FINANCING LOSERS IN COMPETITIVE MARKETS

by

Andrew B. Abel
George J. Mailath

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RODNEY L. WHITE CENTER FOR FINANCIAL RESEARCH
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6367

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Abstract

Projects with negative expected value cannot obtain financing in competitive capital markets if all potential investors are risk neutral and have identical beliefs about the distribution of the project's net revenue. We present a series of examples with heterogeneous beliefs in which it is possible for a project to obtain financing even though all investors in the project believe, conditional on the project being undertaken, that the project has a negative expected value. An important feature of the examples is that the differences in beliefs are due only to differences in information, and are not simply arbitrary unexplained differences in opinions.

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[2]Both authors would like to thank the National Science Foundation for financial support. Abel is in the Finance Department, The Wharton School and Mailath is in the Economics Department, University of Pennsylvania, Philadelphia, PA 19104.

I. Introduction

Projects are financed in competitive capital markets by selling securities to investors to raise capital. Because securities are claims on a project's net revenue, risk neutral investors will not buy the securities of projects that have negative expected net revenue. This result--that projects with negative expected value cannot obtain financing--holds only if all potential investors have identical subjective distributions for the project's net revenues. If investors have different subjective distributions for net revenues, then it is possible for a project to obtain financing even though all investors in the project believe, conditional on the project being undertaken, that the project has a negative expected value.

In this paper we develop an information-theoretic model of project financing and present examples in which all investors believe that a project has a negative expected value if undertaken, and yet the project is financed by these investors. While it is crucial that investors have heterogeneous beliefs, it is noteworthy that in this framework the differences in beliefs are due only to differences in information and are not simply arbitrary unexplained differences in opinions. Investors receive different pieces of information and are sophisticated in their use of this information to obtain Bayesian posteriors. If the investors could credibly pool their information, they would agree on the distribution of the project's net revenues. However, in our model there is no mechanism by which investors can pool their information, either directly or indirectly through information-revealing behavior. Thus, the differences in beliefs persist in the security market equilibrium, which makes it possible for projects with negative expected value to obtain financing.

The model we develop is best viewed as a model of initial public offerings. An entrepreneur has access to a project with uncertain net revenues and must obtain external financing. The entrepreneur offers a group of securities for sale. The prices of these securities, as well as their payoff characteristics, are set by the entrepreneur. Investors, who are each so small that the capital market is competitive, decide which, if any, securities to subscribe to. If investors believe that a security is priced too high, they will not subscribe to the security, and the project does not proceed. If investors think that the security is priced correctly, or even underpriced, then they will subscribe to the security. Because the price of the security is set before investors act in the capital market, the price of the securities cannot reveal information possessed by investors.

Why will investors subscribe to securities of a project that they think has a negative expected value? The answer is that although securities are claims on a project's net revenues, the value of subscribing to a security can differ from the expected value of the project's net revenue. We explore two sources of a wedge between the expected value of a project and the expected value of subscribing to its securities. First, if a security is oversubscribed, the available securities are allocated to subscribers according to a random rationing scheme. Because the probability of acquiring a security after subscribing to it differs in different contingencies, there is a wedge between the expected value of subscribing to a security and the expected value of the project's net revenues, even if there is only one type of security. We present an example with one type of security in which investors not only believe that the project has negative value, but also know that all the other investors believe that the project has a negative value. The rationing in this example (and in some of our other examples) is perverse, in that securities with negative value are rationed while securities with positive value are not.

A second wedge arises if a project is financed by two or more different types of securities. Although the bundle of all securities is a claim on the project's net revenues, different types of securities are different state-contingent claims a project's net revenues. With heterogeneous beliefs, different investors will subscribe to different securities and thus finance the project even though no single investor would want to hold a bundle of all of the project's securities. In the example illustrating this wedge, only securities with positive value are rationed.

II. An Information Theoretic Model

A. The Project and the Securities

Consider a risk-neutral entrepreneur who wants to raise funds to finance a project with a publicly known cost C and unknown (present value of) revenue $x \geq 0$. We denote by $\Phi \equiv \{F_s : s=1,\ldots,S\}$ the finite set of possible distribution functions of the project's revenue x. Let $\rho(s)$ be the probability that the project's revenue distribution is F_s, so that ρ is the prior distribution on Φ.

The entrepreneur has no funds of his own to invest and therefore must raise funds by selling securities.[3] The entrepreneur offers N types of securities for sale. Security 1 is a residual claim on the project's net revenues and thus is equity. The entrepreneur retains a fraction σ of the firm's equity, and the remaining fraction $(1-\sigma)$ is sold to external investors. The entrepreneur does not hold any of the other securities; they are sold entirely to external investors.

[3]The entrepreneur offers securities through a (risk-neutral) investment bank. Rather than study the relationship between the entrepreneur and his investment bank, we will treat the entrepreneur and the investment bank as a single unit with access to the same information.

Each security is characterized by a price and a payoff function. A unit (Lebesgue) measure of each security is offered for sale to external investors. The total amount of capital raised by selling security n to external investors is p_n. Since the entrepreneur retains a fraction σ of the project's equity, p_1 is the amount paid by external investors for $(1-\sigma)$ of the project's equity (security 1). Since all of the remaining securities are sold entirely to external investors, p_n is the total amount paid by external investors for all of security n, $n = 2,\ldots,N$.

Each security specifies the aggregate payment to the owners of the security as a function of revenue. In particular, $y_n(x)$, $n = 1,\ldots,N$, is the amount received by the owners of security n when revenue is x. Because external investors own only a fraction $(1-\sigma)$ of the first security, they pay p_1 for this security and receive a payoff $(1-\sigma)y_1(x)$. For the remaining securities, external investors pay p_n and receive $y_n(x)$.

In order for the project to be undertaken, the funds raised by the sale of securities, $\sum_n p_n$, must be sufficient to cover the known cost of the project, C. Letting ϕ be the amount of excess capital raised by the sale of securities, the project will be undertaken only if

(1) $$\phi \equiv \sum_n p_n - C \geq 0.$$

If excess capital is raised in the public offering, then the entrepreneur can take π (as a supernormal salary, for instance) subject to the restriction that

(2) $$0 \leq \pi \leq \phi.$$

After the project's revenue, x, is realized, the project has funds equal to $x + \sum_n p_n - C - \pi$. These funds are distributed to the owners of the securities according to the payoff functions y_n. Because all of the available funds are distributed we have

(3) $$\sum_n y_n(x) = x + \sum_n p_n - C - \pi \geq 0.$$

The amount of funds available on the right hand side of (3) is nonnegative because $x \geq 0$ and because (1) and (2) imply that $\sum_n p_n - C - \pi \geq 0$. Since the amount of funds available is nonnegative, it is feasible to require that the payoff functions satisfy $y_n(x) \geq 0$, for all $x \geq 0$, $n = 1,\ldots,N$. This restriction corresponds to limited liability: no security owner can be forced to pay an additional amount after purchasing the security.

