OPTIMAL DYNAMIC TRADING WITH LEVERAGE CONSTRAINTS

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I. INTRODUCTION

We solve for the optimal dynamic trading strategy of an investor who faces two constraints. The first constraint is a limitation on his ability to borrow for the purpose of investing in a risky asset, i.e, the market value of his investments in the risky asset X, must be less than an exogenously given function of his wealth $\overline{X}(W)$. The second constraint is the requirement that the investor's wealth be non-negative at all times, i.e., $W_t \geq 0$. We assume that the investor has constant relative risk aversion A, and that the value of the risky asset follows a diffusion with drift $\mu+r$ (where r is the risk free rate) and per unit time variance σ^2. In the absence of the first constraint, $X \equiv (\mu/\sigma^2)*W/A$. We prove that in the presence of the above constraints the optimal investment is $X \equiv \text{Min}[(\mu/\sigma^2)*W/\alpha,\ \overline{X}(W)]$. The coefficient α is not in general equal to A, and represents the extent to which the investor alters his strategy even when the constraints are not binding because of the possibility that the constraints will become binding in the future.

If $A<1$, then $\alpha<A$, and if $A>1$, then $\alpha>A$. Thus, for example, a risk neutral investor (i.e., $A=0$) will not lever his wealth to the highest feasible amount $\overline{X}(W)$, but instead acts as if he is risk averse at low levels of wealth. The fact that the shape of the investment opportunity set depends on wealth causes the risk aversion of the indirect utility function to differ from the risk aversion of the direct utility function, even at levels of wealth where the investment opportunity set is independent of wealth.

We present a number of applications of our results. The most direct application is for an investor who must put up margin for his investment in stocks or futures. Another example involves an investor who can borrow on personal account and use some of the proceeds to invest in the risky asset.

A more complex application involves a pension Fund which has a liability stream which it must finance out of the capital of the Fund. The Fund cannot borrow to lever its investment in the risky asset. Further, the market value of the Fund's investments must not fall below the value of its liabilities. Black and Jones (1987), Perold (1986) and Black and Perold (1987) have recommended a Constant Proportion Portfolio Insurance (CPPI) strategy for the management of a Fund with the above constraints. Their recommendation is exactly of the same form as the solution to our optimal control problem except that they do not explain how the investor should choose the proportionality factor or equivalently the coefficient α. Our analysis shows that the effective degree of risk aversion α, of the Fund, cannot be separated from the constraints which impinge on the Fund. In particular, we compute the benefits to a Fund from using our strategy, relative to the benefit of using a myopic CPPI strategy which sets alpha equal to A. We show that the benefits from using the correct alpha are on the order of 25% of the value of the Fund. That is, it can be extremely costly to choose a strategy which does not adjust today for the possibility that leverage constraints may be binding in the future.

Another application of our result concerns risky project selection by a firm subject to borrowing constraints. Consider a project with returns uncorrelated with the market, and an expected return in excess of the risk free rate. One might guess that a firm would act in a risk neutral fashion toward the project and fully lever all of its capital for investment in the project. Our result implies that the firm will not do so if it cannot be certain that it can raise new capital in the future. For example a firm with 1$ in equity which can borrow 2$ at the risk free rate to invest in a project with an expected return higher than the risk free rate may not borrow the full 2$ because of the possible inability to borrow 4$ in the future which would be required to keep its investment in the project a constant proportion of its capital.

Section 2 presents a simple example of the risk neutral case to show how the leverage and solvency constraints interact to cause the indirect utility function of wealth to be sufficiently concave as to cause the risk neutral investor to act in a risk averse manner. The fact that he cannot take advantage of high levels of wealth by investing the same proportion of wealth at high levels of wealth, lowers his marginal utility of wealth when wealth rises. Section 3 sets up the general framework. Section 4 presents analysis, while Section 5 presents applications and conclusions. Proofs appear in the Appendix.

II A SIMPLE EXAMPLE

To understand how leverage constraints affect optimal investment strategies, consider the example of a risk neutral investor who has access to the following investment opportunity set. The investor may put money into a project will double it in the "good event" (probability .1) or reduce it by 10% in the "bad event" (probability .9). Hence, the expected return is 1% i.e., .1*100%+.9*(-10%). The investor has access to 3 independent draws. His initial net wealth, W_0 is $100.

Assume that the investor can borrow at a zero interest rate to invest in the project but must repay his loan with probability one. If we let X_t be the dollars invested in the project and W_t be the net wealth at time t, then the no bankruptcy constraint is the condition that $W_{t+1} \geq 0$ if the bad event occurs, i.e. $.9X_t \geq X_t - W_t$, since $X_t - W_t$ is the amount owed at t+1 and $.9X_t$ is the capital remaining if the bad event occurs. Hence the no bankrupcy constraint is

$$X_t \leq 10W_t \quad \text{(solvency)}. \tag{2-1}$$

It is east to verify the fact that if (2-1) is the only constraint, then the investor will follow a fully leveraged strategy whereby he always invest 10 times his wealth in the risky project.

Now assume that in addition to (2-1), there is an absolute limit of $900 on how much he can borrow. Hence his investment must satisfy the additional constraint

$$X_t \leq W_t + 900. \tag{2-2}$$

Constraints (2-1) and (2-2) can be combined:

$$X_t \le \bar{X}(W_t) \equiv \text{Min}(10W_t,\ W_t+900). \tag{2-3}$$

It might be conjectured that the risk neutral investor will continue to lever as much as feasible, i.e. to the point where $X_t=\bar{X}(W_t)$. We will show below that this constrained fully leveraged strategy (CFLS) is <u>not</u> optimal, even for a risk neutral investor.

Let $V_t(W)$ be the maximized value of the expected utility of final wealth when there are t periods to go and when current wealth is W. $V_t(W)$ satisfies the Bellman equation:

$$V_{t+1}(W) = \underset{X_{t+1}}{\text{Max}}\ [.1V_t(W+X_{t+1}) + .9V_t(W-.1X_{t+1})] \tag{2-4}$$

subject to $X_{t+1} \le \bar{X}(W)$.

$V_0(W) \equiv W$.

It is immediate that $X_1(W)=\bar{X}(W)$, and

$$V_1(W) = \begin{cases} 1.1W & \text{if} \quad W \le 100 \\ 1.01W + 9 & \text{if} \quad W \ge 100, \end{cases} \tag{2-5}$$

that is, with one period to go, he follows the CFLS strategy. However, this strategy has caused his indirect utility function to become strictly concave as indicated in (2-5) and Figure 1. <u>The fact that he cannot take advantage of high levels of wealth by investing a constant proportion of wealth in the project lowers his marginal utility of wealth when wealth rises</u>.

A slighly tedious but direct calculation shows that $X_2(W)=\bar{X}(W)$, i.e. the CFLS is optimal when only two periods remain. That is, even though $V_1(W)$ is concave, it is not sufficiently concave to cause the investor to deviate from the risk neutral strategy.[1] Direct computation shows that

$$V_2(W) = \begin{cases} 1.21W & \text{if } W \le 100/11 \\ 1.111W+.9 & \text{if } 100/11 \le W \le 100 \\ 1.093W+2.7 & \text{if } 100 \le W \le 1900/9 \\ 1.0201W+18.09 & \text{if } 1900/9 \le W. \end{cases} \tag{2-6}$$

It can be seen from Figure 1 and (2-6) that $V_2(W)$ has become even "more concave" that $V_1(W)$. The fact that high levels of wealth cannot be proportionally invested in the risky asset (because of (2-2)) leads marginal indirect utility to fall.

We now show that $V_2(W)$ is sufficiently concave that $X_3(W)<\bar{X}(W)$ for some W, i.e. CFLS is not optimal when 3 periods remain. Take W=100, then from (2-3), following CFLS would involve setting $X_3(100)=1000$. We show that this is suboptimal by showing that his objective is decreasing at $X_3=1000$: With 3 periods remaining, X_3 is chosen to maximize

$$.1V_2(100+X_3) + .9V_2(100-.1X_3) \equiv H_3(X_3). \tag{2-7}$$

Note that $V_2(W)$ is differentiable at W=1100 and W=0. From (2.6),

$$H_3'(1000) = (.1)(1.0201) + (.9)(-.1)(1.21) = -.00689. \tag{2-8}$$

It follows from (2-8), that there is an X_3 less than $\bar{X}(W)$ which improves upon $X_3=\bar{X}(W)$.[2]

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[1] It can be shown that this statement is true for any parameter values, i.e. if the investor is risk neutral the CFLS strategy is optimal when there are two periods remaining.

[2] Dybvig (1988) offers an interesting analysis of inefficient dynamic strategies. In particular, he show that insufficient diversification over time leads to inefficiency. Dybvig presents a general criterion for a strategy to be

3. THE MODEL

A. The investment problem:

We consider the decision problem of an investor with an horizon of investment of T years and who seeks to maximize the expected utility of his final wealth. The investor can distribute his funds between two assets. One asset (e.g. a bond) is riskless with rate of return r. The other asset is a stock with value p_t. We assume that:

ASSUMPTION 1: $$dp_t/p_t = (\mu+r)dt + \sigma db_t \tag{3-1}$$

where μ and σ are positive constants and b_t is a normalized brownian motion. We assume that there are no transaction costs involved in buying or selling these financial assets.

