PRICING PHYSICAL ASSETS INTERNATIONALLY: A NON LINEAR HETEROSKEDASTIC PROCESS FOR EQUILIBRIUM REAL EXCHANGE RATES

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Abstract

Transferring physical capital and transferring production and sales activities from one country to the other, typically entails large adjustment costs. The model of this paper features two homogeneous stocks of physical capital located in two different countries, separated by an "ocean". The two physical stocks are optimally invested in a random production process yielding real returns, or consumed by local residents, or transferred abroad. Retro-fitting, transferring and re-building capital equipment, and increasing production and sales abroad either takes time (during which capital is idle) or consumes real resources. Under proportional transfer costs, trade imbalances, consumption imbalances and capital imbalances between the two locations are shown to be persistent. The stochastic process for the deviation from the Law of One Price (LOP) is obtained. By construction, this process is compatible with financial market efficiency and with the possibility of (costly) trade in commodities. Whereas empirical studies have found no evidence against the hypothesis that LOP deviations follow a martingale, the theoretical process which is found differs markedly from a martingale: the drift is non linear and mean reverting. But the behavior of the conditional variance more than offsets the reverting effect of the drift and the conditional probability of a move away from Parity is greater than the probability of a move toward Parity. When some price barriers are reached, however, reversion is triggered. We decompose the real-interest rate differential into an expected price change and a risk premium for which a very simple expression is found. The behaviors over time of the rate differential and of its components are examined.

"The exchange rate is determined
in the stock market",
R. Dornbusch (1976a)

INTRODUCTION

Many, perhaps most, goods have an asset character. Some are storable, some can be invested in a production process, others are consumer durables. When goods have an asset character, the dynamic behavior of their price is not simply driven by current flow supply and demand. Rather, their price tends to follow a quasi-martingale process -- like the prices of securities in an efficient market,-- which must be determined jointly with the equilibrium in the capital market. As Roll's (1979) "*ex ante* PPP" hypothesis was meant to point out, this rule applies, in particular, to the relative price of storable goods located in two different countries and, by extension, to physical capital and to any commodity which is a close substitute to physical capital. The implications of this insight for equilibrium models of the international economy have not, so far, been worked out.

In this paper, we present a streamlined "real" macroeconomic model which contains the following features. Two countries are populated with identical investors and endowed with identical technologies. At any given time, the investors of both countries only consume goods physically available in their own country. Transferring goods and/or physical capital from one country to the other is costly. Uncorrelated productivity shocks affect the output of the two countries. We solve for the joint equilibrium in the goods and capital market and obtain the behavior of capital transfers from one country to the other (which is the balance of trade), that of consumption, investment, the "real exchange rate" -- defined as the relative price of physical capital between the two countries, -- and interest rates in both countries.

Empirically, deviations from PPP (arising mostly from deviations from the

Law of One Price) have been found to follow a process which cannot be distinguished statistically from a martingale (Rogalski and Vinso (1978), Roll (1979), Adler and Lehman (1983), Huizinga (1986)).[1] This raises a conundrum. One should be unwilling to admit the possibility that deviations from the LOP can forever wander away from the zero mark: beyond a point, surely, spatial arbitrage becomes profitable and goods, which are non-traded when the deviation is small, get to be traded when the deviation is larger than the cost of trading. Hence there should be <u>some</u> reversion tendency in the LOP deviation. There is an apparent conflict between the LOP reversion produced by trade (even costly trade) and the efficient-market argument in favor of a martingale. The model presented in this paper resolves this conundrum by producing a stochastic process for LOP deviations which is compatible with market efficiency <u>and</u> with the possibility of moving goods across the world, while being able also to invest them as productive assets in more than one country.

The real exchange rate is shown to exhibit mean reversion. However, this is a reversion <u>towards</u> parity, not <u>to</u> parity. Indeed, the behavior of the time-varying conditional variance of the exchange rate more than offsets the reversion in the mean. Except at some boundary points, the conditional probability of the deviation widening further is always larger than the probability of it narrowing down. The exchange rate spends most of the time away from parity, close to the borderline situation where a transfer of capital from one country to the other is imminent and, indeed, takes place periodically. This means that deviations from PPP, even though they do not last forever, nonetheless last <u>a "very long time"</u>. For practical purposes, therefore, the message of the <u>ex ante</u> PPP hypothesis comes out unscathed. The theory helps to explain why econometric attempts at discerning mean reversion

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[1] Abuaf and Jorion (1989) is one recent exception.

in real exchange rates have been largely unsuccessful; the fully specified process which we exhibit should serve to guide further econometric investigations.

This model extends and builds on the existing literature in at least four ways:

-Models of international asset pricing[2] have focused on the fact that investors of different countries consume goods available in their country of residence and, therefore, when Purchasing Power Parity does not hold (at the level of consumer prices), evaluate real returns from their investments differently. These models, however, are partial equilibrium in nature since they do not incorporate explicitly variations in the several national rates of interest and in the distribution of wealth across countries. Above all, they take as given the stochastic processes for the prices of goods. The model we develop here is a general equilibrium one.

-In the closed-economy context, the macroeconomic literature on capital formation has emphasized an interpretation of Tobin's q theory (cf. Tobin (1961, 1969), Brainard and Tobin (1968)) based on costs or delays incurred when installing capital (cf Eisner and Strotz (1963), Gould (1968), Lucas (1967a and b), Mussa (1977), Treadway (1969), Kydland and Prescott (1982)). This literature has recently been extended to stochastic settings, notably by Pindyck (1982) and Abel (1983, 1985), and applied to the Finance field by Brennan and Schwartz (1985), Majd and Pindyck (1987) and Myers and Majd(1987). A symmetric literature considers costs incurred when dismantling or retro-fitting capital, the limiting case being the one where investment is irreversible (Nickell (1974), McDonald and Siegel (1986), Pindyck (1988), Bertola (1987)). Such a specification also causes Tobin's q to be variable. The deviation from the LOP derived here is in the nature of a Tobin's q which

[2]See Adler and Dumas (1983).

reflects both kinds of costs of adjustment. While Tobin's q is traditionally defined as the price of installed capital relative to its market (or consumption) value, the deviation from the Law of One Price is defined here as the price of physical capital located in one country relative to the price of capital located in the other,[3] when it is costly to transfer capital between the two places in either direction.[4]

-The international macroeconomic literature purporting to explain PPP deviations has traditionally emphasized monetary aspects and/or sticky goods prices. These models offer an alternative rationale for mean reversion: the Dornbusch (1976b) "overshooting" model features an economy with non optimizing behavior and postulated price stickiness where a monetary shock occurs as a surprise and is never to be repeated. Recently,[5] papers by Krugman (1987, 1988) provided another rationale for nominal exchange rate reversion, based on Central banks' intervention taking the form of "trigger strategies", as when the exchange rate is maintained within a band. This generates a behavior for nominal exchange rates which is close to the one we produce here for real ones.[6] The results obtained here derive from physical investment decisions and from the international-equilibrium behavior of underlying economic fundamentals, without the social contrivance of Central banks intervening to

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[3] Black (1973) is a forerunner of this approach.

[4] In order to simplify, no distinction is made within each country between investment goods, installed or not installed, and consumption goods. Furthermore, I assume constant returns to scale (no rents). Hence, by construction, within each country, the traditional Tobin q is equal to 1. For a model which incorporates the domestic Tobin q as well as differences in the price of capital between countries, but under certainty, see Murphy (1988).

[5] Some time after the first draft of the present paper was written and presented at the NBER Summer Institute.

[6] Papers in the same vein include Miller and Weller (1988), Hsieh (1988) and Klein (1989).

keep the exchange rate within a band.[7] Aizenman (1984) and Daniel (1986) are models of price setting by firms, in the presence of random exchange rates, where price stickiness is either assumed or generated by pricing costs. In setting their prices, firms, there again, follow "trigger strategies" and induce some reversion in the real exchange rate.[8] The model of Stockman and Svensson (1987) has in common with the present paper that is a stochastic optimizing model of international capital flows, but with money and without deviations from the LOP.

-Much of the methodology of the present paper is borrowed from the Finance literature dealing with portfolio choice under transactions costs.[9] Grossman and Laroque (1987) consider fixed transactions costs while Constantinides (1986) studied a problem of proportional costs, but -- as a way of obtaining an approximate answer, -- arbitrarily setting consumption as a fixed function of one kind of asset, whereas, in the present model, consumption is chosen optimally as a variable fraction and drawn out of the two stocks of goods. The optimal choice of consumption will turn out to play a central role in our model.