B. The valuation of securities

Because all investors are risk-neutral, the value of any security is the expected value of its payoff minus the amount paid for the security. If the distribution of revenue is known to be F_s, then the expected payoff to external owners of security 1 is $(1-\sigma)\int y_1(x)dF_s(x)$. Since external investors pay p_1 in aggregate for security 1 (equity), their valuation of this security is

(4) $$v_1(s) = (1-\sigma)\int y_1(x)dF_s(x) - p_1.$$

External investors receive all of the payoffs to securities $2,\ldots,N$, because the entrepreneur does not retain any fraction of these securities. The value of security n to external investors is

(5) $$v_n(s) = \int y_n(x)dF_s(x) - p_n, \quad n = 2,\ldots,N.$$

The entrepreneur's expected net benefits come from two sources. First, the entrepreneur retains a fraction σ of security 1; second, the entrepreneur may receive a supernormal salary π if excess capital is raised. Thus, if the entrepreneur knows the distribution is F_s, the expected value of the entrepreneur's net benefits, $v_E(s)$, is

(6) $$v_E(s) = \sigma\int y_1(x)dF_s(x) + \pi.$$

Equations (4)-(6) give the values of all of the claims on the project. We can relate the values of these claims to the expected net revenue of the project, which is $\int(x-C)dF_s(x)$ if the distribution is F_s. First, observe from

(3) that $x - C = \pi + \sum_n[y_n(x)-p_n]$, for all x. Integrating both sides with respect to the distribution function F_s, we obtain $\int(x-C)dF_s(x) = \pi + \sum_n\int[y_n(x)-p_n]dF_s(x)$. Using equations (4)-(6) to simplify, we obtain

(7) $$\int(x-C)dF_s(x) = v_E(s) + \sum_n v_n(s).$$

Equation (7) states that the expected value of the project equals the total valuation of all of the securities held by external investors plus the value of the claims held by the entrepreneur.

C. Complete and incomplete subscription

When the entrepreneur offers securities for sale, each investor knows the prices and payoff functions of the securities and, based on his/her subjective distribution for x, decides which of the securities to subscribe to. An investor may subscribe to no securities, one security, or many types of securities. Incomplete subscription occurs when there is at least one security for which there are insufficient subscribers. In case of incomplete subscription, funds are not collected from investors and the project does not proceed. Complete subscription occurs when all securities have sufficient subscribers. If a security is oversubscribed, it must be rationed. We use a random rationing scheme in which each subscriber to a security receives one unit of that security with a probability equal to the reciprocal of the measure of investors subscribing to that security. That is, if the measure of subscribers to a security is 2, then each subscriber has a 50% chance of getting the security, because the measure of that security offered for sale is 1.[4]

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[4]An equivalent nonrandom rationing rule in our setting is that each subscriber receives an amount equal to the reciprocal of the measure of subscribers to that security. Since investors are risk-neutral, receiving one unit of a security with some probability is equivalent to receiving that proportion of one unit with certainty. Note that if investors were allowed to revise their subscription after observing how much of each security they were able to buy, the two rationing schemes are not equivalent. The investor would have more information under the alternative nonrandom scheme than under the random rationing rule we use in the text.

D. Information structure

Assume that the investors can be assigned to M classes of equal size. Specifically, a typical investor class, denoted by i, is a continuum of unit (Lebesgue) measure of investors. All investors in a class are identical and have identical information. To illustrate the role of the information possessed by different classes, as well as to illustrate some of the definitions that follow, we begin with an example. In the example, $M = 2$ so there are two classes of investors denoted as class I and class II. There are $S = 9$ possible distribution functions for x. The information structure is illustrated in Table 1. Investors in class I only learn which row contains the true distribution and investors in class II only learn which column contains the true distribution. Thus, for instance, if the true distribution is F_6 (row 2, column 3), class I investors learn that the distribution is in row 2 and class II investors learn that the distribution is in column 3.

Table 1

class II investors learn column

class I investors learn row			
	F_1	F_2	F_3
	F_4	F_5	F_6
	F_7	F_8	F_9

The information structure of investor class i is described formally as a partition of Φ, written P_i. In Table 1, the partition of investor class I has three elements: $P_1 = \{\{F_1,F_2,F_3\}, \{F_4,F_5,F_6\}, \{F_7,F_8,F_9\}\}$; the partition of investor class II also has three elements: $P_2 = \{\{F_1,F_4,F_7\}, \{F_2,F_5,F_8\}, \{F_3,F_6,F_9\}\}$. Each investor receives a private signal that indicates which element of the investor's partition contains the true probability distribution, F_s. For example, if the true distribution is F_6, investors in class I learn that the (index of the) true distribution is in {4,5,6}, and investors in class II learn that the (index of the) true distribution is in {3,6,9}.[5] Formally, we let $t_i(s)$ be the element of partition P_i that contains the distribution F_s. Using this formalism, we can describe the information structure when the true distribution is F_6 as: $t_1(6) = \{4,5,6\}$ and $t_2(6) = \{3,6,9\}$.

We use the terms ex ante, interim, and ex post to describe different information scenarios (not different times). Ex ante refers to the scenario in which all investors know only the prior distribution $\rho(s)$ over Φ. Interim refers to the scenario in which each investor has received the signal that identifies which element of his/her partition contains the true distribution. Ex post refers to the scenario in which the true distribution, F_s, is known. Investors never see the distribution F_s, although they eventually see the realization of revenues x.

We have already discussed the ex ante scenario in which investors know the prior distribution $\rho(s)$ over Φ, and we have discussed the ex post scenario in which the distribution of revenue is known. The interim scenario requires further elaboration. Suppose that each of the 9 distributions in Table 1 has the same prior probability, so that $\rho(s) = 1/9$ for $s = 1,\ldots,9$. Thus, if the true distribution is F_6, investors in class I know the distribution has an index in {4,5,6}, and update their priors to obtain an interim posterior that assigns a probability of 1/3 to distributions F_4, F_5, and F_6, and zero probability to the other distributions. This interim posterior depends both on the investor class and the true distribution. Formally, when the true distribution is $F_{s'}$, investors in class i know that the distribution is contained in $t_i(s')$ and update their priors to the <u>interim posterior distribution</u> $\rho_i(s|t_i(s'))$, where

[5]We will use F_s and its index s interchangeably.

$$(8) \qquad \rho_i(s|t_i(s')) \equiv \begin{cases} \rho(s)/\sum_{s''\in t_i(s')}\rho(s''), & \text{if } s \in t_i(s'), \\ 0, & \text{if } s \notin t_i(s'). \end{cases}$$

E. Investors decisions and equilibrium

In deciding whether to bid for security n, an investor takes account of the rationing rule which specifies the investor's probability of actually being able to purchase a security for which he/she submits a bid. The probability of obtaining a security depends on how many other investors bid for that security, so the investor must take account of the equilibrium number of bids for security n in deciding whether to submit a bid for that security. To formally describe the role in the investor's decision of the equilibrium number of bids, we introduce three concepts: (1) the security demand correspondence; (2) the market demand for each security; and (3) the set of distribution functions for which there is complete subscription.

1. security demand correspondence

If the true distribution is F_s, investors in class i receive a signal that the (index of the) distribution is in $t_i(s)$. Based on this information, investors in class i choose which, if any, securities to subscribe to. The security demand correspondence, $\xi(s,i)$, describes the set of securities that investors in class i subscribe to when the true (but unknown) distribution is F_s. Formally, $\xi:\Phi\times M \to \mathcal{P}(N)$, where $\mathcal{P}(N)$ is the power set of N.[6] We require that the security demand correspondence respect the information structure in the sense

[6]The power set of N is the collection of all subsets of N. Strictly speaking, since different investors in the same class may choose different securities, it would be more accurate to represent security demands for a class i and a distribution F_s by a vector $(z(N'))_{N'\subseteq N}$ satisfying $z(.) \geq 0$ and $\sum_{N'\subseteq N} z(N') = 1$ (the proportion of investors choosing the set of securities N' is z(N')). The security demand correspondence described in the text assumes that all investors in a class choose the same set of securities. Note that a security can only be subscribed to by some, but not all, investors in a class if that security has a zero value. The more restrictive formulation in the text suffices for our purposes.

that if class i's partition pools F_s with $F_{s'}$ ($s' \in t_i(s)$), then $\xi(s',i) = \xi(s,i)$, i.e., it is measurable with respect to the investors' information partitions.