The investor's strategy must satisfy two constraints. First, his final wealth must be non-negative. Second, the dollar amount X_t invested in the risky asset at t, must be less than or equal to an exogenous function $\bar{X}(W_t)$ of his wealth W_t at t. We assume that:

ASSUMPTION 2: $$\bar{X}(W) = k(W+\bar{L}) \tag{3-2}$$

where k and $\bar{L}$ are non-negative constants.

The formulation above is general enough to encompass several interesting examples. For instance, if the investor has access to a fixed credit line $\bar{L}$,

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optimal for some utility function in a frictionless market. By contrast, our goal is to derive the optimal dynamic strategy for a particular utility function in the presence of leverage constraints. Nevertheless, his examples of inefficient strategies are similar in spirit to the example presented herein.

then $\bar{X}(W_t)=W_t+\bar{L}$. If the investor needs only to put down a certain fraction f of his stock purchases and can borrow the remaining at the risk free rate r, then $\bar{X}(W_t)=(1/f)W_t$.

We will use the following notation: W_t is the wealth to invest at date t. $\bar{X}(W_t) = k(W_t+\bar{L})$ is the maximum dollar amount that can be invested in the risky asset at date t. X_t is the actual amount invested.

With this notation, the two constraints above can be written as:

$$W_t \geq 0 \qquad \text{for all } t, \tag{3-3}$$

$$X_t \leq \bar{X}(W_t) \qquad \text{for all } t. \tag{3-4}$$

The dynamics of W_t are given by:

$$dW_t = rW_t\, dt + \mu X_t\, dt + \sigma X_t\, db_t. \tag{3-5}$$

The objective of the investor is to maximize the expected utility of the final wealth, $Eu(W_T)$. We assume that the utility function exhibits constant relative risk aversion:

ASSUMPTION 3:

$$u(W_T) = \frac{1}{1-A} W_T^{1-A} \qquad \text{for some } A\geq 0,\ A\neq 1.$$

The logarithmic utility (A=1) will be analyzed as a separate case in section 4.

B. Conditions for optimality:

The investor maximizes $Eu(W_T)$ subject to the dynamics (3-5) and to constraints (3-3) and (3-4). Let $J(W,t;k,\bar{L})$ be the value function, i.e. the supremum of the expected utility that the investor can achieve given that his wealth at time t is W and that $\bar{X}(\cdot)$ is characterized by the parameters k and $\bar{L}$. Let $X(W,t;k,\bar{L})$ be the dollar amount invested in the risky asset as a function of wealth, time and of the parameters k and $\bar{L}$.

Assuming that the value function, $J(\cdot)$ is twice differentiable in W and once continously differentiable in t, $J(\cdot)$ satisfies the partial differential equation below, which is known as the Bellman equation:

$$0 = J_t + rW J_W + \max_{X\leq\bar{X}(W)} \{ \mu X J_W + \frac{\sigma^2}{2} X^2 J_{WW}\} \tag{3-6}$$

where J_t, J_W, J_{WW} denote the partial derivatives of J. To understand (3-6), note that $J(W,t;k,\bar{L})=\max_X E_t J(W_{t+\Delta t},t+\Delta t;k,\bar{L})$. Therefore,

$$0 = \frac{1}{\Delta t} \max_X E_t[J(W_{t+\Delta t},t+\Delta t;k,\bar{L})-J(W,t;k,\bar{L})].$$

If we take the limit as $\Delta t\rightarrow 0$ of the right hand side then we obtain the drift of $J(W_t,t;k,\bar{L})$, and thus (3-5) follows. J must also satisfy:

$$J(W,T;k,\bar{L}) = \frac{1}{1-A} W_T^{1-A} , \tag{3-7}$$

$$J(0,t;k,\bar{L}) = 0 \quad \text{if} \quad A<1 , \tag{3-8}$$

$$\lim_{W\rightarrow 0} J(W,t;k,\bar{L}) = -\infty \quad \text{if} \quad A>1. \tag{3-9}$$

Without the cash flow constraint $X_t\leq\bar{X}(W_t)$, the investor's problem reduces to the standard problem studied by Merton (1971). Proposition 3-1 summarizes the results of Merton. Merton proved that the optimal strategy consists in investing a fixed proportion of the wealth in the risky asset. This proportion ($\mu/A\sigma^2$) increases when the excess return on stocks over the risk free rate, μ, increases, when the volatility σ decreases and when the relative risk aversion A decreases.

Proposition 3-1. **(Merton 1971) If there is no credit constraint (formally $\bar{X}=\infty$), the optimal investment is:**

$$X_t^{\infty} = \frac{\mu}{A\sigma^2} W_t . \tag{3-10}$$

The value function is:

$$J^{\infty}(W,t) = e^{\eta(T-t)} \frac{W^{1-A}}{1-A} \quad \textbf{with} \tag{3-11}$$

$$\eta = (r + \frac{\mu^2}{2\sigma^2 A}) (1-A). \tag{3-12}$$

If $\bar{X}(W)=k(W+\bar{L})\geq (\mu/A\sigma^2)W$, then the leverage constraint (3-4) is not binding and the solution to (3-5) is the solution to the unconstrained problem (i.e. (3-10)-(3-12)).

<u>ASSUMPTION 4</u>: We assume in the sequel that the leverage constraint is binding for at least some values of W i.e:

$$\mu > A\sigma^2 k \tag{3-13}$$

Since $\bar{X}(.)=k(W+\bar{L})$, it follows from (3-10) and (3-13), that the cash flow constraint (3-4) will be binding, typically when the wealth of the investor is large. The inequality (3-13) will hold if the expected return on stocks in excess of the risk free rate μ is large, if the volatilty σ is low, if the availability of credit is poor or if the risk aversion of the investor is small.

In the presence of leverage constraints, one natural candidate for an optimal strategy is the myopic strategy where the investor invests the minimum of two quantities: (a) what he would have invested ignoring the leverage constraint and (b) the maximum level of investment $\bar{X}(W)$. Formally the myopic investment strategy is defined by:

$$X^{Myopic}(W) = \min[(\mu/A\sigma^2)W,\ \bar{X}(W)] = \min[(\mu/A\sigma^2)W,\ k(W+\bar{L})]. \qquad (3\text{-}14)$$

When following the myopic strategy, the investor ignores the cash flow constraint until it is binding. We will show that the **myopic strategy is not optimal in general**: The optimal investment while the credit constraint is not binding is affected by the fact that the credit constraint may be binding in the future. However, as shown below, when $\bar{L}=0$, the myopic strategy is indeed optimal, because under Assumption 4, the leverage constraint is binding for all W.

Proposition 3-2. **If $\bar{L}=0$, the optimal investment is**

$$X(W,t;k,0) = kW, \qquad (3\text{-}15)$$

the value function is:

$$J(W,t;k,0) = e^{\lambda(T-t)} \frac{W^{1-A}}{1-A} \quad \text{with} \qquad (3\text{-}16)$$

$$\lambda = (1-A)\ (r+\mu k-\frac{A\sigma^2}{2}k^2). \qquad (3\text{-}17)$$

Proof: $J(W,t;k,0)$ defined above satisfies the Bellman equation (3-6) and the boundary conditions (3-7)-(3-9) ///

The next Proposition provides useful comparisons between $J(W,t;k,0)$, $J^{\infty}(W,t)$ and $J(W,t;k,\bar{L})$.

<u>Proposition 3-3.</u> Let $J(W,t;k,\bar{L})$ be the value function if the cash flow constraint is given by $\bar{X}(.)=k(W+\bar{L})$:

i] $J(W,t;k,0) \leq J(W,t;k,\bar{L}) \leq J^{\infty}(W,t)$ (3-17)

ii] $J(W,t;k,\bar{L}) \leq J(W+\bar{L},t;k,0)$ (3-18)

<u>Proof</u>: (3-17) follows from the fact that the set of feasible strategies increases when $\bar{X}(.)$ increases, W being constant.

(3-18) follows from the fact that a investor with wealth $W+\bar{L}$ and constraint $\bar{X}(W)=kW$ will have a larger feasible set than a investor with wealth W and constraint $\bar{X}(W)=k(W+\bar{L})$.///

Proposition (3-3) implies that the expected utility, $J(W,t;k,\bar{L})$ is bounded even if the investor is risk neutral (note that when A=0, J^{∞} is infinite while $J(W,t;k,\bar{L})$ is not by Proposition (3-1)). Furthermore, from (3-17) and (3-18) it follows that the 'growth rate' of $J(\cdot)$ as (T-t) goes to infinity is λ, the 'growth rate' of $J(W,t;k,0)$. To be precise, we define $V(W,t;k,\bar{L})$ as the "discounted" value function when t years are left to invest i.e.

$$V(W,t;k,\bar{L}) \equiv J(W,T-t;k,\bar{L})\, e^{-\lambda t} \tag{3-19}$$

where λ is defined by equation (3-16).

