AN EXACT SOLUTION TO THE PORTFOLIO CHOICE PROBLEM UNDER TRANSACTIONS COSTS

by

Bernard Dumas
Elisa Luciano

41-89

RODNEY L. WHITE CENTER FOR FINANCIAL RESEARCH
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6367

First draft: February 1989

Last draft: December 1989

Typos corrected: February 1990

AN EXACT SOLUTION TO THE PORTFOLIO CHOICE PROBLEM UNDER TRANSACTIONS COSTS***

Bernard Dumas*

Elisa Luciano**

* Wharton School of the University of Pennsylvania, Centre HEC- ISA and National Bureau of Economic Research. Dumas gratefully acknowledges the financial support of the A. Shoemaker chair at the Wharton school.

** Universita Cattolica, Milano, Italy.

*** We are grateful to Jean-Luc Vila and to participants of an NBER workshop for helpful comments made on an earlier draft, and to Francisco Delgado for his diligent and effective research assistance. This paper is the first part of a document entitled: "Bid-ask Option Pricing under Transactions Costs," which was first presented at an International Conference held at Centre HEC-ISA in June 1988.

ABSTRACT

Much of financial theory neglects transactions costs. Perhaps the most successful implementation of it -- i.e. continuous-time portfolio choice and option pricing -- is downright inconsistent with the existence of any transactions cost at all. Nonetheless prima facie evidence from the trade is that transactions costs are a source of concern for portfolio managers. The presence of practically any friction in financial markets qualitatively changes the nature of the optimization problem; for it produces the need sometimes to do nothing and sometimes to act, an issue which, of course, does not arise in frictionless situations.

The investor considered here does not consume along the way. He accumulates wealth until some terminal point in time. At that point he consumes all. His objective is to maximize the expected utility derived from that terminal consumption. We postpone the terminal point infinitely far into the future to obtain a stationary portfolio rule. The optimal portfolio policy which we find is in the form of two control barriers between which portfolio proportions are allowed to fluctuate before some trade is resorted to.

We show how to calculate these two barriers exactly.

1. Introduction

Much of financial theory neglects transactions costs. Perhaps the most successful implementation of it -- i.e. continuous-time portfolio choice and option pricing -- is downright inconsistent with the existence of any transactions cost at all. For instance, the hedging strategy proposed by Black and Scholes to value derivative assets is not rationally feasible when portfolio adjustment is not costless. Similarly, other reasonings adopted to justify alternative pricing methods[1] do not seem to be immediately applicable when transaction costs impinge on investment returns. When they are applied, straightforward continuous adjustment of the portfolio composition would lead to infinite transactions costs in a finite time period.

Nonetheless prima facie evidence from the trade is that transactions costs are a source of concern for portfolio managers. We include in this category of cost not only brokerage fees, but also any cost of analysis, information cost and any expense incurred in the process of deciding upon and placing an order. Delays in execution which cause prices at which one trades to be different from those at which one planned to trade may in some way be included as well. Recent literature on these questions include Perold (1988), Amihud and Mendelson (1988) and Hasbrouck and Schwartz (1988).

The assumption we make here is that these costs are of a proportional nature, although there would be no added analytical difficulty in assuming that they are fixed in nature, or of a mixed character. Proportionality means that the cost incurred is proportional to the size of the trade.

The presence of practically any friction in financial markets qualitatively

[1] See for example Cox, Ross and Rubinstein (1979).

changes the nature of the optimization problem; for it produces the need sometimes to do nothing and sometimes to act, an issue which, of course, does not arise in frictionless situations. Recently Constantinides (1986) has proposed an approximate solution to the portfolio choice problem under transactions costs (see also Taksar et. al. (1988) which treats the logarithmic special case, Davis and Norman (1987) and Kamin (1975)). The investor in Constantinides' problem maximizes the expected value of his infinite-horizon utility function. Portfolio policies are computed numerically under the assumption (this is the nature of the approximation) that the investor in each period consumes a fixed proportion of his wealth.

In the present paper we examine a somewhat simpler problem.[2] The investor considered here does not consume along the way. He accumulates wealth until some terminal point in time. At that point he consumes all. His objective is to maximize the expected utility derived from that terminal consumption. We postpone the terminal point infinitely far into the future to obtain a stationary portfolio rule. In contrast to Constantinides, our alternative leads to an exact solution.[3] The assumption that the investor does not consume as he goes and cares about the achieved terminal wealth seems quite realistic when modelling the behavior of a trader employed by a financial institution.

The paper is organized as follows: in section 2 we set up the portfolio-selection model. In section 3 we derive the necessary conditions

---

[2] A similar formulation has independently been adopted in Grossman and Vila (1989).

[3] Our solution is not a special case of Constantinides' solution. Since he imposes that the investor's rate of consumption out of wealth be constant over time, one might think that setting that rate of consumption equal to zero would produce our solution. It does not: Constantinides' solution is degenerate in that case.

which must be satisfied when it is optimal to refrain from trading, as well as the optimality conditions which must prevail at the random times when trading takes place. In section 4 we obtain an analytical solution. In section 5 we run some comparative statics exercises. Section 6 concludes and suggests applications.

2. The portfolio selection model

We analyze the behavior of an investor, whose planning horizon is infinite and whose objective is to maximize the expected utility of his final consumption. No consumption takes place along the way, and it is assumed that his preferences are represented by a utility function u(c), where c is terminal consumption.

Assumption 1: The utility function is of the power form:[4]

[1] $$u(c) = c^{\gamma}/\gamma,$$

with γ constant, $\gamma < 1$.

Assumption 2: Given this functional form, we can formulate the investor's problem as:

[2] $$\lim_{T \to +\infty} \max E_0 \; e^{-\beta T} \frac{1}{\gamma} [c(T)]^{\gamma};$$

where E_0 is the expectation operator conditional on time-0 information, T is the horizon and β is a discount rate which will be chosen later to insure

---

[4] This functional form implies constant relative risk aversion equal to $1 - \gamma$. The characteristics of HARA functions are listed, for example, in Merton (1971).

boundedness of the optimized value of the objective function.