Table 2
A security demand correspondence for a security offer with one security

investor class I
$\xi(s,I)$

{1}	{1}	{1}
{1}	{1}	{1}
∅	∅	∅

investor class II
$\xi(s,II)$

{1}	{1}	∅
{1}	{1}	∅
{1}	{1}	∅

Table 2 shows a security demand correspondence for a security offer with one security for the example in Table 1. The matrix on the left shows the securities that investors in class I subscribe to and the matrix on the right shows the securities that investors in class II subscribe to. For example, if the true distribution is F_5, which is located in row 2, column 2, then investors in both classes subscribe to security 1, while only investors in class I subscribe to the security if the true distribution is F_3.

2. the market demand function

The market demand function for security n, $Q_n(s,\xi)$, indicates the measure of investors (i.e., the "number" of investors) that subscribe to security n. The measure of investors that subscribe to security n depends on the signal received by each investor class, which depends on the true distribution, and on the security demand correspondence, ξ. In equilibrium, all investors in a class choose the same set of securities and $Q_n(s,\xi)$ will be the number of investor classes that subscribe to security n.

3. complete subscription

An investor's subscription to a security can be satisfied only in the case of complete subscription. Thus, it is important for investors to know when there will be complete subscription and when there will be incomplete subscription. Let $D(\xi)$ be the subset of Φ in which there is complete subscription when investor behavior is described by the security demand correspondence ξ. Formally, $D(\xi) \equiv \{s: \forall n, \exists i \text{ s.t. } n \in \xi(s,i)\}$. We will often suppress the dependence of D on ξ for notational convenience.

F. The value of subscribing to a security

When the true distribution is $F_{s'}$, investors in class i learn that the distribution is in $t_i(s')$, and they know that the project will proceed only if the distribution is in $t_i(s') \cap D$. The <u>D-interim posterior</u> of investors in class i is the posterior distribution over the set of distributions Φ, where the posterior is based on t_i and is conditional on the project proceeding. The D-interim posterior distribution $\rho_i(s|t_i(s'))$ is given by:

$$(9) \qquad \rho_i(s|t_i(s'),D) \equiv \begin{cases} \rho(s)/\sum_{s'' \in t_i(s') \cap D} \rho(s''), & \text{if } s \in t_i(s') \cap D, \\ 0, & \text{if } s \notin t_i(s') \cap D. \end{cases}$$

The D-interim posterior is used by investors in class i to evaluate the expected value of the project's revenues x, conditional on the project being financed. This conditional expected value will figure importantly in our definitions of efficiency.

What is the expected net value to an investor of subscribing to security n, conditional on the information that the distribution is in $t_i(s')$? The probability that the distribution is $F_s(x)$ is given by the interim posterior $\rho_i(s|t_i(s'))$ in (8). If s is not in D, the project is not financed and the subscription has zero net value. However, if s is in D, an investor has a probability of $1/Q_n(s,\xi)$ of obtaining security n. If the investor succeeds in obtaining security n, it has a value of $v_n(s)$. Thus, if the true distribution F_s is in D, the expected value of subscribing to security n is $v_n(s)/Q_n(s,\xi)$. Weighting the expected value of the subscription by the conditional probability that F_s is the true distribution, the expected value of subscribing to security n is

$$(10) \qquad V_n(t_i(s'),\xi) \equiv \sum_{s \in t_i(s') \cap D} \{v_n(s)/Q_n(s,\xi)\}\rho_i(s|t_i(s')).$$

If the expected value of subscribing to security n is positive, then all investors in class i subscribe to security n; if the expected value of subscribing to security n is negative, none of the investors in class i subscribes to security n. Note that the expected value of subscribing to security n depends on the security demand correspondence; investors take account of the market demand for a security in deciding whether to subscribe to the security.

G. Security Market Equilibrium

Having completed the discussion of investor behavior we can now define a competitive security equilibrium.

Definition: *A security offer, which specifies* π, σ, p_n, *and* y_n, $n = 1,\ldots,N$, *and a security demand correspondence* ξ *constitute a* *__(competitive) security equilibrium__* *if*

(a) $\sum_n p_n \geq C$,

(b) $\sum_{s \in D(\xi)} v_E(s)\rho(s) \geq 0$, *and*

(c) *if for all* $F_s \in \Phi$, *for* $n = 1,\ldots,N$,

(i) $V_n(t_i(s),\xi) > 0 \Longrightarrow n \in \xi(s,i)$ *and*

(ii) $V_n(t_i(s),\xi) < 0 \Longrightarrow n \notin \xi(s,i)$.

Condition (a) states that the funds raised by selling securities to external investors are sufficient to cover the cost of the project. Condition (b) states that the entrepreneur has a nonnegative expected value from the project and its associated securities. Condition (c.i) states that if the expected value of subscribing to security n is positive for investors in class i, they will subscribe to security n. Condition (c.ii) states that if the expected value of subscribing to security n is negative for investors in class i, they will not subscribe to security n.

We can illustrate a competitive security equilibrium for the example with 9 possible distributions, one security (equity), and two classes of investors. The information structure is given by Table 1, with each F_s occurring with equal probability. We assume that the entrepreneur retains no equity and does not earn a supernormal salary so that $\sigma = \pi = 0$. Under these assumptions, the expected value of the project equals the expected value of equity (i.e., $\int(x-C)dF_s(x) = v_1(s)$ for all s). The security demand correspondence in Table 2, along with the security valuations in Table 3a, constitute a competitive security equilibrium.

Table 3a
value of security,
$\int(x-C)dF_s(x) = v_1(s)$

8	2	-4
2	-6	3
-4	3	10

Table 3b
market demand function
from Table 2, $Q_1(s,\xi)$

2	2	1
2	2	1
1	1	0

Table 3c
value of subscribing,
$v_1(s)/Q_1(s,\xi)$

4	1	-4
1	-3	3
-4	3	–

Note that if the distribution is F_5 (row 2, column 2), there will be complete subscription and both investors will subscribe to the security, despite the fact that it has a valuation of -6. Why do both investors subscribe to the security in this case? The answer is that the rationing rule indicates that if the distribution is F_5, a subscriber has only a 50% chance of obtaining the security. The rationing is perverse in the sense that the security is rationed when it has a negative value, but not necessarily when it has a positive value. Table 3c lists the value of subscribing to the security, which takes into account the rationing implied by the market demand in Table 3b. Note that the expected value of subscribing to the security is positive (1/3) for each class of investor when the distribution is F_5. There is incomplete subscription in F_9, indicated by the dash, and so the project is not undertaken in that case.

III. Notions of Efficiency

The general question motivating this research is whether a competitive security equilibrium does an effective job of choosing the appropriate projects to finance. Does it allow for projects with positive expected net revenues to obtain financing, while at the same time preventing projects with negative net expected values from obtaining financing?[7] To pose this question more sharply, we need to introduce various concepts of efficiency.

