THE SUSTAINABILITY OF BUDGET DEFICITS IN A STOCHASTIC ECONOMY

by

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Abstract

The paper derives the government budget constraint and studies the sustainability of deficits in a stochastic, dynamically efficient economy. Contrary to the intuition based on certainty models, policies with permanent expected primary deficits can be sustainable. Even an infinite string of realized primary deficits does not necessarily provide evidence against sustainability.

Moreover, one has to be careful in discounting future fiscal variables. Even if the government finances deficits by issuing safe debt, the safe interest rate cannot be used in transversality conditions and in computing present values.

The stochastic setting allows one to reconcile dynamic efficiency with a safe interest rate below the average rate of economic growth. Evidence that the U.S. government has run average primary deficits and that government bond returns have been below the growth rate over long periods combined with evidence on dynamic efficiency from Abel et.al. (1989) suggests that the sustainability results for the stochastic, dynamic efficient economy are highly relevant for assessing U.S. fiscal policy. (JEL: 321)

1. Introduction

High federal budget deficits have been a central concern for economists and policy-makers in recent years. Over fiscal years 1980-1988, the primary deficit of the federal government has averaged 50 billion dollars.[1] Intuition based on certainty models suggests that primary deficits cannot be sustained for ever (McCallum (1984)). A number of recent empirical studies have examined more formally whether the U.S. fiscal policy satisfies an intertemporal budget constraint; see, e.g., Hamilton and Flavin (1986), Kremers (1989), and Wilcox (1989). Both Kremers and Wilcox conclude that recent fiscal policy does not appear to be sustainable; Hamilton and Flavin disagree.[2]

I will argue that the literature on sustainability has made simplifying assumptions about the discounting of future government payments and receipts that are implausible and require a reexamination of the issue. A careful treatment of discounting is crucial in this context, because interest rates on "safe" government bonds have frequently been below the average rate of economic growth. This suggests either dynamic inefficiency or an important role for risk aversion. Abel et.al. (1989) demonstrate convincingly that the U.S. economy is dynamically efficient and they show that the safe interest rate can be below the growth rate in a stochastic, dynamically efficient economy populated by risk averse individuals.

This paper studies the government's intertemporal budget constraint and sustainability of deficits in an explicitly stochastic model. The model is a simple Lucas (1978) style endowment economy with income uncertainty. Dynamic efficiency is a maintained assumption.

The stochastic model yields the most striking results when the government issues safe debt in a situation where the safe interest rate is below the average growth rate of the economy. In an example where the government stabilizes the debt-income ratio, primary deficits are positive on average. In an example where the government sets taxes equal to (or slightly below) non-interest spending in "most" states of nature and simply issues new debt to cover principal and interest of the outstanding debt, the probability that the government will ever have to run surpluses in the future is close to zero. Thus, contrary to the intuition derived under certainty or in a risk-neutral framework, primary deficits do not necessarily provide evidence against sustainability.

The examples also demonstrate that the discounting of future government debt, taxes, and spending is a non-trivial task, even if deficits are financed by safe debt. The correct discount rates typically differ from the safe interest rate and the return on government bonds (cf. Hamilton and Flavin (1986), Wilcox (1989)). In a world with income uncertainty, deficit financing with safe debt implies that tax rates will have to be high in states of nature where income is low in order to back the fixed payments to bondholders. Future tax payments are highly uncertain, which suggests that the common, familiar government practice of issuing essentially safe debt securities may not be as "safe" and straightforward as it appears at first sight.

Historical data on U.S. interest rates, growth rates, and budget deficits suggest that the case of dynamic efficiency combined with an interest rate on government bonds below the average growth rate is highly relevant for the United States. First, the promised interest rate as well as the ex-post return on government bonds have frequently been below the growth rate of the economy. (The 1980s have been an interesting exception.) Second, average U.S. primary deficits have been positive over long periods. The U.S. government has apparently exploited the difference between average growth and safe interest rates in the past. The government should be able to continue to run substantial deficits, <u>if</u> future interest and growth rates match the historical averages over the 20-th century. However, deficits will be more tightly constrained, if real interest rates stay as high as they have been in the 1980s. A key question for assessing current fiscal policy is whether the growth-interest rate differential will be positive or negative in the future.

The paper starts with a brief review of the relevant U.S. data in Section 2. Section 3 sets up the model and develops the intertemporal budget constraint under uncertainty and the associated transversality conditions. Two examples are presented in Section 4. Example

1 shows that permanent expected primary deficits are sustainable. Example 2 shows that an infinite string of realized primary deficits does not necessarily provide evidence against sustainability. Numerical computations illustrate the size of deficits that can be sustained. Conclusions are summarized in Section 5.

2. Deficits, Growth, and Interest Rates in the U.S.

This section presents some data on U.S. fiscal policy and on the U.S. economy that will motivate the theoretical analysis below.

Starting with fiscal policy, Table 1 displays average primary U.S. deficits and the number of years in which deficits were run, each for several sample periods. The primary deficit is simply the difference of federal outlays excluding interest payments and federal receipts, sometimes called the net-of-interest deficit.[3] The sample periods 1929-88 (start of NIPA accounts) and 1954-88 (postwar period) are popular in macroeconomic analysis, 1960-1984 is the sample period used in studies of fiscal policy by Hamilton and Flavin (1986) and Wilcox (1989). Very recent and longer run data are displayed for comparison.

The long run data show that surpluses were frequent and that the average budget balance was positive in the 19-th century. But deficits occurred more and more often in the 20-th century. Since 1929, primary deficits occurred in more than half of the years and the deficit was 1.77% of GNP on average. The deficits in the 1980s were high, but the average was actually below the 1929-1988 average (which included World-War II).

Regardless of how one evaluates the 1980s, the important result from this table is the observation that primary deficits are neither rare nor a recent phenomenon. Over the period 1800-1988 as a whole, over the 20-th century, and over most 20-th century subsamples, the federal government did not cover its spending on average, even when interest payment are properly excluded.

This observation is difficult to reconcile with standard theories of how fiscal policy operates. Standard models like those in Blanchard–Fischer (1989) or McCallum (1984) demand that the government services and/or retires initial debt by running primary surpluses. A government that does not run primary surpluses to service its debt would violate the intertemporal budget constraint. Even if deficits are positive in some periods, as justified, e.g., by Barro's (1979) tax smoothing theory, surpluses in other periods should be high enough that the average budget balance will still be positive. But the government has in fact been able to operate with budgets that showed a negative balance on average over the past two centuries.

A superficial explanation of why the government has been able to do this is found in the relation between government bond returns and economic growth. Table 2A displays averages of GNP-growth and government bond returns as well as several other interest rate measures for the same sample periods.[4] Table 2B shows the differences between average growth rate and average returns. The key observation from this table is that the difference has been positive for all sample periods except for the 19-th century and the 1980s. For long periods, the government has been able to rely on growth to take care of its debt without any need for debt service.

The implications for future fiscal policy and for economic analysis depend on why the government bond return has been below the growth rate. Two possible explanation have straightforward implications. First, if the explanation were dynamic inefficiency (see Diamond (1965)), the government could presumably continue to count on a positive growth-interest rate gap to run deficits. However, Abel et.al. (1989) have shown that in a stochastic world the difference between safe interest rate--if one interprets T-bills as essentially safe debt--and growth rate is not informative about dynamic efficiency. They provide convincing evidence against dynamic inefficiency for the U.S.

Second, one may argue that the positive values in Table 2B are due to unexpectedly low return realizations or unexpectedly high growth, i.e., that the ex-ante expected return on government debt never was below the expected growth rate. If the data are only a historical accident, standard theory can be applied to assess current and future policy. Indeed, growth rates and returns are volatile enough that many data points in Table 2B are not significantly positive.[5] It is moreover plausible that the low real bond returns for 1929-80 are related to the fact that inflation and nominal yields were trending upward over this period, generating capital losses on these nominal bonds. On the other hand, most point estimates in Table 2B are positive and some growth-return differences are significant, e.g. for T-bills over 1954-1988. Even if the low returns on T-bonds are in part due to unexpected capital losses, the low returns on short-term T-bills cannot be explained away. Thus, it seems that the difference between growth rate and ex-ante safe interest rate has been positive for extended periods of time. At least, this possibility should not be dismissed a priori.[6]

Overall, neither dynamic inefficiency nor sampling error provide appealing explanations for the data in Table 2. The U.S. economy appears (1) to have a safe interest rate that has frequently been below the rate of economic growth and (2) to be dynamically efficient. Since these two characteristics are incompatible in a certainty model, but not with uncertainty and risk aversion (see Abel et.al (1989)), a stochastic model is needed to study the sustainability of government deficits. The next section will therefore derive the government budget constraint in a stochastic economy. The model will allow the growth-interest rate difference to take positive or negative values and it will include risk neutrality and certainty as special cases.

