ASYMPTOTIC ARBITRAGE OPPORTUNITIES IN VARIOUS MODES OF CONVERGENCE AND THE APPROXIMATE LINEAR PRICING RELATION IN ASSET MARKET

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Abstract

The three basic elements of the arbitrage pricing theory (APT) are the linear factor structure of asset returns, the nonexistence of asymptotic arbitrage opportunities, and the approximate linear pricing relation. This paper explores the necessary and sufficient conditions of the approximate linear pricing relation by systematically examining the associations among these three elements. The generalization evolves around various modes of stochastic convergence that characterize the nature of asymptotic arbitrage opportunities and around assorted assumptions about the idiosyncratic risks in the linear factor structure. This study is exhaustive in the sense that all modes of convergence are used in defining the asymptotic arbitrage opportunities. This study also allows researchers to know the trade-off between the linear factor structure assumption and the no-asymptotic-arbitrage condition while keeping the approximate linear relation intact . Our generalization of the APT may enhance the understanding about the arbitrage pricing mechanism and the stochastic nature of the underlying economy.

I. Introduction

The Arbitrage Pricing Theory (APT) has generated extensive research interests since it was first introduced by Ross in 1976. The APT is a one period model in which investors share the same belief that the stochastic properties of capital assets returns are consistent with a linear factor structure. If there are no asymptotic arbitrage opportunities, then the expected returns on these capital assets are approximately linearly related to the factor loadings. The three basic elements of the APT are the linear factor structure of asset return generating process, the nonexistence of asymptotic arbitrage opportunities, and the approximate linear pricing relation (hereafter, ALPR). The purpose of this paper is to investigate the associations among these three elements in the construction of the APT.

In this paper, the APT is generalized in two major aspects. First, the idiosyncratic risks of the linear factor structure are assumed in the literature to have finite variances and be either orthogonal or weakly uncorrelated.[1] In this study, we relax the finite variance and orthogonality (or weak uncorrelatedness) assumptions by considering more general structures for the idiosyncratic risk. The lacunary system and the convergence system are two examples. Second, the no-asymptotic-arbitrage condition is usually defined in the literature as the nonexistence of portfolios which cost nothing, and

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[1] Ross (1976) and Huberman (1982) assumed the idiosyncratic risks to be orthogonal. Ingersoll (1984) assumed the variance-covariance matrix of the idiosyncratic risks to be uniformly bounded. Chamberlain and Rothschild (1983) assumed that only K eigenvalues of the returns' variance-covariance matrix become unbounded. Stambaugh (1983) assumed that it is possible to decompose the returns' variance-covariance matrix as $\Sigma = \mathbf{BB'} + \mathbf{D} - \mathbf{A}$ where $\mathbf{B}$ is the matrix of factor loadings, $\mathbf{D}$ is a diagonal matrix with bounded elements and $\mathbf{A}$ is nonnegative definite. It can be shown all these three assumptions are identical. And this condition will be called "weak uncorrelatedness" hereafter.

have positive returns for sure in the limit as the variances of their random returns converges to zero. In this paper, the no-asymptotic-arbitrage condition is defined in terms of (1) convergence in p^{th} mean, or (2) almost sure convergence, or (3) convergence in probability.[2] The relationship between modes of convergence and assumptions about the idiosyncratic risks in the linear factor structure are studied. Specifically, This research will allow people to understand the trade-off between the linear factor structure assumptions and the no-asymptotic-arbitrage condition while keeping the ALPR intact. Namely, we show how the match of various definitions of no-asymptotic-arbitrage condition and assorted assumptions about the idiosyncratic risks can give rise to the same conclusion: a finite bound on the Euclidean norm of the pricing error vector.