A project is said to be <u>ex post inefficient at F_s</u> if

(11) $$\int (x-C)dF_s(x) < 0.$$

Note that we determine whether a project is ex post inefficient (according to our definition) before the revenue x is realized.

If the distribution function of revenue were common knowledge, then only ex post efficient projects would be financed and undertaken. Because the distribution function is not common knowledge, we need to introduce alternative definitions of inefficiency to incorporate the information structure of the model. We will not be concerned with the inefficiency of the project <u>per se</u>, but rather with the inefficiency of the security equilibrium. A security equilibrium is said to be ex ante inefficient if, conditioning only on the fact

[7]Our definitions of efficiency rely on the risk neutrality of the agents. Projects with negative expected value will have some insurance value if the payoffs to the project are negatively correlated with the market portfolio and if some agents are risk averse.

that the project is undertaken, the expected value of revenue minus cost is negative. Formally, a security equilibrium is <u>ex ante inefficient</u> if

$$\sum_{s\in D}\left\{\int(x-C)dF_s(x)\right\}\left\{\rho(s)/[\textstyle\sum_{s'\in D}\rho(s')]\right\} < 0. \tag{12}$$

Conversely, a security equilibrium is ex ante efficient if, conditioning only on the fact that the project is undertaken, the expected value of revenue minus cost is nonnegative. Formally, a security equilibrium is <u>ex ante efficient</u> if

$$\sum_{s\in D}\left\{\int(x-C)dF_s(x)\right\}\left\{\rho(s)/[\textstyle\sum_{s'\in D}\rho(s')]\right\} \geq 0. \tag{13}$$

We can also evaluate the efficiency of a security equilibrium conditional on information received by investors. A security equilibrium is said to be D-interim inefficient at F_s, according to investors in class i if the conditional expected value of net revenue is negative. In forming the conditional expectation, investors in class i use their information, $t_i(s')$, and condition on the project being undertaken. Formally, a security equilibrium is <u>D-interim inefficient at F_s, according to investors in class i</u> if

$$\sum_{s\in t_i(s')\cap D}\int(x-C)dF_s(x)\rho_i(s|t_i(s'),D) < 0. \tag{14}$$

If investors in class i view the security equilibrium as D-interim inefficient, then based on their available information, they think that when the project is undertaken, it will on average incur losses.

The notions of ex ante inefficiency and ex post inefficiency are unambiguous, in that all classes of investors agree on the questions of whether a security equilibrium is ex ante inefficient and whether it is ex post inefficient. By contrast, the notion of D-interim inefficiency is investor-dependent and signal-dependent: for a given distribution F_s some investors may view the security equilibrium as D-interim inefficient while other investors do not: also a given investor may view a particular security equilibrium as D-interim inefficient or not depending on the signal, $t_i(s)$, he or she receives.

The various definitions of inefficiency involve $\int(x-C)dF_s(x)$. However, rather than calculating this integral directly, we can determine whether a security equilibrium is inefficient by using the valuations $v_E(s)$, and $v_n(s)$, n = 1,...,n. Recall from equation (7) that $\int(x-C)dF_s(x) = v_E(s) + \sum_n v_n(s)$, so that we can replace $\int(x-C)dF_s(x)$ by $v_E(s) + \sum_n v_n(s)$ in the definitions of inefficiency.

Before examining various security equilibria, we state the following theorem, which is proved in the Appendix.

Theorem 1: *Competitive security equilibria are ex ante efficient.*

This theorem formalizes the notion that only projects with ex ante nonnegative expected net revenue will be able to obtain financing from rational investors.

However, as we now show using the example in Tables 2 and 3, rational investors can provide financing in equilibria that are D-interim inefficient. The equilibrium is D-interim inefficient according to both classes when the distribution is F_5. Recall that the value of the project is equal to the value of the security. Both classes are willing to subscribe to the security because, due to rationing, the value of *subscribing* to the security (1/3 > 0 for each class) is higher than the expected value of the security (-1/3 < 0 for each class). The equilibrium is ex ante efficient, however, having an expected value (conditional on financing) of 1/2.

Although the project is financed when the distribution is F_5, we cannot conclude that there is overinvestment. When the distribution is F_9, the project has a positive expected value according to the interim posterior of both investors and yet the project in that case is not financed.

IV. The Role of Rationing in D-Interim Inefficiency with 2 Classes

In this section we consider a version of the model with two securities and two classes of investors which further illustrates the role of rationing.

Definition: *A partition pair* (P_1,P_2) *is* ***weakly multiplicative*** *if*, $\forall t_1,t_1'\in P_1$, $\forall t_2,t_2'\in P_2$, $t_1\cap t_2' \neq \emptyset$, $t_1'\cap t_2' \neq \emptyset$, $t_1'\cap t_2 \neq \emptyset$ *implies* $t_1\cap t_2 \neq \emptyset$.

Table 4 shows two partition pairs--one weakly multiplicative and one non-multiplicative. Investors in class I learn the row and investors in class II learn the column. An x in a cell denotes that there is a possible distribution in that row and column; a blank cell indicates that there is no possible distribution in that row and column.

Table 4
Partition Pairs

weakly multiplicative

x	x		
x	x		
		x	x
		x	x

non-multiplicative

x	x	x	
x	x	x	
		x	x
		x	x

The definition of multiplicative is weak in the sense that a stronger requirement that most of our examples (but not those in Table 4) satisfy is

$\forall t_1 \in P_1$, $\forall t_2 \in P_2$, $t_1 \cap t_2 \neq \emptyset$.[8] If each investor class receives an independent signal on the true distribution, then the implied partition pair is strongly multiplicative.

In the Appendix we prove:

Theorem 2: *Suppose that there are two classes of potential investors and that there are two securities in the security offer. If the partition pair is weakly multiplicative and if the security equilibrium does not involve rationing, then investors in the project do not believe the equilibrium is D-interim inefficient.*

Note that even though D-interim inefficiency is impossible under the assumptions of Theorem 2, it is still possible that ex post inefficient projects are financed.

The example in Table 3 and the example in Section V use perverse rationing in a central way to generate D-interim inefficiencies. Recall that perverse rationing occurs when a security is rationed when it has negative value and not when it has positive value. Theorem 2 suggests that without perverse rationing, examples of D-inefficiencies with two classes of investors and two securities cannot arise. This is not true for the case of more than two classes of investors. We present an example in Section VI in which the information partitions are strongly multiplicative, the rationing is never perverse (only securities with positive value are rationed and all securities with a positive value are rationed), and yet there is a distribution for which all investors believe the equilibrium is D-interim inefficient.

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[8]This is implied by $\forall t_1, t_1' \in P_1$, $\forall t_2, t_2' \in P_2$, $t_1 \cap t_2' \neq \emptyset$, $t_1' \cap t_2 \neq \emptyset$ implies $t_1 \cap t_2 \neq \emptyset$ and $t_1' \cap t_2' \neq \emptyset$.

V. An Equilibrium Known to be D-Interim Inefficient by All Investors

We have already shown that it is possible for a project to be completely subscribed in a competitive security equilibrium even though all investors believe the equilibrium is D-interim inefficient. In this section we present an example that goes a step farther: a competitive security equilibrium in which there is complete subscription when every investor believes the equilibrium is D-interim inefficient, and every investor knows that every other investor also believes the equilibrium is D-interim inefficient.

