THE CRASH PREMIUM: OPTION PRICING UNDER ASYMMETRIC PROCESSES, WITH APPLICATIONS TO OPTIONS ON DEUTSCHEMARK FUTURES

by

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Abstract

This paper develops distribution-specific theoretical constraints on relative prices of out-of-the-money European call and put options that are also valid for American options on futures. Systematic violations of these constraints by prices of American options on Deutschemark futures are found, indicating that distributions more asymmetric than those of standard models are necessary. An American option pricing model for jump-diffusion processes with asymmetric jumps is developed. Estimates of parameters implicit in prices of options on Deutschemark futures indicate that throughout 1984-87, market participants perceived a remote chance of a substantial crash in the dollar.

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Option pricing literature is premised on symmetry. The continuous processes for which option pricing formulas have been derived range from arithmetic Brownian motion (Bachelier), which is symmetric, to geometric Brownian motion (Black-Scholes), which is log-symmetric, to constant elasticity of variance, which for the originally postulated parameters is intermediate between the other two. Jump-diffusion models based on Merton (1976a) almost invariably assume that jumps are distributed mean-zero, so that the distribution is log-symmetric.

The importance of *asymmetric* distributions, on the other hand, has long been recognized in the international finance literature. The "peso problem" under pegged exchange rates is an issue of asymmetric distributions: the large chance over a finite time horizon of no change in the exchange rate (the peg is maintained) and the small chance of an extreme change (the regime collapses) creates a highly skewed distribution *ex ante*. Models of "bubbles" in floating exchange rate regimes also postulate the presence of asymmetries. The large chance of the bubble continuing to expand and small chance of the exchange rate collapsing back to its "fundamentals"-determined level again implies a highly skewed distribution *ex ante*. As noted by Krasker (1980), skewness creates fundamental difficulties for the reliability of statistical inference when testing international asset pricing models.

This paper examines how the symmetry or asymmetry of the underlying asset's price distribution affects prices of options written on the asset, and presents evidence that distributions more asymmetric than those hitherto considered are more consistent with observed prices of options on Deutschemark futures. Since call options will pay off only when the underlying asset's price is in excess of the exercise price (i.e., the call finishes "in the money"), and puts only when the reverse is true, the instantaneous spectrum of call and put option prices across all exercise prices gives a very direct indication of asymmetries in market participants' aggregate subjective distributions. For instance, an assessed risk of a dollar crash (Deutschemark "boom") will lead to call options on DM futures with exercise prices well above the current futures price ("out-of-the-money" calls) being priced higher than puts with exercise prices well *below* the futures price (out-of-the-money puts); the chance of large upward movements in the DM/$ rate makes the call more likely than the put to finish in the money.[1] This observation leads naturally to a "crash premium" measure of asymmetry, defined as the percentage deviation between out-of-the-money (OTM) calls and puts.

Section I of the paper develops the relationship between OTM call and put option prices for the standard distributional hypotheses: arithmetic Brownian motion, geometric Brownian motion, constant elasticity of variance, and log-symmetric jump-diffusion processes. The relative prices of OTM European options are shown to *order* the processes: OTM European call and put prices are equal under arithmetic Brownian motion and diverge increasingly as one moves through constant elasticity of variance (CEV) processes to the geometric

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[1] Similarly, fears of a stock market crash will lead to out-of-the-money (OTM) puts on stock indexes being priced higher than OTM calls. Whaley (1986) has found such asymmetries; see also MacBeth and Merville (1982), Whaley (1982) and Rubinstein (1985) for evidence of asymmetries in stock options.

Brownian motion and log-symmetric jump-diffusion processes. For the last two processes, European call options x% out of the money are priced x% higher than the corresponding OTM puts.

Furthermore, for the specific case of options on futures, the relative ranking of the distributions is shown to be maintained when one compares *American* OTM options. The above "x% rule" continues to hold for American OTM options on futures when the underlying processes are either geometric Brownian motion or log-symmetric jump-diffusion. Data for American options on Deutschemark futures from the Chicago Mercantile Exchange over February 1984 to January 1987 indicate that crash premiums, though volatile, typically lie *above* the range of the standard models (i.e., higher than the x% rule benchmark), consistent with market assessments of downside risk on the dollar.

Section II examines the hypothesis that the asymmetries are the result of the underlying asset's price following an asymmetric jump-diffusion process. Such processes are notoriously difficult to estimate using actual price data for the underlying asset when jumps are infrequent: as in the case of estimating mean returns, accurate estimation of jump frequencies requires a long history of data. The paper sidesteps the problem by estimating the jump size and jump frequency *implicit* in option prices -- an extension of the implicit parameter approach commonly used for stock return volatility. There being no known analytic solutions for American option prices under jump-diffusion processes, I develop an accurate and inexpensive analytic approximation, an extension of the work of MacMillan (1987) and Barone-Adesi and Whaley (1987). Results from fitting the model to options data using nonlinear least squares reinforces the "stylized fact" of positive asymmetries: expectations of positive jumps are almost invariably found. The positive asymmetries are present both during the period of dollar appreciation prior to March 1985 and the period of dollar depreciation thereafter. With some exceptions, the expected jump component per year (size times frequency) is not large: generally about 2-4%. However, the jump component of the process typically accounts for 10-30% of the implicit conditional variance of the underlying process.

1. Measures of Asymmetry Under Standard Distributional Hypotheses

1.1. European Options

Although most options are "American," i.e., allow early exercise, the usefulness of options data in distinguishing between distributional hypotheses is most apparent in the case of European options. The relative prices of out-of-the-money calls and puts are closely related to the symmetry or asymmetry of the underlying asset's price distribution around the mean at the expiration date of the options, as perceived *ex ante* by market participants. That relationship can be made precise under the standard assumptions of option pricing models.

Consider first the following standard generalization of Black and Scholes' (1973) model:

A1) The asset price follows the stochastic difference equation

$$dS = \mu(\bullet)\, dt + \sigma(S,t)\, dZ \tag{1}$$

where Z is a standard Wiener process, and the instantaneous drift $\mu(\bullet)$ is an arbitrary function of (possibly stochastic) variables.

A2) Markets are frictionless: there are no transactions costs or differential taxes, trading takes place continuously, there are no restrictions on borrowing or selling short.

A3) The instantaneous risk-free interest rate is known and constant.

A4) The "cost of carry" to maintaining a position in the underlying asset is a constant proportion b of the asset price.

Assumption A4) is standard in most option pricing models. For non-dividend paying stocks, $b = r$, the opportunity cost of not holding the risk-free asset. For options on foreign exchange, $b = r - r^*$, the domestic/foreign interest differential. An important characteristic of futures contracts is that the cost of carry equals zero: the collateral requirements of taking positions in futures markets can be met by posting Treasury bills, so there is no opportunity cost to such positions.

Under these assumptions, the payoff to any contingent claim on the asset can be replicated by a continuously adjusted portfolio of the asset and the risk-free bond, and the claim is priced *as if* investors were risk-neutral and the asset price followed the equivalent martingale process

$$dS^* = bS^* \, dt + \sigma(S^*,t) \, dZ \tag{2}$$

with $S^*(0) = S(0) \equiv S_0$. In particular, European calls and puts with exercise price X and time T to maturity are priced at what would be the the discounted expected value of their terminal payoffs if the process followed (2).

Let $c(S_0,T;X)$ and $p(S_0,T;X)$ be the prices at time 0 of European calls and puts, given the current price S_0 for the underlying asset, the option's time to maturity T, and the exercise price X. Similarly, $C(S_0,T;X)$ and $P(S_0,T;X)$ will be the corresponding prices of American options. In options market terminology, the call options are "out of the money" if $X > S_0$, and puts are out of the money if $X < S_0$. Then under assumptions A1 - A4, the European option prices will be

$$c(S_0,T;X) = e^{-rT} E_0 \max[S^*_T - X, 0]$$

$$= e^{-rT} E_0[S^*_T - X | S^*_T \geq X] \ \mathrm{Prob}[S^*_T \geq X] \qquad (3)$$

$$p(S_0,T;X) = e^{-rT} E_0 \max[X - S^*_T, 0]$$

$$= e^{-rT} E_0[X - S^*_T | S^*_T \leq X] \ \mathrm{Prob}[S^*_T \leq X] \ , \qquad (4)$$

where S^*_T, the value of S^* at the expiration of the option, has expected value $E_0 S^*_T = S_0 \exp(bT)$. By arbitrage, $S_0 \exp(bT) = F_{0,T}$, the forward price on the asset.[2]

The key insight from (3) and (4) is that calls and puts *as functions of the exercise price X* are symmetric contingent claims. Symmetries or asymmetries in the distribution of S^*_T around its mean $S_0 \exp(bT) = F_{0,T}$ generate corresponding symmetric or asymmetric relationships between out-of-the-money calls and puts. Since different distributional hypotheses imply different distributions of S^*_T, options data can be used to distinguish between those hypotheses.

A caveat is that one cannot in general make direct inferences from options data about the asymmetry of the *actual* distribution of the terminal price S_T of the underlying asset. Option pricing models place no restriction upon the instantaneous drift $\mu(\bullet)$ of the process, and it is conceivable that movements in the drift could create arbitrarily asymmetric distributions for S_T that are not reflected in the distribution of S^*_T. Additional, model-specific restrictions on $\mu(\bullet)$ will therefore be necessary to determine the precise relationship between the S_T and S^*_T distributions.

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[2] For expositi[illegible]

For some such restrictions, however, the asymmetry of S^*_T around F gives an exact measure of the asymmetry of S_T around E_0S_T. The constant-drift geometric Brownian motion process for S_T assumed by the capital asset pricing model,

$$dS/S = \mu\, dt + \sigma\, dZ, \tag{5}$$

implies that S_T and S^*_T are log-normally distributed with *identical* coefficients of skewness $(e^{\sigma^2 T}+2)(e^{\sigma^2 T}-1)^{1/2}$.[3] The Ornstein-Uhlenbeck process[4]

$$dS = (\alpha+\beta S)\, dt + \sigma\, dZ, \tag{6}$$

with no absorbing barrier at S=0 (of which unlimited-liability arithmetic Brownian motion is a special case) implies that S_T and S^*_T are both normally distributed (conditional on S_0) with identical, zero coefficients of skewness. These examples suggest that the asymmetry of S^*_T around F generally gives a reliable indication of the asymmetry of S_T around E_0S_T, when asymmetry is measured by the coefficient of skewness.

Using (3) and (4), one can immediately ascertain the relationship between out-of-the-money calls and puts for *symmetric* distributions of S^*_T:

Proposition 1: If the terminal distribution of the equivalent martingale measure is symmetric around its mean F = S exp(bT), then European call and put prices are related by

$$c(S,T;F+x) = p(S,T;F-x) \quad \text{for any } S,\ x, \text{ and } T, \tag{7}$$

the third argument of c(•) and p(•) being the exercise price.

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[3] The coefficients of kurtosis will also be identical.

[4] See, e.g., Merton (1971), p. 404.