The outline of the paper is as follows. Sections I and II state and solve the mathematical programming problem which leads to the linchpin of the model: the indirect welfare function for various types of wealths. Section III describes how the goods move across the world in general equilibrium. Section IV features the main result: the process for deviations from the Law of One

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[7]The two effects are potentially complementary, with two cases coming to mind, depending on whether or not the bands imposed by the banks fall within those imposed by the physical system.

[8]In the Aizenman (1984) paper, an appendix considers the case where there are not only costs of adjusting prices but also transportation costs.

[9]Some of the optimality conditions used there (i.e. the "smooth-pasting" conditions) were encountered first when solving the optimal early-exercise problem for American-type options (see, e.g. Merton (1973)).

Price. Section V examines the consequences of LOP deviations for differences in real interest rates between countries.

I. THE MODEL

Consider a world economy populated with consumers who are identical, except for the fact that they live (in equal numbers) in two different geographical locations (countries), with the constraint that they can only consume goods physically available in their country of residence.[10] There is only one good, except for the fact that one must distinguish two versions of this good, depending on its physical location at any given time. In both locations, the good in question can be consumed, invested in a random constant-return-to-scale production process, or shipped. The stocks located "at home" and "abroad" are denoted K and K^* respectively.

The world is perfectly symmetrical: not only have the consumer-investors of both countries the same risk aversion, which is assumed constant,[11] but also their initial endowments are such as to warrant a symmetric treatment (more details below). Furthermore, the production processes of both countries have the same expected rate of return and standard deviation of rate of return, while the output shocks in the two countries are uncorrelated. Transferring capital abroad takes time or consumes resources. This assumption is made as a convenient device to produce LOP deviations and to explore their dynamic behavior in financial-market equilibrium. <u>Transfer costs are assumed proportional</u>.

[10] But they can own and trade stocks of goods physically located abroad.

[11] The assumption of constant relative risk aversion (isoelastic utility) is made in order to reduce the dimensionality of the optimization program.

For reasons of portfolio diversification, consumer-investors of both countries would ideally like the two countries' stocks of goods to be equal. Nonetheless, an imbalance can develop and persist, as a result of cumulated random output shocks, and despite optimal shipments and consumption flows. If and when an imbalance develops between the two stocks of goods, it may not pay to correct it immediately by shipping physical resources from the country where they are more abundant to the country where they are less abundant. Instead, within a range of tolerance, the consumer-investors who are fortunate enough to be in the country where abundance prevails, consume more.[12] The first order of business (this section and the next) is to determine the range of tolerance within which no shipping takes place.

In solving for the general equilibrium of this economy, a computational shortcut is available if one assumes that consumer-investors can achieve a (constrained) Pareto-optimal allocation of consumption, by means of a sufficiently rich (perhaps complete) financial market.[13] The assumption is acceptable considering that the model includes no hindrance to the exchange of securities or goods between individuals,[14] only to the movements of goods between locations: if a person ships goods from one place to the other, he or she suffers the cost of shipping, whether or not he retains ownership of the goods, whereas, if a person sells a stock of goods to another without moving them, no cost is borne, regardless of the country of residence of the buyer and seller.

Under this assumption, the capital-market and goods-market equilibrium can

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[12]In the literature dealing with "the demand for storage" (cf. Brennan (1958)), this advantage is called a "convenience yield".

[13]This is a Pareto optimum constrained by the fact that trade between the two countries is costly.

[14]This is in contrast to the models of Black (1974) or Stulz (1981).

be replicated advantageously by an appropriate central-planning problem.[15] Implicit prices -- which would prevail explicitly in decentralized markets -- can then be obtained from the derivatives of the appropriate indirect welfare function for (the various forms of) wealths.

In this context, a policy is a pair of functions $c(K, K^*)$ and $c^*(K, K^*)$ representing the consumption flows at home and abroad respectively, and a pair of adapted[16] processes X and X^* which stand for the cumulative amounts of physical capital ever to have been shipped from home and from abroad respectively. These processes serve to regulate the joint process for K and K^*.[17]

Assume that all consumers originally start their lives with endowments of goods which are such that the appropriate central-planning welfare function devotes equal weights to the utility levels of the households of both countries.[18][19] Accordingly, the central optimization problem is written as follows:

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[15] This shortcut has been utilized previously by Lucas and Prescott (1971) and Constantinides (1982). Uppal (1988) contains a decentralized version of the present model with explicit derivations of portfolio holdings and financial-market equilibrium.

[16] adapted with respect to the filtration generated by (K_t, K^*_t);

[17] "Regulated Brownian motion" is the terminology introduced by Harrison (1985).

[18] In a Pareto-optimal market, these weights are constant over time and across states of nature.

[19] No attempt is made here to relate the welfare weights to the endowments. Note, however, that these weights reflect not only the relative wealths of the two investors, but also the locational composition of the aggregate physical stocks. For instance, under equal welfare weights, a person residing in the country where goods happen to be more abundant, and enjoying therefore locational advantage, must at that time have lower personal wealth -- whether as a result of endowments or by virtue of prior long-term risk-sharing contracts.

$$(1)\quad V(K, K^*) = \underset{\substack{c,c^* \\ \Omega}}{\operatorname{Max}} E_t \int_t^\infty e^{-\rho(u-t)} \left[\frac{1}{\gamma} c_u^\gamma + \frac{1}{\gamma} (c_u^*)^\gamma \right] du; \quad \gamma < 1;$$

subject to:

$$(2)\qquad dK_t = (\alpha K_t - c_t)\, dt + \sigma K_t dz_t - dX_t + s dX_t^*; \qquad s \epsilon (0,1);$$

$$(3)\qquad dK_t^* = (\alpha K_t^* - c_t^*)\, dt + \sigma K_t^*\, dz_t^* + s dX_t - dX_t^*.$$

where:

ρ is the discount rate of utilities common to all investors;

$1-\gamma$ is the degree of risk aversion common to all investors;

α, σ are the expected value and the standard deviation of the physical rates of return in the constant-return-to-scale production processes;

$s \in (0,1)$ is a shipping-loss factor;

dz, dz^* are two standard white-noise output (or productivity) shocks, which, for simplicity, we take to be independent of each other;

Ω is an open region of the (K, K^*) space;

X, X^* are two adapted, non-negative, right-continuous and non decreasing stochastic processes which increase only when (K, K^*) exits from Ω (enters its closed complement).[20]

The boundary of Ω should be viewed as a barrier or trigger point: when (K, K^*) hits it (from the inside of Ω), an impulse (a shipment) dX or dX^* adjusts the two capital stocks and brings them back inside Ω. We seek the optimal positioning of the barrier.

Considering the linear nature of the constraints and the isoelasticity of

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[20] Harrison (1985) shows that these processes have finite variation.

the period utility function, the solution for $V(K, K^*)$, if it exists, must be homogeneous of degree γ and the solutions for X and X^* have the property that, if policy (X, X^*) is optimal for initial conditions (K, K^*), then policy $(2X, 2X^*)$ is optimal for initial conditions $(2K, 2K^*)$. Furthermore, because the transfer costs considered here are strictly proportional, one could prove -- but we will simply assume -- that it is optimal for shipments (the increments dX and dX^*) to be infinitesimal in size; i.e. <u>the optimal X and X^* are continuous processes</u>.[21] This is an example of what Harrison (1985, page 105) calls a "barrier policy".

The "homogeneity" and continuity properties -- in addition to the symmetry of the problem -- imply that the optimal transfer policy can be described by a single number $\lambda > 1$ with the following meaning:

$$X \text{ increases only when } K/K^* = \lambda;$$

$$X^* \text{ increases only when } K^*/K = \lambda.$$

In the (K, K^*) space, there exists a cone Ω delimited by two rays of slopes λ and $1/\lambda$ (see figure 1). When the existing combination of capital places the international economy on the ray of slope λ, a decision is made instantaneously to transfer capital from the home to the foreign country ($dX > 0$). The amount of capital transferred is very small and is such as to directionally push the economy back inside the cone (along a direction of slope $-1/s$). The other ray of slope $1/\lambda$ triggers the reverse transfer of capital (along a direction of slope $-s$).

FIGURE 1 GOES HERE

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[21] If transfer costs were a concave function, as when there is a fixed-cost component, two barriers would be needed: the barrier which triggers action and the barrier one moves to (as in an [s, S] policy). The regulator would no longer be an infinitesimal one: shipments would be of finite size. If transfer costs were a convex function, of sufficient degree of convexity, no barrier policy would be optimal: action would be called for at all times.

Inside the cone, $dX = dX^* = 0$; no shipping takes place. That does not mean, however, that the two stocks of capital move of their own accord, because consumption in each country is optimized, given the stocks of goods available.