From Propositions 3-2 and 3-3, if follows that:

$$\frac{W^{1-A}}{1-A} \leq V(W,t;k,\bar{L}) \leq \frac{(W+\bar{L})^{1-A}}{1-A}\ . \tag{3-20}$$

Hence V is bounded independently of t. The growth rate of the expected utility is independent of the constant $\bar{L}$ and can therefore be computed by setting $\bar{L}=0$

(Proposition (3-2)) We have not been able to compute the discounted value function $V(W,t;k,\bar{L})$ (unless $\bar{L}=0$) associated with the finite horizon case. However, we have been able to do so for the investment problem with infinite horizon, i.e. under the definition

$$V(W;k,\bar{L}) \quad = \quad \lim_{t\to\infty} V(W,t;k,\bar{L}). \tag{3-21}$$

Proposition 3-4. **$V(W;k,\bar{L})$ exists and is continuous for $W>0$.**

Proof: (See appendix A).

The solution to the infinite horizon problem is given in the next Section.

4. A STATIONARY PROBLEM

The Bellman equation for the finite horizon problem is a partial differential equation with two boundary conditions. As far as we know, there is no explicit solution to this equation. Furthermore, the fact that the problem is time-dependent makes it difficult to analyze. The key idea of this part is to consider a limiting problem "far away" from T, by computing $V(W;k,\bar{L})$ (see proposition 3-4). The limiting problem is time-independent and therefore it is easier to solve.

A. Long-run criteria.

Suppose that the objective of the manager is to maximize:

$$\liminf_{t\to\infty} \; E\left[e^{-\lambda t} \frac{W_t^{1-A}}{1-A} \right] \tag{4-1}$$

subject to:

$$W_t \geq 0 \qquad \text{and} \qquad X_t \leq k(W_t+\bar{L}), \tag{4-2}$$

where λ is given in (3-17). The objective (4-1) deserves some comments. We are considering a investment problem where the investor maximizes the expected utility of final wealth and where the horizon is infinite (i.e very long). The term $e^{-\lambda t}$ is a scaling factor which guaranties that the limit in (4-1) is not trivial (i.e is not 0 or infinity). Note that for every finite horizon, the presence of the scaling factor $e^{-\lambda t}$ does not affect the optimal policy. Another possible interpretation, in the case where the investor is acting on behalf of another agent, is that the investor is evaluated according to his long-run performance compared to his **expected** long-run performance with a simple "proportional" strategy (X=kW). Indeed, if the investor follows the proportional strategy, his expected performance is:

$$E \frac{W_t^{1-A}}{1-A} = e^{\lambda t} \frac{W_0^{1-A}}{1-A} \quad . \qquad \text{(see proposition 4-2)}$$

Criterion (4-1) is equivalent to maximizing the ratio of expected utilities:

$$\frac{\text{Expected utility under strategy } X(W)}{|\text{Expected utility under strategy } k*W|}$$

i.e. maximizing

$$\liminf_{t\to\infty} \frac{E\left\{ \frac{W_t^{1-A}}{1-A} \right\}}{\left| e^{\lambda t} \frac{W_0^{1-A}}{1-A} \right|} \tag{4-3}$$

Let $R(W;k,\bar{L})$ be the value function associated with the stationary problem (4-1). R is the value function associated with a well defined optimization problem and can be interpreted that way. However, the following proposition states that the stationary problem is the limit of the non stationary one. This gives another interpretation of R: $R(W;k,\bar{L})$ is the limit of the finite horizon "discounted" value function, $V(W;k,\bar{L})$.

<u>Proposition 4-1.</u> i] $$\frac{W^{1-A}}{1-A} \le R(W;k,\bar{L}) \le \frac{(W+\bar{L})^{1-A}}{1-A}$$

ii] $$R(W;k,\bar{L}) = V(W;k,\bar{L}) = \lim_{t\to\infty} V(W,t;k,\bar{L})$$

<u>Proof</u>: see appendix B.

We now derive the Bellman equation associated with problem (4-1) and derive sufficient conditions for optimality.

B. The Bellman equation:

Let $V(W;k,\bar{L})$ be the value function. By the optimality principle:

$$V(W;k,\bar{L}) = \max E\,[e^{-\lambda\tau}V(W_\tau;k,\bar{L}) \mid W_0=W\,], \tag{4-4}$$

for any stopping time τ.

By assuming that V is twice continuously differentiable (C^2) in W and by letting τ go to zero, (4-4) yields the Bellman equation:

$$\lambda\, V(W;k,\bar{L}) = rW\, V_W(W;k,\bar{L}) + \max_{X\leq k(W+\bar{L})} \{\, \mu X\, V_W(W;k,\bar{L}) + \frac{1}{2}\,\sigma^2 X^2 V_{WW}(W;k,\bar{L})\,\} \tag{4-5}$$

where V_W and V_{WW} denote partial derivatives of V.

Since problem (4-1) is stationary, $V(\cdot)$ is time independent. Hence, the Bellman equation is an ordinary differential equation as opposed to a partial differential equation. The optimal investment in the risky asset is:

$$X(W;k,\bar{L}) = \min \left\{ -\frac{\mu V_W(W;k,\bar{L})}{\sigma^2 V_{WW}(W;k,\bar{L})} \;;\; k(W+\bar{L}) \right\} \tag{4-6}$$

The next proposition gives sufficient conditions for a function to be the value function.

Proposition 4-2. Let $V^*(W;k,\bar{L})$ be a function defined for $W>0$ such that:

i] V^* is twice continuously differentiable in W.

ii] V^* satisfies equation (4-5).

iii] $\dfrac{W^{1-A}}{1-A} \leq V^*(W;k,\bar{L}) \leq \dfrac{(W+\bar{L})^{1-A}}{1-A}$.

iv] If W_t is the wealth process under the investment (4-6) then:

$$\lim_{t\to\infty} e^{-\lambda t} E \left\{ \frac{(W_t+\bar{L})^{1-A}}{1-A} - \frac{W_t^{1-A}}{1-A} \right\} = 0.$$

Then V^* is the value function V associated with problem (4-1) and $X(W;k,\bar{L})$ defined by (4-6) is the optimal investment.

Proof: (see appendix C).

Remark: If A<1, then $\left\{ \frac{(W_t+\bar{L})^{1-A}}{1-A} - \frac{W_t^{1-A}}{1-A} \right\}$ is less than $\frac{\bar{L}^{1-A}}{1-A}$ and $\lambda>0$ so (iv) is a trivial condition.

We now solve for the optimal investment $X(W;k,\bar{L})$.

C. Optimal investment

From equations (4-6), it follows that the optimal investment will be equal to $(\mu/\sigma^2)*V_W/(-V_{WW})$ in the unconstrained domain U defined by

$$U \equiv \{ W \text{ such that } \frac{\mu}{\sigma^2} \frac{V_W}{-V_{WW}} < k(W+\bar{L}) \}$$

and to $k(W+\bar{L})$ in the constrained domain C defined by

$$C \equiv \{ W \text{ such that } \frac{\mu}{\sigma^2} \frac{V_W}{-V_{WW}} > k(W+\bar{L}) \}.$$

In Proposition 4-3 below, we show that U is of the form $]0,W^*[$, i.e. the leverage constraint in not binding for low levels of wealth and binding thereafter. Furthermore, we show that the value function V exhibits constant relative risk aversion in U and that

$$V(W) = K_0 \frac{W^{1-\alpha}}{1-\alpha} \quad \text{in W belongs to U} \qquad (4\text{-}7)$$

where K_0 is a positive constant and α is the positive root of the second degree equation:

$$r\,\alpha^2 + (\lambda - r + \frac{\mu^2}{2\sigma^2})\,\alpha - \frac{\mu^2}{2\sigma^2} = 0 . \tag{4-8}$$

<u>Proposition 4-3.</u> **Assume that the value fuction is C^2.**

i) The unconstrained domain U is an interval $]0,W^*[$ and in U, V is of the type $K_0 \frac{W^{1-\alpha}}{1-\alpha}$ where α is the positive root of equation (4-8).

ii) The optimal investment is:

$$X(W) = \min\,[\,\frac{\mu}{\alpha\sigma^2}\,W,\ k(W+\bar{L})] \tag{4-9}$$

iii) α lies between 1 and A.

iv) For each W, X is larger (respectively smaller) than the myopic strategy if A is larger (respectively smaller) than 1.

v) X increases with μ; X decreases with A and σ.

vi) X increases with r if $A>1$ and decreases with r if $A<1$.

vii) If $A=0$, then $\alpha>0$ which implies that even a risk neutral investor will not invest $k(W+\bar{L})$ in stocks for low values of W.

<u>Proof</u>: see appendix D.

We have not been able to prove that the value function is indeed twice continuously differentiable for any value of μ, σ, r and A. Proposition 4-3' below lists the cases where smoothness is rigourously proven.

Notation: $\delta = \mu/k\sigma^2 - A$.

Proposition 4-3'. Assume that r=0[3]. If A>1 and A+2δ>Aδ or if A<1 then the value function is C^2.

Proof: see appendix E.