Proof: If S^*_T is distributed symmetrically around its mean F, then

$$c(S,F+x;T) = e^{-rT}\int_{F+x}^{\infty} [S^*_T-(F+x)]\ f(S_T^* - F)\ d(S^*_T-F)\ ,$$

where $f(S^*_T-F)$ is the probability distribution function of S^*_T-F. Symmetry implies $f(-x) = f(x)$. Substituting $z_T \equiv 2F-S_T^*$ yields

$$c(S,T;F+x) = e^{-rT}\int_{-\infty}^{F-x} [(F-x) - z_T]\ f(F-z_T)(dz_T)$$

$$= e^{-rT}\int_{-\infty}^{F-x} [(F-x) - z_T]\ f(z_T-F)\ dz_T$$

$$= p(S,T;F-x)\ .$$

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An implication of (7) is that the relevant concept of "out-of-the-money" in comparing calls and puts is relative to the *forward* price F = S exp(bT) rather than relative to the spot price S on the underlying asset. This measure of "moneyness" is standard in the literature,[5] albeit not in options markets, and is completely intuitive: only the terminal distribution matters for pricing European calls and puts, and F is the mean of that distribution. I will henceforth use this definition of "moneyness." In the case of options on futures (b=0), considered below, the two versions of "moneyness" are identical.

It is important to realize that (5) has nothing to do with the European put-call parity relationship

$$c(S,T;X) = p(S,T;X) + e^{-rT}(Se^{bT} - X)\ , \tag{8}$$

which is an arbitrage relationship between puts and calls with *identical* exercise prices, and holds regardless of distribution. The importance of (7) is that it holds *only* for symmetric terminal distributions. If the distribution is asymmetric, then call and put prices for symmetric exercise

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prices will diverge.[6]

For the major distributions studied hitherto, that divergence takes systematic forms. The positively skewed log-normal distribution used in the Black-Scholes model has a *positive divergence* between out-of-the-money calls and puts:

$$c(S,T;F+x) > p(S,T;F-x) \text{ for } x > 0, \tag{9}$$

with out-of-the-money defined relative to the forward rate rather than the spot rate. In fact, the degree of divergence can be quantified for this distribution.

Proposition 2: If the underlying asset's terminal distribution is log-normal, then European call and put prices are related by

$$c(S,T;\ F(1+x)\ ) = (1+x)\ p(S,T;\ F/(1+x)\ ) \quad \text{for any } S,\ x, \text{ and } T. \tag{10}$$

Proof of proposition 2 follows directly from inspecting the generalized Black-Scholes formulas

$$c(S,T;X) = e^{-rT}\ [\ Se^{bT}\ N(d_1)\ -\ X\ N(d_2)\ ] \tag{11}$$

and

$$p(S,T;X) = e^{-rT}\ [\ X\ N(-d_2)\ -\ Se^{bT}\ N(-d_1)\ ]\ , \tag{12}$$

where $d_1 \equiv [\ln(Se^{bT}/X) + 0.5\sigma^2T] / \sigma\sqrt{T}$, $d_2 = d_1 - \sigma\sqrt{T}$, and $N(\bullet)$ is the cumulative normal. ∎

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[6] Using put-call parity, one can interpret Proposition 1 as a statement about the relationship between out-of-the-money and in-the-money European calls when the distribution is symmetric:

$$c(S,T;F+x) = c(S,T;F-x) + e^{-rT}(F-x - F)$$

When x is close to 0 (as is observed in options markets), (10) can be rewritten as

$$c(S,T;F(1+x)) = (1+x)\,p(S,T;F/(1+x))$$
$$\simeq (1+x)\,p(S,T;F(1-x)) \quad . \qquad (13)$$

That is, call options with exercise prices 5% higher than the forward rate should be priced about 5% higher than put options with exercise prices 5% below the forward rate, if the underlying asset's price follows geometric Brownian motion. If the asset price followed unlimited-liability arithmetic Brownian motion, then from (7) the call and put options would have identical prices.[7]

Both geometric and arithmetic Brownian motion are special cases of the broader class of *constant elasticity of variance* (CEV) diffusions, of the form

$$dS = \mu(\bullet)\,dt + \sigma S^{\rho}\,dZ \ . \qquad (14)$$

Setting $\rho=0$ yields arithmetic Brownian motion, while $\rho=1$ yields geometric Brownian motion. The first studies of CEV processes restricted ρ to between 0 and 1, with $\rho=1/2$ especially studied.[8] For such intermediate values of ρ, there is a relationship between OTM calls and puts intermediate between those of Propositions 1 and 2.

Proposition 3: If the underlying asset's price follows the process (14), then out-of-the money call and put prices, as functions of state variables and

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[7] Given limited liability, this distributional assumption is admittedly implausible, although it is of some historical interest in the option pricing literature. I use it because 1) it is a simple example of a symmetric distribution, and 2) it provides a useful lower bound for (16a) below.

[8] See, e.g., Cox and Ross (1976).

relevant parameters,[9] are related by

$$c(S,T;Fk,k^{1-\rho}\sigma,\rho,b) = k\ p(S,T;F/k,\sigma,\rho,b) \quad \text{for } k>0 \text{ and any } \rho \tag{15}$$

and satisfy the following inequalities:

$$\text{a) } c(S,T;F(1+x),\sigma,\rho,b) > p(S,T;F(1-x),\sigma,\rho,b) \text{ for } \rho>0,\ x>0 \tag{16a}$$

$$\text{b) } c(S,T,F(1+x),\sigma,\rho,b) < (1+x)\ p(S,T;F/(1+x),\sigma,\rho,b) \text{ for } \rho<1,\ x>0 \tag{16b}$$

$$\text{c) } c(S,T,F(1+x),\sigma,\rho,b) > (1+x)\ p(S,T;F/(1+x),\sigma,\rho,b) \text{ for } \rho>1,\ x>0 \tag{16c}$$

with inequalities reversed for $x<0$.

Expression (15) is proved in Appendix I for $\rho<1$ via manipulation of the generalized European CEV option pricing formula. That it also holds for $\rho>1$ follows from the useful property also proved in the appendix that a European call with CEV parameter $\rho<1$ corresponds to a European put with CEV parameter $\rho^* = 2 - \rho > 1$, cost of carry $b^* = -b$, and a modified volatility parameter:

$$c(S,T;F(1+x),\sigma,\rho,b) = (1+x)\ p(S,T;F/(1+x),\sigma F^{2(1-\rho^*)},\rho^*,b^*) \tag{17}$$

(16b) and (16c) follow from (15), since option prices are increasing functions of the volatility parameter. (16a) also follows from the (15) for $\rho>1$ (since $F/(1+x) > F(1-x)$ and $p(\bullet)$ is increasing in the strike price), and has been confirmed numerically for $\rho<1$.

The final distribution-specific relationship is for the jump-diffusion process studied by Press (1967) and Merton (1976a,b), among others. This models asset prices as generally following geometric Brownian motion, but occasionally being hit with a Poisson-distributed "event," in which the asset

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[9] In the expressions below, the arguments are the asset price, the time to maturity, the strike price, the volatility parameter, the CEV parameter, and the cost of carry, respectively.

price jumps discretely. The percentage jump is itself generally modelled as a random variable, and the typical assumptions are that it has a zero mean and that one plus the percentage jump has a log-normal distribution.[10] Merton (1976) shows that it is no longer possible to replicate an option with a continuously-adjusted two-asset portfolio of risk-free bond and the underlying asset, so that restrictions either on distributions or on attitudes towards risk become necessary in deriving option values. The standard assumption is that jumps represent idiosyncratic, diversifiable risk to which investors are indifferent. Under this assumption, or the assumption that investors are in fact risk-neutral, European option pricing formulas can be derived, and the following relationship holds:

Proposition 4: If asset prices follow the log-symmetric jump-diffusion process described above, and if investors are effectively risk-neutral with regard to jump risk, then the call-put relationship of Proposition 2 holds:

$$c(S,T;F(1+x)) = (1+x)\, p(S,T;\, F/(1+x)) \quad \text{for any } S,\ x, \text{ and } T. \tag{18}$$

Proof of the proposition is in Appendix II.

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[10] See Merton (1976a,b) and Ball and Torous (1983,1985). The rationale behind the choice of mean-zero jumps is that this distribution is used primarily as a fat-tailed "fix-up" of geometric Brownian motion, to accord with the observed leptokurtosis of asset returns.

1.2. American Options on Futures

The symmetric call-put relationships derived above for European options will not hold in general for American options. A simple counter-example is perpetual options on a non-dividend paying stock: the call will never be exercised early, whereas the put will be exercised early if the stock price falls low enough. More generally, the problem is that the early-exercise feature of American options is symmetric around the *spot* price S_0 whereas, as noted above, the European options are claims symmetric around the *forward* price $F_{0,T} = S_0\exp(bT)$. For $b > 0$, the range of spot prices on the underlying asset for which the call will be exercised early is smaller than the early-exercise range for puts. For $b < 0$, the reverse is true.[11] The resulting asymmetries in early exercise premiums (the premiums being defined as the difference between American and European option prices) implies that comparing American call and put option prices cannot in general be used as a simple diagnostic of asymmetries in distributions.

In the case of American options on futures, however, the cost of carry equals zero, the forward price on the futures contract equals the futures price, and there is no intrinsic asymmetry introduced by moving from European to American options. In this particular case, the above propositions continue to hold, and relative prices of *American* calls and puts can distinguish among competing distributional hypotheses.

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[11] From the logic of equivalent martingale measures, the call and put are priced as if the underlying asset price drifts upward at expected rate b. Positive drift increases the attractiveness of holding on to the call as opposed to exercising it early, but *decreases* the attractiveness of holding on to the put.

Proposition 5: For American options on futures, under assumptions A1 - A4,

1) If the futures price follows unlimited-liability arithmetic Brownian motion, then

$$C(S,T;S(1+x)) = P(S,T;S(1-x)) \quad \text{for any } S,\ T, \text{ and } x, \tag{19}$$

and the critical exercise prices for calls and puts below (above) which the call (put) will be exercised immediately are symmetric around the current futures price S:

$$S - X^*_c = X^*_p - S > 0. \tag{20}$$

2) If the futures price follows a) geometric Brownian motion or b) the log-symmetric jump-diffusion process,[12] then

$$C(S,T;S(1+x)) = (1+x)\, P(S,T;S/(1+x)) \quad \text{for any } S,\ T, \text{ and } x, \tag{21}$$

and the critical exercise prices are log-symmetric around S:

$$S / X^*_c = X^*_p / S \geq 1. \tag{22}$$

3) (conjectured) If the futures price follows the CEV process (14), then

$$C(S,T;Sk,k^{1-\rho}\sigma,\rho,b) = k\, P(S,T;S/k,\sigma,\rho,b) \quad \text{for } k>0 \text{ and any } \rho \tag{23}$$

and the following inequalities are satisfied:

a) $C(S,T;S(1+x),\sigma,\rho,b) > P(S,T;S(1-x),\sigma,\rho,b)$ for $\rho>0$, $x>0$ (24a)

b) $C(S,T,S(1+x),\sigma,\rho,b) < (1+x)\, P(S,T;S/(1+x),\sigma,\rho,b)$ for $\rho<1$, $x>0$ (24b)

c) $c(S,T,F(1+x),\sigma,\rho,b) > (1+x)\, p(S,T;F/(1+x),\sigma,\rho,b)$ for $\rho>1$, $x>0$ (24c)

with inequalities reversed for $x<0$.

Proof of parts 1) and 2b) is in Appendix II, while 2a) is proved in Appendix I. Expression (23) has been confirmed numerically[13] over the

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[12] And imposing the standard restrictions on distributions or on preferences discussed above in the latter case.

parameter range $0<\rho<1$ and therefore also holds for corresponding values of ρ between 1 and 2, given the correspondence proved in Appendix I between American calls on futures with CEV parameter $\rho<1$ and American puts on futures with CEV parameter $\rho^* = 2-\rho > 1$ and cost of carry $b^* = -b = 0$:

$$C(S,T;S(1+x),\sigma,\rho,0) \;=\; (1+x)\;P(S,T;S/(1+x),\sigma S^{2(1-\rho^*)},\rho^*,0)\;. \qquad (25)$$

3b) and 3c) follow from (23); 3a) also follows from (23) for $\rho>1$, and has been confirmed numerically for $0<\rho<1$.