3. The Model

The analysis uses a simple general equilibrium model of the economy due to Lucas (1978). The economy has a representative consumer that earns an exogenous stochastic stream of dividends Y_t from a "fruit tree" and a government that finances an exogenous path of government spending G_t by lump-sum taxes T_t. I assume that financial markets are complete and that the government has the ability to trade on all markets.

The complete markets assumption allows one to study the sustainability for arbitrary ways of financing the deficits. An important point of the paper will be that the type of liabilities that the government issues has important implications for sustainability. Safe, or approximately safe, debt is one of the financing choices.[7]

To simplify the exposition, I will assume a discrete state-space so that all securities can easily be interpreted as bundles of state-contingent claims. (In the examples, I will sometimes jump to continuous distributions, omitting the analogous derivations.) At any point in time, the state s_t consists of a realization $\tilde{s}_t$ and a history s_{t-1}, $s_t=(s_{t-1},\tilde{s}_t)$. At time t, the conditional probability of state s_{t+n} is denoted by $\pi(s_{t+n}|s_t)$. The expectation with respect to π is denoted by $E_t[\cdot]$. Without loss of generality, let the current period be t=0.

Denote the period-t price of a unit claim on the consumption good in period t+1 and state s_{t+1} by $p(s_{t+1}|s_t)$ and denote the quantity of such claims held by the consumer by $A_{t+1}(s_{t+1})$. Subscripts and/or state designations will be omitted when unambiguous; conditioning on s_0 will also be omitted. Assets held by the representative consumer at the start of period t are A_t, consumption in period t is C_t. The consumer maximizes utility

(1) $$\sum_{t\geq 0} E_0[\beta^t \cdot U(C_t)] = \sum_{i=0}^{\infty} [\sum_{s_t} \pi(s_t) \cdot \beta^t \cdot U(C_t(s_t))]$$

subject to the budget constraints

(2) $$A_t + Y_t - T_t = C_t + \sum_{\tilde{s}_{t+1}} p(s_{t+1}|s_t) \cdot A(s_{t+1})$$

for all periods and states.[8] Utility is strictly concave, increasing, and has a positive rate of time preference β, $\beta<1$.

Denote government debt at the start and end of a period by D_t and $\tilde{D}_t$, respectively. The government budget constraint is

(3) $$D_t + G_t = T_t + \sum_{\tilde{s}_{t+1}} p(s_{t+1}|s_t) \cdot D(s_{t+1}) = T_t + \tilde{D}_t.$$

Equilibrium requires $Y_t = C_t + G_t$ and $A(s_{t+1}) = D(s_{t+1})$ for all s_{t+1}. $0 \leq G_t < Y_t$ is assumed for all periods and states, but, since taxes are lump-sum, $T_t > Y_t$ is not ruled out.

The notation with state-contingent debt at the start of the period is most general. When the "type" of liabilities has been specified (in examples), a notation using end-of-period debt notation is more convenient, because one can define the return on the liability portfolio as $R_{t+1}(s_{t+1}) = D_{t+1}(s_{t+1})/\tilde{D}_t - 1$ and obtains the familiar budget constraint

$$\tilde{D}_{t+1} = G_{t+1} - T_{t+1} + (1+R_{t+1}) \cdot \tilde{D}_t.$$

In particular, safe debt is the special case where the government issues an equal number of claims on each state. Then $R_{t+1}(s_t,\tilde{s}_{t+1}) = r_t$ for all $\tilde{s}_{t+1}$, where $r_t = [\Sigma_{s_{t+1}} p(s_{t+1}|s_t)]^{-1} - 1$ is the safe interest rate, and $D_{t+1}(s_{t+1}) = (1+r_t) \cdot \tilde{D}_t(s_t)$ for all $\tilde{s}_{t+1}$. This formulation has the additional advantage that a continuous state space can be handled with the same notation.

Asset prices and returns are characterized by the individual optimality conditions. The standard period-by-period first order conditions are

(4) $$\pi(s_t) \cdot \beta^t \cdot U'(C_t(s_t)) = \lambda(s_t)$$

(5) $$p(s_{t+1}|s_t) \cdot \lambda(s_t) = \lambda(s_{t+1})$$

for all periods and states, where $\lambda(s_t)$ is the Lagrange multiplier of the budget constraint (2) in state s_t. An interior solution will be assumed.[9] Also, assume that $\sum_{t\geq 0} E_0[\beta^t \cdot U(Y_t-G_t)]$ and $\sum_{t\geq 0} E_0[\beta^t \cdot U'(Y_t-G_t)\cdot(Y_t-G_t)]$ are finite so that utility and the present value of consumption are finite in equilibrium.[10]

For later reference, some notation for specific asset prices will be useful. Denote the marginal rate of substitution between periods t and t+n by

$$u_{t,n} = \beta^n \cdot U'(Y(s_{t+n})-G(s_{t+n}))/U'(Y_t(s_t)-G_t(s_t)).$$

Then the interest rate on a safe asset between periods t and t+1, r_t, is characterized by $1/(1+r_t) = E_t[u_{t,1}]$. The value of a "fruit tree" in period t, which is a claim to dividend income from period t+1 on, is

$$V_t = \sum_{j\geq 1} E_t[u_{t,j} \cdot Y_{t+j}].$$

Assume V_t is also finite.[11] Finally, the price at time zero of a unit claim on state s_t is

$$P(s_t) = \prod_{i=0}^{t-1} p(s_{i+1}|s_i) = \lambda(s_t)/\lambda_0 = \pi(s_t) \cdot u_{0,t}(s_t).$$

A critical issue in the context of sustainability is the specification of transversality conditions. As O'Connell and Zeldes (1988) point out, the "standard" condition that sets the limiting value of net assets to zero, combines two things. First, a necessary condition of individual optimization is that no individual will place himself on the lending side of a Ponzi game. Second, in an economy with a finite number of optimizing agents, no agent will be able to run a Ponzi game on others.

Uncertainty and the existence of a government complicates these arguments. Uncertainty implies that a set of constraints, one for every state s_t, must be imposed to rule out Ponzi games that may start at some point in the future. In addition, since government policy is arbitrary and not necessarily optimizing, two complications may arise. The government might allow individuals to run Ponzi games against the government. And, if the government set taxes in a way that T_t jumps around "erratically" between positive and negative values with increasing amplitude, the relevant limits in the transversality conditions may fail to exists.

These complications cannot be ruled out in general, but they do not seem particularly interesting in this context, will not be needed in the rest of the paper and are therefore excluded in the following proposition. For the interested reader, a general characterization of the transversality conditions is in the appendix.

Proposition 1:

Provided the government does not permit individuals to run Ponzi-games against the government and provided the following limits exist, government policy at time zero must satisfy a set of transversality conditions, one for each state s_t:[12]

(6) $$\lim_{N\to\infty} E_t[u_{0,N} \cdot D_N] = \lim_{N\to\infty} \sum_{s_N|s_t} P(s_N)\cdot D(s_N) = 0,$$

Similarly, individuals must satisfy the transversality conditions

(7) $$\lim_{N\to\infty} E_t[u_{0,N} \cdot A_N] = \lim_{N\to\infty} \sum_{s_N|s_t} P(s_N)\cdot A(s_N) = 0.$$

Moreover, in each state of nature the intertemporal budget constraint (IBC)

(8) $$D_t = \sum_{n\geq t} E_t[u_{t,n} \cdot T_n] - \sum_{n\geq t} E_t[u_{t,n} \cdot G_n].$$

must hold. The analogous constraints apply to individuals.

Proof: See Appendix 1.

The rest of the paper will study fiscal policies that satisfy the IBC (8)--or equivalently, (3) and (6)--at all times. Such policies will be called sustainable. It is straightforward to show that any sustainable policy can be supported as an equilibrium allocation (see the appendix). That is, the IBC is sufficient for a policy to be feasible.[13]

Under the regularity conditions stated in the proposition, the transversality and IBC conditions have a format familiar from non-stochastic models and stochastic models with fixed interest rates (see Blanchard-Fischer (1989), ch.2 and ch.6). But it is important that the values of future debt, assets, taxes, and spending are not generally discounted at a common interest rate. Present values depend on the contingent claims prices $P(s_t)$, which generally vary across time and states.

The IBC (8) requires that the present value of future primary surpluses matches initial debt. Starting with initial debt, the government cannot run deficits in all states of nature, which is an important difference to models with dynamic inefficiency. The next section will show, however, that under some conditions surpluses are only needed in remarkably "few" states of nature.

This completes the description of the model. Many potential complications such as a finite lives, capital accumulation, or heterogeneity are omitted. Ricardian equivalence holds. Dynamic efficiency is a maintained assumption. The primary interest will be in the constraints on government policy imposed by (8) or (3) and (6). The model embeds these constraints in an equilibrium setting that is as simple as possible.

