ARBITRAGE, SHORT SALES AND FINANCIAL INNOVATION

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1. INTRODUCTION

The Arrow-Debreu-McKenzie model of general competitive equilibrium assumes the existence of a complete set of contingent commodities. More recently, economic theorists have developed models of general equilibrium with incomplete markets. (Seminal contributions were made by Diamond (1967), Radner (1972) and Grossman-Hart (1979); Cass (1987) surveys later developments by Duffie, Geanakoplos, Magill, Mas Colell, Polemarchakis, Shafer and others). What both types of models have in common is the assumption that the market structure is exogenous. Recent developments in financial markets illustrate the need to go beyond this assumption. The unprecedented rate of financial innovation makes it clear that market structure is neither constant nor exogenous. This simple observation raises important questions. What determines the nature and extent of financial innovation? Do we have too much or too little innovation? More generally, if market structure is endogenous what determines the set of markets observed?

Some of these questions were discussed in an earlier paper (Allen and Gale (1988), henceforth AG). In that paper we described a simple model of financial innovation. The basic model was in the tradition of Diamond (1968), Radner (1972), and Grossman and Hart (1979). There were assumed to be two time-periods, a finite set of states of nature and a single good. All agents had the same information structure: the state was unknown at the first date and revealed at the second. There was a finite number of types of firms and investors and a continuum of each type. Instead of taking the market structure as given, however, we assumed that firms could choose the securities they wanted to issue. In this way the market structure was endogenized and one could begin to study the process of financial innovation.

Because there is a continuum of firms and consumers, the economy described in AG is assumed to be perfectly competitive. The equilibrium concept is essentially Walrasian. In particular, prices are quoted for every possible security and both firms and consumers take these prices as given.

Firms in the AG model make interrelated production and financial decisions to maximize their market value. The firm's production decision specifies a current input and a vector of future outputs, one for each state of nature. Having chosen a production plan the firm wants to sell the output vector for the maximum possible amount. To do this the firm issues a number of different claims against its future revenues. In effect, the firm splits itself into several pieces and sells each of these pieces to a different <u>clientele</u>. Each piece of the firm (claim) is bought by the type of investor (clientele) that values it most highly. The firm's financial decision-problem is to find the optimal way of splitting itself up and marketing the pieces to different clienteles so as to maximize its present value.

A competitive equilibrium of this model has several nice properties:

(i) Under standard conditions equilibrium exists.

(ii) The equilibrium is constrained efficient. That is, given the technology available for issuing claims and the necessity of using claims to transfer wealth between periods and among states, a planner could not do better than the market.

(iii) The form of the optimal claims is characterized and shown to be particularly simple: in each state of nature the firm's entire output is given to the clientele that values it most.

(iv) The maximum number of claims that is needed to support a constrained efficient allocation can be expressed as a simple function of the number of types of firms and consumers.

Unfortunately, these attractive properties are purchased at the price of a strong assumption. One of the features of the analysis in AG was that we found it necessary to rule out short sales. More precisely, unless there are costs which effectively place a lower bound on short sales, a competitive equilibrium of the model does not exist. The non-existence problem can easily be understood by means of an example. Suppose an entrepreneur has set up a firm and wants to sell it for the maximum possible amount. He can issue a single claim against the firm, that is, sell equity only, or he can issue two claims, for example, issue debt as well as equity. In AG the costs of issuing claims were central to the explanation of why markets were incomplete. We assumed that issuing two types of claims would be more costly than one. Typically we assumed the costs of issuing a claim were fixed costs. That is, if the entrepreneur wanted to issue debt as well as equity he would have to pay a fixed additional amount. The entrepreneur's incentive to issue debt as well as equity is that by issuing the second claim he can increase the value of the firm. In fact, he will wish to issue the second claim only if the resulting increase in the firm's value is greater than or equal to the cost of issuing the debt. A corollary is that, in equilibrium, we will observe firms of the same type issuing different (numbers of) claims only if the firm with the more costly financial structure has a higher market value. For example, a firm issuing equity only would have a lower value than one that issued debt as well as equity. The difference in market value would just compensate for the additional costs of the more complex financial structure. But this difference in values gives rise to an opportunity for arbitrage. Since firms of the same type have perfectly correlated returns it must be possible to construct a riskless arbitrage portfolio by taking a short position in the firm issuing two claims and taking a long position in the firm issuing one claim. Clearly the existence of such a riskless arbitrage profit is inconsistent with competitive equilibrium.

To obtain an equilibrium, then, firms of the same type must choose financial structures with the same cost. What we showed in AG was that typically equilibria with this property do not exist. For example, if all firms issue equity only, then there will be an incentive for one firm to issue debt as well. But if some firms were to issue debt and equity then arbitrage would remove the reward for doing so. We cannot find any vector of prices at which all firms' financial plans are optimal. So unless short sales are constrained, equilibrium cannot exist. It should be noted that unlimited short sales are also the cause of non-existence of equilibrium in models with an exogenous market structure (Hart (1975)). But in such models, non-existence seems to be pathological (see, for example, Duffie and Shafer (1985, 1986)). In any case, the reasons for non-existence are quite different.

In some circumstances it may be appropriate to constrain short sales in this way. In other cases, however, it may be interesting to consider what happens if unlimited shorts sales are allowed. Empirically, short sales seem to be quantitatively unimportant (Pollack (1986)). The reasons are not entirely clear, but presumably imply the existence of substantial costs of taking short positions. On the other hand, markets for stock options and index futures may represent a low-cost substitute for short sales. The main reason for studying models with unlimited short sales, however, is theoretical. The possibility of unlimited short sales is a crucial assumption of several models that have a central place in financial economics. Some examples are Arrow securities (Arrow (1964)) the arbitrage pricing theory of Ross (1976) and the options pricing theory of Black and Scholes (1973).

In the present paper we suggest a way of reconciling the existence of equilibrium with short sales by weakening the price-taking assumption. In AG the auctioneer calls out prices and, taking these prices as given, firms choose their production and financial plans and investors choose their portfolios. This seems reasonable in a model with a continuum of agents but it is an assumption that ought to be justified. We attempt to justify the assumption by analyzing the limiting behavior of finite economies as the number of agents becomes unboundedly large. What we find is that the presence or absence of short sales makes a crucial difference. If unlimited short sales are allowed then even in the limit when the number of firms is very large firms may not behave as price-takers. The non-existence problem arises from the imposition of an inappropriate assumption (price-taking behavior) in the presence of short sales. Relaxing this assumption removes the non-existence problem. It also leads to a number of other important changes in the theory.

We begin our revision of the theory by describing a finite economy. There are two time periods and a single good in each period. Firms are represented by a finite set of entrepreneurs. Each entrepreneur is endowed with an asset that produces a random amount of the good in the second period. The random return of the asset can be thought of as the revenue of the firm. To simplify the story we assume that entrepreneurs only consume in the first period. There is a finite number of risk averse investors who value consumption in both periods. Since entrepreneurs only want to consume in the first period they sell their assets to the investors, who want to consume in both periods. An entrepreneur clearly wants to sell his asset for the maximum possible amount. He may be able to do this by issuing a single claim on the asset, that is, by selling equity. On the other hand, he may find it profitable to issue more than one claim and these may be of varying degrees of complexity. His problem is to choose a financial structure (set of claims issued) to maximize the market value of the asset net of the costs of issuing the claims.

Equilibrium is achieved in the first period. In the second period investors consume the income from their portfolio but there is no trade. Equilibrium in the first period is achieved in two stages. In the first stage, entrepreneurs decide what claims to issue. In the second stage, the given stock of claims is traded on an auction market. The market-clearing prices of claims at the second stage are functions of the stock of claims determined at the first stage. In choosing which claims to issue at the first stage, entrepreneurs anticipate the effect of their choices on the second-stage market-clearing prices. They are playing a strategic, non-cooperative game. Investors on the other hand are assumed to behave like perfect competitors. They take prices on the auction market as given and choose a portfolio to maximize their expected utility subject to the usual budget constraint. (This assumption is made to simplify the analysis and focus attention on the behavior of entrepreneurs. It can be justified by taking the finite number of investors to be representatives of continual of investors of the same type.)

In this model existence is not a problem. The existence of equilibrium can be directly attributed to the strategic behavior of entrepreneurs, in contrast to the price-taking behavior of entrepreneurs in the AG model. Because entrepreneurs anticipate the effect of their financial decision on second-stage prices and hence on the value of their asset, there may be an incentive to choose a more costly financial structure even if there is no difference ex post between the value of assets with more or less costly financial structures.

Grossman-Stiglitz (1980) point out that equilibrium may not exist in a model of financial markets where investors can acquire costly information. If prices reveal all the relevant information then investors will have no incentives to acquire it. But if no one acquires information the price signal cannot be informative. Then rational expectations equilibria will not exist. Grossman-Stiglitz resolve this problem by introducing "noise". The source of the problem is the assumption of price-taking behavior. When a very small number of investors is informed it is not clear why they should take prices as given. More precisely, they should anticipate that their decision to become informed will have an impact on prices. Jackson (1988) has shown that when the price-taking assumption is dropped, investors may have an incentive to acquire information even if prices are perfectly revealing. Essentially, it is because information-acquisition has an impact on prices which raises the value of the investors' endowment. This is another example of how the price-taking assumption can lead to non-existence when small agents can have a large impact on the economy. In both cases the solution is to drop the price-taking assumption.

Strategic, non-price-taking behavior is to be expected, of course, in a model with a finite number of entrepreneurs. Competitive behavior will be observed only in the limit as the number of entrepreneurs becomes unboundedly large. For this reason we are interested in what happens as the number of entrepreneurs and assets becomes large. Here the difference between economies with and without short sales is crucial. Consider first the case where no short sales are allowed. As the economy becomes large, the equilibrium converges under certain conditions to the Walrasian equilibrium studied in AG. In particular, the equilibrium is characterized by the following properties in the limit:

(i) Entrepreneurs become price-takers and maximize the value of the firm at the given prices.

(ii) Equilibrium is constrained efficient, that is, subject to the technology for issuing claims, we observe an efficient amount of financial innovation.

(iii) However, since innovation is costly, markets may remain incomplete. Not every claim that can be produced will be produced in equilibrium.

When unlimited short sales are allowed, however, the picture changes. In the first place,

(i) some entrepreneurs may not be price-takers even when the number of entrepreneurs becomes unboundedly large.

Part of the explanation lies in the fact that when unlimited short sales are allowed, the introduction of a new claim represents a "big" change in the economy. When no short sales are allowed the impact of a single entrepreneur is limited by the amount of the claims he introduces. If the entrepreneur is small relative to the economy then he has no market power and so will behave as a price-taker. When short sales are allowed, the entrepreneur has an additional impact on the economy. By introducing a new claim he allows investors to engage in trades that would not otherwise be possible. The open interest in the claim may be significant even when the entrepreneur is small relative to the size of the economy. Thus even in a large economy, a single entrepreneur may have a significant impact on the economy. If the entrepreneur has significant market power we do not expect him to behave as a price-taker. But this is only part of the story. We have explained why an entrepreneur may not behave as a price-taker when he introduces a claim that did not exist before. It does not explain why the claims that are relevant in equilibrium should not be issued by a large number of entrepreneurs. This is where the possibility of arbitrage becomes important. The entrepreneur bears the cost of introducing a new claim but any investor who takes a short position is competing with him in the supply of it. By arbitraging between assets with different financial structures, investors tend to equalize the value of the assets, thus destroying the incentive of the entrepreneur who chose the more expensive financial structure. In order to preserve the incentive to innovate, there must be a small number of entrepreneurs choosing the more costly financial structure. To be more precise, if only a small number choose a particular financial structure the incentive to innovate may be preserved in one of two ways. There may be imperfect competition which provides an incentive ex ante even though there is no incentive ex post. Or there may be idiosyncratic risk in the asset returns which prevents arbitrage among a small number of assets. In either case, the fact that only a small number of entrepreneur choose a financial structure means that none of them will be a price-taker.

As a result,

(ii) equilibrium may not be constrained efficient.

The lack of efficiency can be attributed to imperfect competition but there is also an externality lurking in the background. An entrepreneur's incentive to innovate comes from the effect that it has on the value of his asset. When there are no short sales the entrepreneur's private interests coincide with the social interest. When unlimited short sales are allowed, however, there is a divergence between private and social gains from innovation. The entrepreneur's attempt to maximize the value of his asset leaves out of account the social value of the extra risk-sharing provided by the short sellers. For this reason we might expect the level of innovation to be too low in equilibrium. However, as we shall see, the level of innovation can also be too high.

Finally, for all of the above reasons,

(iii) markets may be incomplete.

In particular, they may be "less complete" than when short sales are ruled out. For a given set of claims, allowing short sales improves risk-sharing. However, by comparison with an economy in which no short sales are allowed, there are reduced incentives to innovate and so the set of available claims may be poorer.

The rest of the paper is organized as follows. The model is defined in Section 2. Examples illustrating the properties of the model are presented in Section 3. In Section 4 we study the limiting behavior of equilibrium as the economy becomes unboundedly large, under the assumption that no short sales are allowed. In Sections 5 and 6 we carry out a similar analysis under the assumption that unlimited short sales are allowed. Illustrative examples are contained in Section 7. Section 8 briefly considers some extensions. Proofs are contained in Section 9.

2. A MODEL OF FINANCIAL EQUILIBRIUM

We begin by describing a simple economy in which agents can choose the kinds of claims they issue. Time is divided into two periods or dates indexed by $\tau=1,2$. At each date there is a single good, which can be thought of as "income" or "money". Asset returns and prices are measured in terms of this numeraire good. Economic agents are assumed to be either entrepreneurs or investors. Without much loss of generality we can assume there is an equal number N, say, of entrepreneurs and investors.