The economy is characterized on the real side by a unique consumption good, used as numeraire, and on the financial side by two assets. In particular, there is available:

. a riskless asset (cash), whose accumulated amount at time t, denoted x(t), increases in value, if trading does not take place, according to the following equation (*Assumption 3*):

$$dx(t) = rx(t)dt, \tag{3}$$

where r is the rate of interest, assumed to be constant;

. a risky asset with an accumulated value[5] y(t) which follows -- again in the absence of portfolio adjustments -- the stochastic differential equation (*Assumption 4*):

$$dy(t) = \alpha y(t)dt + \sigma y(t)dz, \tag{4}$$

with α and σ constants, dz a white noise. One may question whether such a process for asset prices would be compatible with general equilibrium in a world where financial transactions entail costs. But this issue will have to await further research.

The investor acts as a trader in the markets for borrowing, lending and

---

[5] y is the number of shares held at time t multiplied by the price of each share.

for the risky asset y. In these markets he takes prices as given and chooses quantities without restrictions, but incurs transactions costs.

As far as the cost of exchanging financial assets is concerned:

Assumption 5: the conversion ratio denoted s ($s < 1$) is taken to be the same in either direction: an amount sq^* of y can be purchased by giving up an amount q^* of x; similarly, an amount sq of x can be obtained by selling an amount q of y.

Problem [2] has to be solved subject to the constraints [3] and [4] and to the rules for financial trading.

If there were no transactions costs, problem [2] could be dealt with according to Merton (1969). We could define the derived undiscounted value (or Bellman) function at time t:

$$[5] \quad J^*(W;\ t) \equiv \lim_{T \to +\infty} \max\ E_t\ e^{-\beta(T-t)}\ \frac{1}{\gamma}\ [W(T)]^\gamma;$$

where W would be the investor's wealth:[6] $W = x + y$, with dynamics given by:

$$[6] \qquad dW = (rx + \alpha y)dt + \sigma y dz.$$

We look for a *stationary solution* $J^*(W)$. It must satisfy the Hamilton-Jacobi equation:

---

[6] From now on, we will almost always write x and y instead of x(t) and y(t) for the primitive asset holdings.

$$[7] \quad \underset{x,y}{\text{Max}} \{- \beta J^* + J^*_W(rx + \alpha y) + J^*_{WW}\sigma^2 y^2/2\} = 0,$$
$$\text{s.t. } x+y=W$$

Considering the homogeneity of the objective function [5] and of the constraint [6], J^* must be proportional to W^γ/γ. Writing J^* in the form: $J^*(W) \equiv H \times W^\gamma/\gamma$, and substituting this function into the homogeneous differential equation [7], it is found that there exists a real number $H \neq 0$, if and only if an appropriate choice of β is made:[7]

$$[8] \quad \beta = r\gamma - \delta^* \qquad \text{where:} \qquad \delta^* = - \frac{1}{2} \gamma \frac{(\alpha - r)^2}{(1 - \gamma) \sigma^2}$$

The corresponding optimal portfolio strategy would then be to keep y/x at a value θ^* equal to:

$$[9] \qquad \theta^* = \frac{\alpha - r}{(1-\gamma)\sigma^2 - \alpha + r}.$$

In what follows, we choose parameter values in such a way that:

$$[10] \qquad 0 < \frac{\alpha - r}{(1-\gamma)\sigma^2} < 1.$$

thereby guaranteeing that $x > 0$ and $y > 0$ in the absence of transactions

---

[7] It is not possible to determine the number H on the basis of equation [7] alone. The value of H would fall out of the *terminal condition* in the following manner. First, solve for the finite-horizon value function J(W, t), imposing the terminal condition that, at the horizon point T, one has: $J(W, T) \equiv [W(T)]^\gamma/\gamma$. Then let T go to infinity. When β is chosen as in [8], H = 1.

costs. We assume (*Assumption 6*) that condition [10] suffices to also guarantee $x, y > 0$ in the presence of transactions costs.

3. Regulating the portfolio

When trading costs of the type outlined above are present, Constantinides (1986, section III) has proved in a discrete-time framework and has assumed in a continuous-time one that the optimal investment policy is of the following type: exchange a *small amount* of the riskless for the risky asset whenever the portfolio ratio $\theta \equiv y/x$ falls below some fixed level ℓ to be chosen optimally, and do the reverse if it exceeds the control level u. That the optimal trade size should be infinitely small is a postulate which makes sense considering the proportional nature of the transactions costs. It means that we are seeking the optimal regulator of the portfolio, among the class of infinitesimal regulators.

We must verify that, among infinitesimal regulators, one with fixed control barriers u and ℓ is indeed optimal and we must determine the level of these barriers. This is done below by means of the theory of optimal regulated Brownian motion, as exposited by Harrison (1985).[8]

Portfolio rebalancing is modelled mathematically as follows. Define two processes U and L with these properties:

U and L are continuous non decreasing processes;

U increases only when θ reaches the value u;

[8] This approach is very much related to the optimal-stopping literature (Krylov (1980)) which has already been applied to the field of Finance by Grossman-Laroque (1987). The mathematical theory of the instantaneous control of Brownian motion can be found in Benes, Shepp and Witsenhausen (1980) and Harrison and Taksar (1983). Regulated Brownian motion has been applied to problems in Economics by Bertola (1988), Dixit (1988), Dumas (1988a, b).

L increases only when θ reaches the value ℓ.

In order to take account of portfolio readjustments, equations [3] and [4] are amended as:

[11] $dx = rx\ dt + sdU - dL;$

[12] $dy = \alpha y\ dt + \sigma y\ dz + sdL - dU;$

where 1 - s is the rate of transactions costs and U and sL are interpreted as cumulative sales and purchases of equity since time 0. The stochastic differential equations [11] and [12] reduce to [3] and [4] at such times where no rebalancing takes place ($\theta \neq u$ or ℓ).

The problem of maximizing [2] subject to [11] and [12] has two state variables x and y. Its undiscounted value function J(x, y) --defined exactly as the function J^* was in [5], but subject to the new constraints [11] and [12]-- is once again homogeneous of degree γ. It is assumed (*Assumption 7*) to be twice continuously differentiable in y and once continuously differentiable in x. A transposition of Harrison's[9] results to the present setting leads to the following properties of the value function J:[10]

[13a] $J_x(x, y) = s\ J_y(x, y)$ when $\theta \equiv y/x = \ell;$

[13b] $s\ J_x(x, y) = J_y(x, y)$ when $\theta \equiv y/x = u.$

---

[9] See Harrison (1985) Chapter 5. See also the note by Dumas (1988).

[10] When $\theta = \ell$, x is reduced by dL and y is increased by sdL. When this happens, there can be no jump in the value of the problem (value matching):

$$J(x, y) = J(x - dL, y + sdL).$$

Expanding the right-hand side leads to equation [13a].

