# SUPER CONTACT AND RELATED OPTIMALITY CONDITIONS: A SUPPLEMENT TO AVINASH DIXIT'S: "A SIMPLIFIED EXPOSITION OF SOME RESULTS CONCERNING REGULATED BROWNIAN MOTION"

by

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# SUPER CONTACT AND RELATED OPTIMALITY CONDITIONS:

## A Supplement to Avinash Dixit's:

## "A Simplified Exposition of Some Results Concerning Regulated Brownian Motion"

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## 1. Introduction

Dixit (1988) observed that the mathematical construct of "regulated Brownian motion" developed by Harrison (1985) had proved useful in economic models of decision-making under uncertainty. In a recent note he provided a number of methods for calculating expected discounted payoff functions based on such processes. The purpose of this supplement is twofold:

-determine to what extent the first-degree conditions reached by Dixit (his equations (12) and (13) or (12') and (13')) are simply a consequence of the definition of the expected discounted payoff, or to what extent they can be interpreted as first-order conditions of some optimization problem, as has been suggested in Dumas(1988);

-extend Dixit's treatment to the case where there are fixed costs of regulation as in Grossman-Laroque (1987).

Suppose x follows a Brownian motion that is regulated between two barriers 0 and u. For $0<x<u$, we have as usual:

$$dx = \mu\, dt + \sigma\, dz. \tag{1}$$

At $x=0$, a costless and rewardless regulator dL is applied to stop x from going below 0. At $x=u$ another regulator dU is applied which instantaneously takes x to a level $v \leq u$. Overall the stochastic differential equation for x is:[1]

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[1]dL and dU are not well defined mathematical objects. The stochastic differential equations contained in this note should be viewed as short hand for their integral counterparts. L and U

(1') $$dx = \mu\, dt + \sigma\, dz + dL - dU.$$

The upper regulator dU yields a lump-sum reward dr. The reward function $dr = r(u-v)$ is defined over $]0, +\infty[$ and is chosen to be continuous and differentiable at least once everywhere. The value for 0 is obtained by continuity; at 0 therefore the function is assumed to be right-continuous and to have a right derivative. r() is assumed to be a convex function:

$$r[kx_1 + (1-k)x_2] \geq kr(x_1) + (1-k)r(x_2).$$

Suppose f(x) is a strictly concave bounded flow payoff function, and $\delta>0$ the discount rate. Define the expected discounted payoff or the performance of the u,v policy as:

(2) $$F(x;u,v) = E\, [\int_0^\infty e^{-\delta t} f(x_t)\, dt + dr \mid x_0 = x\, ].$$

where x follows the stochastic differential equation (1'). We need a method for calculating F(x;u,v) and, when u and v are choice variable, a method for optimizing them.

## 2. Calculating F(x;u,v) for given u and v

F(x) is defined as the expected value of an integral the kernel of which is a bounded flow. The trajectory of the associated process F therefore cannot be discontinuous except

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are defined as stochastic processes which are non negative and non decreasing. In addition U is right-continuous; L is continuous and increases only when x=0.

perhaps at the time when $x=u$ and the regulator is applied. At that time, the behavior of F depends on the reward being received. If the reward $dr=r(u-v)$ is finite, a jump dF occurs, with $dF=-dr$. If the reward is infinitely small, no jump occurs.

We write this requirement as follows:

$$(3) \qquad F(u;u,v) = r(u - v) + F(v;u,v).$$

This equation has frequently been labelled the "value-matching" condition.

In the special case considered by Harrison (1985) and Dixit (1988) where the reward dr is infinitely small ($r(0)=0$ but $r'(0)>0$) and also the regulator $dU=u-v$ is of small magnitude, (3) can be re-written and expanded as follows:[2]

$$F(u) = r(dU) + F(u - dU);$$

or: $$F(u) = r'(0)\, dU + F(u) - F'(u)dU;$$

which yields:

$$(3') \qquad 0 = r'(0) - F'(u;u,v).$$

We see that in the special case of infinitesimal moves and rewards, the value-matching condition (which usually involves the function F itself as in (3)) takes the form of a condition

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[2]We suppress the parameter arguments u,v in the first two lines of equations.

involving the first derivative $F'$.[3]

At $x=0$ (where we have placed an infinitesimal costless regulator), Harrison (1985) and Dixit (1988) have shown that one must have:

(4) $$F'(0;u,v) = 0.$$

Inside the zone of no intervention ($0<x<u$), the process $x$ moves of its own accord (following (1)). The expected change in $F$ is brought about by the flow payoff $f(x)$ and the effect of discounting:[4]

(5) $$0 \equiv f(x) - \delta F(x) + F'(x)\,\mu + \tfrac{1}{2}\,F''(x)\,\sigma^2.$$

The simple technical problem one faces is that of solving the ordinary differential equation (5) subject to boundary conditions (3) (or (3')) and (4). Call $V(x)$ a particular solution to (5). Then the general solution is:

(6) $$F(x;A_1,A_2) = V(x) + A_1 \exp(\alpha_1 x) + A_2 \exp(\alpha_2 x);$$

where $A_1$ and $A_2$ are two integration constants and $\alpha_1$ and $\alpha_2$ are

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[3]It would be improper to refer to (3') as a "smooth pasting condition". At least, this label would not describe the correct economic meaning of this equation.

[4]We suppress the parameter arguments $u,v$ in equation (5). That equation is not specific to a particular choice $u,v$.

the two real solutions (of opposite signs) of the characteristic equation:

$$\tfrac{1}{2}\sigma^2\alpha^2 + \mu\alpha - \delta = 0.$$

For given u and v, we can find $A_1$ and $A_2$ from (3) and (4) or:

$$(7)\quad V(u) + A_1\exp(\alpha_1 u) + A_2\exp(\alpha_2 u) = r(u - v) + V(v) + A_1\exp(\alpha_1 v) + A_2\exp(\alpha_2 v);$$

$$(8)\quad V'(0) + A_1\alpha_1 + A_2\alpha_2 = 0.$$

These define functions $A_1(u,v)$ and $A_2(u,v)$, and then we can write the solution for F as $F(x;u,v)$ by substitution into (6).

In the special case of infinitesimal moves and rewards, (3') gives instead of (7):

$$(7')\quad 0 = r'(0) - V'(u) - A_1\alpha_1 \exp(\alpha_1 u) - A_2\alpha_2 \exp(\alpha_2 u).$$

Again, the system (7')-(8) can be solved for $A_1$ and $A_2$ to obtain the solution $F(x;u,v)$.[5]

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[5]What has been done so far does not rely on the assumptions that f() is concave and r() convex. But in the next section we seek to optimize the regulator u,v. A policy of the postulated form would not be optimal if, for instance, r were strictly concave. In that case, it would be optimal to control x at all times and not simply when some barrier is reached. We do not consider that case here.

### 3. Optimizing the regulator

In this section we seek toptimize the choice of u and v when the reward function r() is <u>strictly</u> convex. This is the case, for instance, when a proportional reward is received but is associated with a fixed cost of operating the regulator. We do not consider the questions of existence and uniqueness of the optimum, or indeed whether it is globally optimal to regulate at only one level u. We are only interested in writing first-order conditions for u and v, for the cases where the optimal policy takes the postulated form and the optimum exists. This is in line with the stated goal of this note which is to distinguish conditions such as (3), (3') and (4) above, which are simply offshoots of the definition (2) of the expected discounted payoff, from other conditions which will properly be regarded as optimality conditions.

It could be shown easily, as Dixit (1988) did in a special case, that maximizing the performance F(x;u,v) with respect to u and v is equivalent to maximizing $A_1(u,v)$ <u>or</u> $A_2(u,v)$ with respect to u and v. In other words, the derivative of F(x;u,v) with respect to u or v has the same sign for all values of x. Improving boundary behavior increases the value of the performance index everywhere. A change in u or v either shifts the whole function F(x) up or shifts the whole of it down.

Using the fact that the partials of $A_1$ and $A_2$ must be zero for the optimum to obtain, we differentiate (7) (which holds for any u and v) with respect to u and v, keeping $A_1$ and $A_2$ constant:

(9) $V'(u) + A_1\alpha_1 \exp(\alpha_1 u) + A_2\alpha_2 \exp(\alpha_2 u) = r'(u - v);$

(10) $0 = -r'(u - v) + V'(v) + A_1\alpha_1 \exp(\alpha_1 v) + A_2\alpha_2 \exp(\alpha_2 v);$

or, considering (6):

(11) $F'(u;u,v) = r'(u - v);$

(12) $0 = -r'(u-v) + F'(v;u,v).$

In general, u is not equal to v despite the similarity between these two equations. This is because the function F(x) is not generally strictly concave. The strict concavity is lost because of the assumed strict convexity of the reward function r().