There are 6 equi-probable distribution functions, with indices $\{1,2,3,4,5,6\}$. Investors in class 1 have the partition $P_1 \equiv \{\{6,1\}, \{2,3\}, \{4,5\}\}$ and investors in class 2 have the partition $P_2 \equiv \{\{1,2\}, \{3,4\}, \{5,6\}\}$. Note that the partition pair (P_1,P_2) is not multiplicative. There is only one security (equity) and $\sigma = \pi = 0$ so that $v_1(s) \equiv \int(x-C)dF_s(x)$. The value of the security $(v_1(s))$, the amount subscribed by investor class 1 $(\xi(s,1))$, the amount subscribed by investor class 2 $(\xi(s,2))$, and the market demand for the security $(Q_1(s))$ are shown in Table 5.

Table 5
Non-multiplicative partition pair with one security

P_1 ----> <---------> <---------> <----

P_2 <---------> <---------> <--------->

s	1	2	3	4	5	6
$v_1(s)$	4	-6	4	-5	11	-5
$\xi(s,1)$	0	1	1	1	1	0
$\xi(s,2)$	1	1	0	0	1	1
$Q_1(s)$	1	2	1	1	2	1

It is straightforward to verify that the security demand correspondence in Table 5 is part of a competitive security equilibrium.[9] To examine the question of D-interim inefficiency suppose that the distribution is F_2 and consider an investor in class 1. Despite the fact that $E\{v_1(s)|t_1(2)\} = (1/2)(-6) + (1/2)(4) = -1 < 0$, the expected value of subscribing to the security is $(1/2)(-6/2) + (1/2)(4) = 1/2 > 0$ because the investor will be rationed if the distribution is F_2 (note that $Q_1(2) = 2$). Thus investors in class 1 subscribe to the security when the true (unknown) distribution is F_2 despite the fact that they believe the equilibrium is D-interim inefficient. Similarly, for an investor in class 2, $E\{v_1(s)|t_2(2)\} = (1/2)(4) + (1/2)(-6) = -1 < 0$. Thus, both investors believe the equilibrium is D-interim inefficient.

Now let's examine what investors think that other investors think. When the distribution is F_2, an investor in class 1 knows that the index is either 2 or 3. If the index is 2, then (as we showed above) investors in class 2 believe the equilibrium is D-interim inefficient. If the index is 3, then investors in class 2 know that the index is either 3 or 4. In this case, $E\{v_1(s)|t_2(3)\} = E\{v_1(s)|s=3 \text{ or } 4\} = (1/2)(4) + (1/2)(-5) = -1/2 < 0$. Hence investors in class 1 know that investors in class 2 believe the equilibrium is D-interim inefficient. Similarly, it is easy to check that (again at F_2) investors in class 2 know that investors in class 1 believe the equilibrium is D-interim inefficient.

However, it is not common knowledge that the equilibrium is D-interim inefficient.[10] When the distribution is F_2, investors in class 1 cannot rule out

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[9]The equilibrium is not unique. Another equilibrium is given by class 1 investors only subscribing when s = 4 and 5, and class 2 investors only subscribing when s = 5 and 6.

[10]Intuitively, an event is common knowledge if everyone knows the event occurred, everyone knows that everyone knows the event occurred, everyone knows that everyone knows that everyone knows the event occurred and so on. Aumann [1976] proves the result that agents cannot agree to disagree (i.e., agents cannot have different posteriors that are common knowledge). The importance of

the possibility that investors in class 2 think that investors in class 1 believe the equilibrium in D-interim efficient. In particular, investors in class 1 know that investors in class 2 may have observed {3,4} in P_2, in which case investors in class 2 cannot rule out class 1 having observed {4,5} from P_1. If investors in class 1 did observe {4,5}, they would value the project at $E(v_1(s)|s=4 \text{ or } 5) = 3 > 0$.

VI. Three Classes of Investors

In this section, we present an example that avoids two undesirable features of the previous examples: (1) The examples depended upon perverse rationing; (2) The example in Table 5 depends on a nonmultiplicative information structure. A common formulation of private (or asymmetric) information is to assume that agents receive independent signals, which is inconsistent with a nonmultiplicative information structure. The strongly multiplicative information structure in the example below allows the natural interpretation that investors in different classes receive independent signals.

There are 8 possible distributions with indices {1,2,3,4,5,6,7,8}. There are two securities and three classes of investors. Investors in class 1 have the partition $P_1 \equiv \{\{1,2,3,4\}, \{5,6,7,8\}\}$, investors in class 2 have the partition $P_2 \equiv \{\{1,3,5,7\}, \{2,4,6,8\}\}$, and investors in class 3 have the partition $P_3 \equiv \{\{1,2,5,6\}, \{3,4,7,8\}\}$. This information structure is strongly multiplicative.

The entrepreneur receives a supernormal salary $\pi = 1$ and retains 0.5% of the equity (security 1), so that $\sigma = 0.005$. External investors are offered 99.5% of the equity for a total price of $p_1 = 410$. Security 2 is offered to external

common knowledge assumptions are surveyed in Binmore and Brandenburger [1987] and Geanakoplos [1988].

investors at a total price of $p_2 = 2600$. The rest of the relevant data is given in Table 6. In constructing this table we use $v_E(s) = \sigma[v_1(s) + p_1]/(1-\sigma) + \pi$, which follows from (4) and (6); we also use $\int(x-C)dF_s(x) = v_E(s) + v_1(s) + v_2(s)$, which follows from (7).

Table 6
The valuation of securities and expected net revenues, given F_s

s	$\rho(s)$	$v_1(s)$	$v_2(s)$	$v_E(s)$	$\int(x-C)\ F_s(dx)$
1	0.2	-105	0	2.533	-102.467
2	0.1	-333	345	1.387	13.387
3	0.1	-300	270	1.553	-28.447
4	0.1	312	-303	4.628	13.628
5	0.1	-333	345	1.387	13.387
6	0.1	360	-330	4.869	34.869
7	0.1	312	-303	4.628	13.628
8	0.2	0	-105	3.060	-101.940

The hypothesized market demands for each security, $Q_n(s)$, and the implied values of $v_n(s)/Q_n(s)$ are shown in Table 7. Since there is incomplete subscription if the distribution is F_1 or F_8, the entries for $v_n(s)/Q_n(s)$ for s = 1 and 8 are given by dashes.

Table 7
Hypothesized market demands and value of subscriptions

s	$Q_1(s)$	$v_1(s)/Q_1(s)$	$Q_2(s)$	$v_2(s)/Q_2(s)$
1	0	–	3	–
2	1	-333	2	172.5
3	1	-300	2	135
4	2	156	1	-303
5	1	-333	2	172.5
6	2	180	1	-330
7	2	156	1	-303
8	3	–	0	–

The value to each class of investor of subscribing to each security, computed according to equation (10), is shown in Table 8.

Table 8
Value of subscribing to each security

s	Investor class 1 $V_1(t_1,\xi)$	Investor class 1 $V_2(t_1,\xi)$	Investor class 2 $V_1(t_2,\xi)$	Investor class 2 $V_2(t_2,\xi)$	Investor class 3 $V_1(t_3,\xi)$	Investor class 3 $V_2(t_3,\xi)$
1	-95.4	0.9	-95.4	0.9	-97.2	3
2	-95.4	0.9	0.6	-92.1	-97.2	3
3	-95.4	0.9	-95.4	0.9	2.4	-94.2
4	-95.4	0.9	0.6	-92.1	2.4	-94.2
5	0.6	-92.1	-95.4	0.9	-97.2	3
6	0.6	-92.1	0.6	-92.1	-97.2	3
7	0.6	-92.1	-95.4	0.9	2.4	-94.2
8	0.6	-92.1	0.6	-92.1	2.4	-94.2

Each investor subscribes to security n only when the expected value of subscribing to that security, $V_n(t_i,\xi)$, is non-negative. The implied security demand correspondence is given in Table 9.