The above proposition can be used to construct a measure of asymmetry which, by analogy with the term premium, is named the "crash premium".

Definition: The *crash premium* is defined as the percentage deviation of out-of-the-money calls from correspondingly out-of-the-money puts:

$$CP(S,T,x) \equiv C(S,T,\; S(1+x)\;)\;/\;P(S,T;\; S/(1+x)\;)\;-\;1\quad,\; x>0\;. \qquad (26)$$

The crash premium is a function most importantly of the out-of-the-moneyness parameter x. From proposition 5, the crash premium has *approximately* the following properties:

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[13] American CEV option prices were calculated using the Cox-Rubinstein (1985) binomial option pricing methodology. Option prices were expressed in terms of the transformed state variable $Z \equiv (S^{1-\rho}-1)/(1-\rho)$, which follows a process with state-dependent drift but with state-*independent* volatility. Option prices were calculated only for ρ between 0 and 2, because of difficulties with boundary conditions for values of ρ outside this range.

1) $CP(\bullet,x) \simeq 0$ for unlimited-liability arithmetic Brownian motion processes

2) $0 \leq CP(\bullet,x) < x$ for CEV processes with $0 < \rho < 1$,

3) $CP(\bullet,x) = x$ for geometric Brownian motion processes,

4) $CP(\bullet,x) = x$ for log-symmetric jump-diffusion processes,

5) $CP(\bullet,x) > x$ for CEV process with $\rho > 1$.

In sum: *for options on futures, American calls x% out of the money should be priced between 0% and x% higher than the corresponding OTM puts for any of the first four distributions listed above, regardless of the maturity of the options.*[14]

The approximation enters in 1) and 2), in that the perfectly symmetric arithmetic Brownian motion process places constraints on OTM calls and puts for exercise prices symmetric around the forward (and spot) price:

$$X_c = S(1+x) \qquad X_p = S(1-x) \quad , \tag{27}$$

whereas the above definition uses exercise prices log-symmetric around the spot price, as is appropriate for symmetric log-normal processes:

$$X_c = S(1+x) \qquad X_p = S/(1+x) \simeq S(1-x) + [S(1-x)x^2] \ . \tag{28}$$

A more careful but more cumbersome approach would be to define an additional crash premium measure based on (28), and use both premiums to distinguish among distributional hypotheses. The latter premium would be approximately the same but slightly greater than the premium defined in (27), except for options far out of the money or near maturity.[15]

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[14] The choice of out-of-the-money options (x>0) in defining the crash premium is arbitrary. One could equally define the crash premium as (minus) the percentage deviation between in-the-money call and put options (x<0).

[15] The two crash premium measures of *percentage* deviations in relative prices of OTM calls and puts diverge for deep out-of-the-money options. But *absolute* deviations between OTM calls and puts converge, as both call and

The crash premium as defined is only suitable for options on futures. However, a similar measure can be defined in cases where European option pricing formulas are applicable. For example, for options on non-dividend paying stocks, one could compare effectively European out-of-the-money calls with the out-of-the money European puts imputed (via put-call parity) from effectively European in-the-money calls. In these cases, as noted above, the relevant concept of "out-of-the-money" is relative to the forward rate rather than to the spot rate.

1.3. Tests

The futures options on which I have tested the above symmetry relationships are options on Deutschemark futures, traded on the International Monetary Market since January 24, 1984. I use settlement prices of options, the prices used at the end of the day for settling positions. Settlement prices are based on the daily closing prices of options, unless those were considered not current enough, in which case information from other markets is also used. The major advantage of these prices is simultaneity: option prices for the spectrum of exercise prices are determined simultaneously with each other *and* with the underlying futures' settlement prices.[16] Data on settlement prices are published daily in the *Wall Street Journal*, and are also published in the Chicago Mercantile Exchange's *Statistical Yearbook*. The CME unfortunately does not keep the data on computer tapes.

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put prices go to zero.

[16] The simultaneity of settlement prices on futures and on futures options is evident for options deep in the money. The price of such options approaches the early exercise value given by the futures price/exercise price differential.

Options on DM futures over most of the 1984-87 period were available for March, June, September, and December expiration dates. Intra-quarterly options have also been introduced recently. The last trading day for these options is two Fridays before the third Wednesday of the contract month (that Wednesday being the delivery date of the underlying futures contract). The first trading day is nine months (formerly six) before expiration. The options can be exercised on any trading day, yielding a position in the underlying futures the following business day. Option contracts are available for integer exercise prices (in cents/DM) falling within a +/- 2.5 cents/DM band around the current futures price, plus the contracts opened around previous futures prices. The set of options contracts available for a given maturity therefore depends upon the past movements of the underlying futures price during the history of that maturity of option. For instance, on March 3, 1986, the June 1986 futures settlement price was 45.67 cents/DM, and call options were available at exercise prices 35 to 48 cents/DM. The preponderance of in-the-money calls reflects the appreciation of the Deutschemark over the six months up to March that the June 1986 options had been traded. Prices of options are quoted in cents/DM; contract size is DM 125,000. The data sample used picks one day per month over the February 1984 to January 1987 history of the options, with days picked so that the options have maturities that are (roughly) an integer number of months.[17]

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[17] More specifically, the dates were chosen on the same (or nearest preceding) day of the month as the Monday following the last trading date on the shortest-maturity option. That Monday is when the futures contract would be delivered conditional on the option being exercised on the final trading date. Maturity of options fluctuates with the month; a one-month option can mature in 28 to 34 days. Timing of contracts (as opposed to picking 28-day maturities) was chosen so as to coincide with timing of forward contracts on foreign exchange.

Since options exist only for specific exercise prices, the crash premium measure of asymmetry cannot be implemented directly. For each OTM call with exercise price x% above the futures price, there will not in general exist a corresponding OTM put with exercise price exactly x% below the futures price. However, theoretical distributions imply that options prices are continuously differentiable, monotone, and convex functions of the exercise price. Options prices for desired exercise prices were therefore calculated by quadratic interpolation. The method used[18] was extremely accurate when tested on theoretical distributions. Actual option prices were less well behaved than the theoretical ones; while always monotone, there were instances of small local concavities rather than convexity in the spectrum of option prices for given exercise prices.

Table 1 chronicles the crash premium over February 1984 to January 1987, for options 2%, 4%, and 6% out-of-the-money. Uncontroversially, the premiums are invariably positive, as would be the case for the standard model of geometric Brownian motion. However, for short-term options (maturities of 1-3 months), the premiums are predominantly in excess of the x% rule of geometric Brownian motion. The deviation grows more pronounced for options further out of the money. Taking the three-month options of March 9, 1984 as an example: geometric Brownian motion implies calls 4% out of the money should be priced 4% higher than corresponding OTM puts; instead, they are priced 18% higher. The 6% OTM calls should be priced 6% higher than OTM puts; they are priced 25% higher. One cannot rely too heavily upon the magnitude of percentage

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[18] Interpolations were based on parabolas through the three exercise price/option price points surrounding the desired exercise price, provided the parabola was convex. Otherwise, linear interpolation was used. When two parabolas could be fitted, the average of the two interpolations was used.

deviations between options far out of the money, especially in the case of one-month options, but the *sign* is of interest. In total, three-quarters of the crash premiums on 1-3 month options are more positive than would be the case under geometric Brownian motion. This "stylized fact" is consistent with the CEV model with $\rho>1$, or with the jump-diffusion model with asymmetric jumps discussed below.

Crash premiums for medium-term (4-6 month maturities) conform more closely to the benchmark model of geometric Brownian motion. The deviations from the x% rule are smaller than for short-term options, and occur both above and below. Positive deviations are slightly more frequent, because of the predominance of positive deviations for the 4% OTM crash premiums. In sum, the short-term options imply distributions more positively asymmetric than geometric Brownian motion, whereas the evidence from medium-term options is mixed. This "maturity effect," of decreasing absolute deviations from the x% rule as longer maturities are considered, is characteristic both of CEV option pricing models and of jump-diffusion models with asymmetric jumps. Neither model can explain why the deviations are of mixed sign for medium-term options.

2. The Asymmetric Jump-Diffusion Model

The above measures of asymmetry indicate that distributions more asymmetric than those hitherto generally considered would better explain observed option prices. For this reason, and because the distribution is of interest in its own right, it is posited that the asset price follows a stochastic differential equation with asymmetric, deterministic jumps:

$$dS/S \;=\; [\mu(\bullet) - \lambda k]\, dt \;+\; \sigma\, dZ \;+\; k\, dq, \tag{29}$$

where

$\mu(\bullet)$ is the instantaneous unconditional expected return on the asset,

σ is the instantaneous variance conditional on no jumps,

Z is a standard Wiener process,

k is the percentage jump conditional on the Poisson event occurring,

λ is the frequency of Poisson events,

and q is a Poisson counter with intensity λ:

$\text{Prob}(dq = 1) = \lambda dt$, $\text{Prob}(dq = 0) = 1 - \lambda dt$.

The process resembles geometric Brownian motion most of the time, but on average λ times per year the price jumps discretely, by a predictable amount. The instantaneous mean return is μdt, the instantaneous variance is $(\sigma^2+\lambda k^2)dt$.

The postulated process is a special case of the one studied by Jones (1984) and differs from previous work on option pricing under jump-diffusion processes (Merton (1976a,b), Ball and Torous (1983,1985)) in certain important directions. First, the jumps are allowed to by *asymmetric,* i.e., with non-zero mean. Values of the expected percentage jump size k greater (less) than zero imply that the distribution is positively (negatively) asymmetric relative to geometric Brownian motion. The implication is that the crash premium will be greater than or less than the x% rule, depending on k; for sufficiently negative k, the crash premium will be negative.

Second, the jumps are assumed to be deterministic. Market participants know exactly how much an asset price will jump conditional on the jump taking place. Although admittedly an implausible assumption, the work of Ball and Torous (1985) suggests that the *noisiness* of jumps would have relatively little additional effect on option prices relative to a deterministic-jump option pricing model.[19] And the deterministic-jump assumption has the major advantage of implying an *arbitrage-based* option pricing formula; Merton's strong (and implausible) assumption of diversifiable jump risk is no longer necessary.

Jones (1984) develops the arbitrage-based option pricing model for jump-diffusion processes with deterministic jumps. The basic insight is that the additional source of risk coming from jumps can be hedged by adding an additional asset to the continuously-adjusted portfolio replicating an option. The natural instrument to use is another contingent claim; any option can be replicated by a *three*-asset portfolio consisting of a risk-free bond, the underlying asset, and a *different* option on the asset. The implication (see Jones (1984)) is that options are priced *as if* investors were risk-neutral and the underlying asset price followed the equivalent martingale process

$$dS^*/S^* = (b - \lambda^* k)\, dt + \sigma\, dZ + k\, dq^* \quad , \qquad (30)$$

where b is the cost-of-carry coefficient, σ and k are as before, λ^* is the

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[19] Using time series data, Ball and Torous estimate the parameters of a jump-diffusion process with zero-mean random jumps, and find that the resulting option prices are virtually identical to those implied by a (degenerate) jump-diffusion with zero-mean, deterministic jumps -- i.e., the no-jump Black-Scholes model. Ball and Torous use Merton's option pricing model.

frequency of Poisson events *implicit* under risk neutrality in the price of the option used in the replicating portfolio[20] and is assumed constant,[21] and q^* is a Poisson counter with frequency λ^*:

$$\text{Prob}(dq^*=1) = \lambda^* dt,\ \text{Prob}(dq^*=0) = 1 - \lambda^* dt.$$

Since the choice of option used in the replicating portfolio is arbitrary, *all* options will be priced as if the process were (30).