Conditions (11) and (12) are the first-order conditions we were seeking. They serve to determine the two unknowns u and v. They are generally referred to as the "smooth-pasting" conditions. They indicate that, not only is the time-path of F continuous as one applies the regulator (see section 2), but also, for optimality, the time-path of F' must be continuous: $F'(u) = F'(v)$; the marginal indirect expected payoff function takes the same value at the point one jumps from and at the point one jumps to.[6] Moreover both are equal to the marginal reward received when going from u to v.

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[6]See Grossman-Laroque(1987).

## 4. The case of a purely proportional reward

After examining the situation of strict convexity of the reward function r(), we now consider the limiting situation where the reward is strictly proportional to the distance u-v. To maintain homogeneous notations, the reward per unit distance will be called r'. But r' is now a constant. Moreover r(0)=0; there is no fixed cost of regulation. These conditions plus the assumption that the flow payoff f(x) is a strictly concave function are enough to guarantee the strict concavity of the indirect expected discounted payoff F(x).

But, if F(x) is a strictly concave function, F'(x) is strictly monotonic and equations (11) and (12) above, which are still valid, imply: $u^* = v^*$.[7] In mathematical terms, this means that the optimal process U is continuous. The finding makes intuitive sense: under purely proportional reward to regulation there is no sense in taking discrete actions.[8] Infinitesimal moves with infinitesimal rewards are optimal.

If u=v, however, not only do conditions (11) and (12) merge into one condition, but we have also seen, under equation (3') above, that this same condition holds identically for any choice of the trigger point u. We seem to be left without any condition for the optimal choice of u!

Fortunately, we can re-generate an optimality condition, as

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[7]Stars denote optimal values.

[8]Except perhaps at t=0 if $x_o>u$.

optimizing F is equivalent to finding stationary values for $A_1$ and $A_2$, (3') -- re-written explicitly as (7') above -- can be differentiated with respect to u, keeping $A_1$ and $A_2$ constant, to yield:

$$(13) \qquad 0 = - V''(u) - A_1\alpha_1^2 \exp(\alpha_1 u) - A_2\alpha_2^2 \exp(\alpha_2 u).$$

But we recognize the right-hand side of this equation as F"(u). Hence, we have as an optimality condition:

$$(14) \qquad F''(u;u) = 0.$$

This condition is what Dumas (1988) proposed to call a "super-contact" or second-order smooth pasting condition, because it is expressed in terms of the second derivative of the unknown function F.

In fact (14) is none other than the natural extension of smooth-pasting conditions (11) and (12) to the limiting case of the infinitesimal regulator. Indeed re-write and expand (11) and (12) as follows:

$$F'(u) = r'; \qquad 0 = -r' + F'(u - dU);$$
$$0 = -r' + F'(u) - F''(u)dU;$$

which yield: $0 = F''(u)$ as in (14). We see that in the special case of infinitesimal rewards, where infinitesimal moves are optimal, the smooth-pasting conditions (which usually involve the function F' as in (11) and (12)) take the form of a condition involving the second derivative F".

We have made an attempt at clarifying the geometry of these conditions in the attached Figure. The underlying numerical parameter combination include a purely proportional reward function and is chosen in such a manner that the solution exists.[9] The figure contains three graphs.

The lowermost graph plots F(x;u,v) for u=1 and v=0.4 arbitrarily imposed. This regulator is not infinitesimal and is far from being optimal in this proportional reward situation. The boundary condition which is met by this curve is the value matching condition (3): the vertical distance between points A and B is equal to the reward per unit distance r' times the distance between the trigger point and the target point of the regulator: 1-0.4 = 0.6.

The intermediate graph of the figure plots F*(x;u) for u=1 arbitrarily imposed, v*(u)=0.5 being optimally chosen for the given u. This regulator is still not infinitesimal and is still not optimal. The conditions satisfied by this curve are two in number. The first condition is the value-matching condition which holds between points C and D, similar to the one which held between points A and B. It is not an optimality condition. The second condition is that the line CD (of slope r') is tangent at point D of abscissa v*(u). This is a smooth pasting condition

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[9] $\mu=-0.7$ $\sigma=0.5$ $\delta=0.15$ $r'=1.2$ ; f(x) is chosen in such a way that V(x)=sinx is a solution of the O.D.E. (5) over [0,1]. f(x) has a maximum over that interval.

such as (12) above. It represents a partial condition of optimality.