It follows also from the homogeneity property that the problem and its solution can be re-formulated on the basis of a new variable representing physical imbalance[22] $\omega = \ell n K - \ell n K^*$ and a new function:

$$(4) \qquad I(\omega) = -\gamma \ell n K^* + \ell n[V(K, K^*)].$$

In the next section, we optimize the number λ and obtain a numerical solution for the functions V and I. Before we proceed any further, we establish that <u>the function V exists</u>, in the sense that, for finite values of K and K^*, there are conditions under which the expected integral in (1) converges. This is done by exhibiting an upper and a lower bound for V. Imagine an economy identical to the one described above, except for the fact that there would be no shipping cost (s=1). In that case, the solution $\hat{V}(K + K^*)$ is known exactly. Provided that:

$$(5) \qquad \rho > \gamma\ [\alpha - 1/2\ (1 - \gamma)\sigma^2] \quad \text{and } \gamma < 1,$$

we know, from Merton (1971), that:

[22]"ℓn" stands for "natural logarithm".

(6) $\hat{V}(K, K^*) = \frac{\mu}{\gamma} (K + K^*)^\gamma$ with: $\mu = \left[\frac{\rho - \gamma[\alpha - 1/2\,(1 - \gamma)\sigma^2]}{1 - \gamma}\right]^{1-\gamma}$.

Imposing transfer costs cannot possibly increase the achievable welfare level: $V(K, K^*) \leq \hat{V}(K + K^*)$.

At the other extreme, infinite transfer costs would produce autarky and a welfare level which, under condition (5), is equal to: $\check{V}(K, K^*) = (\mu/\gamma)[K^\gamma + (K^*)^\gamma]$. The restrictions (5) on parameter values are sufficient to ensure existence of $\hat{V}$ and $\check{V}$ and, therefore, V.

II. THE SOLUTION OF THE CENTRAL PLANNING PROBLEM

The definition (1) of the expected discounted utility $V(K, K^*)$ indicates that the conditionally expected rate, at which its value evolves during a small interval of time dt, is dictated by discounting at the rate ρ and by optimal consumption, which procures a rate of utility equal to $c^\gamma/\gamma + (c^*)^\gamma/\gamma$. This principle is embodied in the Hamilton-Jacobi equation characterizing the function V. For values of K and K^* in the interior of the cone, when the solutions c, c^* of the obvious first-order conditions for consumption have been substituted in, the Hamilton-Jacobi partial differential equation can be written as follows:[23]

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[23]See e.g. Merton (1969). We assume that V is twice continously differentiable over $K^*/\lambda \leq K \leq \lambda K^*$. Because we optimize consumption in the two countries, the partial differential equation (7) differs from the p.d.e. obtained by Constantinides (1986), which was linear and admitted a "closed form" solution.

$$(7) \qquad 0 = \left[\frac{1}{\gamma} - 1\right] V_1^{\frac{\gamma}{\gamma-1}} + \left[\frac{1}{\gamma} - 1\right] V_2^{\frac{\gamma}{\gamma-1}} - \rho V$$
$$+ V_1 \alpha K + V_2 \alpha K^* + \frac{1}{2} V_{11} \sigma^2 K^2 + \frac{1}{2} V_{22} \sigma^2 K^{*2};$$
$$K^*/\lambda < K < \lambda K^* \quad .$$

Furthermore, an essential property of the process followed by the expected discounted utility (1) over time is that its sample path must be continuous with probability one; this is the mechanical result of the fact that, with the type of policy assumed here, no discrete cost of benefit ever accrues. When a transfer of capital occurs, the movement to the target position is instantaneous; hence the value of the discounted utility at the arrival point must "match" the value at the departure point:

$$\text{for} \quad K/K^* = \lambda: \qquad V(K, K^*) = V(K - dX, K^* + sdX),$$

or, after expansion:[24]

$$(8a) \qquad 0 = - V_1(K, K^*) + sV_2(K, K^*).$$

Value matching must hold also when transfer is triggered in the opposite direction, at the symmetric point:

$$\text{for} \quad K^*/K = \lambda: \qquad V(K, K^*) = V(K + sdX^*, K - dX^*),$$

or:

$$(8b) \qquad 0 = s V_1(K, K^*) - V_2(K, K^*).$$

The partial differential equation (7) and the boundary conditions (8) would have to hold even if the positioning of the two transfer barriers were

[24]These conditions are identical to those used by Constantinides (1986).

arbitrary, for as long as the regulators are assumed to be infinitesimal. In that case, this set of equations (in addition to the homogeneity property of V; see equation (4) above) would suffice to determine a function $V(K, K^*; \lambda)$.

If, further, the barriers of slopes λ and $1/\lambda$ are to be optimal, a first-order condition known as "smooth pasting" must be satisfied.[25] This requires that marginal utilities, in addition to utility itself, take the same values before and after the transfer of capital. We state:

$$\text{for } K/K^* = \lambda: \quad V_1(K, K^*) = V_1(K - dX, K^* + sdX),$$

$$V_2(K, K^*) = V_2(K - dX, K^* + sdX)$$

which, after expansion, imply:

$$(9a) \quad 0 = - V_{11}(K, K^*) + sV_{12}(K, K^*).$$

$$(9b) \quad 0 = - V_{21}(K, K^*) + sV_{22}(K, K^*)$$

and:

$$\text{for } K^*/K = \lambda: \quad V_1(K, K^*) = V_1(K + sdX^*, K - dX^*),$$

$$V_2(K, K^*) = V_2(K + sdX^*, K - dX^*),$$

or:

$$(9c) \quad 0 = s V_{11}(K, K^*) - V_{12}(K, K^*).$$

$$(9d) \quad 0 = s V_{21}(K, K^*) - V_{22}(K, K^*).$$

In Dumas (1988a), these conditions, which involve the second derivatives of the value function, were referred to as "super-contact" conditions.[26]

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[25] See Krylov (1981), theorem 3, page 39, or Merton (1973) or Grossman-Laroque (1987).

[26] For an exposition in a simpler setting, see Dumas (1988b). Constantinides (1986) seemed unaware of these optimality conditions; he directly maximized his analog of our function $V(K, K^*; \lambda)$ -- which, in his case, was available in closed form, -- with respect to λ. "Super Contact"

The optimal solution of the central planning problem, including the optimal value for the intervention point λ, is obtained by solving the partial differential equation (7) subject to boundary conditions (8) and (9) -- a number of which are redundant, because of homogeneity and symmetry. The solution is calculated by, first, implementing the change of variable and unknown function (4) which exploit the homogeneity property, and, then, by applying a numerical technique.[27]

We now turn to the results. They pertain to the behavior of consumption and investment in both countries and to the behavior of the relative price of capital in both countries.

III. EQUILIBRIUM BEHAVIOR OF THE ALLOCATION OF PHYSICAL CAPITAL

Everything one may want to know about the equilibrium behavior of the economy can be derived from the knowledge of the function $V(K, K^*)$, or $I(\omega)$. In this section, we concentrate on the dynamics of the physical stocks of goods, whereas the main object of our exercise -- the dynamics of prices -- will be considered in the next two sections. As far as physical quantities are concerned, two topics are of interest: the size of the cone of no shipping and

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seems equivalent to the "Principle of Smooth Fit" advanced by some researchers in the field of Operations Research (see Benes et al. (1980), Karatzas and Shreve (1984), Bensoussan et al. (1983)) but it does not seem that a full mathematical theory has been worked out to demonstrate the necessity of these conditions.

[27] Briefly, the numerical technique is a "shooting" one. Pick a trial value for λ, and apply (8) and (9) at the extreme point $\omega = \ell n\lambda$, to get the value of I' and I'' there. Given these initial conditions, the partial differential equation (7) (re-written as an ordinary differential equation for $I(\omega)$) can be continued until the centerpoint, using, for instance, the Runge-Kutta method of order four (see Abramovitz and Stegun (1972), p. 897). By symmetry, at the centerpoint $\omega = 0$, we must have $I'(0) = 0$. If this condition is satisfied, the solution is found; otherwise choose a new trial value for λ and go back to the first step.

the behavior of quantities inside the cone. While no shipping takes place inside the cone, endogenous consumption rates serve to drive the stocks of goods.

A. The tolerated imbalance: comparative dynamics

The size of the cone, i.e. the value of λ, the tolerated imbalance, has been obtained for varying degrees of risk aversion $(1 - \gamma)$ and varying degrees of risk (σ). The results are displayed in table 1.