Equilibrium at the first date is achieved in two stages. In the first stage, entrepreneurs decide what kinds of claims to issue against the assets they own. At the second stage, these claims are traded on a competitive auction market. Asset returns are realized at date 2. The claims issued at date 1 are paid off at date 2 also.

2.1 Entrepreneurs

The set of entrepreneurs is denoted by $J_N = \{1,\ldots,N\}$. Entrepreneurs are assumed to be risk neutral and are only interested in consumption at the first date. They seek to liquidate their assets at the first date for the greatest possible value in terms of the numeraire good. Because markets are incomplete it may be possible to increase the market value of an asset by "splitting" it, ie., by issuing more than one type of claim against it.

Each entrepreneur owns exactly one asset. Each asset produces a random return at date 2: the return to the asset owned by entrepreneur $j \epsilon J_N$ is denoted by $\tilde{Z}_j$. Let $G_N : \mathfrak{R}^N \to \mathfrak{R}$ denote the joint cumulative probability distribution of the random variables $\{\tilde{Z}_j : j \epsilon J_N\}$. That is, for any $z = (z_1,\ldots,z_N) \epsilon \mathfrak{R}^N$,

(2.1) $G_N(z) = \text{Prob}\ \{\tilde{Z}_j \leq z_j,\ j \epsilon J_N\}$.

Asset returns are assumed to satisfy the following conditions.

<u>Assumption 1</u>: (a) $\tilde{Z}_j$ is non-negative w.pr.1 and has finite expected value;

(b) G_N is <u>symmetric</u>, ie., for any permutation $\pi : J_N \to J_N$ and any vector of returns $z \epsilon \mathfrak{R}^N$, we have

$$G_N(z_1,\ldots,z_N) = G_N(z_{\pi(1)},\ldots,z_{\pi(N)}).$$

Symmetry does <u>not</u> imply that assets are perfect substitutes. On the contrary, we will typically want to assume assets have idiosyncratic risk. On the other hand, symmetry does imply that assets make similar contributions to the joint return. In this sense there is a single <u>type</u> of asset. The results in this paper could easily be extended to a finite number of types. (A.1) simplifies the analysis without significantly reducing the insight we obtain from it.

Entrepreneurs issue claims against the assets they own. A claim is a function $t:\Re_+ \to \Re_+$. If the underlying asset shows a return of z units the claim t entitles the bearer to t(z) units of the numeraire. An example of a claim is debt with a face value of R. In that case

(2.2) $$t(z) = \min\{z,R\}.$$

The assumption that the claim is a function of the return to the underlying asset is natural in many cases. But it is restrictive nonetheless. In a later section we consider more general types of claims.

There is assumed to be a finite set of claims T that can be issued. Any $t \in T$ is assumed to be a measurable function. Typically an entrepreneur will issue several claims. These claims are assumed to exhaust the entire returns to the underlying asset. A collection of claims having this property is called a financial structure. There is a finite set S of financial structures. For any $s \in S$, the set of claims associated with the structure s is denoted by $T(s) \subset T$. The condition that the claims exhaust the total return to the asset can be expressed by saying that for any $s \in S$ and $z \in \Re_+$,

(2.3) $$\sum_{t \in T(s)} t(z) = z.$$

Suppose that an entrepreneur issues two claims (t_1,t_2). If t_1 is a unit of debt with face value R then (2.2) and (2.3) imply that t_2 must be a unit of equity satisfying

(2.4) $$t_2(z) = \max\{z-R,0\}$$

for any $z \in \Re_+$.

Issuing claims is assumed to be costly. All entrepreneurs are assumed to have the same cost function $C:S \to \Re_+$. If the entrepreneur chooses a financial structure $s \in S$ then he must pay a cost of $C(s)$ units, measured in terms of the numeraire good, at date 1.

2.2 Investors

There is a finite set K of investor <u>types</u>. Unlike entrepreneurs, investors are risk averse and want to consume at both dates. They purchase claims at date 1 in order to provide extra consumption at date 2. An investor of type $k \in K$ has a von Neumann-Morgenstern utility function $U_k : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. Note that consumption is not assumed to be bounded from below. Preferences are assumed to satisfy the following standard conditions:

<u>Assumption 2</u>: For every $k \in K$, U_k is concave, strictly increasing and bounded above.

We also impose the following non-standard condition:

<u>Assumption 3</u>: For any sequence $\{(c_n, \tilde{W}_n)\}$, where c_n is a real number and $\tilde{W}_n$ a random variable, $\liminf E[U_k(c_n, \tilde{W}_n)] = -\infty$ whenever $\liminf \text{Prob}[\tilde{W}_n \leq -M] \geq \varepsilon$ for some $\varepsilon > 0$ and arbitrarily large values of M.

What (A.3) says is that expected utility becomes very negative if there is a small amount of mass very far down the tail of the distribution of second-period consumption. (A.3) is implied by (A.2) if, in addition, the utility function is assumed to be additively separable. This property is not shared by some non-additively-separable utility functions, however. A counterexample is given by $U_k(c_1, c_2) = -\exp(-c_1 - c_2)$. All investors are assumed to have a zero endowment of the numeraire good at each date. This is simply a normalization in view of the fact that consumption is not bounded below.

Let $\Delta_N(K)$ denote the set of distributions on K, that is

(2.5) $\Delta_N(K) = \{\alpha: K \to \Re_+ | \Sigma\alpha(k)=1$ and $\alpha(k)=i/N$ for some $i=0,\ldots,N$ and all $k \epsilon K\}$.

Then the distribution of investor types can be represented by $\alpha_N \epsilon \Delta_N(K)$. For each $k \epsilon K$, $\alpha_N(k)$ is the fraction of investors of type k.

2.3 Exchange Equilibrium

An economy of size N is denoted by E_N, where E_N is identified with the array $(\{T(s)\}_{s\epsilon S}, \{\tilde{Z}_j\}_{j\epsilon J_N}, \{U_k\}_{k\epsilon K}, C, \alpha_N)$. An equilibrium for E_N is defined in two stages. First, we define equilibrium in the auction market, given the claims to be traded. Then we describe how claims are issued in the first stage in anticipation of the auction market equilibrium at the second stage.

Equilibrium in the auction market is assumed to be symmetric. Investors of the same type are treated symmetrically. Entrepreneurs are also treated symmetrically if they choose the same financial structure. More precisely, if two entrepreneurs choose the same structure then their claims will be priced the same way in equilibrium. If these conditions are satisfied then a further restriction can be imposed without loss of generality. It can be assumed that if two entrepreneurs have chosen the same structure, then for each claim t issued by them an investor will demand equal quantities from each entrepreneur. Since this symmetry property is a restriction of investors' choices we must show that it is indeed consistent with optimality.

Let s_j denote the financial structure chosen by entrepreneur j for every $j \epsilon J_N$. For any entrepreneur $j \epsilon J_N$ and any claim $t \epsilon T(s_j)$ let $p(t,j)$ denote the price of the claim and let $d(t,j)$ denote the quantity demanded by some investor, in the auction-market equilibrium. Suppose that two entrepreneurs h and i have chosen the same structure, that is, $s_h=s_i$. We assume that $p(t,h)=p(t,i)$ for every $t\epsilon T(s_i)=T(s_h)$. We want to prove that $d(t,h)=d(t,i)$ for every $t\epsilon T(s_i)$ is consistent with optimality. Let $d^0=(d^0(t,j))$ denote the portfolio in question and define a new portfolio $d^1=(d^1(t,j))$ by putting:

$$(2.6)\quad d^1(t,j) = \begin{cases} d^0(t,j) & \forall\ t\epsilon T(s_j),\ \forall\ j\epsilon J_N\backslash\{h,i\}; \\ d^0(t,i) & \forall\ t\epsilon T(s_i),\ j=h; \\ d^0(t,h) & \forall\ t\epsilon T(s_h),\ j=i. \end{cases}$$

In defining d^1 we have simply transposed the quantities of claims issued by the two entrepreneurs. From (A.2) it follows that the probability distribution of the returns to the portfolio d^0 is the same as the probability distribution of the returns to the portfolio d^1. Certainly the cost of the two portfolios is the same. Now define a third portfolio $d^2=(d^0+d^1)/2$. Since d^2 is a convex combination of d^0 and d^1, it must be weakly preferred by any risk averse investor. If the utility function were <u>strictly</u> concave in second-period consumption, d^2 would be strictly preferred to d^0 and d^1 whenever the latter were not identical. In any case, this argument shows that without loss of generality we can assume that $d(t,h)=d(t,i)$ for all $t\epsilon T(s_h)=T(s_i)$. This restriction is assumed in the definition of equilibrium given below.

The set of distributions on S is denoted by $\Delta_N(S)$ where

$$(2.7)\quad \Delta_N(S) = \{\beta:S\to\mathbb{R}\,|\,\Sigma\beta(s)=1 \text{ and } \beta(s)=i/N \text{ for some } i=0,\ldots,N \text{ for every } s\epsilon S\}.$$

In an economy of size N the cross-sectional distribution of financial structures is represented by $\beta\epsilon\Delta_N(S)$. $\beta(s)$ is the proportion of entrepreneurs choosing the structure $s\epsilon S$. It is assumed that the auction market equilibrium depends only on the distribution of structures; this is a further symmetry property imposed on equilibrium.

The claims traded in the auction market are indexed by $A=S\times T$. The preceding argument shows it is enough to distinguish claims by the payoff function and financial structure. Let $A(\beta)$ denote the set of <u>available</u> claims where $A(\beta)$ is defined by

(2.8) $A(\beta) = \{(s,t)\epsilon A|\beta(s)>0 \text{ and } t\epsilon T(s)\}$.

A <u>portfolio</u> is a function $d:A\rightarrow\mathbb{R}$, where d(a) is the investor's position in claims of type $a\epsilon A$. Note that d(a) is the total position in claims of type a: if n entrepreneurs issue claims of this type the investor demands d(a)/n units from each. As usual, a negative value of d(a) represents a short position. The set of portfolios is denoted by D and the set of available portfolios is denoted by $D(\beta)$, where

(2.9) $D(\beta) = \{d\epsilon D|d(a)=0 \text{ if } a \not\epsilon A(\beta)\}$.

Let $Y_N = (S)^N$ denote the set of assignments of financial structures to entrepreneurs. That is, if $y\epsilon Y_N$ and $y=(y_1,\ldots,y_N)$ then $y_j\epsilon S$ is the structure chosen by entrepreneur $j\epsilon J_N$. For any $\beta\epsilon\Delta_N(S)$ and $y\epsilon Y_N$ write $y\sim\beta$ if the assignment y has the cross-sectional distribution β. Let $Y_N(\beta)$ denote the set of assignments with cross-sectional distribution β, that is, $Y_N(\beta) = \{y\epsilon Y_N|y\sim\beta\}$. For any arbitrary choice of $d\epsilon D$ and $\beta\epsilon\Delta_N(S)$ let $W_N(d,\beta)$ denote the random payoff at date 2 from a portfolio $d\epsilon D$ when the cross-sectional distribution of financial structures is $\beta\epsilon\Delta_N(S)$. The number of entrepreneurs choosing a structure $s\epsilon S$ is $\beta(s)N$ so for any claim $a=(s,t)$ the number of units purchased by an investor from any one entrepreneur is $d(a)/\beta(e)N$. Then $W_N(d,\beta)$ is defined by

$$(2.10)\quad W_N(d,\beta) = \sum_{j\epsilon J_N} \sum_{t\epsilon T(y_j)} \frac{d(y_j,t)t(\tilde{Z}_j)}{\beta(y_j)N}$$

for any $(d,\beta)\epsilon D\times\Delta_N(S)$ and any $y\epsilon Y_N(\beta)$. The right-hand side of (2.10) is well defined by virtue of (A.1). Using (2.10) we define an indirect utility function $u_k^N:\Re\times D\times\Delta_N(S)$ by putting:

$$(2.11)\quad u_k^N(c,d,\beta) = E[U_k(c,W_N(d,\beta))]$$

for any $k\epsilon K$ and any $(c,d,\beta)\epsilon\Re\times D\times\Delta_N(S)$. $u_k^N(c,d,\beta)$ is the expected utility derived by an investor of type $k\epsilon K$ from a first-period consumption $c\epsilon\Re$ and portfolio $d\epsilon D$, when the cross-sectional distribution of structures is $\beta\epsilon\Delta_N(S)$.

Define a price system to be a function $p:A\rightarrow\Re_+$ and denote the set of price systems by P. Denote the set of allocations by $X=(\Re\times D)^N$; for any $x\epsilon X$ we write $x=(x_k)=((c_k,d_k))$, where $c_k\epsilon\Re$ is the first-period consumption and $d_k\epsilon D$ is the portfolio chosen by an investor of type $k\epsilon K$. Two notions of equilibrium are used in the sequel. In one investors are allowed to take unlimited short positions. In the other investors are allowed to take short positions not exceeding Q units, where Q is treated as a parameter.

DEFINITION: For any $\beta\epsilon\Delta_N(S)$ define a symmetric exchange equilibrium relative to β [SEE (β)] to be an ordered pair $(p,x)\epsilon P\times X$ satisfying the conditions:

(a) $\Sigma_{k\epsilon K}\, d_k(s,t)\ \alpha_N(k) = \beta(s) \quad \forall\ (s,t)\epsilon A$;

(b) $\forall k\epsilon K$, x_k maximizes $u_k^N(c,d,\beta)$ subject to $(c,d)\epsilon\Re\times D(\beta)$ and $c+p\cdot d=0$.