The uppermost graph of the figure gives the performance of the optimal regulator, which is indeed infinitesimal and is applied at $u^*=v^{**}=0.65$. The conditions satisfied by this curve are two in number. At point E of abscissa $u^*$, the curve has a slope equal to $r'$ (value matching turns into a condition involving the first derivative but is not an optimality condition). At the same point, it has zero curvature (smooth pasting turns into super contact and is an optimality condition).

**5. Related conditions**

Several authors have proposed first-order conditions of optimality which apply mostly to proportional rewards and infinitesimal regulators in the above or similar contexts. While different from those we have proposed above, these conditions are of course related.

Dumas (1988) and Dixit (1988) have proposed a condition involving the level reached by the unknown performance index at the point of intervention. Dumas (1988) observed that the performance index can be extended outside the zone of no intervention to reflect the initial situation where x might conceivably be above the intervention level u. In that case it is best to move x in one shot to the level u (this is the only exception to the infinitesimal character of the regulator). Hence the performance index outside the area is equal to its value on the boundary F(u) plus r' times x-u -- a straight line. Since the super contact condition (14) above says that the function inside the area has a zero second derivative at the point x=u, that condition implies that the second derivative of F (in addition to the function itself and its first derivative, as we saw) is continuous at that point. If that is so, the O.D.E. (5) applies not only inside the area ($0<x<u$) but also at the boundary point $x=u$, where it is verified equally well by the two pieces of F(x), that which is valid inside and that which is valid outside the area.[10] Inserting the latter into the O.D.E., one finds that the performance level at that point is:

$$(15) \quad F(u) = \frac{f(u) + r'\mu}{\delta} .$$

Dixit (1988) interprets this condition by saying:

> "At [the] optimally set barrier, the expected present value of the optimally controlled process is simply the capitalized value of staying at the barrier forever. The reduction in the flow profit that would result from crossing the [] barrier exactly equals the cost of operating the regulator to prevent such crossing.."

At that point the capitalized value of staying at the barrier forever is given by the flow payoff stream f(u) plus the marginal boundary reward which one expect to receive repeatedly if x has an upward drift $\mu>0$. If x has a downward drift ($\mu<0$), however, the term $r'\mu$ reflects the shadow expense which would have to be incurred in order to fight the drift and stay at the boundary.

[10]The last two sentences are distinctly incorrect in the case of strictly convex rewards and non infinitesimal regulators (section 3 above). See Grossman-Laroque (1987).

Bertola (1987) -- studying also the case of proportional (costs) rewards -- writes an optimality condition for u which would read:[11]

$$(16) \quad E\left[\int_0^\infty e^{-\delta t} f'(x_t)\,dt \mid x_0 = u\right] = r'.$$

This condition says that one must equate the the marginal cost and benefit of cutting back x when it reaches the level u. The marginal benefit (right-hand side) is the marginal reward obtained when cutting x. The marginal cost should be computed as the <u>marginal present value</u> of future payoffs foregone because of one unit of x removed at the present time. Instead, the left-hand side of (16) is evidently equal to the <u>present value of the marginal flow payoff</u> foregone because of one unit of x removed at all future times. A proof of equality between these two quantities when x follows a regulated Brownian motion is supplied in the appendix to Bertola (1987). But in fact there is equality between the two marginal quantities only for an <u>optimally</u> regulated Brownian motion. If the reader refers to Bertola's appendix, he will notice that his proof, which is based on an integration by part of the left-hand side of (16), at one point invokes, without apparent justification, a boundary condition

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[11] I am grateful to Francisco Delgado for helpful discussions on the subject matter of this paragraph.

which is none other than our super-contact condition (14).[12] Condition (16) is therefore a derivative condition based on super contact.