TABLE 1 GOES HERE

Observe that, under any level of risk (σ), increasing $1 - \gamma$ reduces the opening of the cone and tightens, therefore, the control exercised by the shipping activity on the balance of capital stocks. Several reasons are present, all of which act in the same direction. First, a larger value of $1 - \gamma$ implies a greater risk aversion, which increases the benefits of diversification and makes it more worthwhile to keep the capital stocks in balance. Secondly, an increase in $1 - \gamma$ is also a decrease in the rate of intertemporal substitution and in the rate of inter-personal substitution; this last, for instance, implies that more social welfare is lost from unequal consumption, so the cone narrows.[28]

As one approaches risk neutrality, the cone of no shipping widens indefinitely and, in the symmetric setup considered here, shipping occurs almost never, for as long as shipping costs are positive. The *ex ante* benefits of diversification procured by shipping no longer matter. Under risk neutrality, consumers have not hedged against the possibility that a physical

[28] I am grateful to a referee for suggesting this line of explanation.

imbalance may develop between the two capital stocks. If such an imbalance does develop, the only regulating mechanism may perhaps be found in consumption (but see below). Shipping does not take place because investors have not put a hedge in place ex ante that could finance such shipping ex post.[29]

The manner in which increasing risk (σ), in table 1, affects the size of the cone is somewhat surprising: it widens it. Increasing risk has an effect opposite to that of increasing risk aversion.[30] An increase in risk, like an increase in risk aversion, does increase the benefits of diversification and makes shipping more desirable. But observe that a wider cone, when risk

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[29] Roll (1979) has developed a partial-equilibrium argument, purporting to demonstrate that deviations from PPP must follow a martingale; this the "ex ante PPP" hypothesis. The argument refered to two populations of risk-neutral speculators who could not engage in international trade of goods but could engage in risky intertemporal trade (e.g. by storing goods in one country or the other), while the nominal exchange rate between the two countries fluctuated exogenously. There was a problem with this argument because, under strict risk neutrality, no equilibrium of the financial market exists. But the Roll situation can be viewed as a limit of the present model as one lets risk aversion tend to zero. The resulting specification is, in some ways, more general than Roll's because it is a general-equilibrium one and because physical investment opportunities are allowed to be risky. The fact that shipping, under symmetry, occurs almost never rationalizes the Roll assumption of a total absence of international trade. In the next section, we verify Roll's martingale result regarding price behavior.

[30] This result is surprising for another reason. In the context of portfolio choice under transactions costs, Constantinides (1986) finds that increasing risk decreases the opening of the cone of no transactions (see his table 3, page 854). But there are several differences between Constantinides' problem and ours. Our problem is symmetric as far as the choice of assets is concerned: we choose between two physical investments of equal volatility but located in different places, whereas his investor faces a choice between a risk and a riskless asset, so that increasing risk in his context meant increasing the risk of one asset only, not both as we do here. Furthermore, our problem is symmetric as far as the origin of consumption is concerned: people of each country consume out (possibly different amounts) of the stock of goods available locally, whereas Constantinides' investors consumed only out of the wealth invested in one asset: the risless one. Finally, Constantinides utilized an approximate optimization procedure based on sub-optimal consumption behavior, while our results are optimal. We have applied the Constantinides' approximate procedure to the present problem and verified that the differences between the two settings account for the difference in the comparative dynamic result.

increases, does not automatically translate into less frequent shipping. The greater volatility makes reaching the boundaries of a given cone more likely; the increase in risk by itself would increase the frequency of shipping.[31] The implication of the numerical result is that it increases it too much. The optimal cone widens in order in order to check that increase.

B. The process for the physical imbalance

The behavior of $\omega = \ell nK - \ell nK^*$ (the allocation of the goods between the two locations) inside the cone, as a result of production shocks and differential consumption rates, can be obtained easily from the knowledge of the value function. Based on (2) and (3), we have:

$$(10) \qquad d\omega = \left[-\frac{c}{K} + \frac{c^*}{K^*} \right] dt + \sigma\sqrt{2}\, d\bar{z}$$

where $d\bar{z} = (dz - dz^*)/\sqrt{2}$ is a standardized white noise. Furthermore (based on (4)):

$$(11a) \quad V_1 = V \frac{I'}{K} \qquad\qquad (11b) \quad V_2 = V \frac{-I' + \gamma}{K^*},$$

and thus:

$$(12a) \quad \frac{c}{K} = \left[\frac{V\, I'}{K^\gamma} \right]^{\frac{1}{1-\gamma}} \qquad (12b) \quad \frac{c^*}{K^*} = \left[V \frac{-I' + \gamma}{(K^*)^\gamma} \right]^{\frac{1}{1-\gamma}}.$$

Substituting (12a, b) into (10) fully determines the stochastic differential equation for ω inside the cone, from the knowledge of the $I(\omega)$ function.

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[31] I am grateful to a referee for suggesting this line of explanation.

The diffusion coefficient in (10) is constant.[32] This has two consequences. First, the instantaneous drift function, which we are about to examine, will, by itself, be representative of the process and will suffice to characterize the behavior of the variable within the cone: the diffusion will not in any way counteract the drift function. Second, the edges do not act as natural boundaries: there is a positive probability of reaching them in finite time. They are "regular boundaries" in the sense of Feller[33]. Because of the nature of the boundaries -- i.e. because they can be reached, -- the process would not be well defined by the stochastic differential equation (10) alone, without the specification of boundary behavior. This behavior has already been discussed: when ω reaches $\pm\ell n\lambda$ (from the inside of the cone) a directional impulse, or reflection, immediately brings the economy back inside the cone. The direction of the reflection in the (K, K^*) space is -s or -1/s, where s is the shipping loss factor.

FIGURE 2 GOES HERE

Figure 2 shows the conditional expected change (drift) of the ω process as a function of the current level of ω for the set of numerical values indicated in the legend. This puts the ω process in the form of an autoregressive process of order 1; the slope of the drift function increases as one approaches the edges of the cone: the process is non linear AR(1). The drift is equal to the relative consumption imbalance (see equation (10)). It is zero at the centerpoint ($\omega = 0$) when the two stocks of goods are balanced. Otherwise, the drift increases as a function of the imbalance, causing the

[32] This was our goal when we chose to define the physical imbalance as $\omega = \ell nK - \ell nK^*$, rather than, e.g., $\omega = K/(K + K^*)$ or any other way.

[33] See Cox and Miller (1965) pages 219ff or Karlin and Taylor (1981) pages 226ff.

boundary points. The stationary probability measure of ω (not shown) is U-shaped, with most of the probability mass on the sides. The capital mix ω spends most of the time near the boundaries where it exhibits a high degree of persistence; crossovers are rare.

Economically speaking, when one of the two countries, after a series of random shocks, has accumulated a larger capital stock than the other, it tends to accumulate proportionately even more capital, and to remain in that position for a long period of time, during which it is a recurrent excess producer and exporter.[35]

IV. EQUILIBRIUM BEHAVIOR OF DEVIATIONS FROM THE LAW OF ONE PRICE

Even though the optimization problem has been formulated as a centralized one, one can infer the prices which would prevail in a decentralized market economy, by examining the first derivatives of the value function $V(K, K^*)$. This was of course the main object of the determination of this function.

A. The deviation from the Law of One Price

Define p as the price of goods located at home relative to goods located abroad, (the price of a unit of K relative to a unit of K^*). It is given by:

[35] In a decentralized economy, the recurrent flow of export is sustained by prior conditional financial claims which have paid off in favor of the importing country. At the origin of times, each country has bet against itself becoming the capital rich location. This was a way of diversifying the locational advantage or handicap, which acts, in the choice of portfolios, as a non-traded asset. Alternatively, if, historically, the process was started near one boundary, the capital poor country had to start with more financial wealth (i.e. with some claims on the capital rich country). Otherwise, the welfare weights could not have been set equal to each other in the associated central-planning problem.

(14) $$p = \frac{V_1(K,\ K*)}{V_2(K,\ K^*)}\ .$$

Because of the homogeneity of V, p is a function of ω only. This function is (recall (11a, b):

(15) $$p(\omega) = \frac{I'(\omega)\ e^{-\omega}}{-I'(\omega)\ +\ \gamma}$$

The Law of One Price prevails when $p = 1$; p can be viewed as "the real exchange rate".

When goods are transferable, p is closer to the number 1 than it would be under autarky; i.e. when K is abunbdant, the price of K is larger than the autarkic price:

$$\text{when } K > K^*, \qquad 1 > p = \frac{V_1}{V_2} > \left[\frac{K}{K^*}\right]^{\gamma-1} = e^{(\gamma-1)\omega}.$$

One could easily show that this property of the price is equivalent to our previous result (13) concerning consumption behavior in relation to the stocks of goods. This provides an alternative rationale for the result.

Even though p is, properly speaking, the price which would prevail in the goods/capital market, it is inconvenient to study the stochastic process of p itself, whose definition is asymmetric in nature: interchanging the two goods changes p into 1/p, which is a non linear transformation. The symmetry will be preserved if, instead, one studies the behavior of the relative deviation from the LOP defined as the natural logarithm of p: ℓnp. Knowing the $I(\omega)$ function, equation (14) provides ℓnp as a function of ω. Since the process of ω is known,[36] it is an easy matter to obtain the process of ℓnp.