Table 1 offers a preview to our analysis.[3] The propositions listed in Table 1 are the fundamental results to be shown in this paper. All blank entries will be filled as corollaries. This paper is a synthesis as well as an extension and complement of two other works about the generality of the APT. Wang and Lee (1988) studied the APT under linear factor structures that allow

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[2] Lee and Wang (1988) show that not all definitions of the no-asymptotic-arbitrage condition imply asset market equilibrium in the sense that, even when one rules out the possibility of asymptotic arbitrage in terms of convergence in quadratic mean, other types of asymptotic arbitrage opportunities may exist. For some investors, these become their free feast. Hence, knowing the relationship between investors' preferences and definitions of the no-asymptotic-arbitrage condition is important in understanding the generality of the APT. In the context of our discussion, convergence in probability is equivalent to convergence in distribution since we are dealing with stochastic convergence to a real constant (see Laha and Rohatgi (1979).

[3] Table 1 provides the basic framework of this paper. It shows that our analysis evolves around the two basic elements of the APT, namely, the idiosyncratic risks of the linear factor structure and the no-asymptotic-arbitrage condition. A complete recapitulation of our general theory is given in Tables 2 and 3 in the final section.

Table 1 Linear Factor Structure, Asymptotic Arbitrage, And Linear Pricing Relation: A Preview

A. The Linear Factor Structure Under Which No Asymptotic Arbitrage Implies An ALPR

	Linear Factor Structure			
No Asymptotic Arbitrage Condition	BLFS & $E(e_i e_j)=0$ $Var(e_i)<\infty$ for all i	BLFS & $E(ee')=\Omega$ $\lVert\Omega\rVert<\infty$	L-type factor structure	C-type factor structure
$\mathbf{w}'\mathbf{1}\text{-->}0$ & $Var(\mathbf{w}'x)\text{-->}0$ ==> $\mathbf{w}'\mathbf{a}\text{-->}0$	$\mathbf{a}\approx\mathbf{B}^*c$ Huberman (1982)	$\mathbf{a}\approx\mathbf{B}^*c$ Ingersoll (1984) Chamberlain-Rothschild (1983)		
$\mathbf{w}'\mathbf{1}\text{-->}0$ & $w'x-w'a\text{-}L_p\text{->}0$ ==>$\mathbf{w}'\mathbf{a}\text{->}0$			$\mathbf{a}\approx\mathbf{B}^*c$ Prop.2	
$\mathbf{w}'\mathbf{1}\text{-->}0$ & $\mathbf{w}'\mathbf{x}-\mathbf{w}'\mathbf{a}\text{-->}0$ a.s. ==>$\mathbf{w}'\mathbf{a}\text{->}0$				$\mathbf{a}\approx\mathbf{B}^*c$ Prop.6

B. The Linear factor Structure Under Which The ALPR implies No Asymptotic Arbitrage

	Linear Factor Structure			
No Asymptotic Arbitrage Condition	BLFS & $\min_i Var(e_i)>0$ $E(e_i e_j) = 0$	BLFS & $E(ee')=\Omega$ $0<\lVert\Omega\rVert$	B-type factor structure	C-type factor structure & $\inf_i \lvert e_i\rvert>0$
$\mathbf{w}'\mathbf{1}\text{-->}0$ & $Var(\mathbf{w}'x)\text{-->}0$ ==> $\mathbf{w}'\mathbf{a}\text{-->}0$	$\mathbf{a}\approx\mathbf{B}^*\mathbf{c}$ Prop. 1	$\mathbf{a}\approx\mathbf{B}^*c$ Prop. 3 (p=2)		
$\mathbf{w}'\mathbf{1}\text{-->}0$ & $w'x-w'a\ \text{-}L_p\text{->}0$ ==>$\mathbf{w}'\mathbf{a}\text{-->}0$			$\mathbf{a}\approx\mathbf{B}^*c$ Prop. 3	
$\mathbf{w}'\mathbf{1}\text{-->}0$ & $\mathbf{w}'\mathbf{x}-\mathbf{w}'\mathbf{a}\text{-->}0$ a.s ==>$\mathbf{w}'\mathbf{a}\text{-->}0$				$\mathbf{a}\approx\mathbf{B}^*\mathbf{c}$ Prop.C3 x_i are indept.