Harrison (1985), in his original treatise on regulated Brownian motion (chapter 6), considers, as we have, a two-sided regulator placed at levels 0 and u, of which only the upper level u is being optimized. One difference with our setting is that the lower regulator at 0, while not being optimized, is nonethless costly with a purely proportional cost. The cost per unit of distance at that point is denoted c'. Furthermore f(x) is particularized to being a linear function: f(x) = hx. Using (7') and (8) (with a right-hand side replaced by c'), Harrison solves explicitly for the function F(x;u):

$$(17)\quad F(x;A_1,A_2) = \frac{h}{\delta}x + \frac{h\mu}{\delta^2} + A_1\exp(\alpha_1 x) + A_2\exp(\alpha_2 x);$$

where $A_1$ and $A_2$ are given by:

$$(18)\quad A_1\alpha_1\exp(\alpha_1 u) + A_2\alpha_2\exp(\alpha_2 u) = r' - h/\delta;$$

$$(19)\quad A_1\alpha_1 + A_2\alpha_2 = c' - h/\delta.$$

Now impose the super-contact condition at x=u:

$$(20)\quad A_1\alpha_1^2\exp(\alpha_1 u) + A_2\alpha_2^2\exp(\alpha_2 u) = 0.$$

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[12]This is condition [***] in his appendix. In Bertola's appendix the left-hand side of (16) above, as a function of u, is denoted f().

Eliminating $A_1$ and $A_2$ between (18-20) leads to:

$$(21) \qquad \frac{\alpha_1 - \alpha_2}{\alpha_1 \exp(-\alpha_2 u) - \alpha_2 \exp(-\alpha_1 u)} = \frac{r' - h/\delta}{c' - h/\delta}.$$

which is precisely Harrison's condition (6.3.13, page 108) obtained by a completely different "policy improvement" reasoning. In the reverse, Harrison himself (see the proof of corollary 6.3.15 page 108) establishes that (21) implies that $F''(u)=0$. In other words, his conditions implies super contact. The two conditions are therefore equivalent in his special case.

Finally it is probably useful to relate the question of optimal regulation which we have been tackling with the simpler problem of "optimal stopping". In that problem a Brownian motion is again allowed to fluctuate between barriers 0 and u but when one barrier is reached a reward is received (or cost incurred) <u>and the program stops</u>. The position of the barrier(s) is to be optimized. This is the problem which is encountered in financial economics when determining the optimal exercise of an option before maturity (see Samuelson and McKean (1967), Merton (1973)). To illustrate the derivation of optimality conditions, consider the same Brownian motion (1) which we have been using with x producing a flow payoff f(x) for as long as the program lasts. Continue to apply to the process a costless regulator dL at x=0. But if and when x reaches the upper value u we stop the program and the person receives a reward r(u). Within the interval $0<x<u$, the unknown performance function F(x) still satisifies the O.D.E. (5) and at the lower barrier 0 the boundary condition is still (4). But the boundary condition at x=u is now: F(u;u)=r(u). Rewriting this condition explicitly, we get:

$$(22)\quad V(u) + A_1 \exp(\alpha_1 u) + A_2 \exp(\alpha_2 u) = r(u).$$

The boundary conditions (4) and (22) suffice to determine the two integration constants $A_1$ and $A_2$.

Now we choose optimally the stopping point u. To this end, differentiate (22) with respect to u, keeping $A_1$ and $A_2$ constant to reflect optimality, as is now customary. The result is:

$$(23)\quad V'(u) + A_1\alpha_1 \exp(\alpha_1 u) + A_2\alpha_2 \exp(\alpha_2 u) = r'(u).$$

$$(24)\ \text{or:}\qquad F'(u;u) = r'(u).$$

Equation (24) is the "smooth-pasting" condition of optimality of Samuelson (1965), McKean (1965), Merton (1973, footnote 60) and Krylov (1980, page 39). It is formally identical to condition (11) above. In essence the problem of optimal stopping is only half of the problem of optimal regulation: in it one only has to decide when (i.e. from where) to move and not where to move to. The latter aspect is imposed by the reward structure. Furthermore optimal stopping is half of the problem of optimal <u>discrete</u> regulation (of section 3). That is why smooth pasting remains a condition involving the first derivative of the performance index, and not the second one as it does when optimizing an infinitesimal regulator.

Conversely optimal regulation can be viewed as an extension of optimal stopping where the optimal stopping reward r(u) is $\text{Max}_v\ F(u;u,v)$ itself. This statement is indicative of the manner in which Grossman-Laroque (1987) extended the optimal-stopping literature to solve a problem of optimal discrete regulation.

## 6. Conclusion

This note put forth two basic ideas:

-value matching is not an optimality condition; smooth pasting is;

-in the case of a discrete regulator value matching is expressed in terms of the performance function itself and smooth pasting in terms of its first derivative. But in the case of an infinitesimal regulator the order of differentiation moves up one notch: the value-matching condition involves the first derivative of the performance function (which misleadingly makes it look like a smooth-pasting condition) and the smooth-pasting condition now involves the second derivative, leading to "super-contact".[13]

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[13]The increase in the degree of differentiation results from the same mathematical rules which cause one to extend a Taylor expansion to the second degree when the first-degree terms cancel out.