Conditions [13a] and [13b] mean that the indirect utility function must present a marginal rate of substitution between y and x equal to 1/s when $y = \ell x$, and equal to s when $y = ux$. They are a mechanical result of the postulate that the optimal policy is of the infinitesimal-regulator type; i.e. they hold the minute regulation is applied at u and ℓ, regardless of whether these quantities are chosen optimally or not.

Considering the homogeneity of the function J, this function satisfies properties [13a] and [13b] along two rays $y = \ell x$ and $y = ux$ *whose slopes are independent of y and x.*

The *optimality of the control limits* u and ℓ will be imposed by means of two necessary conditions, derived in Harrison and Taksar (1983)[11] and labelled "super-contact conditions" by Dumas (1988c). These are extensions to the case of the infinitesimal regulator of the traditional "smooth-pasting conditions" of Samuelson (1965), McKean (1965), Merton (1973, fn60) and Krylov (1980):[12]

[14a] $$0 = -J_{xx}(x, y) + s\, J_{xy}(x, y) = -s\, J_{yx}(x, y) + s^2\, J_{yy}(x, y);$$
$$\text{at } \theta \equiv y/x = \ell;$$

---

[11] proposition 5.11 page 449. See also the discussion which precedes proposition 5.11.

[12] Smooth pasting roughly says that the derivatives of the value function must take the same value at the point where the regulator is applied (trigger point) and at the point to which one arrives as a result of the regulation (target point). For instance, when $\theta = \ell$, x is reduced by dL and y increased by sdL:

$$J_x(x, y) = s\, J_y(x, y) =$$

$$J_x(x - dL, y + sdL) = s\, J_y(x - dL, y + sdL).$$

Expanding terms directly leads to [14a].

[14b] $0 = s^2 J_{xx}(x, y) - s J_{xy}(x, y) = s J_{yx}(x, y) - J_{yy}(x, y);$

at $\theta \equiv y/x = u$.

Considering that the horizon of the objective function [2] is infinite, it is legitimate to restrict our search to the steady-state solution, defined as the solution for which the control limits ℓ and u are independent of time[13], in addition to being independent of y and x. We assume (*Assumption 8*) that these steady-state control limits are identical to those one converges to as the horizon T tends to $+\infty$. Assuming steady state, it is possible to provide an explicit solution for J(x, y) over the domain $\ell x \leq y \leq ux$;[14] this will be done in the next section.

Inside the cone, the Hamilton-Jacobi equation for J -- reflecting the conditionally expected drift of the value of J when x and y obey [3] and [4] -- is:

[15] $$- \beta J + J_x rx + J_y \alpha y + J_{yy} y^2 \sigma^2/2 = 0 \qquad \ell x \leq y \leq ux.$$

The boundary conditions applying to equation [15] are those on the marginal rates of substitution and on the second derivatives written above as [13-14].

---

13 See Constantinides (1986, section III) and Dumas (1988, section 3). As a far as the value function J is concerned, we later choose the discount factor β in such a way that J is stationary as well.

14 Outside the cone the indirect utility function is a power function (with γ exponent) of x + sy or y + sx --depending on the side being considered. This indirect utility function is relevant only if initial conditions place the portfolio outside the cone. During the later course of the optimal program, the portfolio will never be positioned outside the cone.

4. Analytical solution for the optimal policy

We exploit the homogeneity of degree γ of the value function in order to state that:

[16] $$J(x, y) \equiv x^{\gamma} I(\theta),$$

where θ stands for $\theta(t)$, with $\theta(t) \equiv y(t)/x(t)$. Substituting [16] into [15], it is found that the I function must satisfy the following reduced Hamilton-Jacobi equation:

[17] $$\delta I + \theta(\alpha - r) I'(\theta) + \frac{1}{2} \theta^2 \sigma^2 I''(\theta) \equiv 0; \quad \ell \le \theta \le u,$$

where δ stands for:

$$\delta = - \beta + r\gamma.$$

We reserve till later the choice of an appropriate value for β (and δ).

The boundary conditions [11-14] transform into:

[18a][18b] $$I'(\ell) = \frac{\gamma\, I(\ell)}{1 + \frac{\ell}{s}} \frac{1}{s}; \qquad I'(u) = \frac{\gamma\, I(u)}{1 + us} s;$$

[19a][19b] $$I''(\ell) = \frac{(\gamma - 1)\, I'(\ell)}{1 + \frac{\ell}{s}} \frac{1}{s}; \qquad I''(u) = \frac{(\gamma - 1)\, I'(u)}{1 + us} s.$$

The characteristic equation corresponding to [17] (obtained, as usual, by means of the trial solution: $I(\theta) \equiv \theta^k$) is:

[20] $$0 = \delta + k(\alpha - r - \sigma^2/2) + k^2 \sigma^2/2 ,$$

where k is an unknown; the solutions (real or imaginary) are called k_1 and k_2. The discriminant of [20] is:

[21] $$\Delta = (\alpha - r - \sigma^2/2)^2 - 2\,\delta\sigma^2.$$

We define the critical value δ_c of δ for which the discriminant Δ changes sign:

[22] $$\delta_c = \frac{(\alpha - r - \sigma^2/2)^2}{2\,\sigma^2}$$

If later we are led to choose $\delta < \delta_c$, this will cause the discriminant Δ to be positive, leading to the roots k_1 and k_2 being real. If, to the opposite, δ is chosen greater than δ_c the roots will be imaginary.

The general solution form for the unknown function $I(\theta)$ is:

[23] $$I(\theta) = C_1\,\theta^{k_1} + C_2\,\theta^{k_2},$$

where C_1 and C_2 are two integration constants to be chosen in such a way as to satisfy the boundary conditions [18, 19].

It may be more convenient to avoid imaginary numbers. So, we perform the usual groupings defining trigonometric functions to get:

$$[24] \qquad I(\theta) = \theta^{-\lambda} \{ A\ \mathrm{si}[\nu\ \ell n(\theta)] + B\ \mathrm{co}[\nu\ \ell n(\theta)] \},$$

where:

$$\lambda = \frac{\alpha - r - \sigma^2/2}{\sigma^2}\ ; \qquad \nu = \frac{\sqrt{|\Delta|}}{\sigma^2}$$

and where si[] and co[] stand for the sine (resp. hyperbolic sine) and cosine (resp. hyperbolic cosine) functions, used when Δ is negative (resp. positive). A and B are two new constants of integration.