Pricing European options from (30) is straightforward. The terminal distribution of S^*_T is given by

$$\ln S^*_T = (b - \lambda^* k - 0.5\sigma^2)T + \sigma Z_T + n\gamma\ , \qquad (31)$$

where Z_T has a Normal distribution $N(0,T)$, n has a Poisson distribution $P(\lambda^* T)$, and $\gamma \equiv \ln(1+k)$. European calls are priced at what would be the discounted expected value of their terminal payoffs if the terminal distribution were (31):

$$c(S,T;X) = e^{-rT} \Sigma_{n=0}^{\infty} \text{Prob}(n \text{ jumps})\ E_0[\ \max(S^*_T - X, 0)\ |\ n \text{ jumps}\ ]$$

$$= e^{-rT} \Sigma_{n=0}^{\infty} [e^{-\lambda^* T}(\lambda^* T)^n/n!]\ [Se^{b(n)T} N(d_{1n}) - X\ N(d_{2n})]\ , \qquad (32)$$

where

$$b(n) \equiv (b - \lambda^* k) + n\gamma/T,$$

$$d_{1n} \equiv [\ln(S/X) + b(n)T + 0.5\sigma^2 T] / \sigma\sqrt{T}, \text{ and}$$

$$d_{2n} \equiv d_{1n} - \sigma\sqrt{T}\ .$$

European puts have an analogous formula:

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[20] More precisely, λ^* is the cost of crash insurance per unit time. That is, $(\lambda^*\Delta t)\exp(-r\Delta t) \approx \lambda^*\Delta t$ is the cost at each instant of an Arrow-Debreu security that pays off \$1 in the event of a crash occurring within the next instant and \$0 otherwise. Under risk neutrality, such insurance is priced at the actuarially fair rate: $\lambda^* = \lambda$.

[21] Sufficient conditions for λ^* to be constant are given in Bates (1988) and include 1) constant relative risk aversion of the representative investor, and 2) the percentage change in

$$p(S,T;X) = e^{-rT} \Sigma_{n=0}^{\infty} \text{Prob}(n \text{ jumps}) \, E_0[\max(X-S^*_T,0) \mid n \text{ jumps}]$$

$$= e^{-rT} \Sigma_{n=0}^{\infty} [e^{-\lambda^* T}(\lambda^* T)^n/n!] \, [X \, N(-d_{2n}) - Se^{b(n)T} \, N(-d_{1n})]. \quad (33)$$

There are no known analytic solutions for American calls. Finite-difference methods based upon Cox *et al.* (1979) can be used to evaluate option prices accurately for given parameters, but at a significant cost in computer time. That cost becomes prohibitive when one seeks to estimate the parameters implicit in observed option prices. Consequently, I have developed an accurate and inexpensive quadratic approximation for evaluating American options written on jump-diffusion processes. The approximation is an extension on the one developed by MacMillan (1987) and Barone-Adesi and Whaley (1987) for evaluating American options written on geometric Brownian motion processes; details are in Appendix III. The resulting formula for American calls is

$$C(S,T;X) = \begin{cases} c(S,T;X) + X \, A_2 \, [(S/X)/y_c^*]^{q_2} & \text{for } S/X < y_c^*, \\ S - X & \text{for } S/X \geq y_c^*, \end{cases} \quad (34)$$

where $A_2 = (y_c^*/q_2)[1-c_S(y_c^*,T;1)]$, q_2 is the positive root to

$$0.5\sigma^2 q^2 + (b - \lambda^* k - 0.5\sigma^2)q - r/(1-e^{-rT}) + \lambda^*(e^{\gamma q} - 1) = 0, \quad (35)$$

and the critical spot price/exercise price ratio $y_c^* \geq 1$ above which the call is exercised immediately is given implicitly by

$$y_c^* - 1 = c(y_c^*,T;1) + (y_c^*/q_2)[1 - c_S(y_c,T;1)]. \quad (36)$$

Similarly, American puts have values approximated by

$$P(S,T;X) = \begin{cases} p(S,T;X) + X\,A_1\,[(S/X)/y_p^*]^{q_1} & \text{for } S/X > y_p^*, \\ X - S \quad \text{for } S/X \le y_p^*, \end{cases} \tag{37}$$

where $A_1 = (y_p^*/-q_1)[1+p_S(y_c^*,T;1)]$, q_1 is the negative root to (36), and the critical spot price/exercise price ratio $y_p^* \le 1$ below which the put is exercised immediately is given implicitly by

$$1 - y_p^* = p(y_p^*,T;1) + (y_p^*/-q_1)[1+p_y(y_p^*,T;1)]. \tag{38}$$

The parameters q_1, q_2, y_c^*, and y_p^* can be evaluated rapidly via Newton's method for given parameters r, b, σ, k, and λ^*, and for given time to maturity T.

The expressions, although messy, are in fact fairly intuitive. Deep out-of-the-money American call and put option prices approach the European values c(•) and p(•), respectively. Deep in-the-money call and put option prices smoothly approach the early exercise values (S-X) and (X-S), respectively, attaining those values at the critical spot price/exercise price ratio y*. Intermediate values are taken from a smooth, convex curve connecting the two extremes, with the parameters of curvature q_2 and q_1 inversely related to the conditional variance $(\sigma^2+\lambda^* k^2)T$ over the remaining lifetime of the option.[22] The critical early exercise ratios $y_c^*(T)$ and $y_p^*(T)$ are functions of time and of the parameters of the model, excluding exercise price. As the maturity approaches zero, the curvatures approach infinity and early-exercise ratios shrink towards 1. The American call price approaches max(S-X,0), the American put price approaches max(X-S,0).

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[22] That is, the variance of the log of the equivalent martingale measure. Parameters q_2 and q_1 are also affected by total unit cost of carry bT, but are insensitive to other parameters of the model. See Appendix III

Table 2 compares the estimated calls and puts with those calculated by finite-difference methods. The error is typically less than or equal to rounding error, except in the case of extreme parameters.

2.2. Estimation

The procedure discussed above and in Appendix III gives an American option pricing formula as a function of state variables S and T and and parameters X, r, σ, b, k, and λ^*. The first four are known. The instantaneous risk-free rate can be proxied by Treasury bill rates: I use rates derived from the average of bid and ask discounts on Treasury bills maturing close to the maturity of the option. The jump size k, jump frequency λ^*, and standard deviation σ (conditional on no jumps) are not known.

It is possible to derive estimates of σ, k, and λ from the history of prices of the underlying asset. To derive *accurate* estimates requires a long history of data, however; small sample estimates when jumps are infrequent will be strongly affected by whether jumps did or did not take place within the sample. This paper therefore takes the approach of estimating the jump parameters *implicit* in option prices. The parameters estimated are of course those of the equivalent martingale jump-diffusion process, and only under the assumption of diversifiable jump risk will the estimated jump frequency λ^* equal the true jump frequency λ. But the jump size k is the same for both the actual and equivalent martingale jump-diffusion processes, so options data can still be used to see whether market participants perceive jump risk.

The set of settlement prices of options at a given day was assumed to be represented by corresponding model prices plus a random disturbance term:

$$V_j = V(S,X_j,\sigma,k,\lambda^*) + \nu_j \quad . \tag{39}$$

A cross-sectional data sample of calls and puts with identical maturities (typically about 12 data) was used, and the implicit parameters $\sigma(t)$, $k(t)$, and $\lambda^*(t)$ for that day were estimated via nonlinear least squares. The regression yielded estimates of $\sigma(t)$, $k(t)$, and $\lambda(t)$ implicit in option prices for that day. The implicit parameters were not constrained to be constant over time. Two different regressions were run for the two maturity classes considered: short-term (1-3 months) and medium-term (4-6 months) options. The regressions were repeated for each day of the one day per month, February 1984 to January 1987 sample discussed above.[23]

The implicit parameters were estimated using the quadratic hill-climbing software of Goldfeld and Quandt, GQOPT method GRADX. The likelihood function was typically characterized by multiple equilibria. In particular, there was a local maximum at the no-jump solution which, however, was generally dominated by a positive-jump optimum. Multiple starting values were used to ensure that the global maximum was indeed found. Starting with negative jumps typically converged to the no-jump or positive jump optimum, but not always.

To reduce costs, the implicit parameters were first estimated for the Bernouilli jump-diffusion process, premised upon at most one jump during the life of an option. This assumption yields an European call option pricing formula of

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[23] It is unlikely that observed and model prices differ only by additive random error. However, the major alternative source of error in testing options prices -- the nonsimultaneity of futures' and futures options' prices -- presumably has been reduced by the use of settlement prices. If ν_j could be assumed to be normally distributed, then the regressions would yield maximum likelihood estimates of the parameters. But nonnegativity of options prices precludes that assumption.

$$c(S,T;X) = \Sigma_{n=0}^{1} w_n \, [Se^{b(n)T} N(d_{1n}) - X \, N(d_2)] \ , \tag{40}$$

where $w_0 = (1 - \lambda T)$, $w_1 = \lambda T$, and $b(n) = b - \ln(1+\lambda k) + n\gamma/T$.

The early exercise premium was then derived as discussed above and in Appendix III. The estimated optimal parameters from fitting this model were used as starting values for the Poisson jump-diffusion process discussed above. As one would expect, the Bernouilli optimal parameters were close to the Poisson parameters when estimated jump frequencies were low, which was most of the time.[24]

Parameter estimates are given in Table III. The most striking result is that the implicit jumps are almost invariably estimated as positive. This is true for short-term and medium-term option data; for the period of dollar appreciation prior to March 1985 and for the period of depreciation thereafter. According to options prices, the "stylized fact" for the DM/$ rate over February 1984 to January 1987 is downside risk on the dollar.

The second striking result is that parameter estimates fluctuate wildly and there is no particular consistency between the estimates from short-term and from medium-term options. It is possible that this reflects too few degrees of freedom; it is also possible that expectations are in fact volatile. Pending further research, therefore, not too much reliance can be placed upon individual estimates from settlement prices. Future work will use transactions data.

The broad patterns, however, are of substantial but infrequent expected positive jumps. Figures 1 to 3 give measures of the economic significance of

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[24] Ball and Torous (1985) use a similar two-step procedure in estimating distributional parameters from daily asset price changes.

the jump component of the jump-diffusion process. Figure 1 gives expected jumps per year λk (the frequency of jumps times the size) for short-term assets, with an approximation of the 95% confidence interval. This measure, the implicit contribution of jumps to the expected return on the underlying asset, is typically about 2-4% per year, with some notable exceptions. Figure 2 gives the same measure for medium-term options, with roughly the same results.

Figure 3 indicates by how much the jump risk contributes to the overall risk of the underlying asset returns. Although as volatile as the other measures, jump risk is estimated as typically responsible for roughly 10-30% of the conditional variance of asset returns. Using this or the above measures, it is reasonable to conclude that jumps are perceived by market participants as a substantive component of the behavior of the DM/$ futures price.

3. Conclusions and Extensions

This paper has shown that the "stylized fact" of options on Deutschemark futures is that the implicit distribution is more positively asymmetric than all major theoretical distributions used hitherto in option pricing models, including the benchmark model of geometric Brownian motion. Evidence for this assertion came first from the "crash premium" measure of asymmetry based upon relative prices of out-of-the-money calls and puts. Further evidence came from fitting an asymmetric jump-diffusion option pricing model to the options data, with the sign of the jump parameterizing the direction of asymmetry relative to geometric Brownian motion. Estimated implicit jump sizes were almost invariably positive.