Notes: BLFS: basic linear factor structure; $\mathbf{w}$: vector of portfolio weights; $\mathbf{x}$: vector of realized asset returns; $\mathbf{a}$: vector of expected asset returns; $\mathbf{B}$: matrix of factor loadings, $\mathbf{B}^* = (\mathbf{1}, \mathbf{B})$; $\text{-}L_p\text{->}$: convergence in p^{th} mean; --> a.s.: almost sure convergence; $\mathbf{a}\approx\mathbf{B}^*\mathbf{c}$: ALPR; ==>: imply.

the nonexistence of the second moments of the idiosyncratic risks and/or weak dependence among them. There, the no-asymptotic-arbitrage condition is defined in terms of convergence in p^{th} mean. Lee and Wang (1988) discussed the no-asymptotic-arbitrage condition in terms of the convergence in probability. Our focus in this paper is on the no-asymptotic-arbitrage condition defined in terms of almost sure convergence and on the required assumption about the linear factor structure in order to get the approximate linear pricing relation.

The concept of asymptotic-arbitrage in terms of the almost sure convergence is important for three reasons. First, portfolio returns (defined as the period-ending value of investment divided by the beginning value) are not like assets returns which are nonnegative under the doctrine of limited liability. Moreover, the returns of a portfolio could behave strangely if weights are deliberately chosen. Thus, it is quite possible that the return of a portfolio converges almost surely but does not converge in p^{th} mean, or it converges in p^{th} mean but not almost surely. Third, as shown in Lee and Wang (1988), for a certain type of utility functions, only some particular definition of no-asymptotic-arbitrage condition is consistent with the market equilibrium. For example, a portfolio that is not an arbitrage opportunity in the sense of convergence in quadratic mean may be an arbitrage opportunity in the sense of almost sure convergence. To understand fully the generality of the APT, it is important to investigate the APT under this almost sure mode of convergence.

As indicated in Table 1, the discussion of the APT in the literature is limited to the sufficient condition of the ALPR. In this paper, we derive a necessary as well as the sufficient condition for the ALPR for each combination of linear factor structure and no-asymptotic-arbitrage condition. This comprehensive investigation has two empirical implications.[4] First, if the linear pricing relation is not supported by the data, then we could assert that arbitrage opportunities do exist or that the assumption of linear K-factor structure is not tenable according to the data used in testing the APT. If it is the "arbitrage opportunity" that ruins the ALPR, we might infer that transaction costs are so large as to prevent the utility-maximizer from taking these advantages. Secondly, suppose that the ALPR holds, from the if-and-only-if relation we can say there are no arbitrage opportunities in the economy. In addition, the ALPR can be tested by examining the existence of arbitrage opportunities. Given the data available, if we can find just one arbitrage portfolio with zero cost and zero absolute central pth moment (e.g. variance) having a positive mean, then the hypothesis of linear pricing may be rejected.

In this paper, we assume the number of the assets traded in the economy are countably infinite. The price system is normalized by letting each asset cost one dollar. The assets are arranged in a sequence. We always examine the effect on various objects such as arbitrage opportunities and pricing relation as the number of assets, n, increases to infinity.

The rest of paper is organized as follows. Section II introduces the basic Model which is the primary structure of the following discussion. Section III discusses the APT when the no-asymptotic-arbitrage condition is considered in terms of convergence in p^{th} mean instead of convergence in quadratic mean. First, the concept of martingale difference system is introduced. It is shown that if the linear factor structure holds and if the sequence of idiosyncratic

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[4] Testability of the APT has always been a controversial issue in the literature (Shanken, 1982; Dybvig and Ross, 1985). Strictly speaking, it is not testable since it is impossible to obtain estimates of an infinite number assets.

risks is a martingale difference system, then the ALPR holds. The "nonexistence of asymptotic arbitrage in the sense of convergence in p^{th} mean" (hereafter, NACPM) is proved to be a necessary and sufficient condition of the ALPR. Section IV derives the ALPR from the "nonexistence of asymptotic arbitrage in the sense of almost sure convergence" (hereafter, NAASC) under the "C-type factor structure". As in Section III, a special yet interesting case is introduced first for exposition purpose. Specifically, we introduce the concept of weakly-multiplictive type dependence restriction. Then we provide a more general framework (C-type factor structure) where the sequence of the idiosyncratic risks is assumed to be a convergence system. Section V concludes this paper.

