higher than the price in the continuous dealer market. ∎

Proof of Proposition 5 :

By definition, $\zeta = \frac{1}{\gamma M}$, and taking limits we obtain:

$$\lim_{M \to \infty} \frac{1}{\gamma M} = \frac{\rho\theta}{\frac{\tau_i \rho^2}{\psi} - \theta(\theta + \tau_i)}. \tag{A.46}$$

This expression is just the dealer market parameter λ, defined in (A.21). Since $\lim_{M \to \infty} k(M) = 1$, we see that with an infinite number of traders, the existence condition (A.42) for the game Γ_{ca} coincides with the condition for Γ_c. The equilibria of the two mechanisms coincide only in the limit. Note that as $M \to \infty$, $\gamma \to 0$, so that $d_m(p) \to 0$, i.e., each dealer's trade size becomes arbitrarily small, so the allocation of q_i across dealers differs from the market maker system. ∎

Proof of Proposition 6:

(a) Consider an auction N traders. If $\tau > 0$, open entry for dealers implies the equilibrium price is the conditional expectation of $\tilde{v}$ given public information and the price itself, although the finite number of traders will act strategically. Formally, the market is efficient in that:

$$p = E[\tilde{v}|p] \tag{A.47}$$

The proof constructs the Bayes-Nash equilibrium by solving for a trader's best response to the conjectured strategies adopted by other traders and then shows the conjectures are consistent.

Suppose trader $i, (i = 1, \ldots, N)$ conjectures that all other traders (indexed by $h = 1, \ldots, i-1, i+1, \ldots, N$.) adopt strategy functions of the form:

$$q_h(p) = a_0\mu + a_1 y_h - a_2 x_h - bp \tag{A.48}$$

for $h \neq i$, where p is the batch market price and $b > 0$ is a constant. Note that (y_h, x_h) is the private information of trader h and is not known to trader i. In this mechanism, the orders are submitted directly to dealers, so that each dealer observes the net order flow $Q \equiv \sum_i q_i$. Then, a dealer can form a statistic $\hat{y}^N(Q)$ defined as:

$$\hat{y}^N(Q) = \frac{Q + Nbp - Na_0\mu}{Na_1}\,. \tag{A.49}$$

From (A.48), the statistic $\hat{y}$ is simply:

$$\hat{y}^N(Q) = \frac{\sum_i y_i - \left(\frac{a_2}{a_1}\right)\sum_i x_i}{N} \tag{A.50}$$

This statistic is distributed normally with mean v and with precision denoted by $N\pi$, where π is the precision of $\tilde{y} - \frac{a_2}{a_1}\tilde{x}$. Accordingly, using (A.47), the equilibrium price can be written as:

$$p = \xi\mu + (1-\xi)\hat{y}^N \tag{A.51}$$

where we define $\xi \equiv \tau/(\tau + N\pi)$. Rearranging, we obtain:

$$p = \frac{a_1\xi - a_0(1-\xi)}{a_1 - (1-\xi)b}\mu + \frac{1-\xi}{N(a_1 - (1-\xi)b)}Q\,. \tag{A.52}$$

We turn now to the construction of the demand function of trader i, where $i = 1, \ldots, N$. From (A.52), the price functional faced by trader i takes the form: can be written as:

$$p = (\phi_1\mu + \phi_2\sum_{h\neq i} q_h) + \phi_2 q_i \tag{A.53}$$

where $\phi_1 \equiv \frac{a_1\xi - a_0(1-\xi)}{a_1-(1-\xi)b}$ and $\phi_2 \equiv \frac{1-\xi}{N(a_1-(1-\xi)b)}$. Utility maximization implies that:

$$q_i(p) = \frac{E[\tilde{v}|y_i \wedge p] - p - \alpha' x_i}{\alpha' + \phi_2} \tag{A.54}$$

where $\alpha' \equiv \rho\text{Var}[\tilde{v}|y_i \wedge p]$. The conditional expectation depends on price, so to ensure the conjectures are correct and equilibrium exists, we must write this out in full. Given the market clearing price, trader i can compute:

$$\sum_{h\neq i} q_h = \frac{p - \phi_1\mu}{\phi_2} - q_i\,.$$

Now, when $N \geq 2$, trader i can form the statistic:

$$\hat{v}_i \equiv \frac{\phi_2^{-1}(p - \phi_1\mu) - q_i - (N-1)\mu a_0 + (N-1)bp}{(N-1)a_1}\,. \tag{A.55}$$

Note that $\hat{v}_i = (N-1)^{-1}[\sum_{h\neq i} y_h - (a_2/a_1)\sum_{h\neq i} x_h]$ is a sufficient statistic for the information content of the market clearing price. By construction, $\hat{v}_i$ is distributed normally with mean v and precision $(N-1)\pi$, independently of y_i and x_i. Now define:

$$\begin{aligned} \delta_0 &= \frac{\tau}{\tau+\theta+(N-1)\pi} \\ \delta_1 &= \frac{\theta}{\tau+\theta+(N-1)\pi} \\ \delta_2 &= \frac{(N-1)\pi}{\tau+\theta+(N-1)\pi} . \end{aligned}$$

Then, using Bayes' rule, we can write the conditional expectation of trader i as follows:

$$E[\tilde{v}|\Phi_i \wedge p] = \delta_0\mu + \delta_1 y_i + \delta_2 \hat{v}_i . \tag{A.56}$$