FIGURE 3 GOES HERE

The function $\ell np(\omega)$ is displayed in Figure 3. As expected, and as has been imposed by the boundary conditions (8a, b), p reaches the values s and 1/s at the two extremities: at the instant when shipping is activated, the good located in the country of abundance (from where shipping originates) is s times less valuable than the good located in the country where it is scarce. Of course, ℓnp reaches the corresponding values $\pm \ell ns$. Moreover, because the boundaries have been chosen optimally, the slope $p'(\omega)/p$ at those points is zero, by virtue of the super-contact optimality conditions (9a-d). When no shipping takes place, the price is somewhere between s and 1/s, depending on the degree of imbalance in the two stocks of goods. Under perfect balance in the quantities, $\ell np = 0$ and the LOP prevails.

B. The process for the LOP deviation

FIGURE 4 GOES HERE

Figure 4 shows the conditional expected change (drift) and the conditional standard deviation (in fact the signed diffusion coefficient) of the ℓnp process as functions of the current level of ℓnp (which puts the ℓnp process in the form of an autoregressive process of order 1), for the same numerical values as before, and two values of risk aversion $1 - \gamma = 2$ and $1 - \gamma = 0.1$.

The instantaneous drift of the process exhibits conditional mean reversion, since it is negative (see Figure 4) for a positive deviation from the LOP and

[36] See the previous section.

vice versa. This phenomenon is the result of two effects acting in opposite ways. We have noted (see section III) that the underlying physical process ω exhibits a centrifugal tendency, as a result of optimal differences in consumption rates between the two countries which are less than what would be needed to stabilize the capital mix. However, the $\ell np(\omega)$ function, relating the LOP deviation to the quantity imbalance, tapers off as one reaches the boundaries (see Figure 3), reflecting the anticipation that the intervention of shipping will prevent the price from escaping from the interval [s, 1/s]. This second effect dominates the first one and causes the price process overall to exhibit instantaneous mean reversion. In fact, as one approaches the boundaries, the anticipation of shipment also causes the drift to increase faster and faster, and to approach its finite limiting value with, what appears to be, an infinite slope. Not only is the process of deviations from the LOP not a martingale -- even though capital market efficiency (or rational expectations) has been assumed, -- it also is a highly non linear one.

Actually, the mean reversion obtained from the instantaneous drift is not representative of the probabilistic behavior of the process. Given that a monotone, continous, differentiable function links ℓnp to ω, it would be paradoxical that ω should have a centrifugal behavior while ℓnp would be reverting. The paradox fades away when one realizes (see Figure 4) that the process of LOP deviations, in sharp contrast to the process of ω, is strongly heteroskedastic: the conditional standard deviation is larger at the centerpoint than near the boundaries, and becomes zero (with an infinite slope) at the boundaries.[37]

Because of the shape of the diffusion function, the centerpoint acts as a

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[37] It has been noted above that, when boundaries are optimal, the slope of the $\ell np(\omega)$ function is zero at the boundaries, as a result of the super-contact conditions (9); this accounts for a diffusion coefficient of ℓnp equal to zero, even though that of ω is not zero at the same point.

repellent, which more than offsets the effect of the drift. At any point strictly within the cone, the conditional probability of an outward movement is, here again, greater than that of an inward movement, exactly as was the case for the physical imbalance ω. Despite the conditional mean reversion indicated by the drift of ℓnp, no reversion tendency effectively materializes.[38] The drift becomes dominant on the edges only, where the diffusion is null, and where it is the pure product of the impending reflection. Considering the shape of the $\ell np(\omega)$ function noted above (see again Figure 3), and considering the transformation of distances which it induces, the ℓnp process is "near" its boundaries more frequently than did the ω process.[39]

V. THE REAL INTEREST-RATE DIFFERENTIAL

As soon as the relative price of two goods, or two varieties of a good, fluctuates over time, the rate of interest measured in units of one of them is not equal to the rate of interest measured in units of the other.[40] Furthermore, if the fluctuations of the relative price are random, a financial

[38] In effect, the function $\ell np(\omega)$, being a non linear transformation, has reallocated the roles of the drift and the diffusion functions: the centrifugal drift has become a seeming mean reverting one, while the constant variance of ω has become a variable variance for ℓnp. Because the ω process is homoscedastic, the correct qualitative appraisal of probabilistic behavior is provided by the drift function of that process.

[39] Comparing, in Figure 4, the graphs obtained for $1 - \gamma = 2$, and for a low risk aversion $1 - \gamma = 0.1$, provides confirmation of the Roll (1979) conjecture: as one approaches risk neutrality, the drift function approaches the zero axis. As for the diffusion coefficient, it is also progressively reduced, so that the price process is characterized by less agitation as one approaches risk neutrality.

[40] For an explanation of this relationship and a recent empirical investigation of real rate differences across countries, see Cumby and Obstfeld (1984).

asset which would be riskless when its rate of return is measured in units of one of the goods, no longer is when its rate of return is measured in terms of the other. One must, therefore, be careful to distinguish four quantities which are conceptually quite different:

-quantity 1 (denoted r): the rate of interest on an asset which is riskless in terms of K (the good located at home). This is the own rate measured in units of K itself;

-quantity 2: the expected value of the rate of return on this same K-riskless asset but measured in terms of K^* (the good located abroad).[41] Quantities 1 and 2 differ only by the expected value of the change in the relative price; i.e, they differ if and only if the price does not follow a martingale process;

-quantity 3 (denoted r^*): the own rate of interest on an asset which is riskless in terms of K^*. Quantities 2 and 3 differ by a "risk premium";[42]

-quantity 4: the expected rate of return on the K^*-riskless asset measured in terms of K. Quantity 4 differs from quantity 3 by the expected price change. Quantity 4 differs from quantity 1 by a risk premium.

In what follows, the spread between quantity 1 (denoted r) and quantity 3 (denoted r^*) is measured and is called "the" real interest-rate differential. But this total (diagonal) spread is broken down into a component related to the expected price change and one which constitutes a risk premimum. In this way, the field is completely covered and the other quantities can be reconstructed, if one so desires.

It has been shown by Cox, Ingersoll and Ross (1985) that, in a market

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[41]This is the expected "real" rate of return from the K-riskless asset seen from the point of view of someone (a resident of the foreign country) who consumes the good located abroad.

[42]In what follows, the term "risk premimum" does not necessarily mean that the designated quantity would be equal to zero under risk neutrality, only that it would be zero in the absence of risk.

economy, the riskless interest rate is equal to the discount factor of utilities (ρ in our notation) minus the conditionally expected rate of change in the (undiscounted) indirect utility of wealth. It is also true that, in each country, the rate of interest is related to the expected rate of return α on productive assets:[43]

$$(16a)\quad r = \rho - \frac{1}{V_1}\frac{Ed(V_1)}{dt}; \qquad (16b)\quad \alpha = r + \frac{1}{dt}\frac{dK}{K}\frac{dV_1}{V_1};$$

$$(16c)\quad r^* = \rho - \frac{1}{V_2}\frac{Ed(V_2)}{dt}; \qquad (16d)\quad \alpha = r^* + \frac{1}{dt}\frac{dK^*}{K^*}\frac{dV_2}{V_2};$$

where, as we recall: $dK/K = \sigma dz$ and $dK^*/K^* = \sigma dz^*$. Equations (16b and d) explain how, under uncertainty, the two real rates of interest can be different in the two countries, even though the technological coefficients are equal. Equations (16a) and (16c) imply:

$$(17)\qquad r^* - r = \frac{1}{V_1}\frac{Ed(V_1)}{dt} - \frac{1}{V_2}\frac{Ed(V_2)}{dt}.$$

Based on the definition (14) of p, the following identities are all equally true:

$$(18)\qquad \frac{dV_1}{V_1} - \frac{dV_2}{V_2} = \frac{dp}{p} + \frac{dp}{p}\frac{dV_2}{V_2},$$

or:

$$(19)\qquad \frac{dV_2}{V_2} - \frac{dV_1}{V_2} = \frac{d(1/p)}{1/p} + \frac{d(1/p)}{1/p}\frac{dV_1}{V_1},$$

or yet:

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[43] In the following equations, E is the expected value operator conditional on the current values of K and K*.

$$(20)\qquad \frac{dV_1}{V_1} - \frac{dV_2}{V_2} = d(\ell np) + d(\ell np)\left[\frac{1}{2}\,\frac{dV_1}{V_1} + \frac{1}{2}\,\frac{dV_2}{V_2}\right].$$