II. The Basic Model

The APT is composed of three basic elements, namely, the linear factor structure of asset return generating process, the no-asymptotic-arbitrage condition, and the approximate linear pricing relation. In this section we set up the fundamental framework in our analysis. The three elements of the basic APT model are introduced in a natural sequence.

II.A. The Basic Linear Factor Structure

Definition: The asset return, **x**, is said to satisfy the **basic linear factor structure** if the following conditions hold.

$$\mathbf{x} = \mathbf{a} + \mathbf{B}\mathbf{f} + \mathbf{e} , \qquad (1a)^{5}$$

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5 Variables **x**, **f**, and **e** are random and the sequences **x**, **a**, **B**, **f**, and **e** all depend on n. To avoid messy notations, unless the omission can cause confusion, we do not label the randomness, neither do we index the order of

$E(\mathbf{e}) = \mathbf{0}$ (1b)

$E(\mathbf{f}) = 0$ (1c)

The vectors $\mathbf{x}$, $\mathbf{a}$, and $\mathbf{e}$ are each n x 1 in dimension and represent the realized returns, expected returns and (nonobservable) "residual" portions of the returns, respectively. The e_i, which is often called idiosyncratic risk or residual in the literature, measures the uncertainty unexplained by the common factors. The expected return of i^{th} asset, a_i, is assumed to be bounded for all i. $\mathbf{f}$ is a K x 1 vector of nonobservable values of the common factors. The second moment of f_k is assumed to exist for all k. $\mathbf{B}$ is the n x K matrix of bounded factor loadings, i.e., $|b_{ik}| < \infty$ for all i and k. We do not, right now, impose any assumption on the stochastic properties of the idiosyncratic risks, e_i. The assumptions on the idiosyncratic risks will be added later in differentiating various types of factor structures introduced. Assumption (1b) is always possible through appropriate choice of expected returns, $\mathbf{a}$. Assumptions (1c) can be met by subtracting any factor means. Thus both are innocuous.

Besides linearity assumption, the most important assumptions about the linear factor structure in the literature are (1) finiteness of the second moments and (2) cross-sectional orthogonality (or weak correlatedness) among the idiosyncratic risks. These two assumptions will be relaxed in a variety of ways in this paper.

II.B. No-Asymptotic-Arbitrage Condition

Definition: **No-asymptotic-arbitrage** condition holds if, for any $\mathbf{w} \in \mathbf{R}^n$, whenever

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sequence.

(i). cost of portfolio, $\mathbf{w}$, converges to zero: $\mathbf{w}'\mathbf{1} \to 0$, and

(ii). random return of $\mathbf{w}$ converges in some stochastic mode to its expected value: $\mathbf{w}'\mathbf{x} - \mathbf{w}'\mathbf{a} \overset{?}{\to} 0$,

then, (iii). the expected value of portfolio return converges to zero:

$\mathbf{w}'\mathbf{a} \to 0$,

where $\overset{?}{\to} 0$ indicates the convergence to zero in a certain stochastic mode, $\mathbf{w}$ is an n-element vector of portfolio weights, w_i, and $\mathbf{1}$ is a vector with n 1s.[6]