4. Primary Deficits

This section will explore how much discretion the government has to run deficits without violating the intertemporal budget constraint. It is well known that a policy that features persistent expected primary deficits cannot be sustainable in models that assume risk neutrality or certainty; see, e.g., McCallum (1984) or Blanchard-Fischer (1989). Since risk neutrality (linear utility) implies $P(s_t) = \pi(s_t)\cdot\beta^t$, this also follows from equation (8). However, with risk aversion state-prices $P(s_t)$ will generally not be proportional to the probabilities of the corresponding states. Permanent expected primary deficits turn out to be possible in the stochastic model with risk aversion, as this section will show. Moreover, sustainable policies may display infinite strings of realized primary deficits with high probability.

4.1. Permanent Expected Primary Deficits

In practice, government bonds are typically nominal claims with different maturities, which makes their ex post real payoffs somewhat uncertain. In the examples I will abstract from inflation and interest rate uncertainty and assume that the government issues only safe claims.[14] Formally, this means that the government issues equal amounts of state-contingent claims on each state and does not use its ability to vary the amounts across states of nature.

Two simplifying assumptions are imposed on the economic environment. Income follows a geometric random walk and the ratio of government spending to income is constant. Of course, many macroeconomic time series are autocorrelated in some way. But since this paper is concerned with the long run properties of the economy, such short-term dependencies should not be critical. The question of stationarity (in levels versus growth rates) remains important, however. Here a unit root in the log-endowment is a maintained assumption. Concerning government spending, the optimal taxation literature (e.g., Barro (1979)) has devoted considerable attention to the question of how tax policy should respond to predictable changes in government spending. Since the paper has nothing to add to this issue, one may as well exclude it for now. To be precise, the three assumptions on policy and the economy are:

Y1. Endowment growth $y_{t+1} = Y_{t+1}/Y_t - 1$ is i.i.d.

G1. The ratio of government spending to income is a constant:

$$g_t = G_t/Y_t = g, \text{ where } 0 \leq g < 1.$$

D1. The government issue safe one-period debt:

$$D_{t+1}(s_t, \tilde{s}_{t+1}) = (1+r_t) \cdot \tilde{D}_t \text{ for all } \tilde{s}_{t+1}.$$

Assumptions Y1 and G1 imply that the riskfree rate r_t and the dividend yield v_t are constants, denoted by r and v respectively, and that v>0. Assumptions like Y1 and G1 have been used by Abel et.al. (1989; p.13) to illustrate the relation of safe and risky returns.

Given assumption D1, the government has one remaining choice variable, either taxes t_t or the level of debt $\tilde{D}_t$ (the other one being implied by the budget constraint). For this example, assume the government has a fixed target d for the ratio of end-of-period debt to income, $\tilde{d}_t$. In every period and state, taxes are set so that the target is reached, which requires:

T1. Tax policy with debt-income target:

$$T_t(s_t) = G_t + [D_t - d \cdot Y_t]$$

In terms of the tax rate $\tau_t = T_t/Y_t$, policy (T1,D1) implies

$$\tau_t(s_t) = (g - d) + (1+r) \cdot \tilde{D}_{t-1}/Y_t(s_t)$$

(9) $$= g + [(1+r)/(1+y_t(s_t))-1] \cdot d,$$

The ratio of start-of-period debt to income is stochastic, since income growth is stochastic. Tax rates vary over time and across states of nature, being high (low) whenever income growth has a low (high) realization. Noting that $D_{t+1} = (1+r)\cdot\tilde{D}_t = (1+r)\cdot d\cdot Y_t$, we have

$$\lim_{N\to\infty} \sum_{s_N|s_t} P_N \cdot D_N = \lim_{N\to\infty} (1+r)\cdot d\cdot E_0[u_{0,N} \cdot Y_{N-1}]$$

(10) $$= \lim_{N\to\infty} d\cdot Y_0 \cdot \frac{1}{(1+v)^{N-1}} = 0.$$

Thus, the policy is sustainable. Moreover, the fact that the policy maintains a bounded debt-income ratio may be reassuring to readers who feel that government policy is in practice subject to additional constraints.[15]

In light of the empirical evidence on growth and interest rates, it is important that the sign of $E[(1+r)/(1+y_t)-1]$ has not been restricted.[16] Abel et.al. (1989) have shown that the safe interest rate in this economy will be below the expected growth rate, if the variance of income growth is sufficiently large. In that case, the policy implies a positive primary deficit in expectation.

Proposition 2:

a. The expected primary deficit as share of income, $E[g-\tau_t]$, is positive, if and only if

$E[1/(1+y_t)] < 1/(1+r)$.

b. The expected level of the primary deficit, $E[G_t-T_t]$ is positive, if and only if $E[y_t] > r$.

Proof: (a) is immediate from (9) and (b) follows from $E[G_t-T_t] = E[D_t-d \cdot Y_t(s_t)] = d \cdot E[Y_{t-1}] \cdot E[r-y_t]$. ||

Under the conditions of the proposition, an econometrician studying this economy will almost certainly find positive average primary deficits (absolute or relative to income) for any sufficiently long sample period. Thus, if $E[(1+r)/(1+y_t)]<1$, the historical data showing negative budget balances on average (see Table 1) are consistent with the pursuit of a sustainable policy.

Interestingly, governments typically announce plans for future fiscal policy in terms of <u>expected</u> revenue and spending numbers, which are usually based on projections of <u>expected</u> growth and interest rates. If the government follows a policy like T1, such projections of fiscal policy will include planned budget deficits in all periods. Again, this does not provide evidence against sustainability.

It should be noted that the example can be generalized easily. If government debt were not exactly safe and if the spending-income ratio were not constant, nothing would change except that r would have to be replaced by a stochastic return R_t.[17] But as long as $E[(1+R_t)/(1+y_t)]<1$, negative deficit-income ratios will be observed on average under tax policy T1, even without assumptions D1 and G1.

For plausible parameter values, deficits under policy T1 can be quite substantial. For a debt-income ratio d=0.5, income Y_t=\$5 trillion (both close to current U.S. values), and $1-E[(1+R_t)/(1+y_t)] = 2\%$ (close to the historical averages),[18] policy T1 would lead to average primary deficits of about \$50 billion.

Drastically different conclusions about sustainability would be obtained, if one simply used the safe interest rate or the return on government bonds to discount expected debt, as suggested by Hamilton and Flavin (1986) and Wilcox (1989), respectively. In the example, the safe interest rate and the return on government bonds are both the constant r, which yields

$$\lim_{N\to\infty} E_0[\frac{1}{(1+r)^N}\cdot D_N] = \lim_{N\to\infty} d\cdot Y_0\cdot\frac{E[(1+y_t)]^{N-1}}{(1+r)^{N-1}} = \infty,$$

for $E[y_t]>r$. As shown above, the policy is sustainable regardless of the relation between $E[y_t]$ and r. Although debt is a safe claim, it is inappropriate to discount D_N at the safe rate r. The limiting amount of outstanding debt is uncertain, because the accumulation of debt depends on the stochastic path of spending and taxes. Here $D_N(s_N)$ is proportional to $Y_{N-1}(s_{N-1})$. Hence, the dividend yield on the fruit tree is relevant for the correct valuation derived in (10). For large N, the appropriate discount rate is close to the expected return on equity.[19]

More generally, the example shows that the choice of discount rates is a non-trivial issue in testing for the sustainability of government policy. An estimate of the stochastic process of debt is required to determine the correct discount rate. Depending on whether the processes of debt and consumption are positively or negatively correlated (and how much), the discount rate can be above or below the safe rate and above or below the expected return on equity. Most importantly, the discount rate does not necessarily bear any relation to the realized or expected return on government debt (cf. Wilcox (1989)). It will generally not be the safe interest rate (cf. Hamilton and Flavin (1986)) or any other easily determined number.[20]

The same observation applies to the present values of future taxes and government spending. Unless the amount of taxes or spending is fixed in advance and independent of the state of nature--which is difficult to do in a stochastic world--the safe interest rate is not the correct discount rate. Under uncertainty, the intertemporal budget constraint cannot generally be stated in terms of expected surpluses discounted by any one (safe or other) interest rate.

In the example, the discount rate for government spending is exactly the rate of return on equity, since spending is proportional to income. The discount rate on taxes is lower, since the tax rate moves inversely with income and consumption. This difference in discount rates is far from insignificant for the interpretation of fiscal policy, since it turns out to be large enough that the present value of future taxes always exceeds the present value of future income, even when expected deficits are positive. That is, under policy T1, the present value of next period's deficits is

$$\begin{aligned} E_t[u_{t,1}\cdot(G_{t+1}-T_{t+1})] &= E_t[u_{t,1}\cdot(d\cdot Y_{t+1}-D_{t+1})] \\ &= -\, d\cdot Y_t\cdot v/(1+v) \qquad < 0 \end{aligned}$$

regardless of the sign of $E_t[G_{t+1}-T_{t+1}]$. This result should not surprise, since the IBC requires the present value of taxes to exceed the present value of spending for any sustainable policy starting with initial debt, but it shows that correct discounting is important for assessing the burden of future taxes.