Similarly, for any $\beta\epsilon\Delta_N(S)$ and any $Q\epsilon\Re_+$ define a constrained symmetric exchange equilibrium relative to β and Q [CSEE(β,Q)] to be an ordered pair $(p,x)\epsilon P\times X$ satisfying condition (a) and (b') $\forall$ $k \epsilon K$, x_k maximizes $u_k^N(c,d,\beta)$ subject to $(c,d) \epsilon \Re \times D(\beta)$, $c+p \cdot d = C$ and $d(a) \geq -Q$, $\forall a \epsilon A$.

2.4 Existence of Exchange Equilibrium

Standard arguments suffice to prove the existence of a CSEE(β,Q) for any $\beta \epsilon \Delta_N(S)$ and any $Q \epsilon \Re_+$. A SEE(β) is a limiting case of a CSEE(β,Q) in which $Q \to \infty$. This fact is used to prove the existence of a SEE(β) for any $\beta \epsilon \Delta_N(S)$.

THEOREM 1: If (A.1) and (A.2) are satisfied there exists a CSEE(β,Q) for any $\beta \epsilon \Delta_N(S)$ and $Q \epsilon \Re_+$.

For any $Q \geq 0$ let (p^Q, x^Q) denote a CSEE(β,Q) for some fixed β. Choose a sequence of Q's converging to $+\infty$. For some sufficiently large value of Q the short sale constraint is not binding. The corresponding CSEE(β,Q) (p^Q, x^Q) is the required SEE(β).

THEOREM 2: If (A.1) through (A.3) are satisfied there exists a SEE(β) for any $\beta \epsilon \Delta_N(S)$.

2.5 Financial Equilibrium of the Full Model

Consider an economy of size N in which unlimited short sales are allowed. An equilibrium selection is a function $\phi: \Delta_N(S) \to P \times X$ that associates a SEE(β) with every $\beta \epsilon \Delta_N(S)$. Let Φ_N denote the set of equilibrium selections for an economy of size N.

For any $\phi \epsilon \Phi_N$ define a finite game $\Gamma_N(\phi)$ as follows. Let J_N be the set of players and let Y_N be the set of pure strategy profiles. For any $y \epsilon Y_N$ let $MV_j(y,\phi)$ denote the market value of asset $j \epsilon J_N$ in the SEE(β) $\phi(\beta)$, where $y \sim \beta$. Then denote the payoff function of player $j \epsilon J_N$ by $v_j(\cdot,\phi)$ and define it by putting

$$(2.12)\quad v_j(y,\phi) = MV_j(y,\phi) - C(y_j) \quad \forall\ y \epsilon Y_N$$

Then put $\Gamma_N(\phi) = (J_N, Y_N, \{v_j\}_{j \epsilon J_N})$

Since $\Gamma_N(\phi)$ is a finite game it has a Nash equilibrium in mixed strategies. In fact it has a symmetric Nash equilibrium, ie., one in which each entrepreneur chooses the same mixed strategy $\sigma \epsilon \Delta(S)$, where $\Delta(S)$ denotes the set of probability distributions on S. To see this note that for any $y \epsilon Y_N$ and any $\phi \epsilon \Phi_N$ and any permutation $\pi: J_N \to J_N$ the payoff functions satisfy

$$(2.13)\quad v_j(y_1,\ldots,y_N,\phi) = v_{\pi(j)}(y_{\pi(1)},\ldots,y_{\pi(N)},\phi) \ \forall\ j \epsilon J_N.$$

When (2.13) is satisfied a variant of the usual argument suffices to show that there exists a mixed strategy $\sigma^0 \epsilon \Delta(S)$ such that

$$(2.14)\quad \sigma^0 \epsilon \underset{\sigma \epsilon \Delta(S)}{\arg\max} \sum_{y \epsilon Y_N} v_j(y,\phi)\sigma(y_j)[\underset{j \neq i}{\times} \sigma^0(y_i)] \quad \forall\ j \epsilon J_N.$$

In fact, (2.14) defines a symmetric Nash equilibrium.

DEFINITION: A symmetric financial equilibrium (SFE) is an ordered pair $(\sigma,\phi) \epsilon \Delta(S) \times \Phi_N$ such that σ is a symmetric Nash equilibrium of $\Gamma_N(\phi)$.

From the preceding discussion and Theorem 2 we immediately have an existence theorem.

THEOREM 3: If (A.1) through (A.3) are satisfied there exists a SFE for any economy of size N.

One proceeds in exactly the same way for economies in which short sales are constrained. In the sequel we are only concerned with the case in which Q=0, so that no short sales at all are allowed. In this case we write CSEE(β) in place of CSEE(β,0). A constrained equilibrium selection is defined to be a function $\phi:\Delta_N(S)\rightarrow P\times X$ associating a CSEE(β) $(p,x)=\phi(\beta)$ with every $\beta\epsilon\Delta_N(S)$. The set of constrained equilibrium selections for an economy of size N is denoted by Φ_N^0. For any $\phi\epsilon\Phi_N^0$ we can define a finite game $\Gamma_N(\phi)$ exactly as before.

DEFINITION: A constrained symmetric financial equilibrium (CSFE) is an ordered pair $(\sigma,\phi)\epsilon\Delta(S)\times\Phi_N^0$ such that σ is a symmetric Nash equilibrium of $\Gamma_N(\phi)$.

As with the earlier case, Theorem 1 and the symmetry of the payoff functions guarantee the existence of a CSFE.

THEOREM 4: If (A.1) and (A.2) are satisfied then there exists a CSFE for every economy of size N.

3. EXAMPLES

To fix ideas an illustrate the working of the model, we introduce a simple class of examples in this section. Variants of this class are used in Section 7.

3.1 A Class of Asset Returns

The main simplification we adopt is that the returns to the assets $\{Z_j\}$ take on only a finite number of values. Let $\underline{Z}$ be a finite set of positive numbers. We assume that for any $j \epsilon J_N$, Z_j takes values in $\underline{Z}$ with probability 1. We also assume that there is a finite set of macrostates denoted by Ω. These macrostates are simply a convenient way of representing the correlation between the asset returns, but they are naturally interpreted as representing the values of any economy-wide shock ω. For any pair $(z_j, \omega) \epsilon \underline{Z}\times\Omega$, let $\mu(z_j,\omega)$ denote the probability of the event $\{Z_j = z_j\}$ given the value of the macrostate ω. We assume that given the value of ω, the random variables $\{Z_j\}$ are independently distributed. More precisely, let $f(z_1,\ldots,z_N|\omega)$ denote the joint probability density function of the random variables $\{Z_j\}$ conditional on ω. Then we can write

(3.1) $f(z_1,\ldots,z_N|\omega) = \mu(z_1,\omega)\mu(z_2,\omega)\ldots\mu(z_N,\omega)$.

Let $\mu(\omega)$ denote the probability of the macrostate ω.

Two special cases of this class of probability distributions are worth mentioning here. We can represent the case where all asset returns are perfectly correlated by putting $\underline{Z} = \Omega$ and

$$(3.2)\quad \mu(z,\omega) = \begin{cases} 1 \text{ if } \omega = z; \\ 0 \text{ otherwise.} \end{cases}$$

We can represent the case where all asset returns are independent simply by putting $\Omega = \{\omega_0\}$.

3.2 An Example with No Short Sales

Suppose that $\underline{Z} = \{0.5,\ 2.5\}$ and that asset returns are perfectly correlated, i.e., (3.2) is satisfied. The two output levels occur with probability 0.5. Entrepreneurs have a choice of two financial structures: $S = \{s_0, s_1\}$. The first structure consists of equity alone. The second allows the entrepreneur to issue two securities, one giving a claim to the high return and the other giving a claim to the low return. (In AG it is shown that this financial structure is optimal when there are no short sales). Formally,

$$(3.3)\quad T(s) = \begin{cases} \{(0.5,\ 2.5)\} \text{ if } s = s_0; \\ \{(0.5,0),\ (0,2.5)\} \text{ if } s = s_1. \end{cases}$$

Issuing one claim is assumed to be costless; issuing two has a positive cost γ, say. The cost function C is defined by putting:

$$(3.4)\quad C(s) = \begin{cases} 0 \text{ if } s = s_0; \\ \gamma \text{ if } s = s_1. \end{cases}$$

There are assumed to be two types of investors i=1,2. Their preferences are given by:

$$(3.5a)\quad U_1(c_1,c_2) = 5 + c_1 - \exp(-ac_2), \text{ where } a = 10;$$

$$(3.5B)\quad U_2(c_1,c_2) = 5 + c_1 + \ln c_2.$$

In AG it was assumed that there existed a continuum of investors and consumers. It was also assumed that no short sales were allowed. To illustrate the differences between the two models, we make the same assumptions here. In particular, let us assume that the measure of each type of investor is one unit and the measure of entrepreneurs is two.

Table 1a

EQUILIBRIUM IN EXAMPLE 1 WITH INCOMPLETE MARKETS

Group	Equilibrium Demand for Equity	Consumption		Marginal Utility of Consumption		Utility
		State 1	State 2	State 1	State 2	
a	0.29303	0.14652	0.73258	2.31042	0.00658	4.71248
b	1.70697	0.85348	4.26742	1.17167	0.23433	4.64629

The value of a firm = 0.58583

Table 1b

EQUILIBRIUM IN EXAMPLE 1 WITH COMPLETE MARKETS

Group	Consumption		Marginal Utility of Consumption		Utility
	State 1	State 2	State 1	State 2	
A	0.20706	0.38323	1.26113	0.21660	4.75405
B	0.79294	4.61677	1.26113	0.21660	4.64884

The value of a firm = 0.586033

Table 1c

A COMPARISON BETWEEN COMPLETE AND INCOMPLETE MARKETS IN EXAMPLE 1

Change in Group A's Utility	0.04156
Change in Group B's Utility	0.00255
Change in Firm Value	0.00020
Total Change	0.04431

A Walrasian equilibrium for this economy is illustrated in Table 1a. In this equilibrium, all entrepreneurs choose the financial structure s_0. The entrepreneurs behave as price-takers. There is an asset price corresponding to each choice of financial structure, but entrepreneurs do not perceive that [illegible] have any effect on these prices. If an entrepreneur "splits" his firm, i.e., issues two claims, the claim (0.5,0) will be held only by investors of type 1. Its value is 0.57760. The claim (0,2.5) will be held only by investors of type 2. Its value is 0.29292. Thus, the market value of the split asset will be 0.87052. On the other hand, an entrepreneur who only issues equity will find his asset is worth 0.58583. So it is optimal for an entrepreneur to choose the financial structure s_0 as long as the cost of choosing s_1 is greater than the increase in market value. That is, as long as $\gamma > 0.87052 - 0.58583 = 0.28468$.

When $\gamma = 0.28468$, entrepreneurs are indifferent between the two financial structures. At that point, they are prepared to split their assets to increase their market value. As γ falls, the proportion of entrepreneurs choosing the more expensive financial structure grows. At the same time, markets are becoming "more complete". The difference between the market values of assets with the two structures shrinks so that entrepreneurs remain indifferent between the two structures. When the cost parameter γ reaches 0, markets are effectively complete and the market value of assets is independent of financial structure.

3.3 An Example with Unlimited Short Sales

Because asset returns are perfectly correlated, there are only two states of nature. Either all returns are high ($Z_j = 2.5$ for all j) or all returns are low ($Z_j = 0.5$ for all j). Now, the claims associated with the financial structure s_1 are indistinguishable from contingent commodities on these states of nature. Markets are incomplete here because, except in the limiting cases $\gamma = 0$, the supply of these contingent commodities is limited. If unlimited short sales were allowed, however, markets would be complete.

It is easy to see from the example why no equilibrium may exist if short sales are allowed. By selling short the claims (0.5,0) and (0,2.5) and buying the claim (0.5,2.5), an investor could make a riskless arbitrage profit unless the value of a unit of (0.5,2.5) were the same as the sum of the values of a unit of (0.5,0) and a unit of (0,2.5). But in that case, it can never be optimal to issue two claims. On the other hand, if only one claim is issued by all entrepreneurs, it is clear that there will be an incentive to issue two claims if $\gamma < 0.28468$.

The non-existence of equilibrium arises because we have assumed there is a continuum of price-taking entrepreneurs. Suppose instead that we had assumed there were only two entrepreneurs ($N = 2$). If neither entrepreneur issues two claims, the SEE that results is described by the values in Table 1a. If either of these entrepreneurs issues two claims, markets in the second stage SEE are effectively complete. In that case, the no-arbitrage condition requires that the market value of an asset be independent of its financial structure. The equilibrium values of this SEE are shown in Table 1b.

Whenever someone innovates in the first stage, there is no incentive to innovate ex post. That is, the market value of an asset with the financial structure s_0 will be the same as the market value of an asset with the structure s_1. There may be an incentive to innovate ex ante, however. Let MV_I denote the market value of an asset in the SEE where markets are incomplete and let MV_C denote the market value of an asset in the SEE where markets are complete. If no one innovates the market value of an asset is MV_I. If one or more entrepreneurs innovates the market value is MV_C. A single entrepreneur by innovating may increase the market value of his asset by $MV_C - MV_I$. Of course, each entrepreneur would prefer the other to bear the costs of issuing the second claim, since they receive the same benefits ex post. In a mixed strategy equilibrium, the incentive to innovate comes from the possibility that the other entrepreneur will not innovate. If σ is the probability that an entrepreneur innovates, then an entrepreneur will be indifferent between innovating and not innovating if and only if $\sigma MV_C + (1-\sigma)MV_I = MV_C - \gamma$. Thus, in a SFE σ must satisfy

$$(3.6)\quad \sigma = \frac{MV_C - MV_I - \gamma}{MV_C - MV_I}.$$

In the example, $MV_C - MV_I = 0.00020$ so neither entrepreneur will issue two claims unless $\gamma \le 0.00020$. If, say, $\gamma = 0.00005$ then $\sigma = 0.25$. As γ becomes smaller the probability with which each entrepreneur innovates becomes higher. In the limit as γ approaches 0 they each innovate with probability 1.