II.C. Approximate Linear Pricing Relation

Definition: The **approximate linear pricing relation in terms of Euclidean norm (ALPR)** indicates that the expected returns, $\mathbf{a}$, is a linear combination of the factor loadings, $\mathbf{b}_{.k}$, plus an error term, $\mathbf{v}$. And it gives expected returns a with a mean squared error of zero. In other words, for n = 1, 2, ... , there exist $\mathbf{c}' = (c_0, c_1, c_2, \ldots , c_K)$ such that

$$\mathbf{a} = \mathbf{B}^*\mathbf{c} + \mathbf{v}, \text{ and} \quad (3a)$$

$$\lim_{n \to \infty}(1/n)\|\mathbf{v}\|^2 = 0, \text{ or somewhat more strongly,} \quad (3b)$$

$$\lim_{n \to \infty}\|\mathbf{v}\| < \infty, \quad (3c)$$

where $B^* = (1 \;\; B)$; all a, B, and c depend on n, and $\| \; \| \equiv$ Euclidean norm, i.e., $\|v\| = (\Sigma_{i=1}^{n} v_i^2)^{1/2}$.

Note that the way we express the ALPR is in terms of the Euclidean norm of the pricing error vector. While Wang and Lee (1988) examines several concepts of

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[6] Most researches on the APT start with a definition of "arbitrage". However, the APLR is in fact a result of "no-arbitrage". The no-asymptotic-arbitrage condition is implied by the usual definition of arbitrage through a simple logic that not(A and B) ≡ (A ==> not B).

bound on norm of pricing error vector, this paper is concentrated on the bound of the Euclidean norm type, which is commonly used in the literature.

III. The No-Asymptotic-Arbitrage Condition in the Sense of Convergence in P^{th} Mean (NACPM) and the ALPR

III.B The M-Type Factor Structure

In this subsection, we introduce the concept of martingale difference system which is often used by statistician and mathematician in modeling a stochastic process. If a sequence of random variables, $\{x_i\}$, is such that $E(x_i \| x_j) = 0$ for all $j < i$, then $\{x_i\}$ is called a martingale difference system.[7] Instead of requiring the idiosyncratic risks, e_i, to be uncorrelated or weakly uncorrelated as in the literature, we will assume the sequence of e_i to be a martingale difference system throughout this subsection. This assumption simply means that when we add a new asset into the economy, its idiosyncratic risk has a mean of zero conditioned on the idiosyncratic risks of the original assets.

Note that this is weaker than the assumption of independence but stronger than the assumption of zero covariance as in Ross (1976) and Huberman (1982). However, by adopting such a stronger assumption about the idiosyncratic risks, the no-asymptotic-arbitrage condition stated in Huberman (1982) can be

[7] The name suggests that it is generated from the first differences of a martingale process. For detailed discussion of the martingale system, see Chow and Teicher (1978). With some additional assumptions, a martingale difference system is a special case of a lacunary system which will be examined later.

generalized.[8] Specifically, if we further assume that $\sup_i E|e_i|^p < \infty$ for some $p \geq 2$, the no-asymptotic-arbitrage condition can be defined in terms of convergence in pth mean.[9] In this case we show that the no-arbitrage condition is a sufficient condition for the ALPR. In addition, if we assume that $E(\mathbf{ff}')$ $= I_K$ (where I_K is a K x K identity matrix), **e** and **f** are independent, and $\inf_i E|e_i| > 0$ for all i, then the no-arbitrage condition is a necessary condition for the ALPR.[10]

Definition: The linear factor structure is a **M-type factor structure** if the asset return vector, **x**, satisfies the requirement of a basic linear factor structure, i.e, condition (1); and $\{e_i\}$ is a martingale system with $\sup_i E|e_i|^p$ $< \infty$ for all i and for some $p \geq 2$. ∎

Definition: The no-asymptotic-arbitrage condition defined in terms of convergence in p^{th} mean (**NACPM**) satisfies the requirement that

when $\mathbf{w}'\mathbf{1} \to 0$, and $\mathbf{w}'\mathbf{x} - \mathbf{w}'\mathbf{a} \xrightarrow{L_p} 0$, for some $p \geq 1$,

then $\mathbf{w}'\mathbf{a} \to 0$,

where $\xrightarrow{L_p} 0$ denotes the convergence to zero in p^{th} mean. ∎

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[8] The assumption about the distributions of the factors in the linear factor structure can be relaxed too. In other words, if we are concerned only with showing the no-arbitrage condition as a sufficient condition of the linear pricing relation, then, we do not need to assume the existence of the second moments of the factors.