The borderline case where $\delta = \delta_c$, or $\Delta = 0$, -- which is unlikely to obtain -- would generate a different kind of solution:

$$[25] \qquad I(\theta) = C_1\ \theta^k + C_2\ \ell n(\theta)\ \theta^k,$$

where k is the single root of equation [20].

The four boundary conditions [18a,b, 19a, b] would appear to be all we need to solve for the four unknowns A, B, ℓ and u. In fact, as is evident from the conditions [18, 19] and from the form of the general solution [24], the corresponding system of equations is homogeneous in A and B. In order to avoid the trivial solution A = B = 0, we shall have to impose that some determinant be equal to zero. This will provide a restriction on the as yet arbitrary constant δ (see below equation [28]). The two unknowns A and B will remain determined only up to an arbitrary factor.[15]

It is easy, *in order to obtain an equation system for* ℓ *and* u, to eliminate A and B between the pair of equations [18a, 19a] on the one hand and the pair [18b, 19b] on the other. But the same equation system can be obtained by substituting the boundary conditions into the PDE. Defining for convenience:

$$[26] \quad \varepsilon_\ell = \frac{\ell/s}{1+ \ell/s} \qquad \varepsilon_u = \frac{us}{1 + us},$$

we obtain actually the same second-degree equation to be satisfied by ε_u and by ε_ℓ:

$$[27] \quad 0 = \delta + \gamma\, \varepsilon(\alpha - r) + \frac{1}{2}\, \gamma(\gamma-1)\, \sigma^2\, \varepsilon^2.$$

We impose, of course, that the discriminant of this equation be positive, as there can be no physical interpretation to portfolio control limits being complex numbers,[16] and we interpret the lower real root as ε_ℓ and the higher one as ε_u. The control limits ℓ and u are then obtained from ε_ℓ and ε_u by inverting the definitions [26].

---

[15] As in the frictionless case, the value of that factor would fall out of the terminal condition of the finite-horizon problem, as one postpones the horizon to infinity (see footnote #7). The procedure is not implementable in the case where transactions costs are present since we do not have an explicit solution for the finite-horizon problem. Fortunately, the missing factor is irrelevant for all practical purposes.

[16] This restriction sets an upper bound on the unknown value of δ. This upper bound turns out to be equal to δ^*, defined in [8], which is the value of δ which applies in the frictionless case. One is not surprised to find that the value of δ should always be on one side of the frictionless value.

At this stage, we have been able to determine, for any given value of the separating constant δ, the form of the general solution which depends on two unknown constants A and B, as well as the corresponding values of ℓ and u. We must now choose δ in such a way that it is possible to find A and B. This is accomplished by writing that the determinant of the linear system made up of the two boundary conditions [18] is equal to zero. Rewriting [18a, b] in the form:

$$\ell\ I'(\ell) = \gamma\ I(\ell)\ \varepsilon_\ell, \qquad u\ I'(u) = \gamma\ I(u)\ \varepsilon_u,$$

we obtain the following entries for this determinant:

[28]

$$\begin{vmatrix} -\lambda\ \mathrm{si}[\nu\ \ell n(\ell)] + \nu\ \mathrm{co}[\nu\ \ell n(\ell)] - \gamma\ \mathrm{si}[\nu\ \ell n(\ell)]\ \varepsilon_\ell & -\lambda\ \mathrm{co}[\nu\ \ell n(\ell)] - \nu\ \mathrm{si}[\nu\ \ell n(\ell)] - \gamma\ \mathrm{co}[\nu\ \ell n(\ell)]\ \varepsilon_\ell \\ -\lambda\ \mathrm{si}[\nu\ \ell n(u)] + \nu\ \mathrm{co}[\nu\ \ell n(u)] - \gamma\ \mathrm{si}[\nu\ \ell n(u)]\ \varepsilon_u & -\lambda\ \mathrm{co}[\nu\ \ell n(u)] - \nu\ \mathrm{si}[\nu\ \ell n(u)] - \gamma\ \mathrm{co}[\nu\ \ell n(u)]\ \varepsilon_u \end{vmatrix}$$

Equating this determinant to zero generally produces two roots. One of them is the trivial solution $\delta = \delta_c$ which should be disregarded.[17] We have not proven that equating this determinant to zero necessarily produces a second,

---

[17] It should be disregarded on the grounds that, when $\delta = \delta_c$, the solution is of the type [25], not [24]. Such a solution would generate a determinant different from [28]. In order for this other determinant to be equal to zero precisely for the value $\delta = \delta_c$, a very special combination of parameter values would have to be chosen. The analysis of this borderline situation is uninteresting.

non trivial solution for δ.

The non trivial root can be found by numerical iteration. Figure 1 depicts the value of the determinant as a function of δ for the following *numerical base case:*[18]

[29] $\alpha - r = 0.05$; $\gamma = -1$; $\sigma^2 = 0.04$; $s = 0.99$.

This base case in the absence of transactions costs ($s = 1$) would produce an optimal policy $\theta^* = 1.67$ (or a weight on the risky asset equal to $\theta^*/(1 + \theta^*) = 0.625$) and a value of $\delta^* = 0.015625$. Under 1% transactions costs, the root is found to be in the imaginary domain ($\Delta < 0$; $\delta_c < \delta < \delta^*$; $\delta_c = 0.01125$):

$\delta = 0.014713$; $\ell = 0.811094$; $u = 3.848747$;

implying a weight in the portfolio allocated to the risky asset fluctuating between $\ell/(1 + \ell) = 0.44785$ and $u/(1 + u) = 0.79376$, which is a tolerance zone of 27% on the upper side and 28% on the lower side of the theoretical optimum.

5. Comparative analysis

We now examine deviations from the numerical base case [29] in the dimensions of increasing transactions costs, increasing risk aversion and increasing risk.

5.1. Varying the size of transactions costs

Not surprisingly, as is illustrated in Figure 2, increasing the size of

---

[18] We choose here the same base case as in Constantinides (1986).

transactions costs (lowering the parameter s) widens the region of no transactions. This is especially true for low transactions costs: even a value of s extremely close to 1 already causes portfolio control limits appreciably to separate. The reason for this phenomenon is well known; because the Brownian motion is an infinite variation process, costs accompanying frequent rebalancing, even levied at a small rate, would quickly outweigh the benefits of precisely optimal diversification. The applicable caveat is, of course, that we have no reason to believe in the first place that asset prices should follow a process with unbounded variation when trading is subject to transactions costs.