Table 9
Security demand correspondence

s	securities that investor 1 subscribes to $\xi(s,1)$	securities that investor 2 subscribes to $\xi(s,2)$	securities that investor 3 subscribes to $\xi(s,3)$
1	{2}	{2}	{2}
2	{2}	{1}	{2}
3	{2}	{2}	{1}
4	{2}	{1}	{1}
5	{1}	{2}	{2}
6	{1}	{1}	{2}
7	{1}	{2}	{1}
8	{1}	{1}	{1}

The security demand correspondence in Table 9 is consistent with the hypothesized market demands, $Q_n(s)$, in Table 7. Thus, the security demand correspondence is part of a competitive security equilibrium.

Three remarks are of interest at this point. First, there is one distribution function, F_3, for which there is complete subscription despite the fact that all classes of investors believe the equilibrium is D-interim inefficient. The D-interim inefficiency is not due to perverse rationing as in the example in Tables 1-3; in this example, a security is rationed only for values of s for which the security has a positive value.

Second, the financing scheme in this equilibrium dominates single-security (100% equity) financing. A single-security equilibrium will result in the project being financed either when the distribution is F_1 or when it is F_8 because F_1 and F_8 are in different elements of the partition for all investors. If any investor class buys the sole security when the distribution is F_s, then either F_1 or F_8 is in the element of that investor's partition containing F_s; thus the investor will buy the security when the distribution is F_1 or when the distribution is F_8. Thus, the project will be completely subscribed for either F_1 or F_8. But if the project is financed for F_1 or F_8, the project will be ex ante inefficient. Because Theorem 1 rules out financing of ex ante inefficient projects, the project would never be financed by 100% equity.

Third, we have not ruled out the possibility of multiple competitive security equilibria. However, if there are other competitive security they cannot be ex post efficient. The only security demand correspondences consistent with ex post efficiency (described in the Appendix) are not consistent with equilibrium for this example.

VII. Stocks and Bonds

Up to this point we have examined security equilibria by specifying the values of $v_E(s)$, and $v_n(s)$ $n = 1,\ldots,N$ for all distributions in Φ. Using the relation in (7), we examined the expected value of revenue minus costs without explicitly specifying the distributions F_s or the payoff functions y_n. In this section, we interpret security 1 as equity and security 2 as debt, and we relate the valuations $v_n(s)$ to the payoff functions y_n and the underlying distributions F_s.

The entrepreneur raises capital equal to $p_1 + p_2$ by selling securities to external investors. After the entrepreneur receives a supernormal salary π and spends C to undertake the project, the available capital, denoted K, is

(15) $K \equiv p_1 + p_2 - \pi - C.$

After the project's revenue is realized, the total available resources, K + x, are distributed to the holders of securities (including the entrepreneur who holds a fraction σ of the equity).

Bonds promise to pay an aggregate amount R provided that the available funds, K + x, are sufficient to pay this amount. If the available funds are less than R, the bondholders take all that is available. Therefore, the payoff function to bondholders is

(16) $y_2(x) = \min[R, K+x].$

The stockholders are the residual claimants. If the available resources exceed the amount that must be paid to bondholders, then the stockholders receive the excess, K + x - R. However, if the available funds are less than R, then the stockholders receive nothing. Therefore,

(17) $y_1(x) = \max[K+x-R, 0].$

Suppose that all of the distributions in Φ are members of the exponential family. For each s, there is a pair of parameters $\alpha > 0$ and $\beta > 0$ determining the density function $f(x;\alpha,\beta) = (1/\beta)\exp[-(x-\alpha)/\beta]$ for $x > \alpha$. The mean and variance of revenues are $E(x) = \alpha + \beta$ and $Var(x) = \beta^2$. We can calculate the value of bonds to external investors by substituting (16) and the exponential density function into (5). Similarly we can calculate the value of stock to external investors and to the entrepreneur by substituting (17) and the exponential density function into (4) and (6). These calculations, which are straightforward but tedious, yield

(18) $v_1 = (1-\sigma)\beta \exp[-(R-K-\alpha)/\beta] - p_1.$

(19) $$v_2 = K + \alpha + \beta - \beta\exp[-(R-K-\alpha)/\beta] - p_2,$$

and

(20) $$v_E = \sigma\beta \exp[-(R-K-\alpha)/\beta] + \pi.$$

In Table 10 we show the values of α and β, that, when substituted into equations (18) - (20), yield the values of $v_1(s)$, $v_2(s)$ and $v_E(s)$ in Table 6. The characteristics of the securities in this case are $p_1 = 410$, $p_2 = 2600$, $R = 3110$, $\sigma = 0.005$ and $\pi = 1$, and the cost of the project is $C = 3000$. Thus, the security equilibrium that we presented earlier for the case with three classes of investors is an equilibrium when securities have the characteristics in Table 10 and the parameters of the distributions F_s are as given in Table 10.

Table 10
Underlying exponential distributions when
$R = 3110$; $K = 9$; $p_1 = 410$; $p_2 = 2600$; $\sigma = 0.005$; and $\pi = 1$.

s	α	β	E(x)
1	1879.873	1017.660	2897.533
2	2727.570	285.817	3013.387
3	2559.972	411.581	2971.553
4	955.623	2058.005	3013.628
5	2727.570	285.817	3013.387
6	866.142	2168.728	3034.869
7	955.623	2058.005	3013.628
8	1589.983	1308.077	2898.060

Table 11 presents, for each value of s, the conditional expectation $E(x|t_i(s)\cap D)$ and conditional standard deviation $\sqrt{Var(x|t_i(s)\cap D)}$ of revenues for each investor class i, i = 1,2,3. We also show which security investor i subscribes to (each investor always subscribes to only one security).

Table 11
Mean and standard deviation of x conditional on investors' information and complete subscription

	Investor class 1			Investor class 2			Investor class 3		
s	mean	s.d.	security	mean	s.d.	security	mean	s.d.	security
1	2999.5	1223.1	bonds	2999.5	1223.1	bonds	3020.5	1273.7	bonds
2	2999.5	1223.1	bonds	3020.6	1734.0	stock	3020.5	1273.7	bonds
3	2999.5	1223.1	bonds	2999.5	1223.1	bonds	2999.6	1697.2	stock
4	2999.5	1223.1	bonds	3020.6	1734.0	stock	2999.6	1697.2	stock
5	3020.6	1734.0	stock	2999.5	1223.1	bonds	3020.5	1273.7	bonds
6	3020.6	1734.0	stock	3020.6	1734.0	stock	3020.5	1273.7	bonds
7	3020.6	1734.0	stock	2999.5	1223.1	bonds	2999.6	1697.2	stock
8	3020.6	1734.0	stock	3020.6	1734.0	stock	2999.6	1697.2	stock

As we explained in the previous section, all three investor classes believe the equilibrium is D-interim inefficient when s = 3 (the conditional mean of revenue $E\{x|t_i(3)\cap D\}$ is less than the project cost (3000) for all three investor classes). When s = 3, all three investor classes have essentially the same conditional mean of x, yet some of the investors subscribe to bonds while other investors subscribe to stock. In particular, the investors with relatively low conditional standard deviation (investor classes 1 and 2) subscribe to bonds while investors with relatively high conditional standard deviation subscribe to stocks. Why do investors with high conditional variance buy stocks and investors with low conditional variance buy bonds? Stocks provide high payoffs in the event of high revenues x, but the downside risk on stocks is limited. Once x + K falls below the debt repayment R, it does not matter to the stockholder how far it falls. Thus, investors who view the distribution as having a high variance will buy stock if the reason for a low expected value of x is that there is a lot of weight on very bad outcomes and some weight on very good outcomes. By contrast, bondholders bear the risk of low values of x but do not share in the good fortune if x turns out to be extremely high. Investors who view the distribution as having a relatively small variance will buy bonds because their downside risk is limited and they do not care that there is limited upside potential.