Since debt is safe in this example, tax-payers are forced to make relatively large payments in states of nature with low income and consumption in order to back the promised fixed payments to bond-holders. The burden of future taxes (their present value) is larger than it would be if taxes with same expected value were proportional to income. Financing with safe debt seems to impose risks on tax-payers which increase the burden of future taxes. (See Section 4.2 for more comments on risk taking with safe debt).

A fixed, positive real interest rate is also used in studies exploring the co-integration properties of budget variables, e.g., by Hakkio and Rush (1986) and Trehan and Walsh (1988). This assumption is understandable for tractability reasons (which means that more detailed comments are beyond the scope of this paper), but if--as I argued in Section 2--risk aversion is critical for explaining the data, it is a questionable assumption. It is therefore difficult to assess what co-integration implies for sustainability.[21] Of course, error-corrections models may still be very valuable as parsimonious representations of the approximate stochastic structure of these variables.

Overall, example has shown that policy plans that feature permanent expected primary deficits can be sustainable in a stochastic, dynamic efficient economy. Still, a government pursuing policy T1 will almost surely--with probability one--run a surplus in some periods. The next example will show that this is not generally necessary either.

4.2. Realized Primary Deficits

In the previous example, the time series of primary surpluses alternates randomly between positive and negative values. In this section, the example will be modified in a way that there are long strings of primary deficits. Assumptions Y1, G1, and D1 are imposed again. That is, I abstract from autocorrelation and temporary government spending and assume safe debt. Instead of T1, assume that tax policy is:

T2. Tax policy is defined by

$$\tau_t(s_t) = g - \varepsilon \cdot d_t, \quad \text{if } d_t \le d/(1+\varepsilon)$$
$$= g + d_t - d, \quad \text{if } d_t > d/(1+\varepsilon)$$

where $\varepsilon \ge 0$ and $d>0$.

Starting with positive debt $d_0 \le d$, the government runs a primary deficit unless debt threatens to exceed an upper bound on the debt-income ratio, denoted by d.[22] Under this policy, the debt-income ratio will evolve according to the law of motion

$$d_{t+1} = (1+r)/(1+y_{t+1}) \cdot \text{Min}\{(1+\varepsilon) \cdot d_t, d\}.$$

As long as debt remains below the upper bound d, this an exponential random walk with drift $\mu = \log(1+r) - E[\log(1+y_{t+1})] + \log(1+\varepsilon)$. Since the debt-income ratio is bounded by d, sustainability follows as in Example 1 (using (10)).

The policy exhibits primary deficits unless there is a string of low income growth realizations that drives the debt-income ratio up to the bound d. Under several sets of assumptions, there is a positive probability that that will never happen.

Proposition 3:

Let $P = \text{Prob}\{s_t: d(s_t)<d \text{ for all } t\geq 1\}$ be the probability that debt remains below the boundary d for ever. Assume that $\mu>0$ and $d<d_0$.

a. If y_t is log-normal with variance σ^2, then

(11) $$P \geq 1 - (d_0/d)^{\lambda} > 0$$

where $\lambda = \mu/\sigma^2$.

b. If y_t has a distribution such that $\text{Prob}\{s_t: (1+y_t(s_t)) < (1+r)\cdot(1+\varepsilon)\} > 0$, then $P>0$.

Proof: See appendix 2. ||

The value P is the probability that a government running primary deficits of size $\varepsilon \cdot D_t$ will <u>never</u> be forced to switch to surpluses. In the log-normal case, this probability is bounded from below by a number that can be computed directly as function of the drift and variance of the debt-income process. Whenever μ is positive, the probability P is positive. If normality is considered a too restrictive assumption, part (b) shows that this qualitative result is more general.[23]

Some numbers may be useful to illustrate how high the probability P can be for plausible parameter values. To save space, I will focus on policies with small primary deficits so that μ is the (log) difference between growth and interest rate.[24] Table 2 indicates that plausible values for μ are of the order of 1-2 percentage points.[25] In addition, values for the variance of income growth, σ^2, and for the upper limit on the debt-income ratio, d, are needed. Taking GNP as measure of income, the standard deviation of U.S.

GNP has been 5.93% for 1800-1988, 6.42% for 1929-1988, and 2.37% for 1954-1988. The current debt-GNP ratio of about d_0=0.5 is less than half of its historical high of 1.13 in 1946. Thus, a standard deviation σ between 2% and 8% and values for d/d_0 above 2, say between 2 and 4, seem reasonable for the U.S.

For these parameter ranges, Tables 3 computes the probability 1-P that the government will ever have to run surpluses, assuming normality. The Table shows that this probability is close to zero for $\mu \geq 1\%$ and reasonable values of σ and d. The high value σ=10% is included to provide some sensitivity analysis: Since μ and σ enter exponentially, probabilities rise sharply with low μ and high σ. But even for the extreme values of σ=10% (high) combined with $d/d_0=2$ (low), the probability of ever having to run a surplus is only 25%. For values near the historical averages (or rather slightly below to be on the safe side) of μ=1% and σ=6% combined with $d/d_0=3$, this probability is near zero; the probability P of never having to run a surplus is 99.776%.

Intuitively, the table shows how unlikely it is that GNP will decline so much that the debt-GNP ratio would double or triple. Even during the Great Depression real GNP fell only about 30% (1929-33, total decline); something much worse would be needed to force the government to raise the tax rate and run a budget surplus. As long as the gap between average growth and interest rate remains near its historical average, the government can be extremely confident that it will never be asked to run surpluses.

A different way of displaying the results is useful to focus on the growth-interest rate differential. Table 4 displays values of μ implied by (11) for different combinations of P and σ.[26] Table gives the minimal differences of growth and interest rate needed to run a zero surplus with at least probability P. For example, for σ=6% and $d/d_0=3$ the government can be 90% confident that it will never have to run a surplus as long as the average difference between growth and interest is at least μ=0.38%. Panel C illustrates that μ-values very close to zero are enough to obtain a 50-50 chance of no surpluses. The historical growth-interest rate differences in Table 2 are much higher than these values, except for the 1980s.

These results create an obvious marketing problem for policy advisors. How can one (should one?) convince politicians not to run deficits all the time? Two possible arguments against deficits remain. First, forecasts of future interest rate may be so high or forecasts of growth so low that the the historical growth-interest gap does not exist any more. The key question in this context is whether the negative growth-interest difference for the 1980s is a short term phenomenon or sign of a structural break.

Second, one may argue that politicians should not ignore the small risk of running into high debt-income ratios. Under policy T2, the payments for current government spending are, in part, shifted to some future states of nature occurring with low probability, in which the government would have to run large, burdensome (because of low income) surpluses. The government gambles that these states of nature will not be realized.

To assess such gambles, recall that the IBC (8) with $D_0>0$ requires that the expected present value of taxes exceeds the expected present value of spending. If policy-makers are interested in an even distribution of the tax burden over time as share of income, it seems difficult to justify current primary deficits (at least in examples with constant g_t).[27] In particular, current (certain) deficits cannot be justified by refering to future deficits of similar size that are very likely (under T2) or that will occur in expectation (in Example 1), because actual future deficits are uncertain. The more a policy is "rigged" to produce deficits "many" states of nature, the higher must be the size and/or utility cost of surpluses in the other states (noting that the IBC weights surpluses by $u_{0,t}$).

The examples show that the low promised interest rates implied by safe debt allow the government to run frequent deficits without violating the IBC, but they suggest that safe debt can easily lead to a very uneven distribution of surpluses across states of nature. Thus, government deficits, at least when financed with safe debt, may not indicate a sustainability problem as much as an issue concerning the intertemporal distribution of the tax burden.

The fact that primary deficits can be sustained for ever with high probability also creates problems for the empirical analysis of fiscal policy. On the one hand, primary surpluses are needed in some states of nature. That is, a policy that specifies deficits in all possible states s_t does not satisfy the transversality constraint (6). On the other hand, if the government follows a policy like T2 and if $(d/d_0)^{\lambda}$ is sufficiently high, the probability is close to one that the data will show primary deficits for all observed states. Policy T2 is sustainable. It differs from a policy of always running deficits only in how it responds in rare, "extremely bad" situations. Even if one has an arbitrarily long empirical data set showing primary deficits all the time, one cannot necessarily reject sustainability.[28]

5. Conclusions

The paper has derived the intertemporal constraints on government policy in a stochastic, dynamically efficient economy. Two examples are presented to suggest that the intertemporal budget constraint imposes remarkably weak restrictions on primary deficits. In one example, expected (planned) deficits are negative at all times. In the other, infinite strings of primary deficits may be observed with high probability. Both examples exploit the fact that the safe interest rate can be below the average growth rate in a stochastic economy with risk aversion. Such a scenario would be impossible in dynamically efficient certainty models or those with risk neutrality. Since U.S. data show a positive growth-interest rate gap for long periods, a stochastic with risk aversion seems to be highly relevant for an assessment of U.S. fiscal policy. Another reason for taking this case serious is the fact that the U.S. government has run an average deficit over most of its history (e.g., over 1800-1988 and many subperiods).