4. ASYMPTOTIC BEHAVIOR OF CSFE

Equilibria in which no short sales are allowed provide a useful benchmark for the study of equilibria with unlimited short sales. In this section we characterize the limiting behavior, as N becomes unboundedly large, of CSFE. Consider a sequence of finite economies $\{E_N\}_{N=1}^{\infty}$ that are nested in the sense that

$$(4.1)\quad E_N = (\{T(s)\}_{s\epsilon S}, \{\tilde{Z}_j\}_{j\epsilon J_N}, \{U_k\}_{k\epsilon K}, C, \alpha_N)$$

for every N=1,2,.... As N increases the economy grows by the addition of new assets, entrepreneurs and investors. Suppose in addition that

$$(4.2a)\quad \alpha_N(k)N \le \alpha_{N+1}(k)(N+1) \qquad \forall\ k\epsilon K;$$

$$(4.2b)\quad \lim_{N\to\infty} \alpha_N = \alpha_\infty \epsilon \operatorname{int}\Delta(K).$$

If (4.1) and (4.2) are satisfied we call $\{E_N\}$ an <u>increasing sequence of economies</u>. Let $\{E_N\}$ be a fixed arbitrary increasing sequence. The <u>limit economy</u> corresponding to this sequence is denoted by E_∞ and defined by putting

$$(4.3)\quad E_\infty = (\{T(S)\}_{s\epsilon S}, \{\tilde{Z}_j\}_{j\epsilon J}, \{U_k\}_{k\epsilon K}, C, \alpha_\infty)$$

where $J = \cup J_N = \{1,2,\ldots\}$. For every N, let (σ_N,ϕ_N) be a fixed but arbitrary CSFE of E_N. We want to characterize the asymptotic behavior of the sequence $\{(\sigma_N,\phi_N)\}$.

4.1. Convergence of CSEE

We begin by studying the asymptotic behavior of CSEE. Let $\{\beta_N\}$ be a sequence such that $\beta_N \epsilon \Delta_N(S)$ for every N and $\beta_N \to \beta_\infty$ as $N\to\infty$. Then define a sequence $\{(p^N,x^N)\}$ by putting $(p^N,x^N)=\phi_N(\beta_N)$, for every N. In what follows this sequence is taken as given.

Assets are assumed to satisfy the following variant of the "law of large numbers".

Assumption 4: There exists a random variable $\tilde{Z}_\infty$ such that, for any sequence $\{J'_N\}$ with the property that $J'_N \subset J_N$ for every N and $|J'_N|\to\infty$, as $N\to\infty$, $\sum_{j\epsilon J'_N} |J'_N|^{-1}\, t(\tilde{Z}_j) \to E[t(\tilde{Z}_J)|\tilde{Z}_\infty]$ in probability for any $t\epsilon T$.

Assumption 4 is satisfied, for example, whenever $\{\tilde{Z}_j\}_{j\epsilon J}$ is a family of i.i.d. random variables. A more interesting case is given by

$$(4.3) \quad \tilde{Z}_j = \tilde{Z}_\infty + \tilde{\varepsilon}_j \quad (j\epsilon J),$$

where $\{\tilde{\varepsilon}_j\}$ are i.i.d. The purpose of (A.4) is to ensure that when the number of assets is large, investors can eliminate the idiosyncratic risk associated with particular assets by diversifying their portfolios.

(A.4) allows us to define the random return to portfolios in the limit economy. Let $\{y^N\}$ be a sequence such that $y^N \epsilon Y_N$ and $y^N \sim \beta_N$ for every N. Let $J_N(s) = \{j \epsilon J_N | y^N_j = s\}$ for every $s \epsilon S$. Then for any $d \epsilon D(\beta_\infty)$,

$$(4.4) \quad W_N(d,\beta_N) = \sum_{s\epsilon S} \sum_{t\epsilon T(s)} \sum_{j\epsilon J_N(s)} \frac{d(s,t)}{\beta_N(s)N} t(\tilde{Z}_j)$$

$$\to \sum_{s\epsilon S} \sum_{t\epsilon T(s)} d(s,t) \; E[t(\tilde{Z}_j) | \tilde{Z}_\infty]$$

by (A.4), where the convergence in (4.4) occurs "in probability". Notice that the last expression on the right of (4.4) is independent of β_∞. The only role of β_∞ here is to constrain the support of $d\epsilon D(\beta_\infty)$. Equation (4.4) motivates the following definition. For any $d\epsilon D$ put

$$(4.5) \quad W_\infty(d) = \sum_{s\epsilon S} \sum_{t\epsilon T(s)} d(s,t) R_t(\tilde{Z}_\infty),$$

where $R_t(\tilde{Z}_\infty) = E[t(\tilde{Z}_j) | \tilde{Z}_\infty]$ for any $t\epsilon T$. We interpret $W_\infty(d)$ as the random payoff to a portfolio $d\epsilon D$ in the limit economy with infinitely many assets. Then we define an indirect utility function $u^\infty_k : \mathbb{R}\times D \to \mathbb{R}$ by putting:

$$(4.6) \quad u^\infty_k(c,d) = E[U_k(c, W_\infty(d))] \quad (\forall k\epsilon K, \; \forall (c,d)\epsilon \mathbb{R}\times D).$$

With these concepts we can define an equilibrium in the limit.

<u>DEFINITION</u>: A CSEE(β_∞) of E_∞ is an ordered pair $(p,x)\epsilon P\times X$ satisfying:

(a) $\sum_{k\epsilon K} d_k(s,t)\alpha_\infty(k) = \beta_\infty(s) \quad \forall (s,t)\epsilon A;$

(b) $\forall k\epsilon K$, x_k maximizes $u^\infty_k(c,d)$ subject to $(c,d)\epsilon \mathbb{R}\times D$, $c+p\cdot d=0$ and $d\geq 0$.

There is an important difference between the concept of CSEE defined here for the limit economy and the one used earlier for finite economies. In the finite economy version an investor must satisfy the constraint $d\epsilon D(\beta)$. If $\beta(s)=0$ then

$d(a)=0$ for any $a=(s,t)\epsilon A$. In other words, the market is <u>closed</u> for any claim not issued in positive amounts. In the limit economy an investor is only required to satisfy $d\epsilon D$. In other words, all markets are <u>open</u> whether or not the claims are issued in positive amounts. Of course, all markets are required to clear in equilibrium. In the limit economy, $\beta_\infty(s)=0$ implies that $d_k(s,t)=0$ for any $(s,t)\epsilon A$, because of the market-clearing condition (a) and the no-short-sale constraint $d_k\geq 0$. But this is an equilibrium condition, not a constraint on the investors' behavior. In equilibrium there exists a market-clearing price at which investors are willing to take a zero position. This is the difference between an open market and a closed market. The open market has a market-clearing price that equates demand and supply to zero; the closed market does not. The definition of CSEE for the limit economy E_∞ is essentially the one used in AG. It is in the Arrow-Debreu tradition of assuming complete markets (for claims). Since the definition of CSEE in the limit is stronger than the definition of CSEE for finite economies, it is not obvious that the sequence of CSEE $\{(p^N,x^N)\}$ will converge to a CSEE in the limit. Under one further condition, however, this can be shown to be the case.

In order to discuss convergence at all, one must be sure that the sequence $\{(p^N,x^N)\}$ converges to something. The following lemma will be used repeatedly for this purpose.

<u>LEMMA</u>: Let $\{E_N\}$ be an increasing sequence of finite economies satisfying (A.1) through (A.4). Let $\{\beta_N\}$ be a sequence such that $\beta_N\epsilon\Delta_N(S)$ for every N and $\beta_N\to\beta_\infty$. For every N, let (p^N,x^N) be a fixed but arbitrary CSEE(β_N) (resp. SEE(β_N)) of E_N. Then the sequence $\{(p^N,x^N)\}$ is bounded and the sequence $\{W_N(d_k^N,\beta_N)\}$ is uniformly tight for each $k\epsilon K$.

The lemma implies that the sequence $\{(p^N,x^N)\}$ has a convergent subsequence which, without essential loss of generality, we can take to be the original sequence. Then

(4.6) $(p^N,x^N) \to (p^\infty,x^\infty) \in P\times X$,

say. To show that (p^∞,x^∞) is a CSEE(β_∞) of E_∞ a continuity assumption assumption is required.

A price function p is defined on the set of all (conceivable) claims A whereas only claims in $A(\beta)$ are traded. The prices of claims in $A\backslash A(\beta)$ are clearly indeterminate in equilibrium. So we adopt the <u>convention</u> that $p(a)=0$ if $a\in A\backslash A(\beta)$. Any reference to continuity or uniqueness in the sequel should be interpreted with this convention in mind.

(4.7) For any neighborhood V of (p^∞,x^∞) there exists a neighborhood V′ of β_∞ and a number $\bar{N}$ such that $\phi_N(V')\subset V$ for any $N>\bar{N}$.

Note that (4.7) is a property of a particular sequence $\{\phi_N\}$.

<u>THEOREM 5</u>: Let $\{E_N\}$ be an increasing sequence of finite economies satisfying (A.1) through (A.4) and let E_∞ be the corresponding limit economy. Let $\{(\beta_N,\phi_N)\}$ and $\{(p^N,x^N)\}$ be sequences satisfying the following conditions:

(a) $\beta_N\in\Delta_N(S)$, $\phi_N\in\Phi_N^0$ and $(p^N,x^N)=\phi_N(\beta_N)$ for every N;

(b) $\beta_N \to \beta_\infty\in\Delta(S)$ and $(p^N,x^N) \to (p^\infty,x^\infty)\in P\times X$;

(c) (4.7) is satisfied.

Then (p^∞,x^∞) is a CSEE(β_∞) of the limit economy E_∞.

Theorem 5 justifies our definition of equilibrium in the limit by showing that a sequence of CSEE converges to a CSEE in the limit. The key to this result is the observation that, in view of (4.7), small perturbations of β_N ensure that β_N has full support without perturbing equilibrium in the limit. Thus one can assume, without essential loss of generality, that $D(\beta_N)=D$ for all N sufficiently large. We do not have a complete characterization of the conditions under which (4.7) is satisfied. The following theorem provides a sufficient condition, but one that is "too strong".

<u>THEOREM 6</u>: Let $\{E_N\}$, $\{(\beta_N,\phi_N)\}$ and $\{(p^N,x^N)\}$ satisfy the conditions of Theorem 5. Suppose that (p^∞,x^∞) is the unique CSEE(β_∞) of E_∞. Then (4.7) is satisfied.

4.2 <u>Convergence of CSFE</u>

Now we are ready to analyze the asymptotic behavior of the sequence $\{(\sigma_N,\phi_N)\}$ of CSFE. Since $\sigma_N\epsilon\Delta(S)$, $\{\sigma_N\}$ has a convergent subsequence which we take to be the original sequence. For each N, the cross-sectional distribution of financial structures generated by σ_N is a random vector denoted by $\tilde{\beta}_N$. By the weak law of large numbers, $\tilde{\beta}_N$ converges in probability to a constant β_∞. Then condition (4.7) implies that $\phi_N(\tilde{\beta}_N)$ converges weakly to (p^∞,x^∞). In other words, in the limit the CSFE contains a non-stochastic CSEE. This suggests the following definition of CSFE "in the limit".

<u>DEFINITION</u>: A <u>pure CSFE</u> of E_∞ is an ordered triple $(\sigma_\infty,p^\infty,x^\infty)\epsilon\Delta(S)\times P\times X$ such that:

(a) (p^∞,x^∞) is a CSEE(β_∞) of E_∞, where $\beta_\infty=\sigma_\infty$;

(b) $$\sigma_\infty\epsilon \arg\max_{\sigma\epsilon\Delta(S)} \sum_{s\epsilon S} \sigma(s)\left[\sum_{t\epsilon T(s)} p^\infty(s,t)-C(s)\right].$$

The definition of CSFE in the limit differs from the definition of CSFE for finite economies in two ways. First, it assumes the cross-sectional distribution of financial structures is non-random. Second, it assumes price-taking behavior on the part of entrepreneurs. The justification for this definition is given by the following theorem.

THEOREM 7: Let $\{E_N\}$ be an increasing sequence of finite economies satisfying (A.1) through (A.4) and let E_∞ be the corresponding limit economy. Let $\{(\sigma_N,\phi_N)\}$ be a sequence satisfying:

(a) (σ_N,ϕ_N) is a CSFE of E_N, for every N;

(b) $\sigma_N \to \sigma_\infty \epsilon \Delta(S)$ and $\tilde{\beta}_N \to \beta_\infty = \sigma_\infty$ in probability, where $\tilde{\beta}_N$ is the random cross-sectional distribution generated by σ_N for every N;

(c) $\phi_N(\tilde{\beta}_N) \to (p^\infty, x^\infty) \epsilon P \times X$, in probability and (4.7) is satisfied. Then $(\sigma_\infty, p^\infty, x^\infty)$ is a pure CSFE of E_∞.