VIII. Conclusion and Related Literature

We have shown that if investors have heterogeneous private information, then competitive security equilibria will sometimes finance projects that every investor thinks has negative expected net revenue. In this analysis, we have treated the terms of the securities offered as exogenously given, and we have simply asked whether these securities will be fully subscribed. In the example with three classes of investors, the entrepreneur received a supernormal profit and retained a small share of the equity, so that the entrepreneur profited from this security offering. Investors also profited from the security offering (in a D-interim conditional sense) because they willingly purchased the securities after receiving their private information. The next step in this research project is to study the design of the menu of offered securities and their characteristics.

We conclude with a brief discussion of related literature. The closest finance literature deals with the possibility that stock market prices need not reflect fundamentals. Allen and Postlewaite [1989], using a similar information structure to ours, have constructed examples in which the equilibrium time path of prices in a dynamic trading model involves speculation (i.e., because of the possibility of selling a security for a capital gain, traders are willing to pay a higher price for the security than they would if they were obliged to hold the asset forever). In particular, it is possible that all traders know that the price of a security will fall in the future and still be willing to hold or buy the security.

There are also several papers addressing the possibility of trades when there is asymmetric information about the value of the trades (here the most relevant paper is Sebenius and Geanakoplos [1983]). Sebenius and Geanakoplos [1983] take as their starting point the observation that if two people have different probability assessments about the realization of an uncertain event, they can design a contingent agreement such as a gamble that offers each of them a positive expected value. They show that, for any possible contingent agreement, it cannot be common knowledge that both sides wish to accept that agreement. In contrast to that paper, as well as Geanakoplos and Polemarchakis [1982], in our model it is not common knowledge that all investors wish to invest and there is no repeated exchange of information on the willingness to enter an agreement.

Appendix

Proof of Theorem 1: Suppose $((\pi,\sigma,(p_n,y_n)),\xi)$ is a competitive security equilibrium. Define $S_{in} = \{s\in D(\xi):V_n(t_i(s),\xi) \geq 0\}$. Now,

$$\begin{aligned} 0 &\leq \textstyle\sum_i\sum_{\{t_i:t_i\cap S_{in}\neq\emptyset\}}\sum_{s\in t_i\cap D}[v_n(s)/Q_n(s,\xi)]\rho(s) \\ &= \textstyle\sum_i\sum_{s\in S_{in}}[v_n(s)/Q_n(s,\xi)]\rho(s) \\ &= \textstyle\sum_{s\in D}\sum_{\{i:s\in S_{in}\}}[v_n(s)/Q_n(s,\xi)]\rho(s) \\ &= \textstyle\sum_{s\in D}v_n(s)\rho(s), \end{aligned}$$

where the first equality follows from an investor's types t_i being pairwise disjoint, the second from the assumption that $((\pi,\sigma,(p_n,y_n)),\xi)$ is a security equilibrium (so that $\forall s\in D$, $\exists i$ s.t. $s\in S_{in}$), and the third from $Q_n(s,\xi) = |\{i:s\in S_{in}\}|$. This concludes the proof, since $\sum_{s\in D}\rho(s)\int(x-C)F_s(dx) = \sum_{s\in D}\rho(s)\{v_E(s) + \sum_n v_n(s)\} \geq 0$. ∎

Proof of Theorem 2: Since there are two classes of investors, two securities, and no rationing, we can assume for any F_s that all investors in one class subscribe to one security and all investors in the other class subscribe to the other security (otherwise both of the securities have a zero expected value and the equilibrium is not D-interim inefficient). Without loss of generality, we can assume that $\forall s$, $t_1(s)\cap t_2(s) = \{s\}$.[10] We need a preliminary lemma. For $s\in D$, define $I_i(s) \equiv t_i(s)\cap D$, $S_1(s) \equiv \{s''|\ \exists s'\in I_2(s)$ s.t. $s''\in I_1(s')\}$ and $S_2(s) \equiv \{s''|\ \exists s'\in I_1(s)$ s.t. $s''\in I_2(s')\}$.

<u>Lemma</u>: If the partition pair is multiplicative, then $S_1(s) = S_2(s)$ $\forall s\in D$.

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[10]Let $P \equiv \sum_{s'\in t_1(s)\cap t_2(s)}\rho(s')$. Since the two investors, and so the equilibrium, cannot distinguish between any two distribution functions in $t_1(s)\cap t_2(s)$, we can treat the intersection as the single distribution $P^{-1}\sum_{s'\in t_1(s)\cap t_2(s)}\rho(s')F_{s'}$, with prior probability P.

Remark: This lemma asserts that both classes of investors agree on the set of distributions that each class believes the other class believes possible.

Proof of lemma: Suppose that $t_1 \cap t_2' \subset D$, $t_1' \cap t_2' \subset D$, $t_1' \cap t_2 \subset D$, and that for $s \in t_2'$ investor 2 subscribes to, without loss of generality, security 2. Because of full financing (and hence complete subscription) in $t_1' \cap t_2'$, investor 1 buys security 1 in t_1'. Full financing in $t_1' \cap t_2$ requires that in t_2 investor 2 buy security 2. Similarly, full financing in $t_1 \cap t_2'$ requires that in t_1 investor 1 buys security 1. Thus, $t_1 \cap t_2$ is contained in D.

Suppose $s'' \in S_1(s)$. Then $\exists s' \in t_2(s) \cap D$ s.t. $s'' \in t_1(s') \cap D$. Thus, $s'' \in D$ and since $s' \in t_1(s'')$, we have $s' \in t_1(s'') \cap D$. Hence, $\emptyset \neq t_1(s'') \cap t_2(s) \subset D$. Furthermore, since s and $s'' \in D$, we have that $\emptyset \neq t_1(s) \cap t_2(s) \subset D$ and $\emptyset \neq t_1(s'') \cap t_2(s'') \subset D$. Since the partitions are multiplicative, $t_1(s) \cap t_2(s'') \neq \emptyset$, and from the previous paragraph, $t_1(s) \cap t_2(s'') \subset D$. Thus, $\exists s^* \in (t_1(s) \cap D) \cap (t_2(s'') \cap D)$. Since this implies $s'' \in t_2(s^*) \cap D$, $s'' \in S_2(s)$.