The analysis has two major implications for the assessment of current budget policy. First, the continued sustainability of primary deficits depends crucially on the difference between safe interest rates and growth rates in the future. Expected primary deficits can be sustained easily, if this difference is equal to its 20-th century average, but not if real interest rates stay as high as they have been in the 1980s. Thus, the question whether the high level of real interest rates is permanent or temporary is crucial for assessing the sustainability of current fiscal policy.

Second, the paper shows that an assessment of fiscal policy cannot rely on taking expected values and on discounting at the safe interest rate or any other common interest rate. The correct discount rates on future debt, taxes, and spending depend on the stochastic processes of these variables and they are not necessarily related to the safe interest rate or the government bond rate.

In particular, the examples suggest that deficit financing with essentally safe debt is not as "safe" and straightforward as it appears at first sight. In an stochastic environment, safe debt forces the government to raise tax rates in states of nature with low income in order to back the promised fixed payments to bond-holders. The low promised returns on safe debt can be exploited to construct sustainable policies that display primary deficits frequently, but such policies seem to impose risk on future tax-payers. The risks are easy to see in some cases (e.g., Example 2), but they are less obvious in others (e.g., Example 1). Thus, it seems necessary to use a careful analysis, based on a state-contingent claims framework rather than expected values, to assess the intertemporal allocation of the tax burden in a stochastic environment.[29]

Footnotes

[1] See McCallum (1984) or Trehan and Walsh (1988) on why the primary deficit is relevant (Data source: Economic Report of the President).

[2] These three paper provide direct tests of the intertemporal budget constraint. Indirect tests based on implications for co-integration are in Hakkio and Rush (1986) and Trehan and Walsh (1988).

[3] Data sources are described in more detail in the appendix and in Bohn (1989).

[4] Since the real ex ante returns on government debt are not observed, several different proxies are displayed. The T-bill and long-term T-bond returns are ex post returns for two specific securities in the government's portfolio. The yield and interest charge data can be interpreted as capturing more ex ante information on nominal returns, though the real values include ex post information on inflation. None of the series is perfect, but it is reassuring to see that all series show similar average values.

[5] This is the reason why standard errors are not displayed.

[6] An interesting and open question is whether the negative averages for the 1980s indicate a temporary aberration from the 20-th century pattern or a break in the data around 1980; see below.

[7] The examples below will focus on safe debt, though they will suggest that the type of liabilities is important for sustainability issues. The debt management implications are currently being studied by the author, but beyond the scope of this paper.

[8] Note that the trade in contingent claims $A(s_t)$ starts with each individual owning one fruit tree. Fruit trees can be traded as bundles of contingent claims. For example, an individual could sell the fruit tree (which means selling the stream of dividends) by taking asset positions $A(s_t) = -[Y(s_t)+V(s_t)]$ for all states s_t, with $V(\cdot)$ defined below.

[9] The assumption $U'(0) = \infty$ would be sufficient to guarantee this.

[10] The model would be uninteresting, otherwise. These conditions must be introduced as assumptions, because growth and government spending are exogenous. The conditions could fail, e.g., if the growth rate were much higher than the rate of time preference.

[11] Given a finite value of consumption, this is a fairly weak assumption. A sufficient condition is, e.g., that the ratio of government spending to endowment is bounded below one.

[12] To be precise, the conditional expectation refers to the sum $\sum P(s_N)\cdot D(s_N)$ taken over all states s_N that include a particular history s_t; $N \geq t$ can be assumed without loss of generality.

[13] It should be acknowledged that more elaborate models may well imply additional constraints. In particular, IBC is consistent with unbounded debt-income ratios (see McCallum (1984)), which are usually by supported by rising ratios of tax revenues to income. Arbitrarily high income tax rates are feasible with lump-sum taxes, but since lump sum taxes are rarely used in practice, one may wonder whether all policies that satisfy the intertemporal budget constraint are practically feasible. A general discussion of possible complications would be distracting here, but regardless of what position one takes on this point, the examples below do not seem to exploit any obvious "weakness" in the IBC, since they display bounded debt-income ratios.

[14] I do not mean to question the importance of inflation and interest rate uncertainty for other issues (see, e.g., Bohn (1988), (1990)).

[15] See the earlier footnote on feasibility constraints.

[16] One does not have to distinguish conditional and unconditional expectations here because of the i.i.d. assumption.

[17] Even with safe debt, stochastic g_t would imply a variable real rate r. The assumption of constant g_t is clearly unrealistic, since many U.S. deficits were due to temporary, war-related spending. Assumption T1 has some empirical support (if one disregards short-term

movements), since Kremers (1989) finds that the U.S. government has stabilized the ratio of debt or debt service to GNP. If the debt-income ratio is not stabilized, as Barro (1979) suggests, Example 2 below will apply.

[18] Table 2 shows values for $E[y_t-R_t]$, which is close to $1-E[(1+R_t)/(1+y_t)]$. Though the approximate numbers from Table 2 should be sufficient for this rough calibration, exact historical values for $1-E[(1+R_t)/(1+y_t)]$ are given in the appendix Table A1 for completeness.

[19] If one defines the discount rate R by $(1+R)^{-N}\cdot E_0[D_N]=E_0[u_{0,N}\cdot D_N]$ and notes that $(1+v)\cdot E[1+y_t]$ is the expected gross return on equity, then $E_0[D_N]/E_0[P_0\cdot D_N] = (1+r)\cdot E[1+y_t]^{N-1}\cdot(1+v)^{N-1}$ implies that for large N, $(1+R) = (1+r)^{1/N}\cdot\{(1+v)\cdot E[1+y_t]\}^{(N-1)/N}$ converges to the expected return on equity.

[20] Unless, of course, individuals are risk neutral. But the point of Section 2 was that risk aversion is important for explaining the stylized facts.

[21] There are several problems. The safe interest rate may not be the correct discount rate. It may not be positive. And independently from the interest rate issue, the derivations assume stationarity of the first differences of tax and spending levels (not logs), which would lead to non-stationary variances in models with exponential growth like this one. On the other hand, income ratios may be stationary or difference-stationary, and for the special case when the debt-income ratio is independent of consumption growth, it is straightforward to show that any policy with a difference-stationary process for d_t in sustainable. However, (d_t-d_{t-1}) is not generally a fixed linear combination of tax-, spending-, and debt-income ratios.

[22] As in Section 4.1, the example could be generalized. The only difference between this example and T1 is that here the government is typically not stabilizing the debt-income. Given the mixed evidence on whether the U.S. government stabilizes the debt-GNP ratio

(see Barro (1979), Kremers (1989)), it is unclear whether T1 or T2 better characterizes U.S. policy.

[23] Too see that the assumptions are not very restrictive, note that μ will be positive for small values of ε whenever expected income growth (in logs) exceeds the safe interest rate. Moreover, if $\varepsilon=0$, the equilibrium value of r will always such that the condition in (b) is satisfied. Thus, the probability P is always positive, provided the rate of growth exceeds the safe interest rate and provided the government sets a sufficiently small positive ε-value.

[24] That is, I focus on $\varepsilon\approx0$. Implications for $\varepsilon>>0$ can be worked out analogously by considering appropriately reduced values for μ.

[25] Table 2 shows differences of growth and interest rates in levels, not logs. But the differences between levels and logs are small; to verify this, the log-differences are given in Table A1, Panel B in the appendix.

[26] The solutions for μ are much more robust with respect to alternative assumptions about σ than those for P in Table 1, since both μ and σ enter the exponent in (11).

[27] See Barro (1979) for a discussion of the tax-smoothing objective. In this model, intertemporal shifts of taxes have of course no real implications. An extension to models where such shifts have distributional or allocational effects will be an interesting issue for future research. But the model provides hints about the welfare implications, if some agents cared about the intertemporal allocation of taxes. If the spending-income ratio is variable, the tax-smoothing argument suggests that taxes below permanent spending are difficult to justify.

[28] The testability problem raised here is related to, but distinct from, non-stationarity issues that arise even under certainty. The transversality condition (6) restricts the asymptotic behavior of debt. It is consistent with any set of observations over a finite number of periods. To draw inferences about the asymptotic behavior of fiscal policy from a finite

sample, one has to impose some kind of stationarity assumption. This seems particularly justified in this context, because the concept of sustainability would not be meaningful if one allowed policies that depended explicitly on time, e.g., policies featuring deficits up to some exogenous point in time and surpluses thereafter. Thus, it makes sense to restrict attention to sustainable policies that are specified in a way that does not depend on the time period. Neither this nor the previous example relies on time-dependence in this sense. However, almost all conceivable tax policies in a stochastic setting will have to depend on the state of nature in some way. Under policy T2, sample paths of tax rates are either constant or show a switch to higher tax rates at some future time that depends on the random path of d_t. State-dependence seems to mimic time-dependence. In other words, the point of this example is that such state-dependence is difficult to rule out.