This result is analogous to Hart's derivation of value maximizing behavior in a model of monopolistically competitive equilibrium when the number of firms becomes unboundedly large (Hart (1979)). The difference is that we have extended the notion of similar commodities. Hart's assumption of a compact commodity space would not accommodate the idiosyncratic risk in the present model. (See also Jones (1987)).

Theorem 7 provides the link between the present model and the one described in AG. It justifies the assumption of price-taking behavior in the limit and also provides the means for proving the constrained efficiency of CSFE in the limit. Call an allocation $(\beta,x) \epsilon \Delta(S) \times X$ attainable for E_∞ if and only if

$$(4.8) \quad \sum_{k \epsilon K} d_k(a)\alpha_\infty(k) = \beta(e) \qquad (\forall\ a=(s,t) \epsilon A).$$

An attainable allocation (β,x) is said to be constrained efficient for E_∞ if there does not exist an attainable allocation (β',x') such that

$$(4.9)\quad (a)\quad u_k^{\infty}(c_k, d_k) < u_k^{\infty}(c_k', d_k') \quad \forall k \epsilon K;$$

$$(b)\quad \sum_{k \epsilon K} c_k' \, \alpha_{\infty}(k) + \sum_{s \epsilon S} C(s)\beta'(s) < \sum_{k \epsilon K} c_k \alpha_{\infty}(k) + \sum_{s \epsilon S} C(s)\beta(s).$$

Condition (4.9b) says that all entrepreneurs can be made better off under (β', x').

THEOREM 8: If $(\beta_{\infty}, p^{\infty}, x^{\infty})$ is a pure CSFE for E_{∞} then $(\beta_{\infty}, x^{\infty})$ is constrained efficient for E_{∞}.

5. ARBITRAGE PRICING

In the last section we examined the benchmark case of an economy in which no short sales are allowed. In this section we turn to the study of economies with unlimited short sales. Allowing unlimited short sales gives investors the ability to arbitrage among assets with different financial structures. This arbitrage activity restricts the asset prices that can be observed in equilibrium. We present a simple arbitrage result below. In the next section we show that this result has important implications for the nature of equilibrium.

Let $\{E_N\}$ be an increasing sequence of economies, let $\{\beta_N\}$ be a sequence of cross-sectional distributions of financial structures and let $\{(p^N, x^N)\}$ be a corresponding sequence of SEE. That is, for every N, (p^N, x^N) is a SEE(β_N) of E_N. From the Lemma we know that, if (A.1) through (A.4) are satisfied, $\{(p^N, x^N)\}$ has a convergent subsequence. For every N define the market value $MV^N(s)$ of an asset with financial structure $s \epsilon S$ by the equation:

$$(5.1)\quad MV^N(s) = \sum_{t \epsilon T(s)} p^N(s,t) \quad \forall\, s \epsilon S.$$

We are interested in the behavior of market values as the size of the economy becomes very large.

<u>THEOREM 9</u>: Let $\{E_N\}$ be an increasing sequence of finite economies satisfying (A.1) through (A.4). Let $\{\beta_N\}$ be a sequence of distributions satisfying $\beta_N \epsilon \Delta_N(S)$ for every N. Let $\{(p^N, x^N)\}$ be a convergent sequence such that (p^N, x^N) is a SEE(β_N) of E_N, for every N. Let $S^* \subset S$ be the set of structures such that $\beta_N(s)N \to \infty$ as $N \to \infty$ for all $s \epsilon S^*$. Then there exists a constant $\bar{m}$ such that $MV^N(s) \to \bar{m}$ for every $s \epsilon S$ as $N \to \infty$, where $MV^N(s)$ is defined by (5.1).

What the theorem says is that all assets with financial structures in S* have the same market value in the limit. Several points should be noted about this result. First, it is an arbitrage result. If this restriction were not satisfied then investors could make an arbitrage profit, which is inconsistent with the conditions of equilibrium. Second, it is an asymptotic result. In a finite economy arbitrage imposes relatively weak restrictions because of the idiosyncratic component in asset returns. When the number of assets is very large, on the other hand, idiosyncratic risk can be eliminated through diversification and arbitrage imposes quite strong restrictions. Third, this result holds for all but a negligible set of assets. More precisely, if $MV^N(s)$ does not converge to $\bar{m}$ then the number of assets with financial structure s is bounded uniformly in N. However, although these assets are negligible in number they may be important in other senses. We return to this point below.

The intuition behind Theorem 9 is quite straightforward. If it were not true there would exist, in the limit, two infinite sets of similar assets, one of which had uniformly higher market values than the other. Since these two sets of assets are infinite, (A.4) implies that diversification eliminates idiosyncratic risk. By shorting the overvalued set and going long in the undervalued set it is possible to construct an arbitrage portfolio that is riskless in the limit and yields a positive profit. The existence of such a portfolio is inconsistent with equilibrium.

It is tempting to interpret this result in the spirit of the Modigliani-Miller Theorem (MMT). The MMT states that under certain conditions corporate financial structure is irrelevant. More specifically, it implies that a firm's market value will be independent of its financial structure, where financial structure is identified with the debt-equity ratio. The argument is that as the firm increases (decreases) the amount of its debt outstanding, an investor can exactly offset the effect on his own portfolio by increasing (decreasing) the amount of debt he holds. An investor should therefore be indifferent to the firm's debt-equity ratio and the market value of the firm should not be affected by it either.

On the surface this sounds rather like the conclusion of Theorem 9. If we interpret the assets as firms and the claims as securities issued by the firms then the theorem seems to say that the value of the firm is independent of corporate financial structure. There are important differences between the two theorems, however. In the first place, the MMT is a statement about the choices available to the firm. It says that whatever debt-equity ratio the firm chooses, the firm's market value will be the same. Theorem 9, on the other hand, is a statement about the financial structures chosen in equilibrium. It says that in equilibrium the market values of most assets will be the same. Thus, for example, Theorem 9 is consistent with a situation in which all assets have the same financial structure. This need not be a situation in which the financial structure is a matter of indifference, as the MMT claims. In short, Theorem 9 asserts a condition that must be satisfied in equilibrium; it does not assert the irrelevance of financial structure.

A second and more subtle difference is that Theorem 9 leaves out of account any subset of assets that is finite and hence negligible in the limit. By "negligible in the limit" we mean that the fraction of all assets falling into this subset converges to zero as the total number of assets diverges to infinity. Now a set which is negligible in this sense may not be negligible in terms of its importance for risk sharing in the economy. In particular, the open interest in the claims written on this "negligible" set of assets may be non-negligible in the limit. This is not merely a theoretical possibility. It is crucial to understanding the behavior of an economy with unlimited short sales as the number of assets gets very large. We return to this subject below.

6. ASYMPTOTIC BEHAVIOR OF SFE

In Section 4 we showed that, under some conditions at least, as the number of assets becomes unboundedly large the CSFE converge to a Walrasian equilibrium in the limit. When unlimited short sales are allowed, this kind of well behaved convergence does not take place. That, in fact, is the main conclusion of this section. Because the SFE do not converge to a well-behaved Walrasian equilibrium it is not easy to characterize what does happen. We adopt an indirect approach here. We assume that as N approaches infinity certain conditions that would seem to be "necessary" for perfect competition are eventually satisfied. Then we show that this leads to a "contradiction." In this way we demonstrate the practical impossibility of asymptotic efficiency. Exactly what does happen as N approaches infinity is investigated in more detail in Section 7, using examples.

The rest of this section is organized like Section 4. We make analogous assumptions in order to obtain a competitive limit theorem. But in the limit we find that these assumptions cannot consistently be maintained.

6.1 Convergence of SEE

As a prelude we study the asymptotic behavior of SEE. Let $\{E_N\}$ be an increasing sequence of economies and let $\{(\beta_N,\phi_N)\}$ be a sequence such that $\beta_N \epsilon \Delta_N(S)$ and $\phi_N \epsilon \Phi_N$ for every N. Define a sequence of SEE by putting $(p^N,x^N)=\phi_N(\beta_N)$ for every N. From the Lemma we know there exists a convergent subsequence which we take to be the original sequence. Let $\beta_N \to \beta_\infty$ and $(p^N,x^N) \to (p^\infty,x^\infty)$ as $N\to\infty$.

Call the sequence $\{\beta_N\}$ competitive if and only if it satisfies the following condition:

(6.1) for any $s \epsilon S$ either (a) $\beta_N(s)N\to\infty$ as $N\to\infty$ or (b) $\beta_N(s)=0$ for all N sufficiently large.

What (6.1) rules out is cases where $\beta_N(s)N$ remains positive but bounded for arbitrarily large values of N. If (6.1) is satisfied any financial structure chosen by one entrepreneur is eventually chosen by a large number of entrepreneurs. This would seem to be a necessary condition for price-taking behavior in the limit. Unlike the no-short-sale case an entrepreneur issuing a claim which is issued by no one else or only by a few others will have a potentially large impact on the market. By innovating he allows others to go long or short and the open interest may be non-negligible even if the entrepreneur is negligible. Note that condition (6.1) admits the possibility that an entrepreneur may have a large impact on the market by deviating from his equilibrium strategy and choosing a financial structure not previously chosen by anyone.

Suppose $\{\beta_N\}$ is competitive and let S* be the set of structures $s \epsilon S$ such that $\beta_N(s)N\to\infty$ as $N\to\infty$. Let $\{y^N\}$ be a sequence such that $y^N \epsilon Y_N$ and $y^N \sim \beta_N$ for every

N. Let $D(S^*)=\{d\epsilon D \mid d(s,t)=0 \text{ if } s\notin S^*\}$. Note that $D(S^*)=D(\beta_N)$ for sufficiently large N. (A.4) allows us to define $W_\infty(d)$ and $u_k^\infty(c,d)$ for any $k\epsilon K$ and $(c,d)\epsilon \mathbb{R}\times D$ in the usual way. (See (4.4) to (4.6)). Define the limit economy E_∞ by putting $E_\infty=(\{T(s)\},\{\tilde{Z}_j\},\{U_k\},C,\alpha_\infty)$.

DEFINITION: Given a limit economy E_∞, a cross-sectional distribution $\beta_\infty\epsilon\Delta(S)$ and a set $S^*\subset S$ such that supp $\beta_\infty\subset S^*$ define a SEE(β_∞,S^*) of E_∞ to be an ordered pair $(p^\infty,x^\infty)\epsilon P\times X$ such that

(a) $\Sigma_{k\epsilon K}\, d_k(s,t)\alpha_\infty(k)=\beta_\infty(s) \quad \forall\ (s,t)\epsilon A$;

(b) $\forall k\epsilon K$, x_k maximizes $u_k^\infty(c,d)$ subject to $c+p\cdot d=0$ and $(c,d)\epsilon\mathbb{R}\times D(S^*)$.

The support of β_N may be strictly larger than the support of β_∞. That is, there may exist $s\epsilon S$ such that $\beta_N(s)N\to\infty$ and $\beta_N(s)\to 0$ at the same time. Given that there are short sales, investors may be trading claims associated with one of these structures in the limit even if the net supply is zero. This is why the definition of SEE in the limit refers to both β_∞ and S*. The justification for this definition of SEE is given by the following theorem.

THEOREM 10: Let $\{E_N\}$ be an increasing sequence of finite economies satisfying (A.1) through (A.4) and let $\{(\beta_N,\phi_N,p^N,x^N)\}$ be a sequence satisfying:

(a) $\beta_N\epsilon\Delta_N(S)$, $\phi_N\epsilon\Phi_N$, $(p^N,x^N)=\phi_N(\beta_N)$ for every N;

(b) $\beta_N\to\beta_\infty$ and $(p^N,x^N)\to(p^\infty,x^\infty)$;

(c) supp $\beta_N=S^*$ for every N.

Then (p^∞,x^∞) is a SEE(β_∞,S^*) of E_∞.

In the sequel we will need to appeal to a continuity property of the sequence $\{\phi_N\}$. Specifically, we shall assume that

(6.2) for any neighborhood V of (p^{∞},x^{∞}) there exists a neighborhood V′ of β_{∞} and a number M such that $\phi_N(\beta)\in V$ if $\beta\in V'$, $\beta(s)N>M$ for all $s\in S^*$ and $\beta(s)=0$ if $s\notin S^*$.

In other words, <u>any</u> sequence $\{\beta_N\}$ converging to β_{∞} that is competitive with respect to the set S* produces the same SEE in the limit. This property is related to the structure of the equilibrium set correspondence in the limit. Uniqueness of SEE relative to β_{∞} is sufficient; but uniqueness is a very strong assumption to make.

<u>THEOREM 11</u>: Let $\{E_N\}$ and $\{(\beta_N,\phi_N,p^N,x^N)\}$ be sequences satisfying the conditions of Theorem 10. Suppose that (p^{∞},x^{∞}) is the unique SEE(β_{∞},S*) of E_{∞}. Then $\{\phi_N\}$ satisfies property (6.2).