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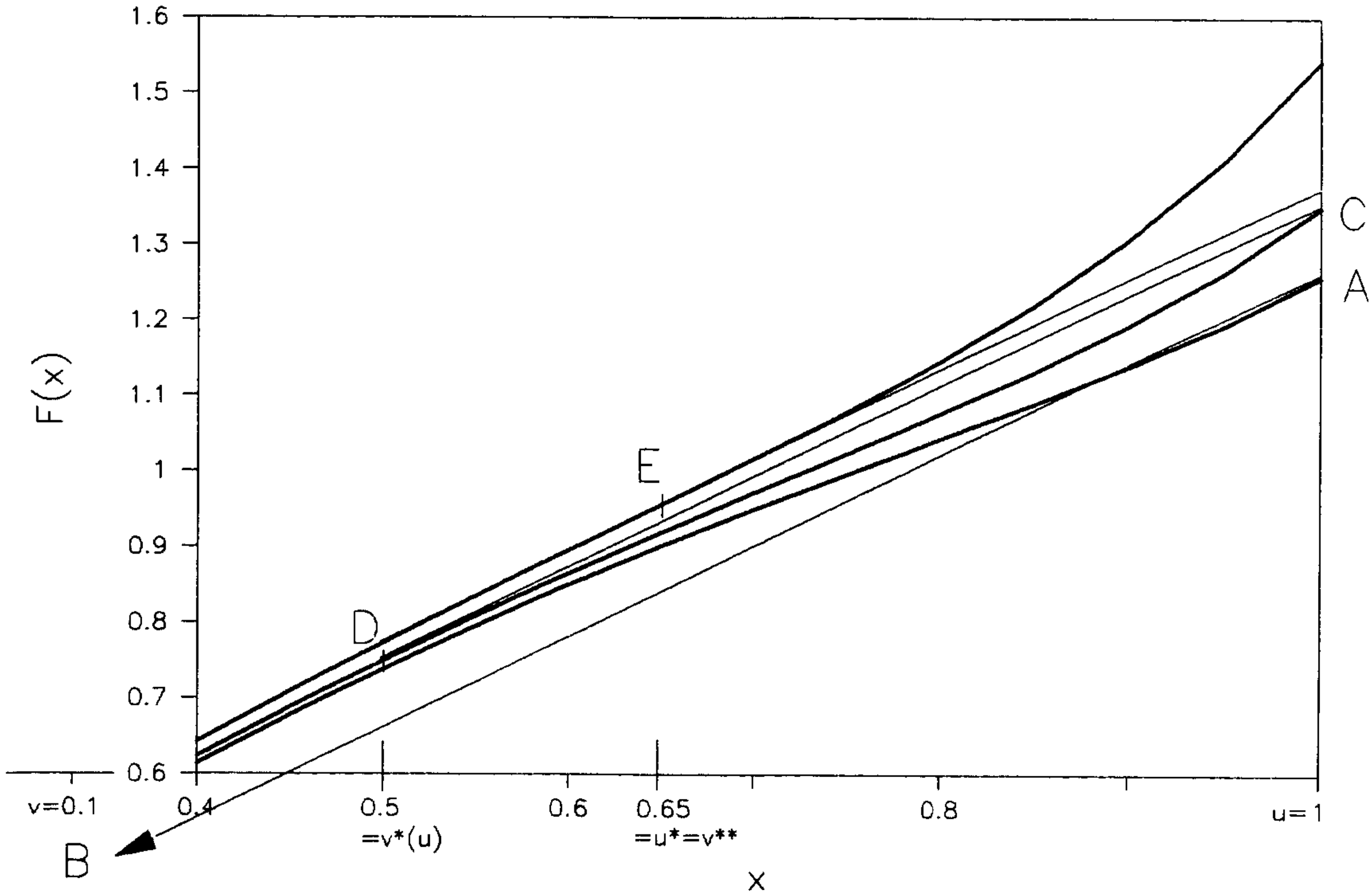
F(x)
1.6
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
v=0.1
0.4
0.5
=v*(u)
0.6
0.65
=u*=v**
0.8
u=1
x
A
B
C
D
E