At all levels of transactions costs, the region of no transactions is markedly wider than that obtained by Constantinides (1986). The difference between the two results is shown on Figure 2. Also, there is no tendency for increased transactions costs to bias the portfolio one way or the other. As is quite visible on Figure 2, this result contrasts with that of Constantinides who found that "transactions costs shift the region of no transactions towards the riskless asset". Both differences in results can be ascribed to the fact that, in Constantinides' model, consumption was taking place along the way, and not just at the terminal time and, furthermore, that consumption came out of the riskless asset only. When a steady flow of consumption expenditures must be met out of the existing cash on hand, there is both less room for fluctuations in the amount of cash available and more need to bias the portfolio in favor of cash, than when consumption is postponed to infinity.

5.2. Varying risk aversion

Figures 3 and 4 give two similar views of the way in which increasing risk aversion affects portfolio choices. Basically, there is very little interaction between risk aversion and transactions costs. Figure 3 provides a measure of the width of the portfolio weight band $[u/(1+u), \ell/(1+\ell)]$ as a function of risk aversion. It is apparent that the width of that band is nearly constant, in fact slowly decreasing as one scans risk aversions ranging from 0.4 to 2.7.

A similar message can be read on Figure 4 which controls for the effect of risk aversion on what would have been the optimal frictionless portfolio (see equation [9]); u/θ^* and ℓ/θ^* are slowly decreasing again as one increases risk aversion.

These results are qualitatively the same as those of Constantinides (1986).

5.3. Increasing risk

Broadly the same conclusions are reached for an increase in risk as for an increase in risk aversion. Figure 5 shows that the width of the band $[u/(1+u), \ell/(1+\ell)]$ is approximately constant around the optimal frictionless value, as σ^2 is gradually increased.

6. Conclusion and applications

The exact solution to the portfolio management problem which we have obtained is in the form of two control barriers. These set the maximum and minimum allowable degrees of imbalance in the portfolio which will be tolerated before any action is taken. We have obtained numerical results for the position of these barriers. The optimal policy identified here calls for wider bands of tolerated imbalance in the portfolio than had been suggested by Constantinides (1986). Also, the band is not biased in favor of cash, and the more so as one increases transactions costs, as was true in Constantinides' model. The comparative static analysis, however, yields conclusions broadly similar to those of Constantinides. Ours are exact results while previous work has only produced approximate solutions.

Several major applications and extensions of our exact solution can be envisaged. The first would be a contribution to the economics of the dealer function.[19] The investor we have considered here trades infrequently and in small quantities. For a price, however, he should be willing to absorb (positively or negatively) finite chunks of the risky asset when a customer comes by who desires to make a finite trade. These prices at which he would be willing to trade finite quantities would be his bid and ask prices. In this theory the determinant of the bid-ask spread would be purely the instantaneous volatility of returns, the size of transactions costs and the inventory of the dealer. A missing determinant, which was examined by Ho and Stoll (1981) would be the rate of random customer arrivals. There is hope that this added variable can also be incorporated in the optimization plan.

A second application would be an extension of the application just mentioned. Ultimately it must pay for the customer to place his order with the dealer rather than transact directly in the market. Also dealers must be allowed to trade with each other. Ultimately the equilibrium dynamics of the price in the presence of frictions must be determined, rather than postulated as they were here.

A third application of this exact solution would be to price derivative assets in the presence of transactions costs, when investors adopt optimal portfolio strategies, in continuous time. Our objective would be to find the

---

[19] On this count see: Treynor (1981), Ho and Stoll (1981), Treynor (1988).

bid- and ask-prices of a European call option in an intertemporal setting.

As has been amply demonstrated by Figlewski (1989), it is not true, in the presence of transactions costs, that a replicating argument would provide the right option price.[20] Instead it would be appropriate to analyze the portfolio strategy of an investor who acts as a trader (and pays transactions costs) on the primitive asset market and as a dealer in the option market.

[20] Recently, Leland (1985) has proposed a replicating strategy for option pricing which implies finite transaction costs and still generates with probability one a payoff equal to that of the option. Unfortunately, the frequency of portfolio revisions in his model is exogenously given, instead of being optimally chosen with respect to transactions costs. As a consequence, his formulation does not match the optimal investment strategy with proportional transaction costs which has been outlined by Constantinides (1979,1986). Merton (1988) also studies the problem of option pricing with transactions costs: he formulates a two-period replicating strategy in a binomial context, but does not extend it to an arbitrary number of periods and, consequently, is not in a position to determine the limiting value of the option price when time is allowed to become continuous. At any rate, pricing by replication is questionable in the current context: when transactions costs are present, an exact replication is not generally the most efficient method of manufacturing an option. Following the purchase or sale of an option, an intermediary would not typically choose to fully offset that transaction by means by sales and purchases of the underlying asset. A strategy of full offset would be unnecessarily costly and would not be pursued. It cannot consequently provide the price of the option under pure competition; it can only provide a lower or an upper bound on the price. In our view the option cannot be priced outside an optimal portfolio-investment framework.

References

Amihud, Y. and H. Mendelson, "Liquidity and Asset Prices: Financial Management Implications," Financial Management, Spring 1988, 5-15.

Benes, V.E., L.A. Shepp and H.S. Witsenhausen, "Some Solvable Stochastic Control Problems," Stochastics, 1980, 4, 39-83.

Bertola, G., "Irreversible Investment," working paper, MIT, July 1987, revised 1988.

Black, F., and Scholes, M. "The Pricing of Options and Corporate Liabilities," Journal of Political Economy, 81(3), May-June 1973, pp. 637-654.

Constantinides, G.M., "Multiperiod Consumption and Investment Behavior with Convex Transactions Costs," Management Science, 25(11), Nov. 1979, pp. 1127-1137.

--------------------, "Capital Market Equilibrium with Transactions Costs," Journal of Political Economy, 94(4), Dec. 1986, pp. 842-862.

Cox, J.C., Ross, S.A., Rubinstein, M., "Option Pricing: a simplified approach," Journal of Financial Economics, 7(2), June 1979, pp.229-263.

Davis, M.H.A. and A.R. Norman, "Portfolio Selection with Transactions Costs," working paper, Imperial College, September 1987.

Dixit, A., "A Simplified Exposition of Some Results Concerning Regulated Brownian Motion," working paper, Princeton University, August 1988.

Dixit, A., "Entry and Exit Decisions of Firms under Fluctuating Real Exchange Rates," unpublished paper, October 1987, revised 1988, forthcoming, Journal of Political Economy.