6.2 <u>Convergence of SFE</u>

Let $\{E_N\}$ be an increasing sequence of finite economies and let $\{(\sigma_N,\phi_N)\}$ be a corresponding sequence of SFE. Since $\sigma_N\in\Delta(S)$ for every N there exists a convergent subsequence of $\{\sigma_N\}$ which we take to be the original sequence. Then $\sigma_N\to\sigma_{\infty}$, say. Choosing a further subsequence if necessary, we can assume without loss of generality that S*=supp σ_N for every N. Let $\tilde{\beta}_N$ denote the random cross-sectional distribution of financial structures generated by σ_N, for every N. The law of large numbers implies that $\tilde{\beta}_N\to\beta_{\infty}=\sigma_{\infty}$ in probability. Call the sequence $\{(\sigma_N,\phi_N)\}$ <u>competitive</u> if and only if, for every $s\in S^*$,

(6.3) $\tilde{\beta}_N(s)N\to\infty$ in probability, i.e., for any M and $\varepsilon>0$, $\text{Prob}\{\tilde{\beta}_N(s)N>M\} \geq 1-\varepsilon$ for all N sufficiently large,

From Theorem 10 it follows that there exists a SEE(β_{∞},S*) of the limit economy. Call it (p^{∞},x^{∞}). Suppose the sequence $\{\phi_N\}$ has the following continuity property:

(6.4) for any neighborhood V of (p^{∞},x^{∞}) there exists a neighborhood V′ of β_{∞} and a number M such that $\phi_N(\beta)\epsilon V$ if $\beta\epsilon V'$, $\beta(s)N>M$ for every $s\epsilon S^*$ and $\beta(s)=0$ for every $s\notin S^*$.

Then we can show that all entrepreneurs choose financial structures with the same cost.

<u>THEOREM 12</u>: Let $\{E_N\}$ be an increasing sequence of finite economies satisfying (A.1) through (A.4). Let $\{(\sigma_N,\phi_N)\}$ be a corresponding sequence of SFE satisfying:

(a) $\{(\sigma_N,\phi_N)\}$ is competitive;

(b) $\sigma_N\to\sigma_{\infty}$ and supp $\sigma_N=S^*$ for every N;

(c) the continuity condition (6.4).

Then there exists a constant κ such that $C(s)=\kappa$ for all $s\epsilon S^*$.

The intuition behind the theorem is the following. We know that $\tilde{\beta}_N$ converges to β_{∞} in probability. The competitive assumption requires $\tilde{\beta}_N(s)N\to\infty$ in probability for every $s\epsilon S^*$. Then the continuity assumption implies that $\phi_N(\tilde{\beta}_N)$ converges in probability to (p^{∞},x^{∞}). For each N, the market value of an asset with financial structure $s\epsilon S$ is random but, by the preceding argument, it converges in probability to a constant $MV^{\infty}(s)$, say.

Suppose that some entrepreneur decided to choose $s\epsilon S^*$ with probability one instead of the mixed strategy σ_N, for each N. Let $\{\tilde{\gamma}_N\}$ denote the random cross-sectional distributions of financial structures that result, for each N. By the preceding argument, $\phi_N(\tilde{\gamma}_N)$ converges in probability to the constant SEE (p^{∞},x^{∞}) and the random market value of $s\epsilon S^*$ converges in probability to $MV^{\infty}(s)$. This implies that a necessary condition for equilibrium is that

(6.5) $s\epsilon S^*$ implies $s \;\epsilon\; \arg\max_{S^*} MV^{\infty}(s) - C(s)$

From Theorem 9, however, we know that since $\{(\sigma_N, \phi_N)\}$ is competitive

(6.6) $MV^{\infty}(s) = \bar{m}$ for every $s \epsilon S^*$.

The conclusion of the theorem is immediate from (6.5) and (6.6).

What Theorem 12 actually says is that all entrepreneurs choose financial structures with the same cost κ. The implications is that they are choosing the structure with the lowest cost. Let $0 \in S$ denote the trivial structure in which only one claim (equity) is issued. Suppose that the trivial structure is the cheapest:

$$(6.6) \quad C(0) < C(s), \quad \forall s \epsilon S \backslash \{0\}.$$

If $0 \epsilon S^*$ then it is obvious that, under the conditions of the theorem, no innovation takes place. However, it may be that $0 \not\epsilon S^*$ if in equilibrium entrepreneurs can get a higher market value by choosing a more complex financial structure. To show that this possibility will not in fact occur two things must be demonstrated. The first is to show that in the limit the market value of an asset with the trivial structure is the same as an asset with a structure $s \epsilon S^*$. The second is that if an entrepreneur deviates from $s \epsilon S^*$ to 0, the equilibrium prices do not change very much. The second fact follows from a slightly strengthened version of the continuity property (6.2). The trivial structure is "spanned" in an obvious sense by any structure $s \epsilon S^*$. Further, the number of entrepreneurs choosing any $s \epsilon S^*$ grows unboundedly large as N approaches infinity. Thus, the risk sharing opportunities for the economy are not significantly altered if an entrepreneur deviates from $s \epsilon S^*$ to 0. The conclusion of (6.2) should continue to hold even under these circumstances.

To show that market value is independent of financial structure one only needs to check the first-order conditions. Suppose that for some $k \in K$, U_k is continously differentiable. Then, for any $s \in S$,

$$\sum_{t \in T(s)} p^{\infty}(s,t) = \frac{E\left[(\partial U_k(c_k^{\infty},W_k^{\infty})/\partial W)\tilde{Z}_j\right]}{E\left[\partial U_k(c_k^{\infty},W_k^{\infty})/\partial c\right]}$$

(Note that by assumption the choice of $\tilde{Z}_j$ is immaterial since idiosyncratic risk has been eliminated.) It follows that, for N sufficiently large, an entrepreneur could increase his payoff by deviating to the lower cost financial structure 0/S*, contradicting the definition of equilibrium.

7. MORE EXAMPLES

In order to give some more of the flavor of what happens as $N \to \infty$, we discuss some simple examples in this section. These examples all fit into the framework outlined in Section 3. The asset returns take on a finite number of values and may be correlated because of the influence of an economy-wide shock ω.

7.1 Perfectly Correlated Returns

We begin with the example described in Section 3.2, but now we allow the size of the economy to be an arbitrary integer N. All entrepreneurs randomize between the two financial structures, choosing the structure with two claims with probability σ. The incentive to innovate comes from the possibility that none of the N - 1 other entrepreneurs will innovate. In equilibrium entrepreneurs must be indifferent between the two financial structure, i.e.,

$$[1 - (1 - \sigma)^{N-1}]MV_C + (1 - \sigma)^{N-1}MV_I = MV_C - \gamma.$$

Solving for the equilibrium value of σ we find that

$$\sigma = 1 - \left(\frac{\gamma}{MV_C - MV_I}\right)^{\frac{1}{N-1}} .$$

Clearly, σ approaches 0 as N approaches ∞. Moreover, it can be shown using L'Hopital's rule that the limit of σN is

$$\ln((MV_C - MV_I)/\gamma).$$

The average number of entrepreneurs who innovate by choosing the financial structure s_1 is finite even though the total number of entrepreneurs is unboundedly large. For example, when $\gamma = 0.00005$ the average number of entrepreneurs issuing two claims is 1.386294 in the limit. Note that the probability that nobody issues two claims is bounded away from zero. In fact, this probability is $(1 - \sigma)^N$, which converges in the limit to $\gamma/(MV_C - MV_I)$.

7.2 <u>Idiosyncratic Returns</u>

Next we consider an example with purely idiosyncratic returns. As before Z_j can take on only two values, 0.5 and 2.5. But now the returns are assumed to be independently and identically distributed. The two values are equally likely. The rest of the assumptions are the same as in the first example.

With N assets there are now 2^N states of nature. Even if every entrepreneur chooses $s = s_1$ and issues two claims, markets will be incomplete. The reason is that claims are contingent only on the returns to the underlying asset.

Nevertheless, this example is straightforward to analyze. Since each asset has only two possible returns, it is always possible to construct a portfolio with a constant payoff across states if one entrepreneur issues two non-trivial claims. Let D denote a constant random variable. Then for any non-trivial claim $t \in T\backslash\{0, id_\Omega\}$ and any asset $j \in J$, we can find numbers r_1 and r_2 such that $D = r_1(Z_j - t((Z_j)) + r_2 t(Z_j)$. This portfolio is effectively debt. For any asset $j \in J$ and any claim $t \in T$, we can find numbers r_1 and r_2 so that $t(Z_j) = r_1 D + r_2 Z_j$. Since any claim can be artificially created out of debt and equity, the Modigliani-Miller theorem holds. All assets have the same market value, independently of their financial structure. This result is only true because the asset returns have two values, of course.

There are only two possible SEE at the second stage. One SEE results if all entrepreneurs choose $s = s_0$. Let the market value of an asset in that case by denoted by MV_I. The other SEE results if at least one entrepreneur chooses $s = s_1$. Let MV_D denote the market value of an asset in this SEE. The analysis of the SFE is similar to the preceding example. Suppose to begin with that there are two entrepreneurs (N = 2). Entrepreneurs will randomize over the two financial structures, choosing $s = s_1$ with probability σ. In equilibrium,

$$\sigma = \frac{MV_D - MV_I - \gamma}{MV_D - MV_I}.$$

It can be shown that $MV_I = 0.56987$ and $MV_D = 0.57406$ so that $MV_D - MV_I = 0.00419$. Entrepreneurs will issue two claims only if $\gamma \leq 0.00419$. For example, if $\gamma = 0.00209$ then $\sigma = 0.5$.

Now consider an economy of arbitrary size N. As N becomes large, investors are able to diversify away most of the idiosyncratic risk. In the limit all risk is eliminated and the return from holding a representative portfolio is 1.5 per asset. The value of an asset is 0.56149 independently of financial structure. In this case it is not worthwhile for a firm to issue more than one claim since it does not alter the second stage equilibrium in any way. In the limit, there is no innovation and there is no need for innovation.

7.3 Imperfectly Correlated Returns

The next example differs from the preceding example in two respects. First, there are assumed to be two macrostates, $\Omega = \{\text{"Good"}, \text{"Bad"}\}$, each of which occurs with probability 0.5. In the good state, an asset's return is 0.5 with probability 0.25 and 2.5 with probability 0.75. In the bad state these probabilities are reversed: the return is 0.5 with probability 0.75 and 2.5 with probability 0.25. Second, in an economy of size N there are N entrepreneurs and N investors of each type.

The case where there are two entrepreneurs can be analyzed in the same way as the example in Section 7.2 above. The main difference is that as N increases the idiosyncratic risk associated with each asset is diversified away but there is still risk attributable to the macrostate. A representative portfolio yields a return of 2 in the good state and 1 in the bad state. In this case, even as N becomes very large it is worthwhile for an entrepreneur to issue two claims.

It is again relatively simple to calculate equilibrium in the limit. By the earlier argument, if anyone innovates, a portfolio equivalent to risk-free debt can be constructed and the Modigliani-Miller theorem holds. There are only two possible SEE in the second stage. One of them occurs if no one innovates; the other occurs if at least one entrepreneur innovates, in which case the equilibrium is the same as if debt and equity were traded.

It follows that, in the limit, if anyone innovates it will be possible to construct portfolios equivalent to contingent commodities defined on the macrostates. It can be shown that $MV_C = 1.21340$ and $MV_I = 1.20845$ so $MV_C - MV_I = 0.00495$ in the limit. As in the perfectly correlated case (Section 7.1,) σN converges to $\ln\left[\frac{MV_C - MV_I}{\gamma}\right]$. For example, if $\gamma = 0.00124$ then the average number of entrepreneurs who innovate in the limit is 1.38629.

7.4 Ex Post Differences in Market Value

All of the examples so far have had the very special property that in the second stage SEE all assets have had the same market value, independently of financial structure. The only incentive to innovate comes from the expectation that the innovation will change the equilibrium. In this section, we consider an example where, ex post, the entrepreneur who innovates achieves a higher market value than the innovator who does not.

There are three possible levels of returns: 1, 2 and 3. All three are equally likely. As in the first examples the number of entrepreneurs and investors is the same. There is no cost if the entrepreneur only issues equity; if he wants to issue a second claim there is a fixed cost $\gamma > 0$. Thus, $T(s_0) = \{(1, 2, 3)\}$ and $T(s) = \{(a,b,c),(1, 2, 3) - (a,b,c)\}$ for any $s \in S\backslash\{s_0\}$, where a,b,c are integers satisfying $0 \leq a \leq 1$, $0 \leq b \leq 2$, $0 \leq c \leq 3$. Investors preferences are the same as before.

It can be shown that when there are two entrepreneurs ($N = 2$) there exists a SFE in which all entrepreneurs randomize between the financial structures {(1, 2, 3)} and {(0, 1, 1), (1, 1, 2)}. Let σ denote the probability that they choose the structure with two claims. If both entrepreneurs innovate, the market value of their assets is 0.55305 in the SEE. If neither innovates the market value is 0.55298. When one innovates and the other does not, the innovator's market value is 0.55340 and the non-innovator's market value is 0.55283.

It is weakly optimal to innovate if and only if

$$0.55305\sigma + 0.55340(1 - \sigma) - \gamma \geq 0.55283\sigma + 0.55298(1 - \sigma)$$

or, rearranging,

$$\sigma \leq \frac{0.00042 - \gamma}{0.00020} .$$

It can be seen that for $\gamma \leq 0.00022$ there exists a pure strategy equilibrium in which both entrepreneurs innovate. In the previous examples, the only symmetric pure strategy equilibria were those in which no one innovated. When there is no ex post incentive to innovate, innovation can occur in a pure strategy equilibrium only if it is asymmetric, because the value of an asset is independent of financial structure in every SEE. For values of γ above 0.00022, mixed strategies are used in equilibrium.

7.5 Efficiency

It was shown in AG that the incentives to innovate were constrained efficient. When unlimited short sales are allowed, this is no longer true. Innovation may not occur when it is socially desirable. More surprisingly, it may occur when it is socially undesirable.