[29] A more complete analysis of different debt management policies and their implications for risk taking is left for future research.

References

Abel, Andrew, Gregory Mankiw, Larry Summers, and Richard Zeckhauser, 1989, "Assessing Dynamic Efficiency: Theory and Evidence," Review of Economic Studies 56, 1-20.

Barro, Robert J., 1974, "Are Government Bonds Net Wealth?" Journal of Political Economy 82, 1095-1117.

__________, 1979, "On the Determination of Public Debt," Journal of Political Economy 87, 940-971.

__________, 1986, "U.S. Deficits since World War I," Scandinavian Journal of Economics 88, 195-222.

Blume, Marshall, and Jeremy Siegel, 1990, "Long-Term Financial and Economic Returns," Working Paper, The Wharton School of the University of Pennsylvania.

Blanchard, Olivier, and Stanley Fischer, 1989, Lectures on Macroeconomics, MIT Press: Cambridge.

Bohn, Henning, 1988, "Why Do We Have Nominal Government Debt?" Journal of Monetary Economics 21, 127-140.

__________, 1989, "Budget Balance through Revenue or Spending Adjustments? Some Historical Evidence for the United States," Rodney White Center for Financial Research (28-89), The Wharton School of the University of Pennsylvania.

__________, 1990, "Tax Smoothing with Financial Instruments," American Economic Review, forthcoming.

Cass, David, 1972, "On Capital Overaccumulation in the Aggregative Neoclassical Model of Economic Growth," Journal of Economic Theory 4, 200-223.

Diamond, Peter, 1965, "National Debt in a Neoclassical Growth Model," American Economic Review 55, 1126-1150.

Hakkio, Craig, and Mark Rush, 1986, Co-Integration and the Government's Budget Deficit, Working Paper, Federal Reserve Bank of Kansas City.

Hamilton, James, and Majorie Flavin, 1986, On the Limitations of Government Borrowing: A Framework for Empirical Testing, American Economic Review 76, 808-819.

Karlin, Samuel, and Howard Taylor, 1975, A First Course in Stochastic Processes, Academic Press: New York.

Kremers, Jeroen, 1989, U.S. Federal Indebtedness and the Conduct of Fiscal Policy, Journal of Monetary Economics 23, 219-238.

Lucas, Robert, 1978, "Asset Prices in an Exchange Economy," Econometrica 46, 1429-1445.

McCallum, Bennett, 1984, "Are Bond-financed Deficits Inflationary? A Ricardian Analysis," Journal of Political Economy 92, 123-135.

O'Connell, and Steve Zeldes, 1988, "Rational Ponzi Games," International Economic Review 29, 431-450.

Trehan, Bharat, and Carl Walsh, 1988, "Common Trends, The Government Budget Constraint, and Revenue Smoothing," Journal of Economic Dynamics and Control 12, 425-444.

Wilcox, David, 1989, "The Sustainability of Government Deficits: Implications of the Present-Value Borrowing Constraint," Journal of Money, Credit and Banking 21, 291-306.

Appendix: Proofs and Data Description

A.1. The Transversality Constraints

Transversality constraints under certainty have been analyzed before in detail, e.g., in Cass (1972) and O'Connell and Zeldes (1988). This section will extend the analysis to the stochastic case. The main complication compared to the certainty case will be to make sure that Ponzi-games are ruled out conditional on every contingency in the future, not only when the expectation is taken currently.

Analogous to the certainty case, a Ponzi game in a stochastic scenario is defined as a situation where an individual i [or government] can borrow without having to repay later. If $Z(s_n)$ is the cash inflow to an economic agent from borrowing and lending in period n, state s_n, then the present value of assets at the start of period N is

$$\sum_{s_N|s_t} P_N(s_N)/P_t \cdot A(s_N) = A_t - \sum_{n=t}^{N-1} \Big[\sum_{s_n|s_t} P(s_n)/P_t \cdot Z(s_n) \Big],$$

where $s_N|s_t$ denotes the sum over all states s_N, $N \geq t$, that include the particular history s_t. Though limits for $N \rightarrow \infty$ do not have to exist, the agent clearly gets something for nothing, if the lim-inf of the present value on the right exceeds current assets. Therefore, I refer to a Ponzi game as a situation when there is a pair of numbers (N,δ) so that

$$\text{(A1)} \qquad \sum_{s_n|s_t} P(s_n) \cdot A(s_n) \leq -\delta$$

for all $n \geq N$, where $N \geq t$ and $\delta > 0$. (This is just a label; the proofs below are valid regardless of whether one finds the definition appropriate.) See O'Connell and Zeldes (1988; p.434) for the analogous definition under certainty. For individual consumers, this definition applies with $A=A^i$ and $Z=C+T-Y$; for the government it applies with $A=-D$, $Z=G-T$.

Since the government is complicating the analysis, it is instructive to discuss the NPG-conditions first for a stochastic economy without government. For Lemmas 1-3 below, assume $T_t=G_t=D_t=0$ for all times and states. The following results hold.

Lemma A1: Without government, the limit

$$\lim_{N\to\infty} \sum_{s_N|s_t} P(s_N)\cdot A^i(s_N) = a^i(s_t)$$

exists for any individual in any state s_t, where $N\geq t\geq 0$, and where $a^i(s_t)=-\infty$ is permitted. Moreover, the limits are related by $a^i(s_t) = \sum_{s_{t+n}|s_t} a^i(s_{t+n})$ for all $n>0$.

Proof: Define wealth $W^i(s_t) = A^i(s_t) + V(s_t)$. Then the budget equation can be written as

$$W^i_t = C^i_t + \sum_{\tilde{s}_{t+1}} p_t(\tilde{s}_{t+1})\cdot W^i_{t+1}(s_{t+1}).$$

Looking n periods ahead, one obtains

$$\sum_{s_{t+n+1}|s_t} P(s_{t+n+1})\cdot W^i_{t+n+1}(s_{t+n+1})$$

$$= \sum_{s_{t+n}|s_t} P(s_{t+n})\cdot W^i_{t+n}(s_{t+n}) - \sum_{s_{t+n}|s_t} P(s_{t+n})\cdot C^i_{t+n}(s_{t+n})$$

Since consumption is non-negative, the sequence defined by $w_n(s_t) = \sum_{s_{t+n}|s_t} P_t(s_{t+n})\cdot W^i_{t+n}(s_{t+n})$ is monotonically declining. Because of monotonicity, it has a well defined limit, denoted by $w(s_t)$ (possibly minus infinity). Since the value of the fruit tree at time t is finite, $a^i(s_t)=w(s_t)$. The relation between the limits follows from $w_N(s_t)= \sum_{s_N|s_t} P(s_N)\cdot W^i(s_N) = \sum_{s_n|s_t}\sum_{s_N|s_n} P(s_N)\cdot W^i(s_N)= \sum_{s_n|s_t} w_N(s_{t+n})$. ||

Lemma A2: Suppose individuals act as if they cannot run Ponzi games now or in the future. That is, they optimize subject to $a^i(s_t)\geq 0$ for all s_t. Then individual optimization in an economy without government implies that $a^i(s_t)=0$ for all s_t.

Proof: Assume for contradiction that $a(s_T)>0$ for some s_T under the optimal consumption plan $\{C^*(s_t)\}$. Then consider the alternative consumption plan $\{C^{**}(s_t)\}$ defined by

$C^{**}(s_t)=C^*(s_t)$ for all states $s_t\neq s_T$,

$C^{**}(s_T)=C^*(s_T)+a^i(s_T)/P(s_T)$ for state s_T, and

$A^{i**}(s_{T+n}) = A^{i*}(s_{T+n}) - a^i(s_{T+n})/P(s_{T+n})$

for all states s_{T+n} that include the specific history s_T. Plan $\{C^{**}(s_t)\}$ increases utility and it is constructed in a way (using Lemma A1) that satisfies constraints $a^i(s_t) \geq 0$ as well as the period by period constraints (2), thereby contradicting optimality of $\{C^*(s_t)\}$. ||

For t=0, $a^i(s_0) \geq 0$ is the standard NPG-constraint. The condition $a^i(s_t) \geq 0$ for t>0 means that individuals believe that they will not be able to start a Ponzi game conditionally on some event in the future. Assumptions $a^i(s_t) \geq 0$ may or may not be justified in equilibrium (see below), but if it were dropped without replacement, unbounded consumption and an infinitely large debt would be the "optimal" strategy.