D/ The cost of myopia

In Proposition 4-3 above, we have shown that the myopic strategy is not optimal. It is interesting to know "how bad" the myopic strategy is. For this purpose, let $H(W;k,\bar{L})$ be the long-run performance if the manager follows the myopic strategy. H is a solution of the differential equation:

$$\lambda H = rW\,H_W + \frac{\mu^2}{A\sigma^2}\,W\,H_W + \frac{\mu^2}{2A^2\sigma^2}\,W^2\,H_{WW} \qquad \text{if } W<\hat{W} \tag{4-10a}$$

$$\lambda H = rW\,H_W + \mu k(W+\bar{L})H_W + \frac{\sigma^2}{2}k^2(W+\bar{L})^2\,H_{WW} \qquad \text{if } W>\hat{W} \tag{4-10b}$$

where $\hat{W}$ is the "switching point" under the myopic strategy $[(\mu/A\sigma^2)\hat{W}=k(\hat{W}+\bar{L})]$:

$$\hat{W} = \frac{A}{\delta}\,\bar{L} \qquad (\delta = \frac{\mu}{k\sigma^2} - A).$$

To simplify the analysis, let us assume that r=0 in which case an explicit solution to (4-10) can be given (see appendix D). We know that H<V (since the myopic strategy is not optimal). To measure the improvement of our strategy over the myopic strategy, let us define the function ϕ:

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[3]The case r=0 is not as restrictive as it sounds. This case will have a natural interpretation in the portfolio insurance problem (see part section 5A).

$$V(W;k,\bar{L}) = H(W(1+\phi(W/\bar{L}));k,\bar{L}). \tag{4-11}$$

Thus the investor is indifferent between starting with wealth W and following the optimal strategy and starting with wealth $W(1+\phi)$ and following the myopic strategy. Since V and H are homogenous with respect to $(W,\bar{L})$, $\phi(.)$ is a function of $W/\bar{L}$. The values of ϕ for different levels of risk aversion A, expected excess return μ, standard deviation σ and level of $W/\bar{L}$ are displayed in the following table.

INSERT TABLE 1

The table above shows that taking credit constraints into account substantially improves the performance. The extent of the improvement is a decreasing function of the risk aversion A and of the ratio $W/\bar{L}$. For instance, in the extreme case where the investor is risk neutral, the myopic strategy requires him to follow a stop-loss strategy that is to invest $k(W+\bar{L})$ in stocks and to switch to bonds whenever his wealth reaches its lower bound 0. Our results show that the stop loss strategy is too risky especially when W is small compared to $\bar{L}$.[4]

[4]See Dybvig (1988) for another interpretation of the inefficiency of the stop loss strategy.

E/ The logarithmic case

The case of the logarithmic utility fucntion ($u(W)=\text{Log}W$) can be studied as the limit of the constant relative risk aversion $\frac{W^{1-A}}{1-A}$ when A tends to 1. It can be seen from equation (4-8) that as A tends to 1, α tends to 1 and hence the optimal strategy defined by equation (4-9) converges towards the myopic strategy ($\alpha \to 1$). The optimal strategy for A=1 can also be derived by a direct analysis of the logarithmic case.

Let us assume that $u(W)=\text{Log}W$ and consider the following optimization problem:

$$\max \left\{ \liminf_{t\to\infty} E\left(\frac{1}{T} \text{ Log } W_T\right)\right\} \tag{4-12}$$

subject to dynamics (3-5) and to constraints (3-3)-(3-4).

Let $\Gamma(W;k,\bar{L})$ be the value function of problem (4-12). Again, we assume that $\mu > k\sigma^2$, i.e. that the credit constraint is binding.

Lemma 4-A: $\Gamma(W;k,\bar{L}) = r + \mu k - k^2\frac{\sigma^2}{2} = \lambda$ **if** $W>0$. **(4-13)**

$\Gamma(0;k,\bar{L}) = -\infty$.

Proof: By the same argument as in proposition (3-3), we have:

$$\Gamma(W;k,0) \le \Gamma(W;k,\bar{L}) \le \Gamma(W+\bar{L};k,0).$$

$\Gamma(W;k,0)$ is the long run performance of the proportional strategy $X=kW$, i.e.:

$$\Gamma(W;k,0) = \lim_{T\to\infty} \frac{1}{T} \text{ E Log } W_T = \lambda + \lim_{T\to\infty} \frac{1}{T} \text{ Log } W = \lambda \quad (\text{if } W>0). \ ///$$

We now consider the following problem which is the version of (4-1) in the logarithmic case:

$$Z(W;k,\bar{L}) = \max \left\{ \liminf_{t\to\infty} E(\text{Log } W_T - \lambda T) \right\} . \tag{4-14}$$

subject to dynamics (3-5) and to constraints (3-3)-(3-4).

Lemma 4-B below extends the results of proposition (4-1) to the logarithmic case.

Lemma 4-B: i] $\text{Log } W \le Z(W;k,\bar{L}) \le \text{Log}(W+\bar{L})$.

ii] $Z(W;k,\bar{L}) = \lim_{t\to\infty} (J(W,T-t;k,\bar{L}) - \lambda t)$ where $J(W,T-t;k,\bar{L})$ is the value function of the finite horizon program.

Proof: The proof is the same as for proposition (3-4) and (3-1). ///

By the optimality principle: $Z(W;k,\bar{L}) = \max \{ E (Z(W_\tau;k,\bar{L}) - \lambda\tau \}$. By letting τ go to zero, and by assuming that $Z(W;k,\bar{L})$ is C^2 in W, we get the Bellman equation:

$$\lambda = rW\, Z_W(W;k,\bar{L}) + \max_{X\le W+\bar{L}} \left\{ \mu X\, Z_W(W;k,\bar{L}) + \frac{1}{2}\, \sigma^2 X^2\, Z_{WW}(W;k,\bar{L}) \right\} . \tag{4-15}$$

Propositions 4-4 and 4-4' below extend the results of proposition 4-3 and 4-3' to the logarithmic case.

Proposition 4-4. If the value function of program (4-3) is C^2, then the Myopic investment strategy is optimal.

Proof: See appendix D.

Proposition 4-4': If r=0 then the function defined by (4-5)-(4-8) is the value function of program (4-3). The Myopic investment strategy is optimal.

Proof: See appendix E.

5. APPLICATIONS

A. Optimal Investment for a Leveraged Constrained Pension Fund:

One direct application of our model is the portfolio choice of an institutional investor subject to: (1) a prohibition against borrowing for the purpose of investing in the risky asset, and (2) subject to meeting a capital requirement to finance exogenous liabilities. For instance, it is not unusual for the manager of a pension Fund to face the following two constraints. First, the Fund has to make some deterministic payments to the participants of the Fund without going bankrupt. These payments are composed of a flow y_t ($0 \le t \le T$) and of a one shot payment Y_T at date T. Second, the Fund cannot borrow to finance its investment in the risky asset i.e., it cannot invest more than the value of the fund in the risky asset.

Notation: F_t is the value of the stocks and bonds owned by the fund at t. L_t is the present value at t of the manager's future liabilities, i.e:

$$L_t = \int_t^T y_s e^{r(t-s)} ds + Y_T e^{r(t-T)} \tag{5-1}$$

$W_t = F_t - L_t$ is the **net** value of the fund.

X_t is the dollar amount invested in the risky asset.

With this notation, the capital requirement and the borrowing constraint can be respectively written as:

$$W_t \ge 0 \quad \text{for all } t, \tag{5-2}$$

$$X_t \le F_t \quad \text{for all } t. \tag{5-3}$$

The dynamics of F_t are given by:

$$dF_t = (rF_t - y_t)\ dt + \mu X_t\ dt + \sigma X_t\ db_t. \tag{5-4}$$

In addition, we have:

$$F_{T(+)} = F_{T(-)} - Y_T \tag{5-5}$$

where $F_{T(-)}$ (respectively $F_{T(+)}$) refers to the fund's value before (respectively after) the payment Y_T. Assuming that the objective of the manager is to maximize the utility of the final value of the fund at T(+), we can redefine the problem in terms of W_t, the net value of the fund. The dynamics of W_t are given by

$$dW_t = rW_t dt + \mu X_t dt + \sigma X_t db_t. \tag{5-6}$$

By definition $W_{T(+)} = W_{T(-)}$. The manager maximizes:

$$Eu(W_T) \text{ subject to: } \quad W_t \geq 0 \quad \text{and} \quad X_t \leq W_t + L_t. \tag{5-7}$$

The results of Section 3 can be applied to solve (5-7) explicitely (when T is very large) for two interesting cases:

<u>case 1</u>: $y_t = \bar{y}$; $Y_T = \bar{y}/r$. In this case, the Fund pays the interest $\bar{y} = rY_T$ as a flow and the principal Y_T at the end. When T is very large, the liabilities are a perpetual payment, one interpretation being a constant volume of pension payed continuously. In this case, $L_t = \bar{L}$. Hence, this case can be dealt with by using our model for k=1, $\bar{L}>0$.