Dixit, A., "Hysteresis, Import Penetration, and Exchange Rate Pass-Through," unpublished paper, November 1987, forthcoming Quarterly Journal of Economics.

Dumas, B., "Pricing Physical Assets Internationally," Working Paper, Wharton School of the University of Pennsylvania, March 1988 (a).

Dumas, B., "Perishable Investment and Hysteresis in Capital Formation," working paper, University of Pennsylvania, November 1988 (b).

Dumas, B., "Super Contact and Related Optimality Conditions," working, University of Pennsylvania, November 1988 (c).

Figlewski, S., "Options Arbitrage in Imperfect Markets," Journal of Finance, December 1989.

Grossman, S.J. and G. Laroque, "Asset Pricing and Optimal Portfolio Choice in the Presence of Illiquid Durable Consumption

Goods," working paper, Princeton University and INSEE (Paris, France), 1987.

Grossman. S.J. and J.-L. Vila, "Optimal Dynamic Trading with Leverage Constraints," working paper, University of Pennsylvania, March 1989.

Harrison, J.M., Brownian Motion and Stochastic Flow Systems, New York; John Wiley & Sons, 1985.

Harrison, J.M. and M.I. Taksar, "Instantaneous Control of Brownian Motion," Mathematics of Operations Research, 8, 3, August 1983, 439-453.

Hasbrouck, J. and R.A. Schwartz, "Liquidity and Execution Costs in Equity Markets," The Journal of Portfolio Management, Spring 1988, 10-16.

Ho, T., Stoll, H., "Optimal Dealer Pricing Under Transactions and Returns Uncertainty," Journal of Financial Economics, 9(1), March 1981, pp.47-73.

Kamin, J. H., "Optimal Portfolio Revision with a Proportional Transactions Cost," Management Science, 21, 11 (July 1975), 1263-1271.

Krylov, N.V., Controlled Diffusion Processes, New York: Springer Verlag, 1980.

Leland, H., "Option Pricing and Replication with Transaction Costs," The Journal of Finance, 40(5), Dec.1985, pp. 1283-1031.

McKean, H.P., Jr., "Appendix: A Free Boundary Problem for the Heat Equation Arising from a Problem in Mathematical Economics," Industrial Management Review, 6, 2 (Spring 1965), 32-39.

Merton, R.C., "Lifetime Portfolio Selection under Uncertainty: the Continuous-Time Case," The Review of Economics and Statistics, 51(3), Aug. 1969, pp. 247-257.

------------, "Optimum Consumption and Portfolio Rules in a Continuous-Time Model,", Journal of Economic Theory, 3(4), Dec. 1971, pp.373-413.

------------, "On the Application of the Continuous-Time Theory of Finance to Financial Intermediation and Insurance," Paper presented at the HEC-ISA Conference, June 29-July 1 1988, Paris.

Perold, A.F., "The Implementation Shortfall: Paper versus Reality," The Journal of Portfolio Management, Spring 1988, 4-9.

Samuelson, P.A., "Rational Theory of Warrant Pricing," Industrial Management Review, 6, 2 (Spring 1965), 41-50.

Taksar, M., Klass, M.J. and Assaf, D., "A Diffusion Model for Optimal Portfolio Selection in the Presence of Brokerage Fees," Mathematics of Operations Research, 13 (2), May 1988, 277-294.

Treynor, J.L, "What does it take to win the trading game?," Financial Analysts

Journal, January-February 1981, 55-60.

Treynor, J.L., "The Economics of the Dealer Function," Financial Analysts Journal, November-December 1987, 27-34.