It is, of course possible that the observed "moneyness bias" in option prices relative to the Black-Scholes benchmark is attributable to other sources of asymmetries than to asymmetric jumps in a jump-diffusion process. Relevant alternative models include stochastic volatility option pricing models such as the ARCH (autoregressive conditional heteroskedasticity models) and GARCH (generalized ARCH), and the CEV model with $\rho>1$. Standard ARCH and GARCH assume,[25] however, that the evolution of volatility is *independent* of the asset price. In continuous-time models, this implies a stochastic differential equation for volatility of the form

$$d\sigma = f(\sigma,t)\,dt + g(\sigma,t)\,dZ_\sigma, \tag{41}$$

with $f(\bullet)$ and $g(\bullet)$ *not* functions of the asset price S, and with $\mathrm{Cov}(d\sigma,dS) = 0$. It is straightforward to modify the proofs to show that the propositions relating prices of out-of-the-money calls and puts will continue to hold under

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[25] See Nelson (1987) for a critical discussion of ARCH and GARCH assumptions.

this assumption -- standard ARCH and GARCH cannot explain the moneyness bias. Such models imply "fat-tailed" distributions that are *symmetrically* fat-tailed.

The modified GARCH of Nelson (1987) allows a non-zero correlation between innovations in volatility and in the asset price. *Positive* correlations (increased volatility as the dollar depreciates), a characteristic also of CEV models with $\rho>1$, would induce a moneyness bias in option prices consistent with the ones observed empirically. There has hitherto been little estimation of such modified GARCH processes for exchange rates, so the sign of the correlation is an open question. One relevant study is that of Melino and Turnbull (1988), who estimate over 1975 to 1986 a *negative* correlation between the dollar/Canadian dollar exchange rate and the volatility of that rate. Since options on Canadian dollars also exhibit positive implicit asymmetries,[26] stochastic volatility models would not appear to be capable of explaining the moneyness biases of such options.

The implications of asymmetries go well beyond ascertaining which distribution best explains option prices. Tests of asset pricing models are invariably tests of a three-fold hypothesis:

1) The theoretical constraint on expected returns across assets is correct,
2) Observations *ex post* are valid instruments for expectations *ex ante, and*
3) distributions are "well-behaved."

But as Krasker (1980) pointed out, skewed distributions are *not* well-behaved. If skewness is present, sample distributions converge only slowly to the asymptotic normal, so that sample moments from small samples lead to

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[26] See Borensztein and Dooley (1987).

misleading inferences.

Thus, estimated implicit distributions from options prices point out a possible explanation for rejections of international asset pricing models, and suggest an alternative. Use of the distributions implicit in option pricing may give a more accurate picture of true distributions. Furthermore, insofar as these implicit distributions do represent true expectations of market participants *ex ante,* using the distributions can reduce the reliance on the rational expectations hypothesis that expectations *ex ante* and realizations *ex post* differ only by unforecastable white noise, and thus lead to more direct tests of asset pricing models. Whether using implicit distributions can make that much difference remains to be seen; but the potential is there.

Appendix I

Properties of Constant Elasticity of Variance (CEV) Option Prices

Lemma: For European CEV option prices in general, expressed as functions of the asset price S, time to maturity T, the strike price, and the list of parameters,

$$c(S,T;Fk,\sigma,\rho,b) \;=\; k\; p(S,T;F/k,\sigma F^{2(1-\rho^*)},\rho^*,b^*) \quad \text{for any } k, \qquad (A.1)$$

where $\rho^* \equiv 2-\rho$, $b^* \equiv -b$, and Fk and F/k are the strike prices of the call and put, respectively, expressed relative to the forward price $F = S\exp(bT)$. The same is also true for American options on futures (b=0):

$$C(S,T;Sk,\sigma,\rho,0) \;=\; k\; P(S,T;S/k,\sigma S^{2(1-\rho^*)},\rho^*,0) \quad \text{for any } k. \qquad (A.2)$$

Proof: For $\rho \neq 1$, CEV option prices are homogeneous of degree 1 in S, X, and $\rho^{1/(1-\rho)}$. Writing the European CEV call price in terms of the forward price instead of the spot price, and dividing by the call's strike price X = Fk yields a normalized call price of

$$\begin{aligned} c'(y,T;1,\sigma',\rho,b) \;&\equiv c(Fe^{-bT},T;X,\sigma,\rho,b)\;/\;X \\ &= c(\;(F/X)e^{-bT},T;1,X^{-(1-\rho)}\sigma,\rho,b)\;, \end{aligned} \qquad (A.4)$$

where $y \equiv F/X$ and $\sigma' \equiv X^{-(1-\rho)}\sigma$. The partials of c' are $c'_T = -bye^{-bT}c_S + c_T$, $c'_F = e^{-bT}c_S$, and $c'_{FF} = e^{-2bT}c_{SS}$. Substituting into (A.3) yields a general p.d.e. for CEV option prices as a function of the (normalized) forward price,

$$-V'_T \;+\; 0.5[\sigma' e^{bT(1-\rho)}]^2 y^{2\rho} V'_{FF} \;=\; rV' \qquad (A.5)$$

for V'=c', which is solved subject to the normalized terminal boundary conditions specific to European calls,

$$c'(y,0;1,\sigma',\rho) \;=\; \max(y-1,0) \quad . \qquad (A.6)$$

(since the normalized forward price F/X equals the normalized spot price S/X when T=0).

Define $h(y,T;1,\sigma',\rho^*,b^*) \equiv yc'(y^{-1},T;1,\sigma',\rho,b)$, where $\rho^* = 2-\rho$ and $b^* = -b$. Its partial derivatives are $h_T = yc'_T$, $h_y = c'-y^{-1}c'_F$, and $h_{yy} = y^{-3}c'_{FF}$, so

$$-h_T + [\sigma' e^{bT(1-\rho)}]^2 y^{2\rho^*} h_{yy}$$

$$= -yc'_T + [\sigma' e^{bT(1-\rho)}]^2 y^{2\rho^*-3} c'_{yy}$$

$$= y \{ -c'_T(y^{-1},T;1,\bullet) + [\sigma' e^{bT(1-\rho)}]^2 (y^{-1})^{2\rho} c'_{FF}(y^{-1},T;1,\bullet) \}$$

$$= y[rc'(y^{-1},T;1,\bullet)] = rh \qquad (A.7)$$

So, $h(\bullet)$ satisfies the p.d.e. (A.5) for $V' = h$ and parameters ρ^* and b^*; and since

$$h(y,0;1,\bullet) = y \max(y^{-1}-1,0) = \max(1-y,0), \qquad (A.8)$$

which is the terminal boundary condition specific to European puts,

$$p'(y,T,1,\sigma',\rho^*,b^*) = h(\bullet) = yc'(y^{-1},T,1,\sigma',\rho,b) \quad . \qquad (A.9)$$

Exploiting the homogeneity of $p'(\bullet)$ and $c'(\bullet)$, (A.9) can be rewritten as

$$c'(1,T;y,\sigma' y^{1-\rho},\rho) = y\ p'(1,T;y^{-1},\sigma'(y^{-1})^{1-\rho^*},\rho) \qquad (A.10)$$

or

$$c'(1,T;y,\sigma'',\rho) = y\ p'(1,T;y^{-1},\sigma'',\rho) \qquad (A.11)$$

where $\sigma'' = \sigma' y^{1-\rho} = \sigma' y^{-(1-\rho^*)}$. Multiplying both sides by F yields

$$c'(F,T;Fy,\sigma'' F^{1-\rho},\rho) = y\ p'(F,T;F/y,\sigma'' F^{1-\rho^*},\rho^*)\ , \qquad (A.12)$$

and redefining $\sigma \equiv \sigma'' F^{1-\rho}$ yields (A.1). ∎

The normalized price of American call options on futures $C'(y,T;1,\sigma',\rho)$ must satisfy (A.5), (A.6), and also the early-exercise "smooth-pasting" boundary conditions

$$C'(y^*,T;1,\sigma',\rho) = y^* - 1 > 0 \qquad (A13a)$$

$$C'_y(y^*,T;1,\sigma',\rho) = 1$$

call option is exercised immediately. It is straightforward to confirm that $H(y,T;1,\sigma',\rho^*) \equiv yC'(1/y,T;1,\sigma',\rho)$ and $y_p^* \equiv 1/y^*$ satisfy the corresponding early-exercise "smooth-pasting" conditions for American put options on futures:

$$H(y_p^*,T;1,\sigma',\rho^*) = 1 - y_p^* > 0 \quad \text{(A.14a)}$$

$$H_y(y_p^*,T;1,\sigma',\rho^*) = -1 \quad \text{(A.14a)}$$

Therefore, $P'(\bullet) = H(\bullet)$, and (A.2) follows. ∎

Setting $\rho = 1 = \rho^*$ yields propositions 2 and 5b for options on geometric Brownian motion processes.

Proposition 3: Again writing the option prices in terms of the forward price instead of the spot price, and including the list of relevant parameters, Proposition 3) specifies that

$$c'(F,T;Fk,k^{1-\rho}\sigma,\rho,b) = k\, p'(F,T;F/k,\sigma,\rho,b) \quad \text{for } k>0,\ \rho<1, \quad \text{(A.15)}$$

where $c'(F,T;X,\sigma,\rho,b) \equiv c(F\exp(-bT),T;X,\sigma,\rho,b)$, similarly for $p'(\bullet)$ and $p(\bullet)$, Fk is the strike price of the call, and F/k is the strike price of the put. Dividing both sides of (A.15) by Fk and exploiting the homogeneity in F, X, and $\sigma^{1/(1-\rho)}$ yields the proposition to be proved:

$$c'(y,T;1,\sigma',\rho,b) = p'(1,T;y,\sigma',\rho,b) \quad \text{(A.16)}$$

where $\sigma' \equiv \sigma F^{\rho-1}$, $y \equiv 1/k > 0$.

This can by shown manipulating the European CEV option pricing formulas for $\rho<1$ given in Cox and Ross (1976) and Cox and Rubinstein (1985).

The generalized European CEV call option pricing formula for $\rho < 1$ is

$$c'(F,T;X,\sigma',\rho,b) = e^{-rT}[F\Sigma_{n=1}^{\infty} g(n,f)G(n+\lambda,x) - X\Sigma_{n=1}^{\infty} g(n+\lambda,f)G(n,x)] \quad (A.17)$$

where $\lambda \equiv 1/[2(1-\rho)]$, $\theta \equiv 2\lambda b/[\sigma^2(e^{bT/\lambda}-1)]$, $f \equiv \theta F^{1/\lambda}$, $x \equiv \theta X^{1/\lambda}$, b is the cost of carry, and $\Gamma(\bullet)$, $g(\bullet)$, and $G(\bullet)$ are the gamma function, gamma density function, and complementary gamma distribution function, respectively:

$$g(n,z) = e^{-z}z^{n-1}/\Gamma(n), \; G(n,z) = \int_z^{\infty} g(n,x)\, dx \; .$$

From put-call parity,

$$\begin{aligned} & c'(y,T;1,\bullet) - p'(1,T;y,\bullet) \\ &= c'(y,T;1,\bullet) - c'(1,T;y,\bullet) - e^{-rT}(y-1) \\ &= e^{-rT}\{ y\, \Sigma_{n=1}^{\infty} g(n,\theta y^{1/\lambda})\, G(n+\lambda,\theta) - \Sigma_{n+1}^{\infty} g(n+\lambda,\theta y^{1/\lambda})\, G(n,\theta) \\ &\quad - [\Sigma_{n=1}^{\infty} g(n,\theta)\, G(n+\lambda,\theta y^{1/\lambda}) - y\, \Sigma_{n=1}^{\infty} g(n+\lambda,\theta)\, G(n,\theta y^{1/\lambda})\,] \\ &\quad - (y-1)\,\} \end{aligned} \quad (A.18)$$

Grouping terms according to whether they are or are not pre-multiplied by y, and using the equalities $G(n,x) = \Sigma_{j=1}^{n} g(j,x)$ and $G(n-1+\lambda,x) + \Sigma_{j=n}^{\infty} g(j+\lambda,x) = 1$ yields after some manipulation the equalities

$$\begin{aligned} & c'(y,T;1,\bullet) - p'(1,T;y,\bullet) \\ &= \Sigma_{n=1}^{\infty} [y\, g(n,\theta y^{1/\lambda})\, g(n+\lambda,\theta) - g(n,\theta)\, g(n+\lambda,\theta y^{1/\lambda})\,] \\ &= \exp(-\theta-y^{1/\lambda})\, \Sigma_{n=1}^{\infty}\{\theta^{2n+\lambda-2}\, y^{(n-1)/\lambda}\, [y-(y^{1/\lambda})^{\lambda}]\,\} / [\Gamma(n)\, \Gamma(n+\lambda)] \\ &= 0 \; . \end{aligned}$$

∎

Corollary: $c(S,T;Fk,k^{1-\rho}\sigma,\rho,b) = k\, p(S,T;F/k,\sigma,\rho,b)$ for $\rho>1$ and $k>0$.