<u>case 2</u>: $y_t = 0$, $Y_T = \bar{L}e^{rT}$. In this case, the payment is made only at T. One

interpretation is that the Fund capitalizes an annuity at time 0 which begins payment at time T. In this case, $L_t=\bar{L}e^{rt}$. We can see, by redefining the variables to eliminate r, that case 2 is a particular case of case 1. Indeed, let $L'_t=L_t e^{-rt}$; $W'_t=W_t e^{-rt}$; $X'_t=X_t e^{-rt}$. The dynamics of W'_t are:

$$dW'_t = \mu X'_t \, dt + \sigma X'_t \, db_t. \tag{5-8}$$

The constraints imposed on the Fund

$$X'_t \leq W'_t + \bar{L} \quad \text{and} \quad W'_t \geq 0. \tag{5-9}$$

Hence this case can be dealt with by using the model in part 1 for k=1; $\bar{L}>0$; r=0.

B. Portfolio Insurance:

The constraint discussed above will "insure" that the value of the fund F_t does not fall below L_t. It thus solves the leveraged constrained optimization problem for a fund desiring to expose its capital to the risky asset but limit its downside risk. It is a formalization of the Constant Proportional Portfolio Insurance strategy suggested by Black and Jones (1987), Perold (1986), and Black and Perold (1987)[5]. Grossman (1988) has argued that this "insurance" will not be feasible when a significant fraction of the population attempts to use it. In particular, the assumption that the volatility σ is constant , will be inconsistent with general equilibrium if such strategies are widely adopted.

[5] See Brennan and Solanki (1981) for the general form of optimal portfolio insurance contracts.

B: Investing on Margins:

The constraint $X \leq k(W+\bar{L})$ is faced by most investors. If an investor's only source of capital is his portfolio wealth, then $\bar{L}=0$ and $1/k$ represents the margin rate. On the other hand, the investor may also be able to borrow against his human capital using an unsecured loan with a credit line denoted by $\bar{L}/k$.

C: Constrained Capital Budgeting:

An implication of our analysis is that a firm with the opportunity to invest in a project with no market risk will not necessarily do so even if the project has an expected return higher than the risk free rate, and even if it has enough capital to make the investment.

To see this, consider the following example: A manager has access to a sequence of projects with a one year horizon. All projects undertaken in the same period are fully correlated: Let $\tilde{R}_t$ be the return on the t-th period project. If the manager invest X_t dollar in the t=th period project, his payoff is $X_t\tilde{R}_t$. We assume the $\tilde{R}_t$'s to be i.i.d. so that projects undertaken at different dates have independent payoffs.

Suppose that the value of the firm's debt is $\bar{D}$, and the value of the firm's equity is E. If the firm cannot issue new securities, our result implies that the value maximizing size of the project as a function of the value of the equity is:

$$X(E) = \min(sE,\ E+\bar{D})$$

where $s=\mu/\alpha\sigma^2$; μ is the expectation of $E(\tilde{R}_t)$; σ^2 is the variance of $\tilde{R}_t$ and α is given by equation (4-8). When E is small, $X(E)<E+\bar{D}$ and the firm will not invest in high yielding projects even though it has the current capital to do so. In effect, it is keeping a reserve to finance investment in future projects.

Similarly, a bank facing a fixed limit to the amount it can borrow from the Federal Reserve Bank will not borrow the maximal amount even if it could make investments with a higher expected return than the risk free rate. In particular if the deposit/equity ratio is high, the bank will not make all the loans which are feasible, because it is effectively risk averse. The risk aversion arises out of its inability to fully benefit from the state of nature in which the loans have a high payout, since it will not be able to lever its wealth for reinvestment.

6. CONCLUSION

We have provided explicit solutions to optimal portfolio problems containing leverage and minimum portfolio return constraints. Numerical solutions show that it is possible to substantially improve upon the myopic strategies which ignore the constraint until it binds. The improvement is associated with the fact that an optimal portfolio deviates from the myopic proportion ($\mu/A\sigma^2$) at wealth levels <u>below</u> the point where the leverage constraint is binding. This deviation occurs because the leverage prevents the indirect utility function of wealth from "inheriting" the constant elasticity of the direct utility function. The cash constraint causes risk aversion to shift towards one.

An interesting fact is that, under an optimal policy, the leverage constraint causes the investor to behave as if had a constant risk aversion $\alpha>0$ in the region where the leverage constraint does not bind (and this region exists even if $A=0$). Hence, even a risk neutral investor will not choose a stop loss strategy (where he invests all of his wealth in the risky asset and sells only when his wealth hits the floor).

In the case where $A<1$, the investor behaves in a more risk averse manner, when he uses his human capital (i.e. unsecured) loans to finance his investment in the risky asset. An implication of this magnified risk aversion is that margin restrictions on borrowing to finance equity investments increase the required rate of return earned by equity in general equilibrium.

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APPENDICES

APPENDIX A

Proof of Proposition 3-4: The discounted value fuction $V(W,t;k,\bar{L})$ has a limit when t tends to infinity. Furthermore, this limit is a continuous function of W for W>0.

- Step 1: $V(W,t;k,\bar{L})$ is non decreasing in t. To see this, let t and h be two non-negative numbers. By the optimality principle:

$$J(W,T-t-h;k,\bar{L}) = \max E\{J(W_{T-h},T-h;k,\bar{L})|W_{T-t-h}=W\}$$

Which implies that:

$$V(W,t+h;k,\bar{L}) = \max E\{V(W_{T-h},h;k,\bar{L})e^{-\lambda t}\ |W_{T-t-h}=W\}.$$

By proposition 3-3:

$$V(W,t+h;k,\bar{L}) \geq \max E\{\ \frac{W_{T-h}^{1-A}}{1-A}\ e^{-\lambda t}\ |W_{T-t-h}=W\} = V(W,t;k,\bar{L}).$$

- Step 2: Since (3-20) shows that $V(W,t;k,\bar{L})$ is bounded, it follows that $V(W;k,\bar{L})=\lim_{t\to\infty} V(W,t;k,\bar{L})$ exists for W>0. Futhermore $V(W;k,\bar{L})$ is a concave function as pointwise limit of concave functions. Thus $V(W;k,\bar{L})$ is continuous for W>0.///

APPENDIX B

<u>Proof of proposition 4-1:</u>

Part (i) follows immediatly from (3-20). We now demonstrate part (ii).

- Step 1: By investing X=kW, the investor gets an expected long run utility of $\frac{W_0^{1-A}}{1-A}$, hence $R(W;k,\bar{L}) \geq \frac{W^{1-A}}{1-A}$.

- Step 2: $R(W;k,\bar{L}) \leq V(W;k,\bar{L})$. Indeed,

$$R(W;k,\bar{L}) = \max \liminf_{t\to\infty} E\{e^{-\lambda t} \frac{W_t^{1-A}}{1-A}\} \quad \text{s.t. (3-2)-(3-4).}$$

Hence:

$$R(W;k,\bar{L}) \leq \liminf_{t\to\infty} \max E\{e^{-\lambda t} \frac{W_t^{1-A}}{1-A}\} = V(W;k,\bar{L}).$$

- Step 3: $R(W;k,\bar{L}) \geq V(W,t;k,\bar{L})$ for every t. Indeed, by the optimality principle:

$R(W;k,\bar{L}) = \max E\{e^{-\lambda t} R(W_t;k,\bar{L}) | W_0=W\}$. Since $R(W_t;k,\bar{L}) \geq \frac{W_t^{1-A}}{1-A}$, we get that: $R(W;k,\bar{L}) \geq V(W,t;k,\bar{L})$.

- Step 4: From step 2 and step 3, R=V which yields $R(W;k,\bar{L}) \leq \frac{(W+\bar{L})^{1-A}}{1-A}$.

APPENDIX C

Proof of proposition 4-2.

Let V^* be a C^2 function satisfying the conditions of proposition (4-2); define $V^*(0;k,\bar{L})=\lim_{W\to 0} V^*(W;k,\bar{L})$ and

$$X(W) = \min(k((W+\bar{L}); \frac{\mu}{\sigma^2} \frac{V_W^*}{-V_{WW}^*}). \tag{C-1}$$

Let $y(W)$ be a feasible control $0\leq y(W)\leq k(W+\bar{L})$ and let W_t be the wealth process under control y.

$$dW_t = rW_t\, dt + \mu y\, dt + \sigma y\, db_t \qquad \text{if } W_t>0. \tag{C-2}$$

Let $h_t=V^*(W_t)e^{-\lambda t}$. By Ito's lemma:

$$dh_t = -\lambda h_t\, dt + V_W^* e^{-\lambda t}\, dW_t + 1/2\, V_{WW}^* e^{-\lambda t}\, \sigma^2 y^2\, dt \qquad \text{if } W_t>0,$$

$$dht = \{-\lambda V^* + rV_W^* y + \mu V_W^* y + 1/2\, V_{WW}^*\, \sigma^2 y^2\}\, e^{-\lambda t}\, dt + \sigma\, V_W^*\, e^{-\lambda t}\, y\, db_t\ . \tag{C-3}$$

Hence since V^* satisfies the Bellman equation (3-5):

$$dh_t \leq \sigma\, V_W^*\, e^{-\lambda t}\, y(W_t)\, db_t. \tag{C-4}$$

This yields:

$$V^*(W;k,\bar{L}) \geq E\{V^*(W_t;k,\bar{L})e^{-\lambda t} | W_0=W\} \tag{C-5}$$

where the inequality in (C-5) is replaced by an equality if $y(.)=X(.)$. By condition (iii):

$$V^*(W;k,\bar{L}) \geq \lim_{t\to\infty} E\{ \frac{W_t^{1-A}}{1-A}\, e^{-\lambda t} \quad |\quad W_0=W\}. \tag{C-6}$$

So V^* is greater than the value function V. The last thing to show is that $V^*\leq V$ i.e. that the investment strategy X (C-1) generates a performance of $V^*(W;k,\bar{L})$.