Consider the example of Section 7.1, but change the ratio of entrepreneurs to investors to 2:1. In this case an innovation by any entrepreneur leads to complete markets. The increase in investors' utilities from the completion of markets is 0.03782 for typc $i = 1$ and 0.01586 for type $i = 2$. However, the market value of assets declines by 0.00402. Since all agents have transferable utilities, it is clear that welfare can be improved by undertaking the innovation and making transfers at the first date. But entrepreneurs have no incentive to make this innovation, even when the cost of the innovation is zero.

The second kind of inefficiency arises in the same example if we change the ratio of entrepreneurs to investors to 0.25:1. In this case the completion of markets reduces the investors' utilities by 0.00262 for type $i = 1$ and by 0.02128 for type $i = 2$. The market value of assets increases by 0.05617 so that the overall increase in the sum of utilities (ignoring the cost of the innovation) is 0.03225. It can be seen that for $0.03225 < \gamma < 0.05617$ entrepreneurs will have an incentive to innovate -- will innovate with positive probability in a SFE-- but welfare will be reduced.

The possibility of making unlimited short sales does increase the opportunities for risk-sharing. However, as a result of the inefficient incentives it provides for innovation, everybody may be worse off. In the example of Section 7.1, there is some innovation if $0.00020 < \gamma < 0.28468$ and short sales are not allowed, but none if they are. Everyone is better off when no short sales are allowed, even if there are no transfers at the first date. In other words, the CSFE Pareto dominates the SFE.

8. THE POSSIBILITY OF COMPLETE MARKETS

So far, we have restricted the form of the claims that entrepreneurs can issue. Specifically, we have assumed that a claim is a function of the returns to the underlying asset. This restriction seems to accord with what we observe in practice. It is desirable relax this restriction to avoid the suspicion that it alone is responsible for the characteristic incompleteness of markets. Ideally, one would like to allow entrepreneurs to choose from a set of claims that can in principle span the states of nature. The problem is that some bounds on the complexity of claims is needed to make the model tractable. When assets exhibit idiosyncratic risk, the number of states of nature grows as the size of the economy grows. The set of claims needed to span the states of nature may grow without bound as well. If more structure is placed on the model, however, it is possible to find a set of claims that is bounded and yet complete.

Suppose that asset returns are generated by a stochastic structure of the sort introduced in Section 3.1. That is, each asset has a finite set of possible returns $\underline{Z}$, there is a macrostate ω that takes on a finite number of values in Ω and the asset returns are independently distributed conditional on the value of the macrostate ω. In the limit, as N approaches ∞, the idiosyncratic risk associated with each asset can be diversified away. The only uncertainty remaining is associated with the macrostate ω. By conditioning claims on the macrostate ω as well as the return to the underlying asset, we can ensure that the set of available claims is complete. That is, the set of claims is rich enough to provide first best risk-sharing in the limit.

Define a claim to be a function $t:\underline{Z}\times\Omega \to \Re_+$ and let T be a finite set of such claims. It is easy to check that this extension of the model does not affect the existence results in Section 2. The asymptotic results will also have the same flavor. This is not entirely surprising. We have already seen, in the case of assets with perfectly correlated returns, that markets may remain incomplete even though the set of claims T is capable in principle of spanning the set of states of nature. With idiosyncratic risk, the analysis is more subtle, but one can generally rely on the argument that, without some incompleteness of markets, there is no incentive to innovate. More precisely, it may not be feasible for an entrepreneur to issue claims that are independent of z. In that case, his claims have some idiosyncratic risk attached to them. In order for markets to be complete, it is necessary for many entrepreneurs to issue similar claims, so that the idiosyncratic risk can be diversified away. But then there is an incentive for one entrepreneur to become a free rider on the rest. It is not an equilibrium strategy to innovate.

9. PROOFS

9.1 Proof of Theorem 1

In any CSEE(β,Q) we have $d_k(a) \geq -Q$ for every $k \in K$ and $a \in A$. Market-clearing requires

(9.1) $$\sum_{k \in K} d_k(e,f)\alpha_N(k) = \beta(e) \quad \forall (e,f) \in A.$$

Without essential loss of generality we can restrict d_k to lie in a compact, convex set $\hat{D}$, say. If $\{(c^n,d^n)\}$ is a sequence such that $d^n \in \hat{D}$ and $c^n \to -\infty$ then (A.2) implies that $u_k^N(c^n,d^n,\beta) \to -\infty$. Since $u_k^N(0,0,\beta) > -\infty$ it follows that without loss of generality we can assume that c_k is bounded below. The investors' budget constraints and the market-clearing condition (9.1) imply that $\Sigma_{k \in K} c_k \alpha_N(k) \leq 0$ in equilibrium. Then without loss of generality c_k can be assumed to lie in a compact interval. From this point onward, standard methods can be used to establish the existence of a CSEE(β,Q). Current consumption can be chosen as the numeraire because U_k is strictly increasing.||

9.2 Proof of Theorem 2

For every integer $Q\geq 0$ there exists a CSEE(β,Q) according to Theorem 1. Call it (p^Q,x^Q). The strategy of the proof is to show that, for sufficiently large Q, the short-sale constraint is no longer binding and hence that (p^Q,x^Q) is a SEE(β). In order to do this it is necessary to eliminate "wash trades", ie., trades that do not affect the final allocation. If (p,x) is a CSEE(β,Q) write $x=(x_k)=((c_k,d_k))$ and $d=(d_k)$. For any $d'=(d'_k)$ write $d'<d$ if and only if the following conditions are satisfied:

(9.2a) $d_k(a)d'_k(a)\geq 0$ and $|d'_k(a)|\leq|d_k(a)|$, $\forall\ a\epsilon A,\ k\epsilon K$;

(9.2b) $\Sigma_{k\epsilon K}\ d_k\alpha_N(k) = \Sigma_{k\epsilon K}\ d'_k\alpha_N(k)$;

(9.2c) $W_N(d_k,\beta) = W_N(d'_k,\beta)$, $\forall\ k\epsilon K$;

(9.2d) $d\neq d'$.

We call (p,x) <u>essential</u> if there does not exist $d'<d$. Now suppose that for some Q, $d<d^Q$, where $x^Q=((c^Q_k,d^Q_k))$ and $d^Q=(d^Q_k)$. We claim that (p^Q,x) is a CSEE(β,Q) where $x=((c^Q_k,d_k))$. To see this note that from (9.2c) we must have $p^Q\cdot d_k\geq p^Q\cdot d^Q_k$ for every $k\epsilon K$. Otherwise investors are not maximizing expected utility. Then (9.2b) implies that $p^Q\cdot d_k=p^Q\cdot d^Q_k$ for every $k\epsilon K$. Then the fact that (p^Q,x) is a CSEE(β,Q) follows from the definition and inspection of (9.2). The existence of an essential CSEE(β,Q) follows from the continuity of the relations in (9.2) and from Zorn's lemma. Then, without essential loss of generality, we can assume that (p^Q,x^Q) is essential for every Q.

The next step is to show that the sequence $\{W_N(d^Q_k,\beta)\}$ is uniformly tight for every $k\epsilon K$. (A.3) and the definition of equilibrium imply that for every $\varepsilon>0$ there exists a number $M>0$ such that

(9.3) $\text{Prob}\ \{W_N(d^Q_k,\beta)\geq -M\}\geq 1-\varepsilon \quad \forall\ k\epsilon K,\ \forall Q.$

Otherwise $\liminf u_k^N(c_k^Q,d_k^Q,\beta) = -\infty < u_k^N(0,0,\beta)$, a contradiction. The attainability condition (9.1) implies that

$$(9.4) \quad \Sigma_{k\epsilon K} W_N(d_k^Q,\beta)\alpha_N(k)N = \Sigma_{j\epsilon J_N} \tilde{Z}_j \qquad \forall Q.$$

Since $\tilde{Z}_j$ is integrable, for any $\varepsilon>0$ there exists a number $M>0$ such that

$$(9.5) \quad \text{Prob}\{\Sigma_{j\epsilon J_N} \tilde{Z}_j \leq M\} \geq 1-\varepsilon \ .$$

Putting (9.3), (9.4) and (9.5) together we obtain

$$\text{Prob } \{-M \leq W_N(d_k^Q,\beta) \leq M\} \geq 1-\varepsilon \qquad \forall\ k\epsilon K,\ \forall Q$$

for any $\varepsilon>0$ and some M. In other words, $\{W_N(d_k^Q,\beta)\}$ is uniformly tight for every $k\epsilon K$.

Suppose next that, contrary to what is to be proved, $\{d^Q\}$ is unbounded. Along some subsequence (which can be taken to be the original) $||d^Q||\to\infty$. Let $\bar{d}^Q=d^Q/||d^Q||$ for each Q and choose a subsequence so that (in the same notation) $\bar{d}^Q\to d^\infty$. By construction and the fact that $\{W_N(d_k^Q,\beta)\}$ is uniformly tight for every $k\epsilon K$, d^∞ has the property that $W_N(d^\infty,\beta)=0$ for $k\epsilon K$, $\Sigma_{k\epsilon K} d_k^\infty\alpha_N(k)=0$ and $d^\infty\neq 0$. Then $(d^Q-d^\infty)<d^Q$ for Q sufficiently large, contradicting the assumed essentiality of (p^Q,x^Q).

Thus $\{d^Q\}$ is bounded. For sufficiently large values of Q the short-sale constraint is not binding and (p^Q,x^Q) is a SEE(β).||

In the sequel we shall always assume that equilibria are essential.

9.3 Proof of the Lemma

Let $\{(p^N,x^N)\}$ be a sequence of SEE(β_N). The case where (p^N,x^N) is a CSEE(β_N) is similar -- in fact easier because of the short-sale constraint -- and will not be given here.

Note first of all that for any $k\epsilon K$ and $\varepsilon>0$ there exists a number $M>0$ such that

(9.6) $\text{Prob}\{W_N(d_k^N,\beta_N)\geq -M\}\geq 1-\varepsilon \quad \forall\ N=1,2,\ldots$

Otherwise (A.3) implies $\liminf u_k^N(c_k^N,d_k^N,\beta_N)=-\infty < u_k^N(0,0,\beta_N)$, a contradiction. The attainability condition of equilibrium requires

(9.7) $\Sigma_{k\epsilon K} W_N(d_k^N,\beta_N)\alpha_N(k)N = \Sigma_{j\epsilon J_N} \tilde{Z}_j \quad \forall\ N=1,2,\ldots.$

Since $N^{-1}\Sigma_{j\epsilon J_N} \tilde{Z}_j$ converges in probability to $\tilde{Z}_\infty$ by (A.4), (9.7) implies that for any $\varepsilon>0$ there exists a number M such that

(9.8) $\text{Prob}\{W_N(d_k^N,\beta_N) \leq M\} \geq 1-\varepsilon \quad \forall\ k\epsilon K,\ \forall\ N=1,2,\ldots.$

(9.7) and (9.8) imply that $\{W_N(d_k^N,\beta_N)\}$ is uniformly tight for every $k\epsilon K$.

Suppose that, contrary to what we want to prove, there is some $k\epsilon K$ and some subsequence of $\{(p^N,x^N)\}$, which can be taken to be the original sequence, such that $c_k^N\to-\infty$. Choosing a further subsequence if necessary we have $W^N(d_k^N,\beta_N)\to W_\infty$, say. Then (A.2) implies that $u_k^N(c_k^N,d_k^N,\beta_N) = E[U_k(c_k^N,W_N(d_k^N,\beta_N))]\to-\infty$, a contradiction. This proves that $\{c_k^N\}$ is bounded below for every $k\epsilon K$. Since attainability and the budget constraint imply that $\Sigma_{k\epsilon K}\ c_k^N\alpha^N(k)\leq 0$, for every N, it is immediate that $\{c_k^N\}$ is bounded for every $k\epsilon K$.

To show $\{d^N\}$ is bounded, assume the contrary and define $\bar{d}^N=d^N/||d^N||$ for each N. For some subsequence, that can be taken to be the original sequence, $\bar{d}^N\to d^\infty$. One can show that for N sufficiently large, $(d^N-d^\infty)<d^N$, contradicting the essentiality of (p^N,x^N).