Taking expectations as in (17), equation (18) provides the relationship between interest rates which is established by people residing in the foreign country, while (19) provides the same relationship from the point of view of the home country. In order to adopt a symmetric viewpoint, we choose the third, hybrid, expression (20) involving the logarithm; we substitute it into (17), and obtain a decomposition of the real-rate differential:

$$(21)\qquad r^{*} - r = \frac{Ed(\ell np)}{dt} + \frac{1}{dt}\,(\ell np)'(\sigma dz - \sigma dz^{*})\left[\frac{1}{2}\,\frac{dV_1}{V_1} + \frac{1}{2}\,\frac{dV_2}{V_2}\right];$$

where $(\ell np)'$ refers to the derivative of the function $\ell np(\omega)$. However, on grounds of symmetry, we have:

$$\sigma dz\left[\frac{dV_2}{V_2} - \frac{dV_1}{V_1}\right] = \sigma dz^{*}\left[\frac{dV_1}{V_1} - \frac{dV_2}{V_2}\right]$$

so that:

$$(\sigma dz - \sigma dz^{*})\left[\frac{1}{2}\,\frac{dV_1}{V_1} + \frac{1}{2}\,\frac{dV_2}{V_2}\right] = \sigma dz\,\frac{dV_1}{V_1} - \sigma dz^{*}\,\frac{dV_2}{V_2}\,.$$

But the right-hand side of this last equation, by virtue of (16b) and (16d), is none other than $(r^{*} - r)dt$ itself. Substituting this result into (21) we get finally:

$$(22)\qquad r^{*} - r = \frac{Ed(\ell np)}{dt} + (\ell np)'(r^{*} - r);$$

or:

$$\text{(23)} \qquad \frac{Ed(\ell np)}{dt} = [1 - (\ell np)'](r^* - r).$$

Equation (22) gives the real interest-rate differential as a function of the physical imbalance ω. The first term reflects the expected price change (obtained in section 4 above), while the second term is a remarkably simple expression for the risk premium, which says that the risk premium arises from the adjustment of the relative price of capital in relation to capital imbalance.

FIGURE 5 GOES HERE

The rate differential and its components are plotted as figure 5 for the same numerical values of the parameters as before. Observe the following:

-the differential, the expected price change and the risk premium are zero at the centerpoint $\omega = 0$. This is the result of the symmetric definition of the price variable ℓnp, used to reflect the expected price change, and of the symmetry of the break-up (20);

-the rate differential behaves very much like the expected price change. They both increase monotonically with the physical imbalance; they both reach a maximum on the edges of the cone. The risk premium notwithstanding, a large real-rate differential remains an indication of strong expected reversion in the LOP deviation. It can be said equally well that shipping is triggered when the interest rate differential reaches its largest possible value, or when the deviation from the LOP reaches its largest possible value;[44]

[44] In the pre-publication version of Adler and Lehman (1983), an appendix purported to demonstrate the martingale property of real exchange rates, based on the assumption that real interest rates were equal in the two countries. This assumption is unwarranted: we observe here that deviations from PPP and deviations from equality of real rates are part and parcel of the same

-the risk premium behaves in an interesting way: (i) its sign is always such as to increase the absolute value of the interest rate differential; it is equal to zero and changes sign when the physical imbalance ω passes the centerpoint $\omega = 0$; (ii) it is equal to zero again on the edges when the $\ell np(\omega)$ function has a zero slope and the price has a zero volatility; (iii) it reaches its largest absolute value somewhere between the centerpoint and the edges, so that the risk premium and the expected price change do not co-vary in the same way in a neighborhoood of the centerpoint and in a neigborhood of the extremities; (iv) despite the fairly large amount of risk and risk aversion assumed in our numerical example, the risk premium remains small in comparison to the expected price change and its possible fluctuations over time are also small in comparison with those of the expected price change.

Empirical studies, which have been conducted on foreign exchange markets, have produced evidence of discrepancies between <u>nominal forward</u> premia and conditional expected values of subsequent spot exchange rate changes. In Cumby and Obstfeld (1984), for instance, realized spot exchange rate changes were regressed on previous forward premia (or interest rate differtentials). The regression coefficient was practically always found to be significantly <u>smaller</u> than the hypothesized value of 1 (and sometimes to be negative). Equation (23) above provides the theoretical analog of Cumby and Obstfeld's regression. The coefficient is variable and equal to $1 - (\ell np)'$. Since $\ell np(\omega)$ (see Figure 3) is a decreasing function, the theoretical coefficient is <u>larger</u> than 1. In light of expression (23), the empirical findings must be considered extremely odd, for it is hard to see how, in almost any model, ℓnp would not generally be a decreasing function of ω (surely, K must be cheaper

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phenomenon.

when more abundant).[45]

CONCLUSION

If, after a string of positive output shocks, one country happens to have accumulated more capital than the other, the following mechanisms are set in motion:

-the goods or physical capital located in the capital-abundant country become cheaper than in the capital-scarce country. If we agree to call the price ratio "the real exchange rate", this says that the real exchange rate moves away from parity. A partially stabilizing influence at that stage is provided by consumption imbalance: the country with the lowest real exchange value of its capital consumes more. But it does so less than proportionately to the stock of capital located there, so that the capital-abundant country gradually becomes even more capital abundant: consumption alone is not capable of preventing an escape towards the boundaries. Within the cone of no shipment, the conditional probability of an outward move is everywhere larger than that of an inward move;

-beyond a point, if the physical imbalance between the two countries becomes excessive, a decision is made globally to transfer physical resources from the capital-abundant country to the capital-scarce country. This transfer is triggered when the price disparity has become sufficient to compensate for

[45]Fama (1986) invoked fluctuating risk premia in order to rationalize the discrepancy in Cumby and Obstfeld's results. But, in order for risk premia to perform that function, Fama observed, they would have to covary with expected exchange rate changes in a systematic way, and undergo fluctuations of comparable magnitude. The remarks of the previous paragraph indicate that the risk premium, which the present model has produced, does not fit Fama's hypothesized pattern.

the cost of the transfer;

-because the eventuality of the transfer is anticipated by investors, the price differential at all times is capped by the cost of tranferring the goods. As one approaches the point where a physical transfer is going to take place, the price differential tapers off even if the physical imbalance continues to widen;

-the real interest rate (measured in terms of local goods) in the capital-abundant country is lower than in the capital-scarce country, even though the anticipated productivity of each unit of physical capital remains the same in the two countries by virtue of the symmetry in technologies;

-this interest-rate differential is accounted for mostly by the anticipation that the real exchange rate, when a transfer occurs, will revert towards parity. However, a small risk premium also contributes to its determination.

Under sluggish quantity adjustment, the deviations from the Law of One Price do not follow a martingale process. Considering the Markov specification of the model, it is not surprising that the process found should be AR(1). It is also not surprising that the deviation should remain between two boundary values. The more interesting aspects concern the non linearity and heteroskedasticity of this AR(1) process. The conditionally expected change of the deviation is a non linear function of the current deviation (see Figure 4); its sign is such as to produce mean reversion (which is strongest when the deviation is largest, near and at the boundaries). However, this effect is more than offset by the conditional volatility of the deviation which is largest when the deviation is zero and smallest (actually zero) when the deviation is at its largest possible value, near and at the boundaries. As a result of the two effects combined, the probability of an outward move of the relative price is larger than that of an inward move, as was the case for the process of the physical imbalance. This is true everywhere except on the edges where a transfer of capital takes place. Furthermore, the process is most of the time near one of the boundaries, which means that LOP deviations -- and the associated trade imbalances -- typically last "a long time". These are periods of "quiescence", such as those which have been observed in the foreign exchange market; they can be interrupted by rare periods of "turbulence"[46], if an when the international economy, under the influence of a run of random shocks in the right direction, crosses over to the other boundary. At such times, the volatility is indeed larger than when the economy is near the boundaries.

Econometricians will have to determine whether empirical techniques (either in the space or frequency domains) which have been applied in the past to the study of LOP deviations, remain valid for this type of process. Even if they do, there is no doubt that the knowledge of the exact form of the process should help greatly increase the power of the tests.[47] It would, in particular, help in discriminating between the relative roles of non linearities and heteroskedasticity, a task which is close to impossible to accomplish when one uses general-purpose statistical models.[48]

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[46] See Frenkel and Levich (1977).

[47] See Glen (1989). Hsieh (1989) tests for non linearities in exchange rates and finds that a GARCH model is a fair representation of the data. It is not clear to me what relationship may exist between discrete-time GARCH models and the non-linear variable-variance continuous-time type of model which we have produced here.

[48] See Meese and Rose (1989)

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Table 1

The opening of the cone of no shipping as a function of risk and risk aversion. The table gives the value of λ. Numerical values of the parameter, other than γ and σ, are: $\rho = 0.15$, $s = 1/1.22$, $\alpha = 0.11$. The values in this table are obtained by a numerical procedure which has been described in footnote 26.