Proof: For $\rho>1$, define $\rho^* \equiv 2-\rho < 1$ and $b^* \equiv -b$. Then,

$$
\begin{aligned}
c(S,T;Fk,k^{1-\rho}\sigma,\rho,b) &= k\, p(S,T;F/k,(k^{1-\rho}\sigma)F^{2(1-\rho^*)},\rho^*,b^*) && \text{from (A.1)} \\
&= c(S,T;Fk,k^{1-\rho^*}[k^{1-\rho}\sigma F^{2(1-\rho^*)}],\rho^*,b^*) && \text{from (A.15)} \\
&= c(S,T;Fk,\sigma F^{2(1-\rho^*)},\rho^*,b^*) \\
&= k\, p(S,T;F/k,[\sigma F^{2(1-\rho^*)}]F^{2(1-\rho)},\rho,b) && \text{from (A.1)} \\
&= k\, p(S,T;F/k,\sigma,\rho,b) \quad . && \blacksquare
\end{aligned}
$$

Appendix II

Properties of American Options on Futures

1. Unlimited-liability arithmetic Brownian motion (ABM) (Proposition 5.1)

It is straightforward to verify that under unlimited-liability ABM, prices of American options on futures depend on the *difference* between the futures price and exercise price rather than on each separately:

$$C(S,T;X) = C(S-X,T;0) \equiv C'(x,T) \quad \text{(B.1a)}$$

$$P(S,T;X) = P(S-X,T;0) \equiv P'(x,T) \quad \text{(B.1b)}$$

where $x = S-X$. $C'(x,T)$ satisfies the p.d.e.

$$-C'_T + 0.5\sigma^2 C'_{xx} = rC' \quad \text{(B.2)}$$

subject to boundary conditions

$$C'(x,0) = \max(x,0) \quad \text{(B.3)}$$

and early-exercise smooth-pasting conditions

$$C'(x^*,T) = x^* > 0 \quad \text{(B.4a)}$$

$$C'_x(x^*,T) = 1. \quad \text{(B.4b)}$$

Define $H(x,t) \equiv C'(-x,T)$. $H(x,T)$ and $x_p^* \equiv -x^*$ satisfy the boundary conditions and p.d.e. for American puts, so $H(x,T)=P'(x,T)$. Proposition 5.1) then follows. ∎

2. Geometric Brownian motion (Proposition 5.2a)

Proved in Appendix II, as a special case of CEV processes.

3. Jump-diffusion with log-symmetric jumps (Proposition 5.2b)

Option prices under such processes are homogeneous in S and T, so the normalized price of American call options on futures is given by

$$C(y,T;1) \;=\; C(S,T;X)\,/\,X \;=\; C(S/X,T;1) \quad . \tag{B.5}$$

In the Merton (1976) model, adapted for options on futures, $C(\bullet)$ solves the general jump-diffusion p.d.e.

$$-V_T + 0.5\sigma^2 y^2 V_{SS} + \lambda E[V(Se^J,T) - V] \;=\; rV \tag{B.6}$$

for V=C subject to call-specific boundary conditions (A.6) and (A.13). J is a random variable with a Normal $N(-0.5\delta^2,\delta^2)$ distribution, and $[\exp(J)-1]$ is the random percentage change conditional on a jump occurring.

Define $H(y,T) \equiv yC(y^{-1},T;1)$. $H_T = yC_T$, $H_{yy} = y^{-3}C_{SS}$, so

$$-H_T + 0.5\sigma^2 y^2 H_{yy} + \lambda E[H(ye^J,T) - H]$$

$$= \; -yC_T + 0.5\sigma^2 y^{-1} C_{SS} + \lambda\, E[\, ye^J C(y^{-1}e^{-J},T;1) - yC]$$

$$= \; y\{-C_T + 0.5\sigma^2 y^{-2} C_{SS} + \lambda\, E[C(y^{-1}e^J,T;1) - C]\,\}$$

$$= \; y\{rC(y^{-1},T;1)\} \;=\; rH \tag{B.7}$$

where the penultimate line follows from properties of the given Normal distribution:

$$E[e^J C(y^{-1}e^{-J},T;1)] \;=\; \int_{-\infty}^{\infty} C(y^{-1}e^{-J},T;1)\; e^{-J}(2\pi\delta^2)^{-1/2}\; e^{-(J+0.5\delta^2)^2/2\delta}\; dJ$$

$$= \; \int_{-\infty}^{\infty} C(y^{-1}e^{-J},T;1)\; (2\pi\delta^2)^{-1/2}\; e^{-(-J+0.5\delta^2)^2/2\delta}\; dJ$$

$$= \; E[C(y^{-1}e^J,T;1)]. \tag{B.8}$$

Since $H(y,T)$ solves the p.d.e. (B.6), and $H(y,T)$ and $y_p^* \equiv 1/y^*$ satisfy the put-specific boundary conditions (A.8) and (A.14) if $C(y,T)$ and y^* satisfy the call-specific boundary conditions (A.6) and (A.13), $H(y,T) = P(y,T;1)$. Proposition 5.2b then follows. ∎

The same proof can be used to prove Proposition 4 for European options in general under such processes, by using option prices written as functions of the forward price: $c'(y,T;1) \equiv c(y\ \exp(-bT),T;1)$. Such option prices solve the p.d.e. (B.6), and $h(y,T) \equiv yc(1/y,T;1)$ solves the put's terminal boundary conditions if $c'(\bullet)$ solves the call's terminal boundary conditions. ▮

Appendix III

Quadratic Approximation to American Option Values for Jump-Diffusion Processes.

The American call option price $C(S,T;X)$ must meet the boundary conditions

terminal condition: $C(S,0;X) = \max[S - X,0]$ (B.1)

early exercise conditions: $C(S,T;X) = S - X$ for $S \geq S_c^*$ (B.2)

$C_S(S_c^*,T;X) = 1$ (B.3)

where S_c^* is the critical early-exercise price on the underlying asset (relative to X) above which the option will always be exercised immediately; determination of S_c^* is part of the problem. The option must satisfy the Bellman equation in the interior of the no-stopping region:

$$-V_T + (b - \lambda k)SV_S + 0.5\sigma^2 V_{SS} + \lambda[V(Se^{\gamma},T\ X) - V] = rV\ , \quad \text{(B.4)}$$

where $V(S,T;X) = C(S,T;X)$. The Bellman equation states that the call option is priced *as if* it yielded the instantaneous risk-free return given jump-diffusion (30) for the underlying asset price: $E^*(dV) = rV$. From Ito's lemma (extended to jump-diffusions), that expected return is composed of the maturity effect $-V_T$, the diffusion components, and the expected effect of jumps. The European call option price $c(S,T;X)$ solves the Bellman equation subject to terminal condition (B.1).

Given that there exists an analytic solution for the European option price, the problem is to find a good approximation for the early exercise premium

$$\varepsilon_c(S,T,X) \equiv C(S,T;X) - c(S,T;X). \quad \text{(B.5)}$$

Given the linearity of (B.4) in V and its partials, ε_c also must solve (B.4) in the no-stopping interior. The premium is homogeneous in S and X: $\varepsilon_c(S,T,X) = X\ \varepsilon_c(S/X,T,1)$. Furthermore, without loss of generality, the premium can be written as

$$\equiv X\,K(T)\,f(y,K), \quad y \equiv S/X \,, \tag{B.6}$$

where K(T) is an arbitrary function of the time to maturity variable T. Taking the partial derivatives $\varepsilon_S = Kf_y$, $\varepsilon_{SS} = K/X\, f_{yy}$, and $\varepsilon_T = XK_T + XKK_Tf_K$, and substituting into (B.4), one discovers that the function f(y,K) must satisfy the Bellman equation

$$0.5\sigma^2y^2f_{yy} + (b-\lambda k)yf_y - rf[1 + (K_T/rK)(1 + Kf_K/f)] + \lambda[f(ye^{\gamma},K) - f(y,K)] = 0 \tag{B.7}$$

Choosing $K(T) = 1 - e^{-rT}$ simplifies the expression to

$$0.5\sigma^2y^2f_{yy} + (b-\lambda k)yf_y - (r/K)f - r(1-K)f_K + \lambda[f(ye\gamma,K) - f(y,K)] = 0, \tag{B.8}$$

which for calls is solved subject to the boundary conditions

$$f(y,0) = 0 \tag{B.9a}$$

$$f(0,K) = 0 \tag{B.9b}$$

$$f(y_c^*,K) = (y_c^* - 1) - c(y_c^*,T;1) \tag{B.9c}$$

$$f_y(y_c^*,K) = 1 - c_S(y_c^*,T;1) \,. \tag{B.9d}$$

Conditions (B.9c) and (B.9d) require that the early exercise premium *smoothly* approaches the stopped or early exercise value

$$f(y,K) = (y - 1) - c(y,T;1)$$

as y approaches the critical spot price/strike price ratio y_c^* above which the call is always exercised early.

Apart from the term $r[1-K(T)]f_K$, expression (B.6) is an ordinary differential equation in y. The quadratic approximation for the early exercise premium is generated by ignoring this term, and solving the ordinary differential equation subject to the boundary conditions (B.9). The choice of $K(T) = 1 - \exp(-rT)$ ensures that the approximation becomes exact at the extreme boundaries: as $T \to 0$, $(1-K) \to 0$, whereas as $T \to \infty$, $f_K \to 0$.

Under the approximation, (B.8) becomes

$$0.5\sigma^2 y^2 f_{yy} + (b-\lambda k)yf_y - [r/K(T)]f + \lambda[f(ye^{\gamma},K) - f(y,K)] = 0. \quad \text{(B.10)}$$

The general solution to this is of the form

$$f(y) = A_1 y^{q_1} + A_2 y^{q_2}\ , \quad \text{(B.11)}$$

where q_1 and q_2 are the roots to

$$0.5\sigma^2 q^2 + (b-\lambda k-0.5\sigma^2)q - r/K(T) + \lambda e^{\gamma q} = 0\ . \quad \text{(B.12)}$$

One root (q_1) is negative, the other (q_2) is positive. For given values of the parameters r, σ, k,and λ, accurate values of the roots q_1 and q_2 can be rapidly determined from (B.12) via Newton's method. Starting values are obtained from the quadratic equation given by second-order Taylor expansion of $\exp(\gamma q)$ in (B.12):

$$0.5(\sigma^2+\lambda k^2)q^2 + [b - 0.5(\sigma^2+\lambda k^2)] - r/K(T) = 0 \quad \text{(B.13)}.$$

The expression (B.13) indicates that to a second-order approximation, the parameter of curvature q depends on jump parameters λ and k only insofar as they contribute to the average variance per unit time of the underlying process, $\sigma^2+\lambda k^2$. Furthermore, since $r/K(T) = r/[1-\exp(-rT)] \approx 1/T$, the parameter of curvature q is insensitive to the interest rate.