Substituting (A.56) into (A.54), and rearranging, we see that q_i has the conjectured form where the coefficients solve:

$$a_0 = \frac{\delta_0 - \delta_2[(N-1)a_1\phi_1\phi_2]^{-1} - \delta_2 a_0 a_1^{-1}}{\alpha' + \phi_2 + \delta_2[(N-1)a_1]^{-1}} \tag{A.57}$$

$$a_1 = \frac{\delta_1}{\alpha' + \phi_2 + \delta_2[(N-1)a_1]^{-1}} \tag{A.58}$$

$$a_2 = \frac{\alpha'}{\alpha' + \phi_2 + \delta_2[(N-1)a_1]^{-1}} \tag{A.59}$$

$$b = \frac{-\delta_2[(N-1)a_1\phi_2]^{-1} - \delta_2 a_1^{-1} b + 1}{\alpha' + \phi_2 + \delta_2[(N-1)a_1]^{-1}} . \tag{A.60}$$

The system consists of four equations in nine unknowns, $a_0, a_1, a_2, b, \phi_1, \phi_2, \delta_0, \delta_1$, and δ_2. The parameters ϕ_1 and ϕ_2 are functions of the a's and b while the δ's depend on π which in turn depends on $\frac{a_2}{a_1}$. To ensure a closed-form solution, we must express the ratio a_2/a_1 in terms of the original parameters of the model, ρ, ψ, θ, τ, and N. From the definition of a_1 and a_2, we have:

$$\frac{a_2}{a_1} = \frac{\alpha'}{\delta_1} = \left(\frac{\rho}{\tau+\theta+(N-1)\pi}\right) \Big/ \left(\frac{\theta}{\tau+\theta+(N-1)\pi}\right) = \frac{\rho}{\theta}.$$

Hence, $\pi = \omega$, where ω was defined in equation (A.11). Using this result, we can directly determine the values of δ_0, δ_1, and δ_2. Now consider ϕ_1. If the market clears without dealer participation ($Q = 0$), then by symmetry, the price should be μ, which implies that $\phi_1 = 1$. Alternatively, since ϕ_1 does not depend on N, setting $N = 1$ should correspond to the limiting equilibrium of the game Γ_{ca} as shown by Proposition 5, which implies that $\phi_1 = 1$. This implies that in equation (A.52) $b = a_0 + a_1$. Substituting in the equation for ϕ_2, the solution can be obtained from the system (A.57)–(A.60) by first solving simultaneously for a_1 and b and then computing the remaining parameters. A sufficient condition for these coefficients are well-defined is $\theta < \rho^2\psi^{-1}$ when $N > 2$.

It is easy to show that the auction system can function where continuous markets fail. Consider an extreme case where $\tau = 0$, i.e., there is no public information. In this case, $M = 0$, since dealers cannot make expected profits if they absorb any net order flow. We now require the N traders to set a price that clears the market (i.e., determine a price such that $Q = 0$) without dealer intervention.

The proof is a straightforward repetition of our proof technique above except that equation (A.47) is replaced with the market clearing condition $Q(p) = 0$, and we set $a_0 = 0$. We now summarize the results of this exercise: The demand function of trader i is linear of the form:

$$q_i = a_1(y_i - p) - a_2 x_i$$

where:

$$\begin{aligned} a_1 &= \frac{\bar{a}\theta}{(N-1)\omega + \theta} \\ a_2 &= \frac{\bar{a}\rho}{(N-1)\omega + \theta} \end{aligned}$$

and $\bar{a}$ is a constant given by:

$$\bar{a} = \frac{((N-1)\omega + \theta)[(N-2)\theta - 2(N-1)\omega]}{(N-1)\rho\theta}$$

and ω was defined in (A.11). From this, the equilibrium price is:

$$p^* = \frac{1}{N}\sum_i^N \left[y_i - \left(\frac{\rho}{\theta}\right) x_i \right] .$$

This price is the price that prevails in a Walrasian auction. Equilibrium exists if $\theta < \frac{\rho^2(N-2)}{\phi N}$. This argument demonstrates the auction system can function in markets where a continuous system is not defined.

(b) Using (A.47), we see the mechanism Γ_a is semi-strong form efficient. This condition implies that in a sequence of periodic auctions, indexed by t, with N_t traders in each auction, the auction prices follow a martingale. To show this, note that

$$E[\tilde{p}_{t+1}|p_t] = E[E[\tilde{v}|p_{t+1}]\,|p_t] = E[\tilde{v}|p_t] = p_t .$$

(c) From equation (A.51), the unconditional price variability is:

$$\text{Var}[\tilde{p}_i] = \xi^2\tau^{-1} + (1-\xi)^2 \text{Var}[y^N(\tilde{Q})] \tag{A.61}$$

which is simply $(\tau + N\omega)^{-1}$. Now consider a sequence of N transactions in the mechanism Γ_c. Using proposition 1, the variance of these prices is monotonically decreasing over time as market makers learn. Consequently, of this sequence, p_N has the lowest variance which is $(\tau + N\omega)^{-1}$. It follows that the unconditional variance of the auction price is (weakly) less than the unconditional variance of the entire sequence of corresponding prices $p_1, \ldots, p_N$ in the dealer market. ∎