Figure 1. Locating the root of the system

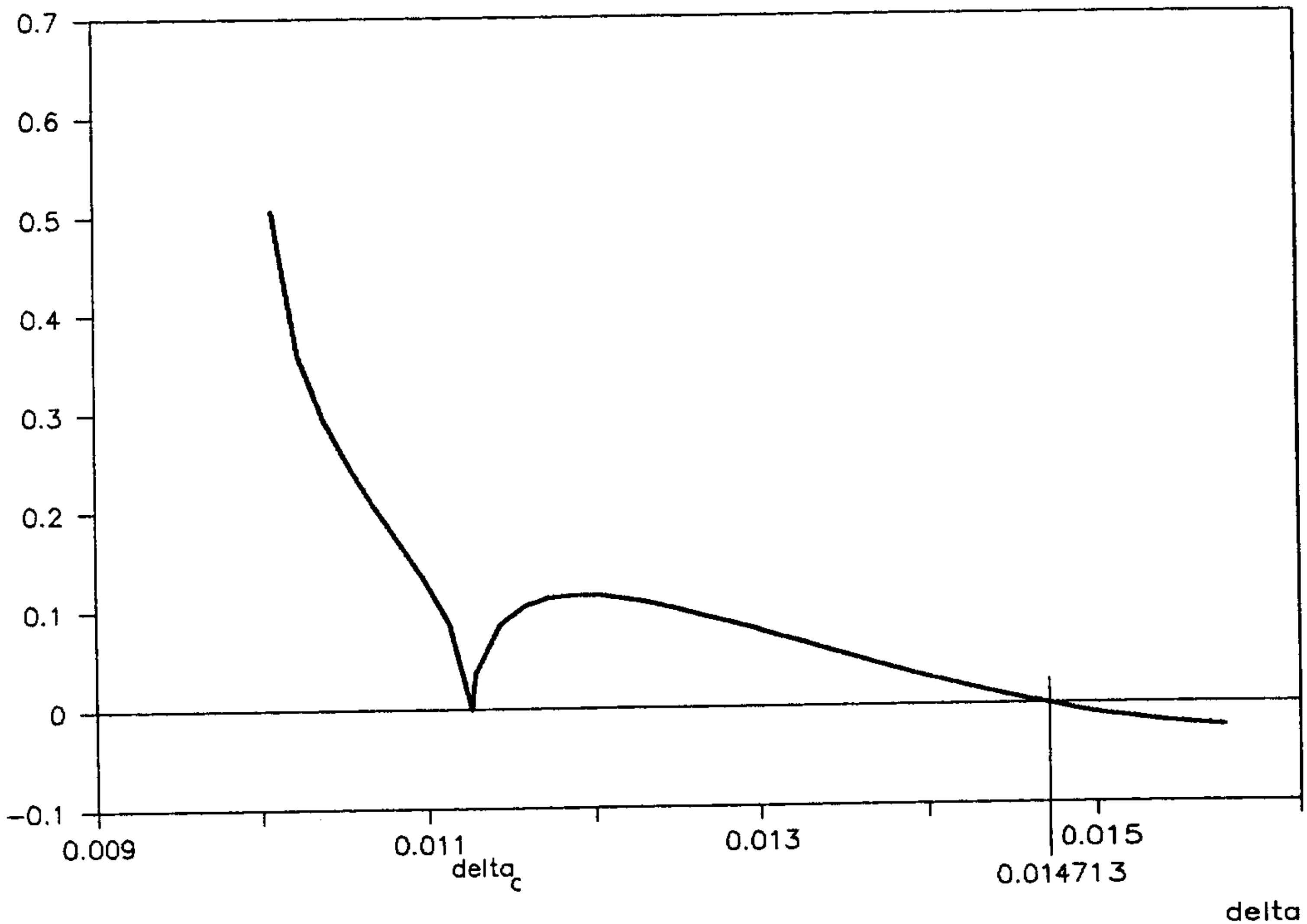

Figure 2: The effect of transactions costs on portfolio choice

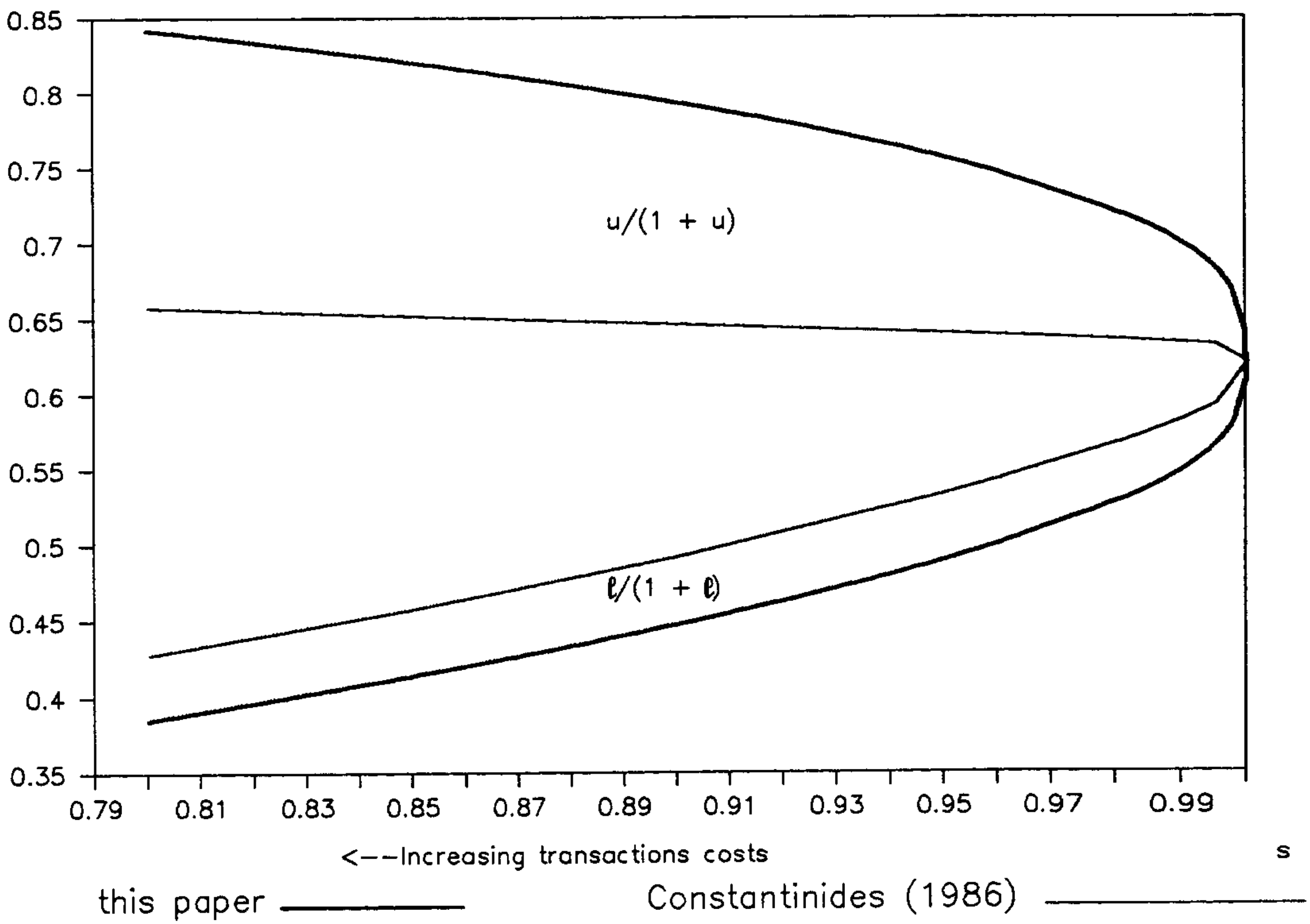

Figure 3: The effect of risk aversion on portfolio choice