Lemma A3: Ponzi games do not exist in the economy without government. That is, individual plans satisfy $a^i(s_t)=0$ for all individuals i and all states s_t.

Proof: Market equilibrium implies $\sum_i A^i(s_t)=0$. Assume for contradiction that $\sum_{s_n|s_t} P(s_n)\cdot A^k(s_n) \leq -\delta<0$ for some individual k, some s_t, and all $n \geq N$. Then $0 < \delta \leq \sum_{i \neq k} \sum_{s_n|s_t} P(s_n)\cdot A^i(s_n)$ for all $n \geq N$, which implies that $a^j(s_t)>0$ for at least one other individual j. But that is impossible by Lemma A2. ||

Lemma A3 justifies the NPG-condition for each individual's optimization problem. Equation (7) in Proposition 1 holds in an economy without government. Overall, Lemmas A1-A3 are a straightforward generalization of the certainty case (see O'Connell and Zeldes (1988), Cass (1972)).

The existence of a government complicates the analysis in two ways. First, taxation complicates the individual optimization problems in a way that some limits may not exist. With arbitrary taxation, Lemma A1 cannot be maintained. The problem is that by setting $T(s_t)<<0$ in some states and $T(s_t)>>0$ in others, the government can add and subtract arbitrarily large asset balances to private portfolios in a way that the limits $a^i(s_t)$ may fail to exist. (On the other hand, if $T(s_t) \geq 0$ for all s_t, Lemma A1 would apply with "C" replaced by "C+T" in the proof). As result, one has to work in general with lim-sup and lim-inf conditions.

Second, the government may try to run Ponzi games or permit others to run them. Since government policy is arbitrary and not necessarily optimizing, two cases must be distinguished. If the government decides not to be on the lending side of a Ponzi game, private and public Ponzi games can be ruled out. On the other hand, it is possible that the government might permit private Ponzi games by volunteering to be on the lending side. The general results are as follows.

Lemma A4: Suppose individuals optimize subject to

(A2) $$\limsup_{N\to\infty} \sum_{s_N|s_t} P(s_N)\cdot A^i(s_N) \geq 0$$

for all s_t. Then individual optimization implies that

(A3) $$a^*(s_t) = \liminf_{N\to\infty} \sum_{s_N|s_t} P(s_N)\cdot A^i(s_N) \leq 0$$

for all s_t.

Proof: Note that limits for different states are linked by $a^*(s_t) \leq \sum_{s_{t+n}|s_t} a^*(s_{t+n})$, since $\sum_{s_N|s_t} P(s_N)\cdot A(s_N) = \sum_{s_n|s_t} \sum_{s_N|s_n} P(s_N)\cdot A(s_N)$ for $N\geq n\geq t$ (cf. Lemma A1). Assume for contradiction that $a^*(s_t)>0$ for the optimal consumption plan $\{C^*(s_t)\}$. Then consider the alternative consumption plan $\{C^{**}(s_t)\}$ defined by

$$C^{**}(s_n) = C^*(s_n) \text{ for all states } s_n\neq s_t,$$

$$C^{**}(s_t) = C^*(s_t)+a^*(s_t)/P(s_t) \text{ for state } s_t, \text{ and}$$

$$A^{**}(s_{t+n}) = A^*(s_{t+n}) - a^*(s_t)/P(s_{t+n})\cdot a^*(s_{t+n})\cdot\{\sum_{s_{t+n}|s_t} a^*(s_{t+n})\}^{-1}$$

for all states s_n that include the specific history s_t. Plan $\{C^{**}(s_t)\}$ increases utility and satisfies all constraints (details are as in Lemma A2), thereby contradicting optimality of $\{C^*(s_t)\}$. ||

Lemma A4 generalizes Lemma A2 to the economy with government. The following lemma generalizes Lemma A3.

Lemma A5: The government cannot run a Ponzi game. That is, government policy must satisfy

(A4) $$\limsup_{N\to\infty} \sum_{s_N|s_t} P(s_N)\cdot[-D(s_N)] \geq 0$$

for all states s_t.

Proof: Since individuals satisfy Lemma A4, there will be no lenders to support a borrowing strategy that violated (A4). ||

This establishes the key transversality constraint on the government as inequality. As noted before, one cannot exclude the possibility that the government allows others to run Ponzi games.

Lemma A6: If government policy is such that

(A5) $$\liminf_{N\to\infty} \sum_{s_N|s_t} P(s_N)\cdot[-D(s_N)] \leq 0$$

for all states, then no individual can run a Ponzi game. On the other hand, if

$$\liminf_{N\to\infty} \sum_{s_N|s_t} P(s_N)\cdot[-D(s_N)] > 0$$

for some state s_t, Ponzi games cannot be ruled out.

Proof: The first part is analogous to Lemma A3, the second follows from the fact that the government would be willing to be on the lending side of a Ponzi game. ||

Proposition 1 collects the special cases in the lemmas.

Proof of Proposition 1: Since existence of the limits and inequality (A5) are assumed, equation (6) follows from Lemma A5. Given (A5), Lemma A6 implies that (A2) holds. Again assuming existence of the limit, Lemma A4 then proves (7). Finally, (3) and (6) are clearly equivalent to (8). ||

Sufficiency of the government budget constraint (8) (or equivalently, (3) and (6)) was claimed in the text and it can be shown as follows.

Lemma A7: Consider any policy $\{T(s_t),D(s_t)\}$, where initial debt D_0 is taken as given. If the policy satisfies (8) for all s_t at given prices

$$P(s_t) = \pi(s_t)\cdot b^t \cdot \frac{U'(Y(s_t)-G(s_t))}{U'(Y(s_0)-G(s_0))}$$

and if all individuals i start with given identical initial assets $A^i_0=D_0$, then the $\{D(s_t)\}$ can be supported as part of an equilibrium allocation.

Proof: Since (8) holds, $\sum_{n\geq t} E_t[P_n \cdot T_n]$ is well defined for all states, which implies that the limit of $\sum_{s_N|s_t} P(s_N)\cdot A^i(s_N)$ exists and is zero (because of Lemma A6) for all individuals and states. Hence, individual assets are

$$A^i(s_t) = \sum_{n\geq t} E_t[P_n \cdot c^i_n] + \sum_{n\geq t} E_t[P_n \cdot T_n] - \sum_{n\geq t} E_t[P_n \cdot Y_n].$$

Combined with (8), this is equivalent to

$$\text{(A6)} \qquad A^i(s_t) = D(s_t) + \sum_{n\geq t} E_t[P_n \cdot c^i_n] + \sum_{n\geq t} E_t[P_n \cdot G_n] - \sum_{n\geq t} E_t[P_n \cdot Y_n]$$

Moreover, individual first order conditions (4) and (5) at prices $P(s_t)$ imply that

$$c^i(s_0)=Y_0-G_0 \iff c^i(s_t)=Y(s_t)-G(s_t), \text{ for all } s_t$$

hence

$$\sum_{n\geq 0} E_0[P_n \cdot c^i_n] = \sum_{n\geq 0} E_0[P_n \cdot (Y_n-G_n)]$$

$$\iff \sum_{n\geq t} E_t[P_n \cdot c^i_n] = \sum_{n\geq t} E_t[P_n \cdot (Y_n-G_n)] \text{ for any given } s_t.$$

Combined with (A6) and the assumption $A^i_0=D_0$, this equivalence implies $A^i(s_t)=D(s_t)$ for all s_t. In other words, individuals are willing to hold the government debt, policy $\{D(s_t)\}$ is supported as an equilibrium. ||

As a corollary, if one wants to focus on debt policy, the Lemma shows that for debt policy $\{D(s_t)\}$ that satisfies (6) there is a tax policy that supports $\{D(s_t)\}$ as part of an equilibrium allocation, namely the tax policy implied by (3). Thus, any debt policy that satisfies (6) is feasible.

A.2. Proof of Proposition 3

For the log-normal case, the theory of Brownian motion can be applied. Let d^*_t be a Brownian motion on the real line $t \geq 0$ with drift $-\mu$ and variance σ^2 starting at $d^*_0 = d_0$. As long as the path of d^* has not reached the bound d, d^*_t has the same distribution as d_t for all integer numbers $t \geq 0$. The result

$$\text{Prob}(s: d^*_t(s) \leq d \text{ for all } t \geq 0) = 1-(d_0/d)^{\lambda}$$

with $\lambda = -2 \cdot \mu/\sigma^2$ is standard, see e.g. Karlin/Taylor (1975; ch.7). Since the inequality $d^*_t(s) > d$ may be realized at a non-integer value of t, (11) has an inequality.