A symmetric argument shows that $S_2(s)$ is contained in $S_1(s)$. ■

Suppose investor 1 buys security 1 when F_s is the distribution. Then $\sum_{s' \in I_1(s)} \rho(s') v_1(s') \geq 0$ and $\sum_{s' \in I_1(s)} \rho(s')[v_1(s') - v_2(s')] \geq 0$. Investor 2 buys security 2, so $\sum_{s' \in I_2(s)} \rho(s') v_2(s') \geq 0$ and $\sum_{s' \in I_2(s)} \rho(s')[v_1(s') - v_2(s')] \leq 0$. Full financing implies that investor 1 buys security 1 for all $s'' \in I_2(s)$ and so $\sum_{s' \in I_1(s'')} \rho(s') v_1(s') \geq 0$ and $\sum_{s' \in I_1(s'')} \rho(s')[v_1(s') - v_2(s')] \geq 0$. Summing the second inequality over s" yields (since for $s' \in S_1(s)$, $\exists$ unique s" s.t. $s'' \in I_2(s)$, $s' \in I_1(s'')$):

(21) $$\sum_{s'\in S_1(s)}\rho(s')[v_1(s')-v_2(s')] = \sum_{s''\in I_2(s)}\sum_{s'\in I_1(s'')}\rho(s')[v_1(s')-v_2(s')] \geq 0.$$

Similarly, investor 2 buys security 2 in $s'' \in I_1(s)$, so $\sum_{s'\in I_2(s'')}\rho(s')v_2(s') \geq 0$ and $\sum_{s'\in I_2(s'')}\rho(s')[v_1(s')-v_2(s')] \leq 0$. Summing the second inequality over s" yields

(22) $$\sum_{s'\in S_2(s)}\rho(s')[v_1(s')-v_2(s')] = \sum_{s''\in I_1(s)}\sum_{s'\in I_2(s'')}\rho(s')[v_1(s')-v_2(s')] \leq 0.$$

Since $S_1(s) = S_2(s)$, $\sum_{s'\in S_1(s)}\rho(s')[v_1(s')-v_2(s')] = \sum_{s'\in S_2(s)}\rho(s')\times[v_1(s')-v_2(s')]$, which equals 0 from the inequalities in (21) and (22). Therefore, since $\sum_{s'\in I_1(s'')}\rho(s')[v_1(s')-v_2(s')] \geq 0$ for any $s'' \in I_2(s)$, $\sum_{s'\in I_1(s)}\rho(s')[v_1(s')-v_2(s')] = 0$. So investor 1 is indifferent between the two securities. Since the project is divided between the two securities, the project has a (D-interim from investor 1's viewpoint) conditional expected payoff greater than or equal to its cost. A similar argument applies to investor 2. ∎

Proposition: *Suppose ξ is a security demand correspondence such that* $D(\xi) = \{2,4,5,6,7\}$. *Then ξ is either given by*

s	class 1	class 2	class 3
1	{2}	{2}	{2}
2	{2}	{1}	{2}
3	{2}	{2}	Ø
4	{2}	{1}	Ø
5	{1}	{2}	{2}
6	{1}	{1}	{2}
7	{1}	{2}	Ø
8	{1}	{1}	Ø

or the pattern obtained by switching securities 1 *and* 2.

Proof of Proposition: Suppose there is an equilibrium in which there is complete subscription for $s \in \{2,4,5,6,7\}$ only. Now, it is never the case that an investor demands both securities in the same state. If an investor did demand both securities there would be full financing for at least one of s = 1, 3, or 8.

We proceed through a series of claims.

<u>Claim 1:</u> Investors in class I or II subscribe to a security when s = 3.

<u>Proof:</u> Suppose neither class subscribed to either security when s = 3. Complete subscription when s = 2 implies that class II subscribes to one of the securities when $s \in \{2,4,6,8\}$, which we take to be security 1 w.l.o.g., and class III subscribes to 2 when $s \in \{1,2,5,6\}$. But when s = 5, class II does not subscribe to any security, so class I subscribes to security 1 when $s \in \{5,6,7,8\}$. Now, when s = 4, II subscribes to 1 and I does not subscribe to any securities, so that III subscribes to 2 when $s \in \{3,4,7,8\}$. But this implies that, when s = 8, I subscribes to 1 and III subscribes to 2, yielding complete subscription, a contradiction. ■

<u>Claim 2:</u> The same security (which, w.l.o.g., we will take as security 1 from here on) is unsubscribed when s = 1 and 3.

<u>Proof:</u> Since 1 and 3 are in the same element of I's partition, class I investors must subscribe to the same security when s = 1 and when s = 3. The same is true of class II investors. By claim 1, I or II subscribes to some security when s = 3 and so when s = 1. The other security is therefore unsubscribed when s = 1 and 3 (otherwise there would be complete subscription when s = 1 and 3). ■

<u>Claim 3:</u> Class III investors never subscribe to security 1.

<u>Proof:</u> Follows immediately from Claim 2, since that claim implies that class III investors do not subscribe to security 1 when $s \in \{1,2,5,6\}$ or $s \in \{3,4,7,8\}$. ■

Claim 4: Investors in classes I and II subscribe to security 2 when s = 3.

Proof: Clearly if classes I and II subscribe to any security when s = 3, it must be to the same one (otherwise we have complete subscription). So, suppose first that class I does not subscribe to any security when s = 3 (and so also when s = 1, 2, and 4). Then, from claim 1, class II subscribes to security 2 when s∈{1,3,5,7}. Full financing when s = 7 implies that I subscribes to security 1, since III never subscribes to security 1. Full financing when s = 4 requires III to subscribe to security 2 (and II to subscribe to security 1). But then there is complete subscription when s = 8, since I subscribes to 1 and III subscribes to security 2, a contradiction.

Now suppose that class II does not subscribe to any security when s∈{1,3,5,7}. Then I subscribes to security 2 for s∈{1,2,3,4}, and II subscribes to security 1 when s∈{2,4,6,8}. Full financing when s = 7 requires I to subscribe to security 1 when s∈{5,6,7,8} and III to subscribe to 2 when s∈{3,4,7,8}. But this is a contradiction, since it implies complete subscription when s = 8.

∎

Now, since there is complete subscription when s = 2 and 4, class II investors subscribe to security 1 when s∈{2,4,6,8}. Also, since there is complete subscription when s = 5 and 7, class I investors subscribe to security 1 when s∈{5,6,7,8}. Complete subscription when s = 6 implies that class III investors subscribe to security 2 when s∈{1,2,5,6}. Because class I and II investors subscribe to security 1 when s = 8, class III investors do not subscribe to security 2 when s = 8 (or else there would be complete subscription). Thus class III investors do not subscribe to any security when s∈{3,4,7,8}.

∎

References

Allen, Franklin and Andrew Postlewaite [1989], "Speculation and Asset Bubbles in Rational Expectations Equilibria," mimeo, University of Pennsylvania.

Aumann, Robert J. [1976], "Agreeing to Disagree," The Annals of Statistics, 4, 1236-1239.

Binmore, Ken and Adam Brandenburger [1987], "Common Knowledge and Game Theory," ICERD Discussion Paper No. TE/88/167, London School of Economics.

Geanakoplos, John [1988], "Common Knowledge, Bayesian Learning, and Market Speculation with Bounded Rationality," mimeo, Yale University.

Geanakoplos, John and Heraklis Polemarchakis [1982], "We Can't Disagree Forever," Journal of Economic Theory, 28, 192-200.

Sebenius, James and John Geanakoplos [1983], "Don't Bet On It: Contingent Agreements With Asymmetric Information," Journal of the American Statistical Association, 78, 424-426.