By (C-5) and condition (iii):

$$E\left(\frac{W_t^{1-A}}{1-A} \quad e^{-\lambda t} \;\middle|\; W_0=W\right) \leq \quad V^*(W;k,\bar{L}) \quad \leq E\left(\frac{(W_t+L)^{1-A}}{1-A} \quad e^{-\lambda t} \;\middle|_0 W = W\right) \quad . \quad (C-7)$$

(C-7) and condition (iv) yields:

$$V^*(W;k,\bar{L}) = \lim_{t\to\infty} E\left(\frac{W_t^{1-A}}{1-A} \, e^{-\lambda t} \;\middle|\; W_0=W \right).$$

APPENDIX D

Proofs of propositions (4-3) and (4-4).

Let V be the value function, we first prove the following lemma.

Lemma D1: $\lim_{W\to 0} V(W;k,\bar{L}) = 0$ if $A<1$,

$\lim_{W\to 0} V(W;k,\bar{L}) = -\infty$ if $A>1$.

$\lim_{W\to 0} Z(W;k,\bar{L}) = -\infty$ (logarithmic case).

Proof: Let t be a positive number. Let $X(\cdot)$ be a feasible investment strategy. Let $\zeta_X(W)$ denote the first time τ such that $W_\tau=0$ given that $W_0=W$ and that $X(\cdot)$ is followed.

$$V(W;k,\bar{L}) = \sup \quad E[\tilde{V}(W_{\inf(t,\zeta_X)};k,\bar{L})e^{-\lambda \inf(t,\zeta_X)}] \quad \text{over } X(\cdot) \tag{D-1}$$

where $\tilde{V}(W;k,\bar{L}) = V(W;k,\bar{L})$ if $W>0$ and

$= u(W)$ if $W=0$.

If $u(0)=0$ then $V^+(0;k,\bar{L})=\lim_{W\to 0} V(W;k,\bar{L})\geq 0$ (from proposition 3-1i]). Hence letting $W_0=W$ go to zero in (D-1) we get $V^+(0;k,\bar{L})\leq e^{-\lambda t}V^+(0;k,\bar{L})$ and therefore $V^+(0;k,\bar{L})=0$.

If $u(0)=-\infty$ then, since $V(W;k,\bar{L})>-\infty$ for $W>0$, $\text{prob}[W_{\inf(t,\zeta_X)}=0|W_0=W)]=0$. Thus, $\inf(t,\zeta_X)=t$. Therefore if $V^+(0;k,\bar{L})>-\infty$ then by letting W go to zero in (D-1) we get that $V^+(0;k,\bar{L})=e^{-\lambda t}V^+(0;k,\bar{L})$. Since $V^+(0;k,\bar{L})$ must be negative (proposition 3-1i]), we have a contradiction. As a conclusion $V^+(0;k,\bar{L})=u(0)$.///

Lemma D-2: If the value function is C^2, then the investment strategy given by (4-9) is optimal.

Proof: i/ A general solution to the Bellman equation can be given in U (the unconstrained domain). Indeed when:

$$\frac{\mu}{\sigma^2} \quad \frac{V_W}{-V_{WW}} \leq k(W+\bar{L})$$

$$\lambda V = r W V_W + \frac{\mu}{2\sigma^2} \quad \frac{V_W^2}{-V_{WW}} \ .$$

Let us define the variable Y such that (see Karatzas et al. (1986) for a similar transformation):

$$Y^{-\alpha} = V_W \ . \tag{D-2}$$

W(Y) is a smooth function of Y when W belongs to U. Using that

$$V_{WW}(Y) \; W'(Y) = -\alpha \; Y^{-\alpha-1} \tag{D-3}$$

we can rewrite the Bellman equation:

$$\lambda V(W(Y)) = r \; W(Y) \; Y^{-\alpha} + \frac{\mu^2}{2\sigma^2} \quad \frac{1}{\alpha} \; Y^{1-\alpha} \; W'(Y) \ . \tag{D-4}$$

Differentiating (D-4) with respect to Y yields the linear equation below:

$$0 = \left\{ \alpha(r - \frac{\mu^2}{2\sigma^2} - \lambda) + \frac{\mu^2}{2\sigma^2} \right\} W'(Y) - r\alpha^2 \frac{W}{Y} + \frac{\mu^2}{2\sigma^2} \; YW''(Y) \tag{D-5}$$

using (4-8), the general solution to (D-5) is:

$$W = R_1 Y + R_2 Y^{-\xi} \tag{D-6}$$

with: $\xi = \dfrac{2r\alpha^2\sigma^2}{\mu^2}$ (D-7)

where R_1 and R_2 are constant on any connected subset of U.

ii/ U contains some non empty interval $]0,\tilde{W}[$. Otherwise, $\lim_{W\to 0} X(W) \geq k\bar{L}$ and in a neighbourhood of 0, V satisfies the differential equation:

$$\lambda V = r W V_W + \mu k(W+\bar{L}) V_W + \frac{\sigma^2}{2} V_{WW} k^2 (W+\bar{L})^2 \tag{D-8}$$

If A>1, and $\lim X(W)>0$, $E \dfrac{W_t^{1-A}}{1-A} e^{-\lambda t} = -\infty$. The same reasoning holds in the logarithmic case.

If A<1 then $\lim_{W\to 0} V(W) = 0$ (from lemma (D-1) and by (D-8),

$\lim_{W\to 0} X(W) \leq \lim_{W\to 0} \dfrac{\mu}{\sigma^2} \dfrac{V_{WW}}{V_W} \leq \dfrac{\bar{L}}{2}$ which yields a contradiction.

Hence, in an interval $]0,\tilde{W}[$, $W=R_1 Y$ and $X = \mu/\alpha\sigma^2 W$. Therefore $\tilde{W}=\bar{L}[(\mu\sigma^2/k\alpha)-1]^{-1}$.

iii/ We now prove that $U=]0,\tilde{W}[$. Per absurdum, suppose that there exists an interval $]W_1,W_2[$ included in U such that $(X(W_1)/k)-W = (X(W_2)/k)-W = \bar{L}$. We define: $Y_1=Y(W_1)$ and $Y_2=Y(W_2)$. For Y in the interval $]Y_1,Y_2[$, the following relationships hold:

$$X(Y) = \frac{\mu}{\alpha\sigma^2} \left\{ R_1 Y - \xi R_2 Y^{-\xi} \right\} , \tag{D-9}$$

$$\Psi(Y) = \frac{X(Y)}{k} - W(Y) = R_1 \left(\frac{\mu}{\alpha\sigma^2} - 1 \right) Y - R_2 \left(\frac{\mu}{\alpha\sigma^2} \xi + 1 \right) Y^{-\xi} , \tag{D-10}$$

$$V(Y) = \frac{1}{\lambda} \left\{ R_1 \left(r + \frac{\mu^2}{2\alpha\sigma^2}\right) Y^{1-\alpha} - R_2 \left(1 + \xi \frac{\mu^2}{\alpha\sigma^2}\right) Y^{-\alpha-\xi} \right\} , \tag{D-11}$$

$$\Psi(Y) \leq \bar{L} \quad , \tag{D-12}$$

$$\Psi(Y_1) = \Psi(Y_2) = \bar{L} \quad . \tag{D-13}$$

If $R_1 > 0$ and $R_2 \geq 0$ then Ψ is increasing which contradicts (D-13).

If $R_1 \leq 0$ and $R_2 \geq 0$ then $\Psi \leq 0$ which contradicts (D-13).

If $R_1 > 0$ and $R_2 \leq 0$ then $X(W) \geq \mu/\alpha\sigma^2 W \geq k(W+\bar{L})$ from the previous section. Hence (D-12) is violated.

If $R_1 \leq 0$ and $R_2 \leq 0$ then W is non-positive.

In all cases, we have a contradiction which implies that there is exactly one switching point between U and C.///

APPENDIX E

Proof of propositions (4-3'),(4-4'): the case r=0.