[9] It can be proved (Rohatgi, 1976; Laha and Rohatgi, 1979) that
(1) $E|X|^p < \infty \iff |x|^{p-1} P\{|X|>x\}$ is integerable over $(0,\infty)$.
(2) $E|X|^p < \infty \iff \sum_{n=1}^{\infty} P\{|X|>n^{1/p}\} < \infty$.
(3) $E|X|^p < \infty \implies n^p P\{|X|>n\} \to 0$.
(4) $n^{p+\alpha} P\{|X|>n\} \to 0 \implies E|X|^p < \infty$ for some $\alpha > 0$.

[10] The assumption that $\inf_i E\|e_i\| > 0$ for all i is weaker than the common assumption in the literature, namely, $Var(e_i) > 0$ for all i.

Proposition 1: Given a M-type factor structure, the NACPM implies the ALPR.

Before proving this proposition, we introduce the following lemma.

Lemma 1: Let $\{e_i\}$ be a sequence such that $E(e_i|e_j) = 0$ for all $j < i$, and $\sup_i E|e_i|^p < \infty$ for some $p \geq 2$, then there exists a positive constant K_p such that for any sequence of real constant, d_i,

$$E|\Sigma_{i=m}^{n} d_i e_i|^p \leq K_p(\Sigma_{i=m}^{n} d_i^2)^{p/2}.$$

Proof: See Appendix A. Q.E.D.

Proof of Proposition 1: Use Lemma 1 as above and Theorem 1 in Wang and Lee (1988). Q.E.D.

Definition: The linear factor structure is a **M'-type factor structure** if, given a M-type factor structure, we have $E(\mathbf{ff}') = \mathbf{I}_K$ where $\mathbf{I}_K$ is a K x K identity matrix; **e** and **f** are independent; and $\inf_i E|e_i| > 0$ for all i. *

Proposition 2: Given a M'-type factor structure, the NACPM is equivalent to the ALPR.[11]

Remark: In Proposition 2, we have to assume the existence of the second moments of the factors, f_k.

Lemma 2: Let $P \geq 2$ and let $\{e_i\}$ be a sequence such that

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[11] If we maintain the same assumptions as in Proposition 2 except $\inf_i|e_i| > 0$, then it is easy to prove that $\lim_{n\to\infty}\|\mathbf{a} - \mathbf{B}^{*}\mathbf{c}\| = 0$ implies the nonexistence of asymptotic arbitrage opportunity, i.e.,

$\mathbf{w}'\mathbf{1} \to 0$ and $\mathbf{w}'\mathbf{x} - \mathbf{w}'\mathbf{a} \xrightarrow{L_p} 0 \implies \mathbf{w}'\mathbf{a} \to 0$.

$E(e_i|e_j) = 0$ for all $j < i$, and $\inf_i E|e_i| > 0$,

then there exists a positive constant $H_p > 0$ such that, for all constant d_i,

$$E|\Sigma_{i=m}^{n} d_i e_i|^p \geq H_p(\Sigma_{i=m}^{n} d_i^2)^{p/2} \text{ for all } n \geq m.$$

Proof: See Appendix A. Q.E.D.

Proof of Proposition 2: Use Lemma 2 as above and Theorem 2 in Wang and Lee (1988). Q.E.D.

Propositions 1 and 2 are, respectively, special cases of Propositions 3 and 4 in next section. This is easily seen, since if the sequence of e_i is a martingale difference system and satisfies $\sup_i E|e_i|^p < \infty$ for some $p \geq 2$ and for all i, then $\{e_i\}$ is a lacunary systems of order p which we now turn to.