The log-normal case yields an explicit number for λ, but the assumption of log-normality may not always be appropriate. To show that the qualitative result does not rely on log-normality, part (b) has been formulated more generally, using martingale theory. The idea is derived from the fact that λ satisfies $E[\exp(-\lambda \cdot \tilde{y}_t)]=1$ in the log-normal case, where $\tilde{y}_t = \log(1+y_t)-\log(1+r)-\log(1+\varepsilon)$. Without normality, the equation $E[\exp(-\lambda \cdot \tilde{y}_t)]=1$ still has a solution $\lambda>0$, because (1) $\lambda=0$ solves this equation, (2) $dE[\exp(-\lambda \cdot \tilde{y}_t)]/d\lambda = -E[\tilde{y}_t]<0$ at $\lambda=0$, and (3) $\lim_{\lambda \to \infty} E[\exp(-\lambda \cdot \tilde{y}_t)] = +\infty$. Note that (3) uses the assumption that $\text{Prob}\{s_t: (1+y_t(s_t))<(1+r)\cdot(1+\varepsilon)\} = \text{Prob}\{s_t: \tilde{y}_t<0\} > 0$.

For any pair of positive numbers (a,b), the i.i.d. assumption together with $\text{var}(\tilde{y}_t)>0$ guarantees that $\text{Prob}\{d^*_T{}^{\lambda} \geq b \text{ or } d^*_T{}^{\lambda} \leq a \text{ for some } T<\infty\}=1$. Therefore, the optional sampling theorem (Karlin/Taylor (1975), ch.6) applies to the time T where the process $d^*_t{}^{\lambda}$ first reaches a or b. That is,

$$d_0^{\lambda} = \text{Prob}\{d^*_T \leq a\} \cdot E[d^*_T{}^{\lambda} | d^*_T \leq a, a<d^*_t<b \text{ for all } t<T\}$$
$$+ \text{Prob}\{d^*_T \geq b\} \cdot E[d^*_T{}^{\lambda} | d^*_T \geq b, a<d^*_t<b \text{ for all } t<T\}.$$

Taking $a \to 0$ and noting that $E[d^*_T{}^{\lambda} | d^*_T \geq b, a<d^*_t<b \text{ for all } t<T\} \geq b^{\lambda}$, one obtains $\text{Prob}\{d^*_T \geq b\} \leq d_0^{\lambda}/b^{\lambda}$. Thus, (11) holds for b=d, hence $\lambda>0$ implies $P>0$.

A.3. Data Description

Data on the federal budget for fiscal years 1800-1988 were compiled in Bohn (1989) from the Historical Statistics of the United States, U.S. budget documents, and miscellaneous other sources. In Table 1, the primary deficit is the difference of federal receipts and non-interest outlays. Since historical GNP data are annual series but fiscal data are collected by fiscal year, deficits were deflated by GNP of the calendar year in which the fiscal year started.

In Table 2, data for 1929-1988 and its subsamples are from different sources than many of the data going further back. For post-1929 data, GNP is from the Commerce Department (WEFA database), T-bill returns (1 month), T-bond returns, and inflation (CPI) are from Ibbotsen (CRSP file). Returns labeled HF and W are the average real rate and average return of government debt as measured by Hamilton-Flavin (1986) and Wilcox (1989), respectively.

The long series for GNP and inflation (GNP-deflator) are explained in Bohn (1989). For all samples, "Int. charge" refers to interest payments on the federal debt divided by debt at the start of the fiscal year. T-bond yields for all samples and the T-bond returns for the long samples are from Blume and Siegel (1990).

Table 2B shows the differences $E[y_{t+1}-R_t]$ in levels, but ratios of discount factors, $1-E[(1+R_t)/(1+y_{t+1})]$, and logarithms, $E[\log(1+y_{t+1})-\log(1+R_t)]$ are relevant in Sections 4.1 and 4.2, respectively. The distinctions are quantitatively minor and therefore ignored in the text to save space. For completeness, data on $1- E[(1+R_t)/(1+y_{t+1})]$ and $E[\log(1+y_{t+1})-\log(1+R_t)]$ are given in Table A1, Panels A and B, respectively.

Table 1: Federal Deficits

Sample	Deficit/ GNP	Number of Deficits
1929-1988	1.77	32/60
1954-1988	0.28	18/35
1980-1988	1.47	9/9
1960-1984	0.39	13/25
1800-1988	0.40	64/189
1800-1899	-0.43	25/100
1900-1988	1.34	30/89

Legend:

Sample periods refer to fiscal years.

Deficit/GNP = Primary federal deficit divided by GNP of the calendar year in which the fiscal year started.

Sources are described in the appendix.

Table 2: A Comparison of Growth and Interest Rates

Panel A: Basic Data

	Growth GNP	T-Bill return	T-Bond return	T-Bond yield	Int. Charge	HF return	W return	Memo: Inflation
1929-88	2.97	0.23	0.96	1.53	0.45			3.33
1954-88	3.10	1.11	0.53	2.01	0.89			4.46
1980-88	2.87	3.62	6.53	5.57	3.73			4.62
1960-84	3.14	0.95	-0.51	1.51	0.44	1.12	2.11	5.36
1800-1899	4.28		6.08	5.10	5.62			-0.44
1900-1988	3.14		-0.18	1.25	0.43			3.29
1800-1988	3.74		3.08	3.27	3.18			1.30

Panel B: Differences between growth and interest rates or returns using . . .

	T-Bill return	T-Bond return	T-Bond yield	Int. Charge	HF return	W return
1929-88	2.74	2.02	1.44	2.52		
1954-88	1.99	2.57	1.08	2.21		
1980-88	-0.75	-3.66	-2.70	-0.86		
1960-84	2.19	3.66	1.64	2.70	2.02	1.04
1800-1899		-1.80	-0.82	-1.34		
1900-1988		3.32	1.89	2.71		
1800-1988		0.66	0.47	0.56		

Legend: The data are described in the appendix.

Table 3: The Probability of Observing a Surplus (1 - P)

Panel A: Growth exceeds interest rate by 1% ($\mu = 0.01$)

St. dev. σ	---	Ratios d/d_0	---
	2	3	4
2%	0.000%	0.000%	0.000%
4%	0.017%	0.000%	0.000%
6%	2.126%	0.224%	0.045%
8%	11.463%	3.228%	1.314%
10%	25.000%	11.111%	6.250%

Panel B: Growth exceeds interest rate by 2% ($\mu = 0.02$)

St. dev. σ	---	Ratios d/d_0	---
	2	3	4
2%	0.000%	0.000%	0.000%
4%	0.000%	0.000%	0.000%
6%	0.045%	0.000%	0.000%
8%	1.314%	0.104%	0.017%
10%	6.250%	1.235%	0.301%

Legend: Table entries are values of 1 - P that satisfy equation (11) for given values of μ, σ, and d/d_0.

Table 4: Implies Values for the Parameter μ

Panel A: Policies with 95% probability of no surpluses

St. dev. σ	---	Ratios d/d_0	---
	2	3	4
2%	0.09%	0.05%	0.04%
4%	0.35%	0.22%	0.17%
6%	0.78%	0.49%	0.39%
8%	1.38%	0.87%	0.69%
10%	2.16%	1.36%	1.08%

Panel B Policies with 90% probability of no surpluses

Std. dev σ	---	Ratios d/d_0	---
	2	3	4
2%	0.07%	0.04%	0.03%
4%	0.27%	0.17%	0.13%
6%	0.60%	0.38%	0.30%
8%	1.06%	0.67%	0.53%
10%	1.66%	1.05%	0.83%

Panel C: Policies with 50% probability of no surpluses

St. dev. σ	---	Ratios d/d_0	---
	2	3	4
2%	0.02%	0.01%	0.01%
4%	0.08%	0.05%	0.04%
6%	0.18%	0.11%	0.09%
8%	0.32%	0.20%	0.16%
10%	0.50%	0.32%	0.25%

Legend: Table entries are values μ that satisfy equation (11) for given values of P, σ, and d/d_0.

Table A1: More Data on Growth and Interest Rates

Panel A: Growth-return differences, computed as 1 - (1 + return)/ (1 + growth), using . . .

	T-Bill return	T-Bond return	T-Bond yield	Int. Charge	HF return	W return
1929-88	2.66	1.96	1.40	2.45		
1954-88	1.93	2.49	1.05	2.14		
1980-88	-0.73	-3.56	-2.63	-0.84		
1960-84	2.12	3.55	1.59	2.62	1.96	1.00
1800-1899		-1.72	-0.79	-1.29		
1900-1988		3.22	1.83	2.63		
1800-1988		0.64	0.45	0.54		

Panel B: Growth-return differences in logarithmics using . . .

	T-Bill return	T-Bond return	T-Bond yield	Int. Charge	HF return	W return
1929-88	2.70	1.98	1.41	2.48		
1954-88	1.95	2.52	1.06	2.16		
1980-88	-0.73	-3.50	-2.60	-0.83		
1960-84	2.15	3.61	1.60	2.66	1.98	1.01
1800-1899		-1.71	-0.78	-1.28		
1900-1988		3.27	1.85	2.66		
1800-1988		0.64	0.45	0.54		

Legend: The data are described in the appendix.