Boundary condition (B.9a) rules out the negative root solution; $A_1 = 0$ for calls. Conditions (B.9c) and (B.9d) pin down the critical early-exercise ratio y_c^* and the coefficient A_2. Since $f(y) = A_2 y^{q_2}$ and $f'(y) = A_2(q_2/y)y^{q_2}$, from (B.9c) and (B.9d) we find that y_c^* is the *implicit* solution to

$$y_c^* - 1 = c(y_c^*,T;1) + (y_c^*/q_2)[1 - c_S(y_c^*,T;1)], \quad \text{(B.14)}$$

and A_2 is given by

$$A_2 = (y_c^*/q_2)[1 - c_S(y_c^*,T;1)] \quad \text{(B.15)}$$

(32); the summation is continued until further terms would make no difference in accuracy.

A similar expression holds for the quadratic approximation to the early exercise premium on American puts: The positive root is ruled out; $f(y) = A_1 y^{q_1}$. Solving this subject to the boundary conditions

$$1) f(y_p{*}) = (1-y_p{*}) - p(y_p{*},T;1) \tag{B.16a}$$

$$2) f_y(y_p{*}) = -1 - p_S(y_p{*},T;1) \tag{B.16b}$$

yields the critical early-exercise spot price/exercise price ratio y_{p*} as the implicit solution to

$$1 - y{*} = p(y{*},T;1) + (y{*}/-q_1)[1 + p_S(y{*},T;1)] \quad . \tag{B.17}$$

The coefficient A_1 is given by

$$A_1 = (y_c{*}/-q_1)\ [1 + p_S(y_c{*},T;1)] \ . \tag{B.18}$$

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Table 1

Crash Premium for Options on Deutchemark Futures:
Percentage Deviation of OTM Call Prices From Corresponding OTM Put Prices

$CP(S,T,x) \equiv [\ C(S,T;\ S(1+x)\ )/P(S,T;\ S,T,\ S/(1+x)\ )\ -\ 1]$, in percent

Out-of-the-moneyness parameter x at 2% increments.

Geometric Brownian motion implies $CP(\bullet,x) = x\%$.

Short-term options (1-3 months)

Date	Maturity (months)	x: 2%	4%	6%
8402	(1)	2%	9%	55%
8403	(3)	4	18	25
8404	(2)	7	20	81
8405	(1)	-14	-31	19
8406	(3)	3	14	33
8407	(2)	-1	-1	
8408	(1)	-6	-2	
8409	(3)	7	10	8
8410	(2)	3	5	13
8411	(1)	14	58	142
8412	(3)	0	2	7
8501	(2)	9	39	
8502	(1)	7	70	
8503	(3)	4	9	
8504	(2)	2	5	7
8505	(1)	6	6	4
8506	(3)	6	9	8
8507	(2)	1	4	
8508	(1)	11	27	
8509	(3)	2	4	8
8510	(2)	-3		
8511	(1)	1	4	26
8512	(3)	6	12	
8601	(2)	1	1	
8602	(1)	23	84	
8603	(3)	5	11	
8604	(2)	1	2	
8605	(1)	3	5	
8606	(3)	5	6	
8607	(2)	3	5	
8608	(1)	3		
8609	(3)	1	0	
8610	(2)	4	9	
8611	(1)	7		
8612	(3)	1		
8701	(2)	6		
Frequencies: > 0 :		**86%**	**87%**	**100%**

Medium-term options (4-6 months)

Maturity (months)	x: 2%	4%	6%
(4)	21%	28%	
(6)			
(5)	7	9	
(4)	-3	5	
(6)			
(5)	13		
(4)	3	5	11
(6)			
(5)	2	3	1
(4)	7	11	15
(6)			
(5)	0	1	0
(4)	3	6	-5
(6)	2	4	9
(5)	7	3	-1
(4)	2	2	1
(6)	3	2	2
(5)	-3	-1	
(4)	0	5	
(6)	5	7	8
(5)	1	0	1
(4)	2	4	12
(6)	5	6	
(5)	2	5	
(4)	7	10	
(6)	2	1	
(5)	1	1	
(4)	1	1	
(6)	3	2	
(5)	1	3	
(4)	1	-1	
(6)	1	2	
(5)	3	4	
(4)	1		
(6)	1		
(5)	1	4	
Frequencies: > 0 :	**90%**	**93%**	**75%**

Table 2

Theoretical American Futures Options Values under Asymmetric Jump-Diffusion Processes

Exercise Price X = 100. Parameters: r = 0.06, σ = 0.20, T = 0.25.

		Call Options			Put Options		
			American C(S,T;X)			American P(S,T;X)	
Jump Parameters	Futures Price S	European c(S,T;X)	Finite Diff.	Quadratic Approx.	European p(S,T;X)	Finite Diff.	Quadratic Approx.
k = 0.10 λ=1.0	80	0.12	0.12	0.12	19.82	20.02	20.01
	90	1.02	1.02	1.02	10.87	10.91	10.91
	100	4.35	4.36	4.37	4.35	4.35	4.36
	110	11.02	11.08	11.08	1.17	1.17	1.17
	120	19.91	20.09	20.09	0.21	0.21	0.21
k = -0.10 λ=1.0	80	0.06	0.06	0.06	19.76	20.00	20.00
	90	0.90	0.90	0.90	10.75	10.81	10.81
	100	4.39	4.39	4.40	4.39	4.40	4.40
	110	11.21	11.24	11.25	1.35	1.36	1.36
	120	20.05	20.18	20.18	0.34	0.34	0.35
k = +0.40 λ=0.1	80	0.33	0.33	0.33	20.03	20.10	20.12
	90	1.18	1.18	1.19	11.03	11.04	11.05
	100	4.31	4.33	4.35	4.31	4.31	4.32
	110	10.95	11.01	11.04	1.10	1.10	1.10
	120	19.88	20.07	20.09	0.18	0.18	0.18
k = -0.40 λ=0.1	80	0.05	0.05	0.05	19.75	20.00	20.00
	90	0.83	0.83	0.83	10.68	10.74	10.80
	100	4.35	4.35	4.35	4.35	4.37	4.41
	110	11.43	11.43	11.44	1.58	1.59	1.61
	120	20.48	20.48	20.51	0.78	0.79	0.80

Table 3

Estimates of Implicit Parameters Under Alternative Distributional Hypotheses
Options on Deutschemark Futures -- Pooled Calls and Puts

First entry: estimates for short-term options (1-3 months)
Second entry: estimates for medium-term options (4-6 months)

Derived variables: λk is expected jumps per year;
$R \equiv \lambda k^2/(\sigma^2 + \lambda k^2)$ is the fraction of conditional variance attributable to the jump component of the process

		Geometric Brownian Motion		Jump-diffusion					Derived Variables	
Date	N	σ	SEE	σ	k	λ	SEE	F	λk	R
8402	10	0.084	0.011	0.080	-0.046	0.35	0.012	0.50	-0.016	0.102
	10	0.089	0.029	0.084	0.568	0.01	0.027	1.98	0.005	0.290
8403	16	0.099	0.096	0.051	-0.054	2.47	0.095	1.12	-0.134	0.741
	4	0.112	0.061	0.010	0.096	1.37	0.004		0.131	0.992
8404	16	0.109	0.028	0.097	0.126	0.21	0.017	13.00**	0.027	0.264
	10	0.118	0.023	0.093	0.041	3.28	0.018	3.93	0.135	0.394
8405	15	0.109	0.016	0.096	-0.069	0.62	0.012	7.35*	-0.043	0.244
	15	0.110	0.031	0.100	0.092	0.27	0.024	5.37*	0.025	0.188
8406	15	0.118	0.039	0.105	0.186	0.11	0.023	13.46**	0.021	0.258
	0									
8407	16	0.118	0.023	0.112	0.232	0.05	0.017	7.32*	0.011	0.169
	10	0.117	0.051	0.082	0.099	0.70	0.026	14.23**	0.069	0.500
8408	17	0.129	0.017	0.120	-0.148	0.17	0.016	2.59	-0.026	0.208
	14	0.130	0.026	0.124	0.475	0.01	0.019	6.31*	0.006	0.149
8409	17	0.131	0.026	0.125	0.252	0.03	0.023	3.22	0.008	0.117
	4	0.130	0.038	0.056	-0.035	11.47	0.055	0.20	-0.397	0.812
8410	19	0.175	0.015	0.171	1.179	0.00	0.010	14.49**	0.004	0.139
	10	0.175	0.021	0.174	0.982	0.00	0.022	0.50	0.001	0.025
8411	12	0.149	0.030	0.111	0.058	3.12	0.014	18.95**	0.182	0.463
	10	0.170	0.039	0.151	0.226	0.17	0.022	10.93**	0.039	0.279
8412	10	0.143	0.027	0.136	0.236	0.05	0.027	1.05	0.013	0.137
	5	0.153	0.067	0.127	0.754	0.03	0.042	4.05	0.024	0.534

*Rejection at 5% level of no-jump hypothesis (k *or* λ = 0)
*Rejection at 1% level of no-jump hypothesis (k *or* λ = 0)

Table 3 (cont.)

Estimates of Implicit Parameters Under Alternative Distributional Hypotheses

	G.B.M.			Jump-Diffusion					Derived Variables	
Date	N	σ	SEE	σ	k	λ	SEE	F	λk	R
8501	11	0.125	0.028	0.108	0.105	0.40	0.015	13.73**	0.043	0.279
	10	0.138	0.029	0.130	0.175	0.09	0.029	1.13	0.015	0.136
8502	11	0.114	0.015	0.107	0.129	0.15	0.011	4.75*	0.019	0.178
	11	0.129	0.050	0.113	0.143	0.22	0.048	1.45	0.031	0.256
8503	12	0.159	0.031	0.148	0.410	0.04	0.018	12.70**	0.016	0.234
	11	0.164	0.047	0.144	0.296	0.09	0.032	6.76*	0.028	0.284
8504	12	0.178	0.010	0.174	0.370	0.02	0.008	4.85	0.007	0.082
	11	0.178	0.023	0.174	-0.161	0.05	0.026	0.20	-0.008	0.043
8505	12	0.181	0.016	0.177	2.203	0.00	0.015	1.67	0.005	0.263
	12	0.189	0.014	0.186	0.120	0.09	0.015	0.20	0.011	0.037
8506	12	0.171	0.014	0.165	0.123	0.15	0.014	1.13	0.018	0.076
	12	0.166	0.022	0.156	0.031	3.17	0.024	0.06	0.099	0.111
8507	10	0.160	0.055	0.153	0.240	0.02	0.011	104.25**	0.005	0.049
	10	0.158	0.030	0.158	0.003	0.26	0.033	0.00	0.001	0.000
8508	10	0.132	0.010	0.093	0.009	99.23	0.009	1.82	0.942	0.509
	10	0.149	0.018	0.136	0.049	1.67	0.019	0.69	0.082	0.180
8509	12	0.158	0.013	0.151	0.145	0.11	0.010	4.94*	0.016	0.094
	12	0.155	0.028	0.146	0.048	1.28	0.031	0.12	0.061	0.121
8510	10	0.149	0.015	0.144	-0.157	0.08	0.014	2.20	-0.013	0.090
	10	0.159	0.013	0.157	0.169	0.02	0.015	0.14	0.004	0.027
8511	11	0.114	0.012	0.111	0.510	0.01	0.011	1.85	0.005	0.181
	11	0.141	0.015	0.136	0.307	0.03	0.012	4.31	0.008	0.114
8512	10	0.126	0.019	0.116	0.108	0.25	0.009	17.18**	0.027	0.180
	12	0.129	0.022	0.123	0.215	0.04	0.019	2.85	0.009	0.116

Table 3 (cont.)