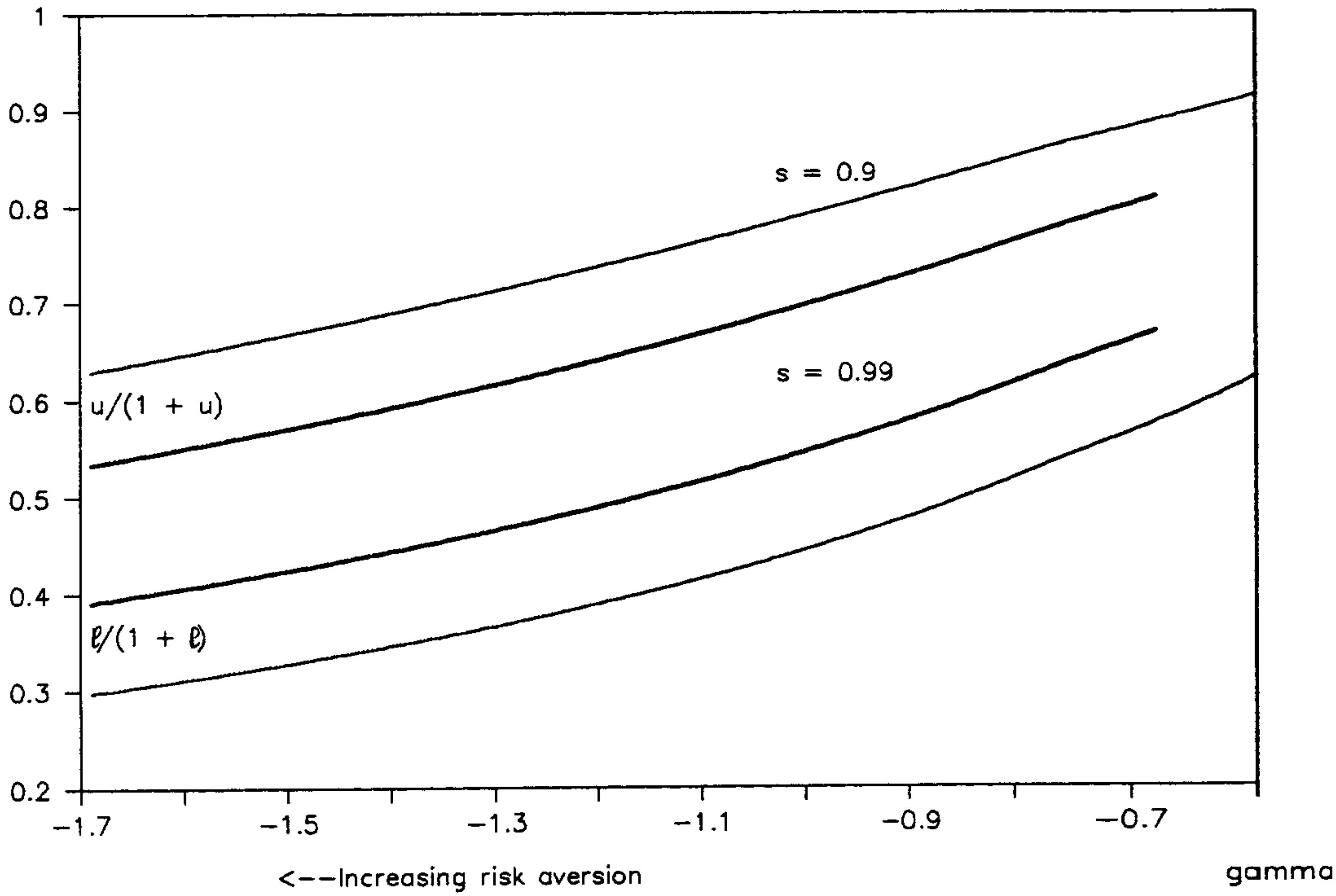

Figure 4: The effect of risk aversion on portfolio choice relative to the frictionless case

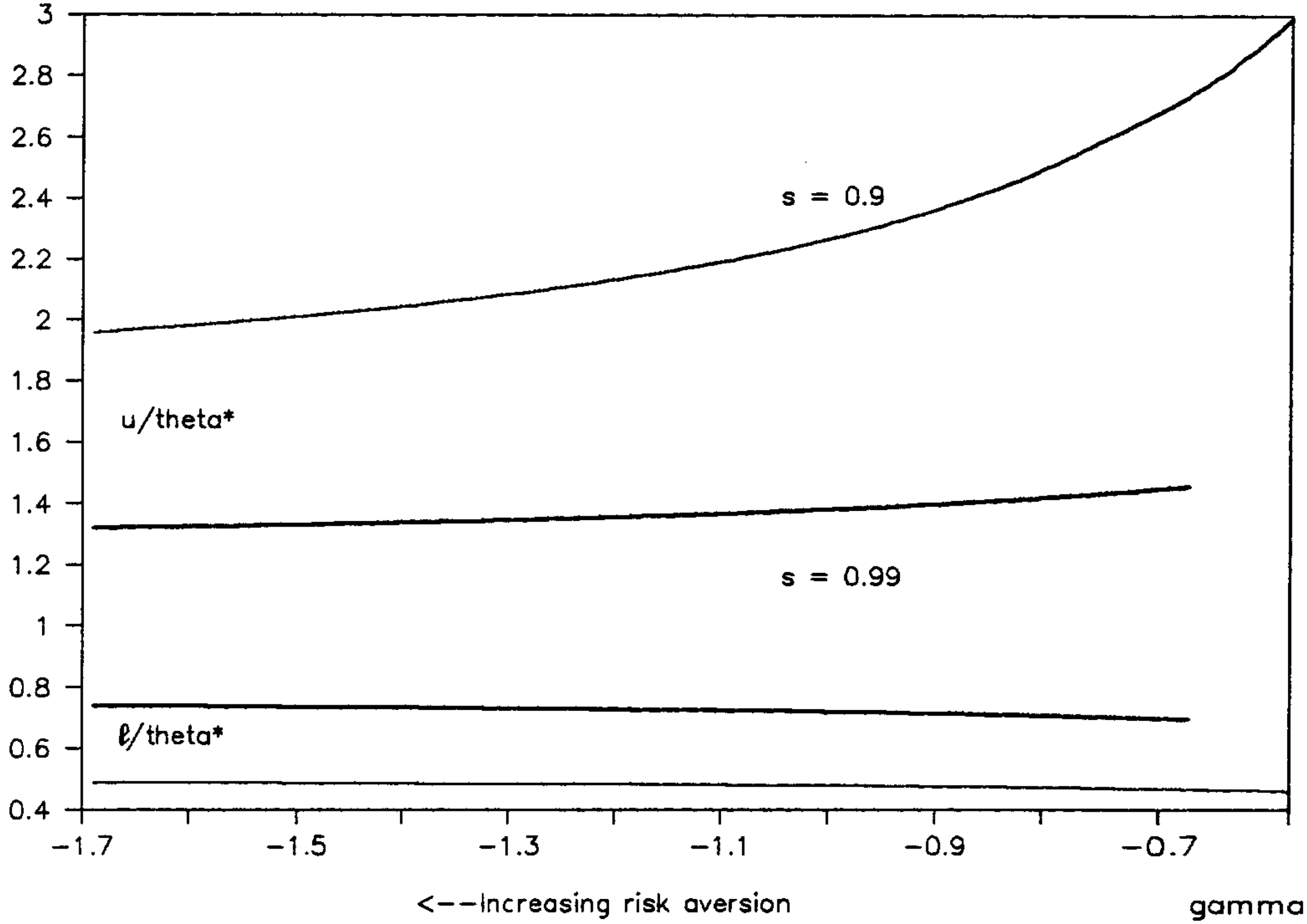

Figure 5: The effect of risk on portfolio choice