When r=0, it is possible to solve explcitely for the value function V of problem (4-1). The method is to "guess" a solution and to show that the conditions of the verification theorem (proposition (4-2)) hold. First, we know that if the value function V is twice continuously differentiable, then the unconstrained U is an interval $]0,W^*[$ and that in this interval V is of the form

$$V = K_0 \frac{W^{1-\alpha}}{1-\alpha} \quad \text{with} \tag{E-1}$$

$$\alpha = \frac{\dfrac{\mu^2}{2\sigma^2}}{\dfrac{\mu^2}{2\sigma^2} + \lambda} = \frac{(A + \delta)^2}{A + 2\delta + \delta^2} \tag{E-2}$$

with $\delta=(\mu/k\sigma^2)-A$. The optimal investment in U is:

$$X(W;k,\bar{L}) = \frac{\mu}{\alpha\sigma^2} W = k \frac{A + 2\delta + \delta^2}{A + \delta} W \tag{E-3}$$

The "switching" point (where the constraint is met) is:

$$W^* = \left[\frac{\mu}{k\alpha\sigma^2} - 1 \right]^{-1} \bar{L} = \frac{A + \delta}{\delta + \delta^2} \bar{L}. \tag{E-4}$$

Furthermore, when r=0, the general solution of the Bellman equation in C is:

$$V(W;k,\bar{L}) = K_1 \frac{(W+\bar{L})^{1-A}}{1-A} + K_2 (W+\bar{L})^{-(A+2\delta)} \tag{E-5}$$

From Proposition 4-1, $W^{1-A}/1\text{-}A \leq V(W;\bar{L},k) \leq (W+\bar{L})^{1-A}$. Hence:

$$K_1 = \lim_{W\to\infty} (1\text{-}A)V(W;\bar{L},k)/W^{1-A} = 1 \quad \text{and } K_2 \leq 0.$$

In addition, if V is twice continuously differentiable then by continuity of V and V_W at W^*:

$$K_0 = \frac{2\delta + 1}{\delta + 1} * \frac{(W^* + \bar{L})^{-A}}{W^* - \alpha}, \tag{E-6}$$

$$K_2 = - \frac{\delta}{(1+\delta)(A+2\delta)} * (W^* + \bar{L})^{1+2\delta}. \tag{E-7}$$

Proposition 4-3′ gives conditions under which (E-1)-(E-7) is indeed the value function when r=0.

Similarly, we can compute the performance of the myopic strategy H by solving the differential equation (4-10).

$$\text{If } W \leq \hat{W} \text{ then } H = K_3 \frac{W^{1-\omega}}{1-\omega}; \tag{E-8}$$

$$\text{If } W \geq \hat{W} \text{ then } H = \frac{(W+\bar{L})^{1-A}}{1-A} - K_4 (W+\bar{L})^{-(A+2\delta)}; \tag{E-9}$$

$$\text{where } \omega = (\frac{1}{2}+A) - \left\{ (\frac{1}{2}+A)^2 - \theta \right\}^{1/2}; \tag{E-10}$$

$$\theta = 2A \left\{ 1 - \frac{(1-A)A(\delta+A/2)}{(A+\delta)^2} \right\} \tag{E-11}$$

Finally, it is also possible to solve for the value function Z of the logarirthmic problem.

$$Z(W;k,\bar{L}) = \frac{1+2\delta}{(1+\delta)^2} \text{ Log } W + B \qquad \text{if } W \le W^* = \frac{\bar{L}}{\delta}. \tag{E-12}$$

$$Z(W;k,\bar{L}) = \text{Log }(W+\bar{L}) - D\,(W+\bar{L})^{-(1+2\delta)} \qquad \text{if } W \ge W^*. \tag{E-13}$$

$$\text{With : } D = \frac{\delta}{(1+\delta)(1+2\delta)} \,(W^*+\bar{L})^{1+2\delta} \tag{E-14}$$

$$B = \text{Log }(1+\delta) + \frac{\delta^2}{(1+\delta)^2} \text{ Log } W^* - \frac{\delta}{(1+\delta)(1+2\delta)}. \tag{E-15}$$

We will now show that the functions V and Z computed above are solution to the control problems and that the function H is the performance of the myopic strategy.

Lemma E1: Let $F_A(W,t)$ be defined as:

$$F^A(W,t) = \left\{ \frac{(W+\bar{L})^{1-A}}{1-A} - \frac{W^{1-A}}{1-A} \right\} e^{-\lambda t} \qquad \text{if } A \ne 1,$$

$$F^1(W,t) = \text{Log}(W+\bar{L}) - \text{Log}W \quad .$$

If the investor follows the myopic strategy and W_t is the resulting wealth process: $\lim_{t\to\infty} E\, F^A(W_t,t) = 0$.

If the investor follows strategy (4-9) and $A \le 1$ or $(2\mu-\alpha\sigma^2)\alpha > A\mu$ then $\lim_{t\to\infty} E\, F^A(W_t,t) = 0$.

<u>Proof of lemma E1</u>: i) If $A<1$, then $F^A(W,t) \le \frac{\bar{L}^{1-A}}{1-A} e^{-\lambda t}$ and the lemma is trivial.

ii) $A>1$: In this case $A>\alpha>1$. Suppose that the manager follows the investment strategy X(W) and call $F=F^A(W_t,t)$. By Ito's lemma:

$$dF(t) = \left\{ -\lambda F + \mu X F_W + \frac{\sigma^2}{2} X^2 F_{WW} \right\} dt + \sigma X F_W \, db(t). \tag{E-16}$$

$$E(dF(t)) = - E(F(t)\ g(W))\ dt$$

Where $$g(W) = \frac{\lambda F_W - \mu X F_W - (\sigma^2/2) X^2 F_{WW}}{F} \qquad \text{(E-17)}$$

Note that g(.) does not depend on t. We will show that g(W) is bounded away from zero ($g(W) \geq \bar{g} > 0$). This implies that $E(F(t) | F(0)) \leq F(0) e^{-\bar{g}t}$ which tends to zero when t tends to infinity. g(W) is a continuous function. To prove that it is bounded away from zero we need to prove that :

$$g(W) > 0;\ \lim_{W\to 0} g(W) \neq 0 \text{ and } \lim_{W\to\infty} g(W) \neq 0.$$

$$\lim_{W\to 0} g(W) = k^2 \frac{(A-1)\delta^2}{2A} \sigma^2 > 0$$ if X is the myopic strategy.

$$\lim_{W\to 0} g(W) = (A-1) \left(\frac{\mu}{\alpha\sigma^2} - k\right) \left\{ \mu - \frac{A\sigma^2}{2} \left(\frac{\mu}{\alpha\sigma^2} + k\right) \right\}$$ if X is the strategy (4-9).

Lim g(W)>0 under condition E-18 below:

$$A + 2\delta > \delta A\ . \qquad \text{(E-18)}$$

We now consider the behavior of g when $W\to\infty$.

$$\left\{ (W+\bar{L})^{1-A} - W^{1-A} \right\} \sim (1-A)\ W^{1-A} \operatorname{Log}\left(1+ \frac{\bar{L}}{W}\right) \quad \text{when } W\to\infty.$$

$$\left\{ (W+\bar{L})^{-A} - W^{-A} \right\} \sim (-A)\ W^{-A} \operatorname{Log}\left(1+ \frac{\bar{L}}{W}\right) \quad \text{when } W\to\infty.$$

$$\left\{ (W+\bar{L})^{-A-1} - W^{-A-1} \right\} \sim (-A-1) W^{-A-1} \operatorname{Log}\left(1+ \frac{\bar{L}}{W}\right) \quad \text{when } W\to\infty.$$

Hence, $\lim_{W\to\infty} g(W) = k\mu - Ak^2\sigma^2/2 > 0$ if X is the myopic strategy strategy or strategy (4-9).

We must now prove that g(W) is non negative.

$$g(W) = \frac{1}{F}\left\{(W+\bar{L})^{1-A} H_1 + W^{1-A} H_2\right\} e^{-\lambda t} \quad \text{where}$$

$$H_1 = k\mu - \frac{k^2 A\sigma^2}{2} - \mu k \frac{X}{k(W+L)} + \frac{k^2 A\sigma^2}{2}\left\{\frac{X}{k(W+L)}\right\} \quad \text{and} \qquad \text{(E-19)}$$

$$H_2 = -k\mu + \frac{k^2 A\sigma^2}{2} + \mu \frac{X}{W} - \frac{A\sigma^2}{2}\left(\frac{X}{W}\right)^2 . \qquad \text{(E-20)}$$

Since $X \le k(W+\bar{L})$, and $\mu > kA\sigma^2$, H_1 decreases with $X/k(W+\bar{L})$ and

$$H_1 \ge k\mu - \frac{k^2 A\sigma^2}{2} - k\mu + \frac{k^2 A\sigma^2}{2} = 0.$$

H_2 is decreasing in $\frac{X}{W}$ iff $\frac{X}{W}$ is greater than $\frac{\mu}{A\sigma^2}$.

Hence if X is the myopic strategy strategy: $H_2(W) \ge H_2(0) = \delta^2\sigma^2/2A > 0$.

If X is strategy (4-9) then $H_2(W) \ge \min(H_2(0), \lim_{W\to\infty} H_2(W))$. Under condition (E-18), $H_2(0)>0$. Furthermore, $\lim H_2(W) = 0$. As a conclusion under condition (E-18), $\lim_{W\to\infty} E\, F^A(W_t, t) = 0$.

iii) A=1: A similar approach yields that $\lim EF^1(W_t, t) = 0$.