Risk aversion $1 - \gamma$	Risk (σ) 0.02	0.1	0.45	0.5
2	1.3447	1.9813	2.647	2.66325
1 (log)				3.2609
1/2		3.3598		4.1831
0.1				14.912
0 (neutral)	∞	∞	∞	∞

Legends for figures

Figure 1: The cone of no shipping. When the stochastic process of (K, K^*) reaches the frontiers of a cone, it receives a directional impulse of direction $-s$ or $-1/s$, where s is the shipping loss factor. On the edges, the slopes of the indifference curves of the value function of the programming problem, $V(K, K^*)$, are also equal to $-s$ or $-1/s$.

Figure 2: Drift of omega process. $\omega = \ell nK - \ell nK^*$ measures physical imbalance. The conditional variance of this process is constant. The figure shows the conditional expected change in ω for the following numerical values: $\rho = 0.15$, $\sigma = 0.5$, $\alpha = 0.11$. The figure demonstrates that the process for the physical imbalance is centrifugal.

Figure 3: The function $\ell np(\omega)$. p is the price of K relative to K^*. It is drawn here for the same numerical values used already for figure 2. It is a decreasing function of the relative amounts of K vs. K^*. The slope of this function is zero at the extremities when the frontier is placed optimally.

Figure 4: The LOP deviation process. The process for ℓnp is more complex than that of ω because the variance is not constant. The drfit and the diffusion functions are drawn here for the same numerical values. Despite the mean reversion produced by the drift function, the process is not a reverting one because of the countervailing effect of the diffusion.

Figure 5: The decomposition of the real rate differential into two components: the expected change in the price of K relative to K^* and a risk premium, measured symmetrically, as in the text, or asymmetrically from the point of view of a resident of the foreign country.

Figure 1: THE CONE OF NO SHIPPING

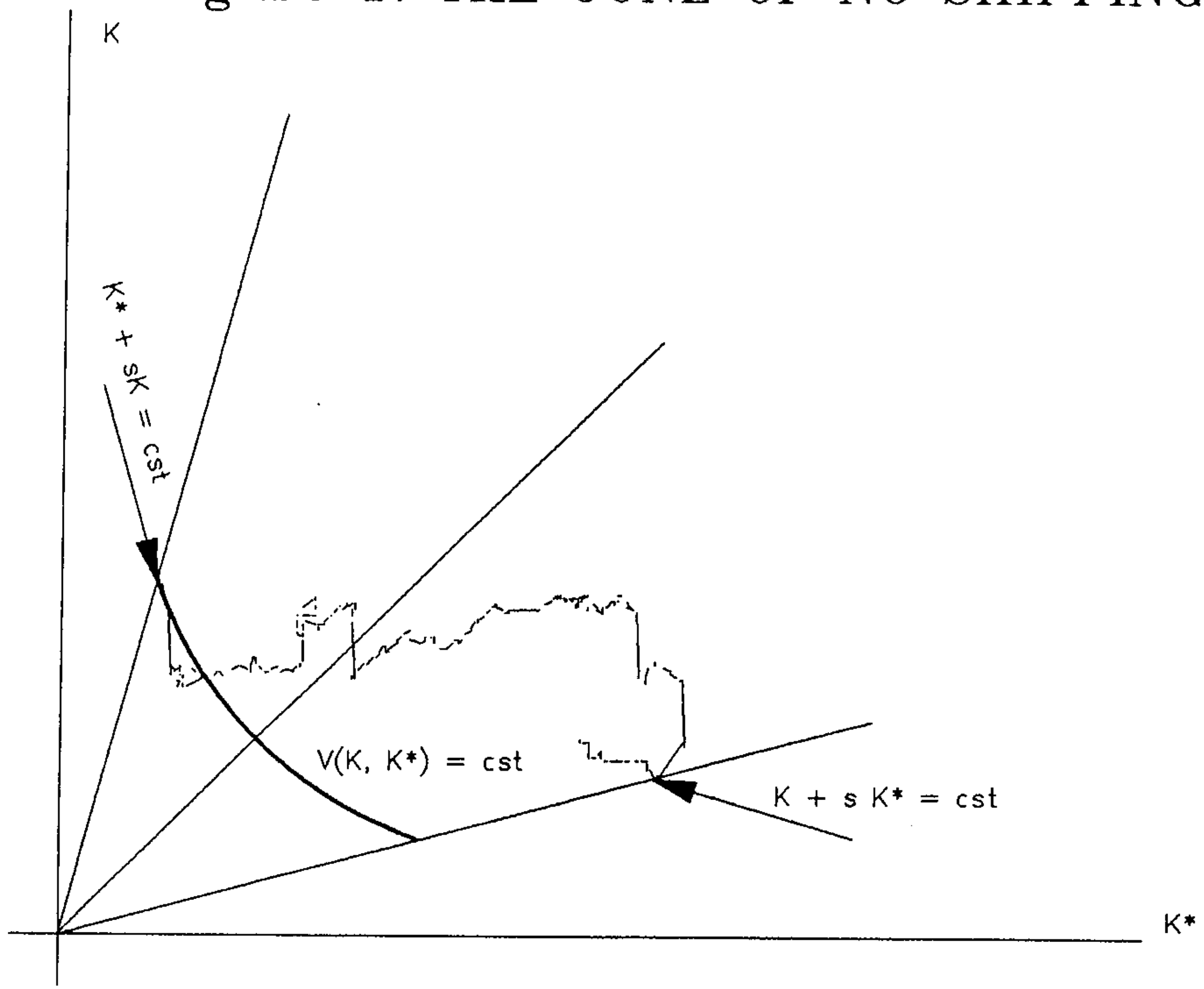

Figure 2: DRIFT OF OMEGA PROCESS

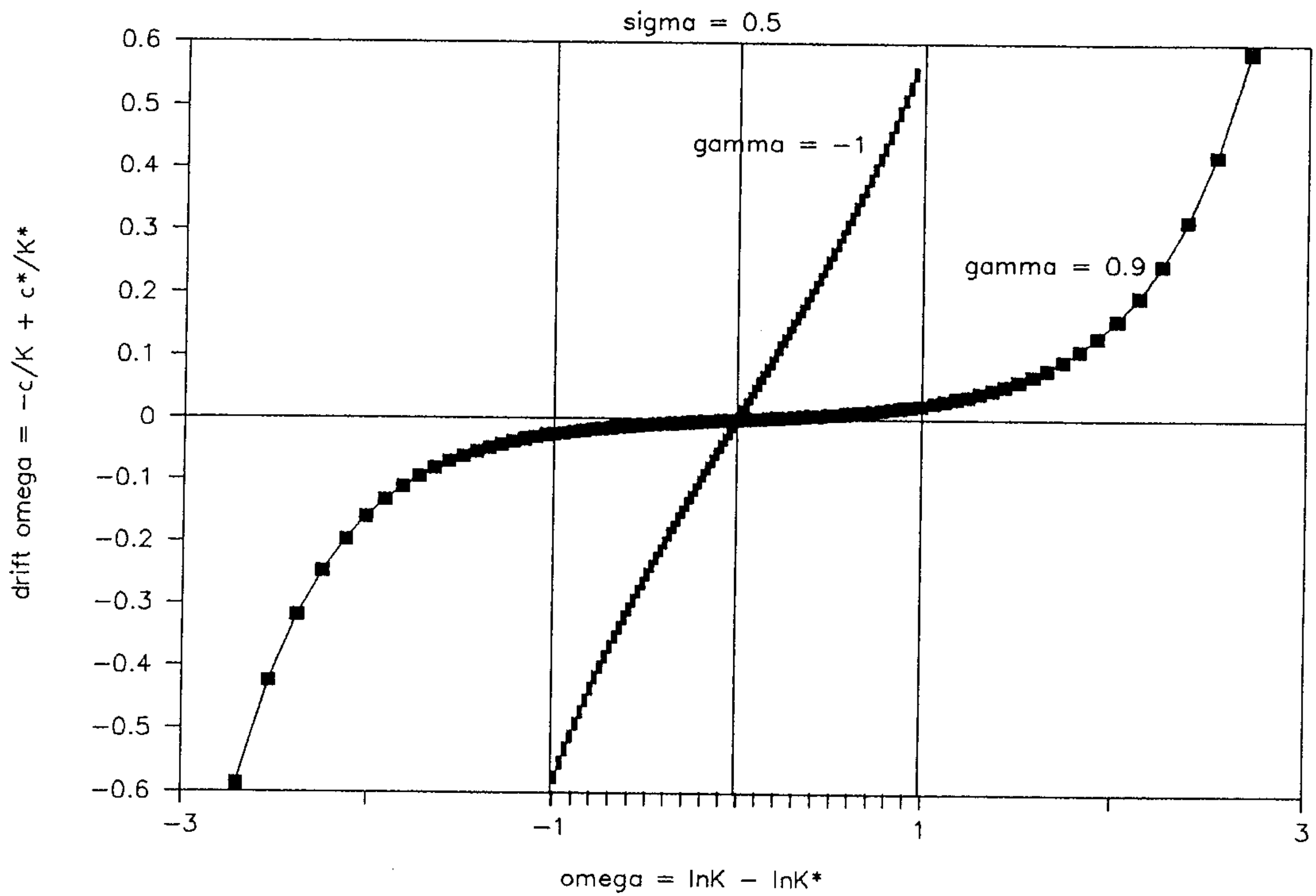

Figure 3: THE FUNCTION lnp(omega)