Estimates of Implicit Parameters Under Alternative Distributional Hypotheses

	G.B.M.			Jump-Diffusion					Derived Variables	
Date	N	σ	SEE	σ	k	λ	SEE	F	λk	R
8601	11	0.117	0.012	0.114	0.387	0.01	0.010	2.69	0.004	0.103
	12	0.124	0.025	0.115	0.177	0.08	0.018	5.93*	0.015	0.162
8602	10	0.107	0.015	0.089	0.036	2.97	0.007	20.51**	0.106	0.322
	11	0.111	0.021	0.102	0.117	0.18	0.012	11.21**	0.021	0.187
8603	12	0.149	0.035	0.137	0.335	0.06	0.016	21.42**	0.019	0.259
	12	0.137	0.048	0.130	0.297	0.03	0.050	0.54	0.009	0.137
8604	12	0.161	0.019	0.156	0.440	0.02	0.016	3.22	0.009	0.138
	12	0.149	0.016	0.149	-0.168	0.01	0.018	0.02	-0.001	0.011
8605	10	0.173	0.017	0.168	1.032	0.01	0.015	2.25	0.012	0.300
	11	0.163	0.019	0.161	0.460	0.00	0.021	0.17	0.002	0.038
8606	12	0.154	0.040	0.141	0.210	0.12	0.031	4.33	0.026	0.214
	12	0.153	0.026	0.143	0.087	0.41	0.026	0.63	0.036	0.131
8607	12	0.146	0.016	0.142	0.332	0.02	0.013	3.42	0.007	0.109
	11	0.144	0.015	0.141	0.201	0.03	0.016	0.96	0.006	0.061
8608	10	0.119	0.011	0.116	1.031	0.00	0.011	1.60	0.005	0.269
	10	0.130	0.010	0.129	-0.345	0.01	0.009	1.93	-0.002	0.035
8609	11	0.133	0.013	0.133	-0.207	0.01	0.015	0.09	-0.001	0.016
	11	0.138	0.018	0.137	0.250	0.01	0.020	0.22	0.003	0.038
8610	11	0.118	0.012	0.114	0.255	0.03	0.006	13.97**	0.007	0.114
	10	0.123	0.015	0.118	0.156	0.06	0.011	4.15	0.010	0.096
8611	12	0.103	0.009	0.101	0.411	0.01	0.006	7.26**	0.004	0.145
	11	0.115	0.013	0.114	0.182	0.01	0.014	0.42	0.002	0.034
8612	12	0.112	0.016	0.110	0.426	0.01	0.015	1.74	0.003	0.098
	11	0.114	0.020	0.113	0.363	0.01	0.021	0.39	0.002	0.049
8701	10	0.108	0.019	0.103	0.440	0.02	0.017	2.19	0.008	0.248
	10	0.110	0.024	0.105	0.116	0.10	0.025	0.89	0.012	0.107

FIGURE 1: Expected Percentage Jumps per Year Implicit in Short-Term Options on DM Futures

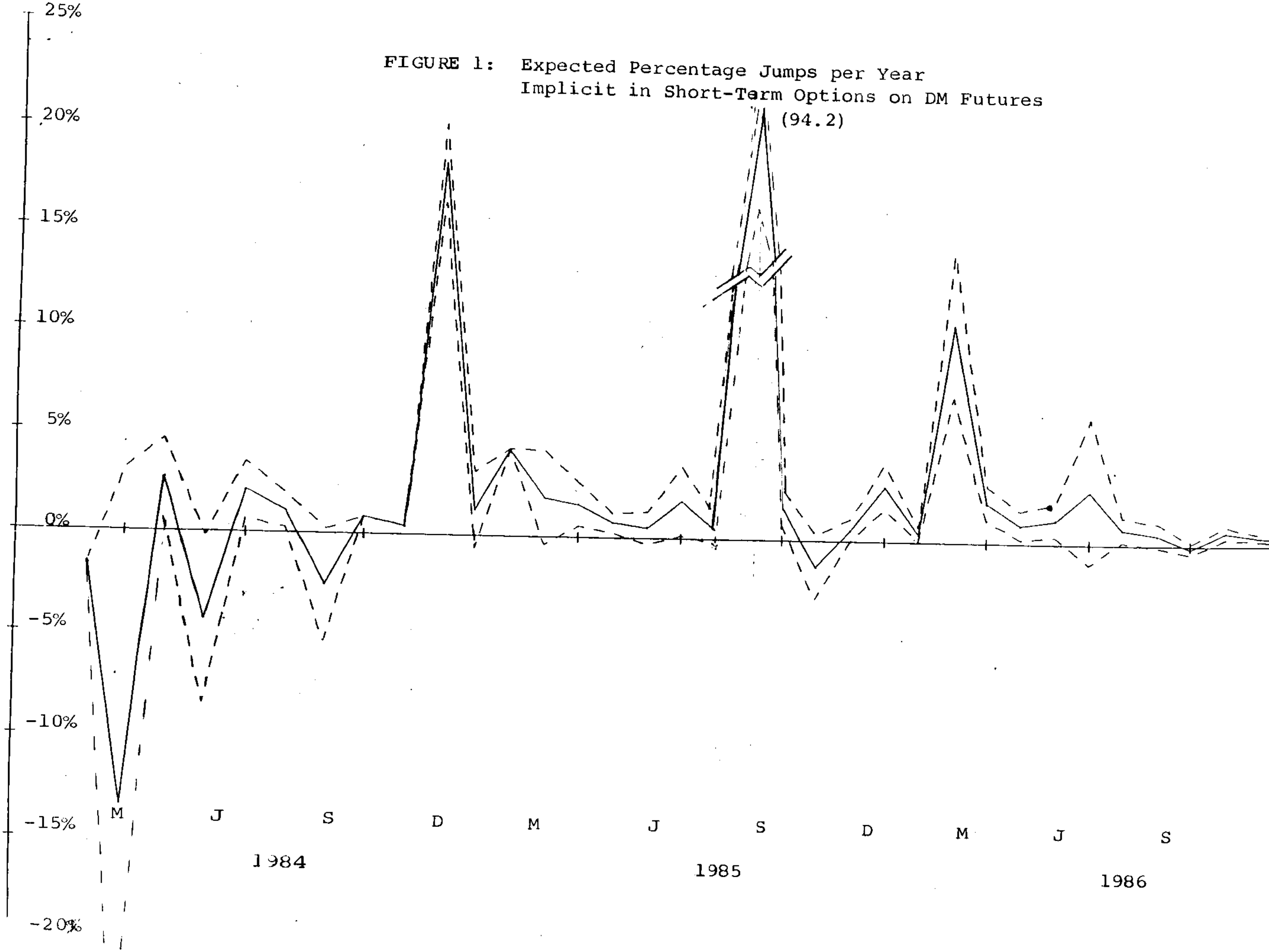

FIGURE 2 : Expected Percentage Jumps per Year
Implicit in Medium-Term Options on DM Futures

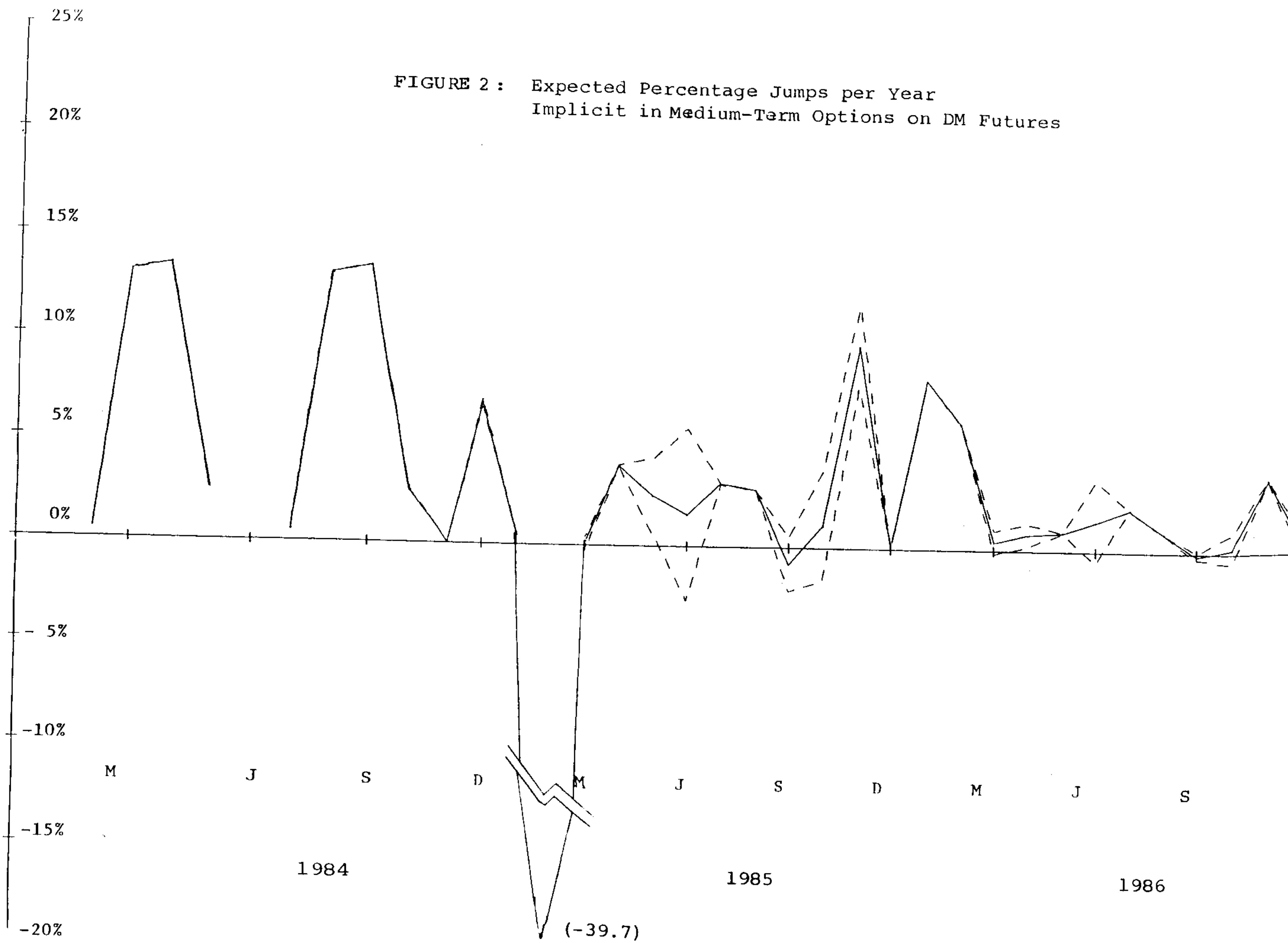

FIGURE 3: Percentage of Implicit Conditional Variance Attributed to Jump Risk