Lemma E2: i] The function V defined by (E-1)-(E-7) satisfies:

$$\frac{W^{1-A}}{1-A} \le V(W) \le \frac{(W+\bar{L})^{1-A}}{1-A}$$

ii] The function H defined by (E-8)-(E-11) satisfies:

$$\frac{W^{1-A}}{1-A} \le H(W) \le \frac{(W+\bar{L})^{1-A}}{1-A}$$

iii] The function Z defined by (E-12)-(E-15) satisfies:

$$\frac{W^{1-A}}{1-A} \le Z(W) \le \frac{(W+\bar{L})^{1-A}}{1-A}$$

Proof of lemma E2: The proof is presented at the end of this appendix.

Lemma E3: If $W \geq W^*$: $\dfrac{\mu V_W}{-\sigma^2 V_{WW}} \leq W+\bar{L}$ and $\dfrac{\mu Z_W}{-\sigma^2 Z_{WW}} \leq W+\bar{L}$.

Proof: From (E-5)-(E-7), we get that:

$$\frac{\mu V}{-\sigma^2 (W+\bar{L}) V_{WW}} = \frac{1 + \frac{\delta}{1+\delta} \left\{ \frac{W^* + \bar{L}}{W + L} \right\}^{(1+2\delta)}}{A + \frac{\delta}{1+\delta} (A+2\delta+1) \left\{ \frac{W^* + \bar{L}}{W + L} \right\}^{(1+2\delta)}} \quad (A+\delta) > 1.$$

The same computation holds for A=1.

Lemma E4: r= 0 i/ If A<1 or A>1 and $A+\delta > A\delta$, the value function is V and (4-9) gives the optimal investment.

ii/ If $A \neq 1$, H is the payoff of the myopic strategy.

iii/ If A=1, Z is the value function and the myopic strategy is optimal.

Proof: i/ & iii/ follow from proposition 4-2.

ii/ can be proven by a argument similar to the one in appendix C (see C-7).

Proof of lemma E2

i/ a/ $V(W) \geq \dfrac{W^{1-A}}{1-A}$ if $W \leq W^*$ we must prove that:

$$K_0 \frac{1}{1-\alpha} W^{1-\alpha} \geq \frac{1}{1-A} W^{1-A} \qquad \text{(E-21)}$$

Since α lies between 1 and A, is suffices to verify (E-21 for $W=W^*$. That is by (E-4) and (E-6):

$$\frac{(2\delta+1)(A+2\delta+\delta^2)}{(\delta+1)(1-A)(A+2\delta)} \; \frac{W^*}{W^* + \bar{L}} \geq \frac{1}{1-A} \left(\frac{W^*}{W^* + \bar{L}} \right)^{1-A} = \frac{1}{1-A} \left(1 - \frac{\delta+\delta^2}{A+\delta^2+2\delta}\right)^{1-A}$$

Since $\frac{1}{1-A}(1+x)^{1-A} \leq \frac{1}{1-A} + x$ (by concavity of $\frac{1}{1-A}(1+x)^{1-A}$), a sufficient condition is (by using that $\frac{W^*}{W^* + \bar{L}} = \frac{A+\delta}{A+2\delta+\delta^2}$):

$$(1+\delta)^2(A+2\delta) > (A+2\delta) + \delta^2 \tag{E-22}$$

which is verified.

If $W \geq W^*$, we must check that:

$$\frac{(W+\bar{L})^{1-A}}{1-A} + K_2(W+\bar{L})^{-(A+2\delta)} \geq \frac{W^{1-A}}{1-A}.$$

Since $\frac{(W+\bar{L})^{1-A}}{1-A} - \frac{W^{1-A}}{1-A} \geq \bar{L}\,(W+\bar{L})^{-A}$

it suffices to show that:

$$\bar{L} \geq (W^* + \bar{L})^{1+2\delta}\,(W+\bar{L})^{-2\delta}\,\frac{\delta}{(1+\delta)(A+2\delta)} = \Delta_1(W)$$

As $\Delta_1(W) \leq \Delta_1(W^*)$ we must check that:

$\bar{L} \geq (W^* + \bar{L})\,\frac{\delta}{(1+\delta)(A+2\delta)}$ that is $(1+\delta)(A+2\delta) > (A+2\delta)+\delta$ which holds.

b) $V \leq \frac{(W+\bar{L})^{1-A}}{1-A}$ (E-23)

When $W \geq W^*$ (E-23) is obvious given (E-7). When $W \leq W^*$ we must prove that

$$\frac{(W+\bar{L})^{1-A}}{1-A} \geq \frac{1}{1-\alpha}\left(\frac{2\delta+1}{\delta+1}\right)(W^*+\bar{L})^{-A}\,W^{*\alpha}\,W^{1-\alpha}.$$

that is:

$$\frac{1}{1-A} \geq \frac{1}{1-A}\left\{\frac{(2\delta+1)(A+\delta)}{(\delta+1)\,(A+2\delta)}\right\}\left[\frac{W^*+\bar{L}}{W+L}\right]^{1-A}\left[\frac{W}{W^*}\right]^{1-\alpha}. \tag{E-24}$$

The right hand side of (E-24) is increasing for $W \leq W^*$ (take the logarithm and differentiate) therefore we need only to show that:

$\frac{1}{1-A} \geq \frac{1}{1-A}\,\frac{(2\delta+1)\,(A+\delta)}{(\delta+1)\,(A+2\delta)}$ which holds.

ii) a/ $H \geq \frac{W^{1-A}}{1-A}$

For $W \leq \hat{W}$ we must prove that $K_3 \frac{W^{1-\omega}}{1-A} \geq \frac{W^{1-A}}{1-\omega}$. It suffices to prove it for $W=\hat{W}$ i.e.

$$\frac{1}{1-A} \quad \frac{1+2\delta}{A+2\delta+\frac{A+\delta}{A}(1-\omega)} \geq \frac{1}{1-A}\left\{\frac{\hat{W}}{\hat{W}+\bar{L}}\right\}^{1-A} = \frac{1}{1-A}\left\{1-\frac{\delta}{A+\delta}\right\}^{1-A}.$$

A sufficient condition is that:

$$\frac{1}{1-A} \quad \frac{1+2\delta}{A+2\delta+\frac{A+\delta}{A}(1-\omega)} \geq \frac{1}{1-A} - \frac{\delta}{A+\delta} \qquad \text{(E-25)}$$

Since ω lies between 1 and A (E-25) holds. Indeed it suffices to check (E-25) for $\omega=1$ and $\omega=A$ (by monotonicity of homographic functions) which is straightforward.

For $W \geq \hat{W}$, we must prove that:

$$\frac{(W+\bar{L})^{1-A}}{1-A} - \frac{W^{1-A}}{1-A} \geq K_4 \quad (W+\bar{L})^{-(A+2\delta)}$$

Since:

$$\frac{(W+\bar{L})^{1-A}}{1-A} - \frac{W^{1-A}}{1-A} \geq \bar{L} \quad (W+\bar{L})^{-A}$$

we must prove that:

$$\bar{L} \geq (W+\bar{L})^{-2\delta} \quad K_4$$

which needs to be verified only for $W=\hat{W}$.

We must now check that:

$$1 > \frac{\frac{\delta}{(A+\delta)\ (1-A)} - \frac{1}{1-\omega}}{1+\frac{A+2\delta}{1-\omega} * \frac{A}{A+\delta}} \qquad \text{(E-26)}$$

(E-26) can be verified by using the fact that ω lies between A and $\frac{A^2+\delta}{A+\delta}$ which is straightforward to verify.

iib/ $H \leq \frac{(W+\bar{L})^{1-A}}{1-A}$.

As in i-b/ it suffices to check that the inequality holds for $W=\hat{W}$. Since H is continuous this is equivalent to showing that K_4 is non-negative which in turn results from the fact that ω lies between A and $\frac{A^2+\delta}{A+\delta}$.

iii/ a/ $Z \geq \text{Log}W$. For $W \leq W^*$, we must check that:

$$- \frac{\delta^2}{(1+\delta)^2} \quad \text{Log}W + B \geq 0$$

which needs to be proved only for $W=W^*$ that is:

$$\text{Log }(1+\delta) \geq \frac{\delta}{(1+\delta)(1+2\delta)}$$

which holds .

For $W \leq W^*$ it suffices to prove that :

$$\bar{L} \geq E\ (W^*+\bar{L})^{-1-2\delta} \quad \text{which is straightforward.}$$

b/ $Z \leq \text{Log}(W+\bar{L})$.

For $W \leq W^*$, $\text{Log}(W+\bar{L}) - Z(W)$ is non increasing. Therefore it suffices to verify the inequality for $W=W^*$ which is also straightforward.

TABLE 1

$r = 0$

$k = 1$

A = relative risk aversion.

μ = expected excess return on stock.

σ = standard deviation of stock returns.

$W/\bar{L}$ = net wealth/credit line.

ϕ = improvement of the optimal strategy over the Myopic Strategy.

A	μ	σ	$W/\bar{L}$	ϕ
0	10%	20%	10%	65%
0	5%	20%	50%	9%
0	10%	20%	50%	7%
0	20%	20%	50%	2%
.5	5%	20%	10%	10%
.5	10%	20%	10%	25%
.5	20%	20%	10%	35%
.9	10%	20%	10%	5%
2	10%	20%	10%	5%

Figure 1

Vi(W)-W as a function of W, i=0,1,2