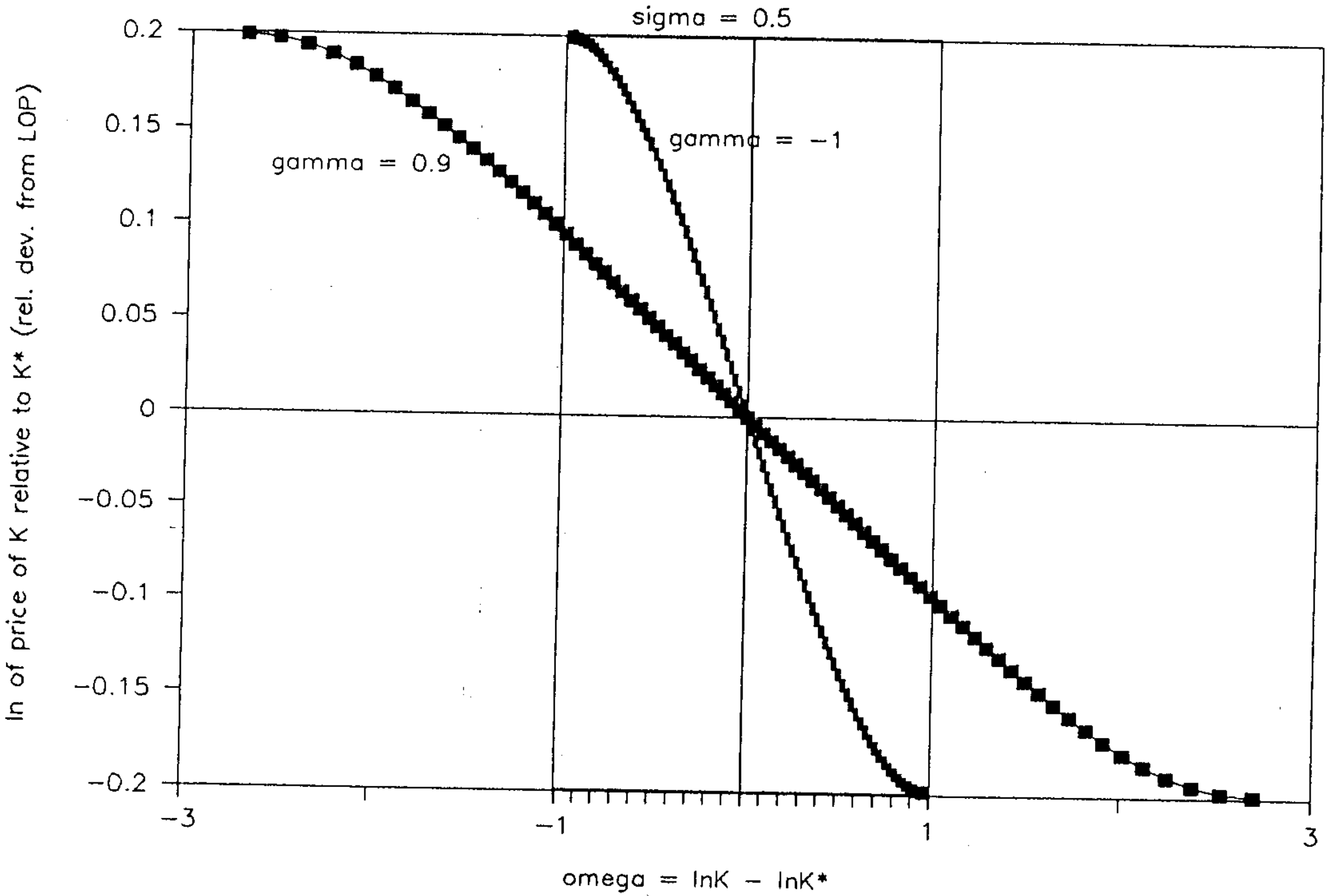

Fig. 4: LOP DEVIATION PROCESS

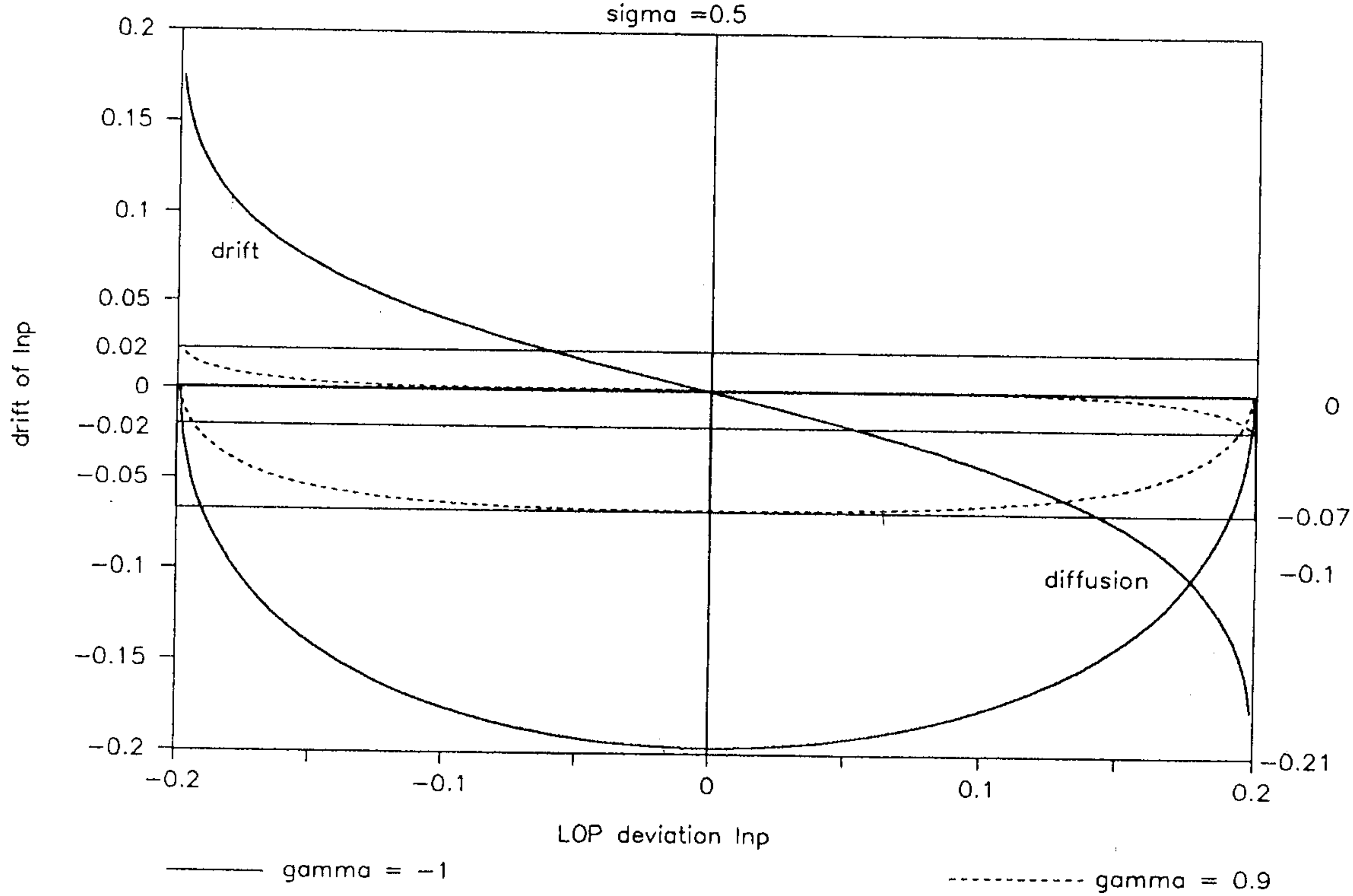

Figure 5: THE REAL RATE DIFFERENTIAL