It remains to show $\{p^N\}$ bounded. If not, then the wealth of some entrepreneurs grows without bound. Attainability implies that $c_k^N\to-\infty$, along some subsequence, for some $k\epsilon K$, a contradiction.||

9.4 Proof of Theorem 5

From the conditions of the theorem and part (a) of the definition of CSEE for E_N it is immediate that (p^∞, x^∞) satisfies part (a) of the definition of CSEE for E_∞. To show that part (b) of the definition is also satisfied, we assume the contrary and obtain a contradiction. Suppose then that, for some $k \epsilon K$, there exists $(c_k, d_k) \epsilon \mathbb{R} \times D$ such that $c_k + p^\infty \cdot d_k = 0$, $d^k \geq 0$ and $(c_k, d_k) > u_k^\infty(c_k^\infty, d_k^\infty)$. Choose a sequence $\{\hat{\beta}_N\}$ converging to β_∞ so that, for every N, $\hat{\beta}_N \epsilon \Delta_N(S)$ and $\hat{\beta}_N(s) > 0$ for every $s \epsilon S$. Then $d_k \epsilon D(\beta_N) = D$, for every N. Let $(\hat{p}^N, \hat{x}^N) = \phi_N(\hat{\beta}_N)$ for every N. The continuity property (4.7) (condition (c) of the theorem) implies that $(\hat{p}^N, \hat{x}^N) \rightarrow (p^\infty, x^\infty)$. By (A.4), $u_k^N(-\hat{p}^N \cdot d_k, d_k, \hat{\beta}_N) \rightarrow u_k^\infty(c_k, d_k)$. But $u_k^N(-\hat{p}^N \cdot d_k, d_k, \hat{\beta}_N) \leq u_k^N(\hat{c}_k^N, \hat{d}_k^N, \hat{\beta}_N)$ for every N and $u_k^N(\hat{c}_k^N, \hat{d}_k^N, \hat{\beta}_N) \rightarrow u_k^\infty(c_k^\infty, d_k^\infty)$ so $u_k^\infty(c_k, d_k) \leq u_k^\infty(c_k^\infty, d_k^\infty)$, a contradiction.||

9.5 Proof of Theorem 6

The proof is by contradiction. Suppose there exists a neighborhood V of (p^∞, x^∞) and a sequence $\{\hat{\beta}_N\}$ converging to β_∞ such that, for arbitrarily large values of N, $\hat{\beta}_N \epsilon \Delta_N(S)$ and $\phi_N(\hat{\beta}_N) \notin V$. Then there is no essential loss of generality in putting $(\hat{p}^N, \hat{x}^N) = \phi_N(\hat{\beta}_N)$ for every N and assuming that $(\hat{p}^N, \hat{x}^N) \notin V$ for every N. The Lemma implies that $\{(\hat{p}^N, \hat{x}^N)\}$ has a convergent subsequence which, without essential loss of generality, we can take to be the original sequence. Then $(\hat{p}^N, \hat{x}^N) \rightarrow (\hat{p}^\infty, \hat{x}^\infty) \neq (p^\infty, x^\infty)$ say. By Theorem 5, $(\hat{p}^\infty, \hat{x}^\infty)$ is a CSEE$(\hat{\beta}_\infty)$ of E_∞ contradicting the assumption of uniqueness.||

9.6 Proof of Theorem 7

Let $s \epsilon S$ be a fixed but arbitrary financial structure and let $\tilde{\gamma}_N$ be a random vector with values in $\Delta_N(S)$ representing the random cross-sectional distribution of financial structures generated when N-1 entrepreneurs play σ_N and one entrepreneur plays $s\epsilon S$ with probability 1. Let λ_N denote the probability distribution of $\phi_N(\tilde{\gamma}_N)$ for every N. From condition (b) of the theorem, $\tilde{\gamma}_N \to \beta_\infty$ in probability and so from condition (c) of the theorem, $\phi_N(\tilde{\gamma}_N) \to (p^\infty, x^\infty)$ in probability. Then

$$(9.9) \quad \int \Sigma_{t\epsilon T(s)} p(s,t)\, d\lambda_N \to \Sigma_{t\epsilon T(s)} p^\infty(s,t).$$

Since $s\epsilon S$ is arbitrary, (9.9) implies that $(\sigma_\infty, p^\infty, x^\infty)$ satisfies part (b) of the definition of CSEE(β_∞) for E_∞. Otherwise, σ_N could not be a best response for some sufficiently large N.

Part (a) of the definition of CSEE(β_∞) for E_∞ is, of course, satisfied in view of Theorem 5.||

9.7 Proof of Theorem 8

Suppose that (β_∞, x^∞) is not constrained efficient. Then there exists an allocation (β, x) that is attainable for E_∞ and satisfies (4.9). From the definition of CSFE we have

$$(9.10) \quad \sum_{s\epsilon S} \beta_\infty(s) \Big(\sum_{t\epsilon T(s)} p^\infty(s,t) - C(s)\Big) \geq \sum_{s\epsilon S} \beta(s) \Big(\sum_{t\epsilon T(s)} p^\infty(s,t) - C(s)\Big).$$

The definition of CSFE and (4.9a) imply that

$$c_k + p^\infty \cdot d_k > c_k^\infty + p^\infty \cdot d_k^\infty \qquad \forall\ k\epsilon K.$$

Summing this inequality and using the attainability condition gives

$$(9.11)\quad \sum_{k\in K} c_k\alpha_\infty(k) + \sum_{s\in S}\ \sum_{t\in T(s)} p^\infty(s,t)\beta(s) >$$

$$\sum_{k\in K} c_k^\infty\alpha_\infty(k) + \sum_{s\in S}\ \sum_{t\in T(s)} p^\infty(s,t)\beta_\infty(s).$$

Subtracting (9.10) from (9.11) gives

$$\sum_{k\in K} c_k\alpha_\infty(k) + \sum_{s\in S} C(s)\beta(s) > \sum_{k\in K} c_k^\infty\alpha_\infty(k) + \sum_{k\in K} C(s)\beta_\infty(s) \ ,$$

contradicting (4.9b).

9.8 Proof of Theorem 9

The proof is by contradiction. Suppose, contrary to what is to be proved, that there exist structures s_0 and s_1 in S* such that $MV^N(s_0)\to m_0$ and $MV^N(s_1)\to m_1$ as $N\to\infty$ and $m_1>m_0$. Define a sequence of portfolios $\{\delta^N\}$ for some fixed but arbitrary investor-type $k\epsilon K$ as follows. Put

$$\delta^N(s,t) = \begin{cases} d_k^N(s,t) & \text{if } s\epsilon S\setminus\{s_0,s_1\} \\ d_k^N(s,t)-1 & \text{if } s=s_1 \\ d_k^N(s,t)+1 & \text{if } s=s_0 \end{cases}$$

for every $(s,t)\epsilon A$. Then by (4.4)

$$W_N(\delta^N,\beta_N) = W_N(d_k^N,\beta_N) + |J_N(s_0)|^{-1} \sum_{j\epsilon J_N(s_0)} \tilde{Z}_j$$

$$-|J_N(s_1)|^{-1} \sum_{j\epsilon J_N(s_1)} \tilde{Z}_j$$

$$\rightarrow W_\infty(d_k) + E[\tilde{Z}_j|\tilde{Z}_\infty] - E[\tilde{Z}_j|\tilde{Z}_\infty] = W_\infty(d_k^\infty),$$

where (p^∞, x^∞) denotes the limit of $((p^N, x^N))$ and $x^\infty = ((c_k^\infty, d_k^\infty))$. From the budget constraint, however, (γ^N, δ^N) is an affordable choice for the investor of type k, where γ^N is defined by

$$\begin{aligned}\gamma^N &= -p^N \cdot \delta^N \\ &= c_k^N - MV^N(s_0) + MV^N(s_1) \\ &\geq c_k^N + (m_1 - m_0) - \varepsilon\end{aligned}$$

for N sufficiently large. Therefore,

$$\begin{aligned}u_k^N(\gamma^N, \delta^N, \beta_N) &\geq u_k^N(c_k^N + (m_1 - m_0) - \varepsilon,\ \delta^N,\ \beta_N) \\ &= E[U_k(c_k^N + (m_1 - m_0) - \varepsilon,\ W_N(\delta^N, \beta_N))] \\ &\rightarrow E[U_k(c_k^\infty + (m_1 - m_0) - \varepsilon,\ W_\infty(d_k^\infty)].\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary and $u_k^N(c_k^N, d_k^N, \beta_N) \rightarrow E[U_k(c_k^\infty, W_\infty(d_k^\infty))]$ we have a contradiction of the equilibrium condition $u_k^N(\gamma^N, \delta^N, \beta_N) \leq u_k^N(c_k^N, d_k^N, \beta_N)$ for some N sufficiently large.||

9.9 <u>Proof of Theorem 10</u>

The attainability condition $\Sigma_{k \epsilon K}\ d_k^N(s,t)\alpha_N(k) = \beta_N(s)$ holds for every $(s,t) \epsilon A$ and every N. So in the limit the analogous condition $\Sigma_{k \epsilon K}$ $d_k^\infty(s,t)\alpha_\infty(k) = \beta_\infty(s)$ holds for every $(s,t) \epsilon A$.

To show that condition (b) holds, note that for every $k \epsilon K$ and every N, $c_k^N + p^N \cdot d_k^N = 0$ and $(c_k^N, d_k^N) \epsilon \Re \times D(\beta_N) = \Re \times D(S^*)$ for every $k \epsilon K$. Now suppose that, contrary to what we want to prove, there exists a type $k \epsilon K$ and a pair $(c,d) \epsilon \Re \times D(S^*)$ such that $c+p^{\infty}\cdot d=0$ and $u_k^{\infty}(c,d)>u_k^{\infty}(c_k^{\infty},d_k^{\infty})$. By continuity, we can assume that $c+p^{\infty}\cdot d<0$ without loss of generality and this implies that $c+p^N\cdot d<0$ for N sufficiently large. From (A.5) and the definition of W_{∞} we have

$$E[U_k(c_k^N,W_N(d_k^N,\beta_N))] \to E[U_k(c_k^{\infty},W_{\infty}(d_k^{\infty}))]$$

and

$$E[U_k(c,W_N(d,\beta_N))] \to E[U_k(c,W_{\infty}(d))],$$

so for N sufficiently large $u_k^N(c_k^N,d_k^N,\beta_N)<u_k^N(c,d,\beta_N)$, contradicting the definition of SEE(β_N). Thus, $(c_k^{\infty},d_k^{\infty})$ is in fact optimal for $k\epsilon K$ and that completes the proof.||

9.10 <u>Proof of Theorem 11</u>

The proof is by contradiction. Suppose that, contrary to what is to be proved, there exists a neighborhood V of (p^{∞},x^{∞}), a sequence $\{\hat{\beta}_N\}$ converging to β_{∞} and a sequence $\{(\hat{p}^N,\hat{x}^N)\}$ such that $(\hat{p}^N,\hat{x}^N) = \phi_N(\hat{\beta}_N)$ and $(\hat{p}^N,\hat{x}^N)\notin V$ for every N. The Lemma ensures that $\{(\hat{p}^N,\hat{x}^N)\}$ is bounded and so has a convergent subsequence that can be taken to be the original sequence, without essential loss of generality. Then $(\hat{p}^N,\hat{x}^N)\to(\hat{p}^{\infty},\hat{x}^{\infty})$, say, where $(\hat{p}^{\infty},\hat{x}^{\infty})\neq(p^{\infty},x^{\infty})$. By Theorem 10, $(\hat{p}^{\infty},\hat{x}^{\infty})$ is a SEE(β_{∞},S*) of E_{∞}, contradicting the assumption of uniqueness.||

9.11 <u>Proof of Theorem 12</u>

Since $\tilde{\beta}_N$ converges to β_{∞} in probability, for any neighborhood V' of β_{∞} there exists a number N_0 such that:

(9.12) Prob $\{\tilde{\beta}_N\epsilon V'\}\geq 1-\varepsilon$ for any $N>N_0$.

The competitiveness condition (6.3) implies that for any M and $\varepsilon>0$ there exists a number N_1 such that

(9.13) $\text{Prob}\ \{\tilde{\beta}_N(s)N \geq M,\ \forall\ s\epsilon S^*\} \geq 1-\varepsilon$ for any $N>N_1$.

Let V be an arbitrary neighborhood of (p^∞, x^∞) and let V' be the neighborhood of β_∞ and M the number such that $\phi_N(\beta)\epsilon V$ if $\beta\epsilon V'$ and $\beta(s)N>M$ iff $s\epsilon S^*$. The existence of β_∞ and M are guaranteed by (6.4). Then from (9.12) and (9.13) it is clear that for any $\varepsilon>0$ there exists a number N_2 such that

(9.14) $\text{Prob}\ \{\phi_N(\tilde{\beta}_N)\epsilon V\} \geq 1-\varepsilon$ for any $N>N_2$.

Let $MV^N(s,\tilde{\beta}_N)$ denote the (random) market value of an asset with financial structure $s\epsilon S^*$ in the SEE($\tilde{\beta}_N$). Then it is immediate from (9.14) that $MV^N(s,\tilde{\beta}_N) \to MV^\infty(s)$ in probability where $MV^\infty(s) = \Sigma_{t\epsilon T(s)}\ p^\infty(s,t)$.

We can duplicate this argument for the case where some entrepreneur deviates from σ_N and chooses some fixed but arbitrary $s\epsilon S^*$ for every N. Let $\{\tilde{\gamma}_N\}$ denote the resulting sequence of (random) cross-sectional distributions of financial structures. Then $\{\tilde{\gamma}_N\}$ satisfies (6.3) and hence $\tilde{\gamma}_N$ converges in probability to β_∞. By the preceding argument we can show that for any $\varepsilon>0$ and any neighborhood V of (p^∞, x^∞) there exists N_3 such that

$\text{Prob}\ \{\phi_N(\tilde{\gamma}_N)\epsilon V\} \geq 1-\varepsilon$ for any $N>N_3$.

Hence

(9.15) $MV^N(s,\tilde{\gamma}_N) \to MV^\infty(s)$ for any $s\epsilon S^*$.

For any fixed entrepreneur $j\epsilon J$, equilibrium requires that if s, $s'\epsilon S^*$ then

(9.16) $E[MV^N(s,\tilde{\beta}_N)-C(s)\,|\,y_j=s] = E[MV^N(s',\tilde{\beta}_N)-C(s')\,|\,y_j=s']$

From (9.15), however, (9.16) implies

(9.17) $MV^{\infty}(s)-C(s) = MV^{\infty}(s')-C(s')$ for any s, $s'\epsilon S^*$.

The conditions of the theorem imply that there exists a sequence $\beta_N \to \beta_\infty$ such that $\beta_N(s)N \to \infty$ if $s\epsilon S^*$ and $\beta_N(s)=0$ if $s \not\epsilon S^*$ and the corresponding sequence of equilibria $(p^N, x^N) = \phi_N(\beta_N) \to (p^\infty, x^\infty)$. Then Theorem 3 implies $MV^{\infty}(s)=MV^{\infty}(s')$ for any s, $s'\epsilon S^*$. Then the theorem follows immediately (9.17).||

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