III.C. The L-Type Factor Structure

The generalization of the APT in the context where the sequence of idiosyncratic risks is a lacunary system of order p for some $p \geq 1$ is discussed in detail by Wang and Lee (1988). We summarize the relevant results here for the sake of completeness. All proofs and explanations are omitted.

Definition: Given $p > 0$, a sequence of real-valued random variables $\{e_i\}$ is called a **lacunary system** of order p, or an S_p system, if there exist a positive constant K_p such that for any sequence of real constants d_i,

$$E|\Sigma_{i=m}^{n} d_i e_i|^p \leq K_p(\Sigma_{i=m}^{n} d_i^2)^{p/2} \qquad \text{for all } n \geq m.$$

If the system $\{e_i\}$ is an S_p system for every $p > 2$, then it is called an S_∞ system.

■

Definition: The linear factor structure is a **L-type factor structure** if the asset return vector, **x**, satisfies the requirement of a basic linear factor structure, i.e, condition (1); and $\{e_i\}$ is a lacunary system of order p for some $p \geq 1$. ▪

Proposition 3: Given a L-type factor structure, the NACPM implies the ALPR.

III.D. The B-Type Factor Structure

Now let's turn to the necessary condition.

Definition: A sequence of random variables $\{e_i\}$ is said to satisfy the **Bessel inequality** if there exists $M > 0$ such that for all constants d_i

$$E|\Sigma_{i=m}^{n} d_i e_i|^2 \geq M\ \Sigma_{i=m}^{n} d_i^2 \qquad \text{for all } n \geq m.$$ ▪

Definition: The sequence of random variables $\{e_i\}$ which is a lacunary system of order $p \geq 2$ and satisfies the Bassel inequality is called a **Banach System** (Banach, 1930). ▪

Definition: The linear factor structure is a **B-type factor structure** if the asset return vector, **x**, satisfies the requirement of a basic linear factor structure, i.e, condition (1); $E(\mathbf{ff}') = \mathbf{I}_k$; **e** and **f** are independent; and $\{e_i\}$ is a Banach system. ▪

Proposition 4: Given a B-type factor structure, the NACPM and ALPR are equivalent.

IV. The Almost Sure No-Asymptotic arbitrage Condition and the APT

The lacunary factor structure as well as the NACPM allow us to relax the assumptions on the finiteness of second moments of the idiosyncratic risks and the orthogonality (or weak uncorrelatedness) among them. To establish the validity of the ALPR given a L-type factor structure, we need to rely on the concept of convergence in p^{th} mean. As discussed in the introduction, it is useful to examine the no-arbitrage condition under other mode of convergence. A natural question arises from this exercise is the "cost" of using different mode of convergence in defining the no-arbitrage condition. How much restriction need to be imposed on the linear factor structure so as to match the no-asymptotic-arbitrage condition in terms of almost sure convergence (NAASC) without losing the ALPR? In this section, more types of factor structure are introduced, together with the NAASC, to derive the same ALPR. These structures do not impose strong assumptions on the orthogonality or the existence of second moments.

IV.A. W-Type Factor Structure

In this subsection we first introduce the concept of "weakly multiplictive type dependence restriction". The "weak multiplictiveness" refers to any form of restriction on the product moment $E\{e_{i(1)}e_{i(2)}\ldots e_{i(p)}\}$ of order p for all $1 \le i(1) < i(2) < \ldots < i(p)$. Longnecker and Serfling (1978) suggested the conditions for three different weakly multiplictive types. The first two can be characterized as orthogonality related dependence restrictions. The third one is similar to the first two, except it is motivated by examining the structure of the product moments of a Gaussian sequence.

Definition: A sequence of random variables $\{e_i\}$ satisfies **Condition A_p** (weakly multiplictive dependence restriction of type A_p) with respect to an even integer p if $E(e_i{}^p) < \infty$, for all i, and there exists a symmetric function f with p - 1 arguments such that

$$|E(e_{i(1)} \ldots e_{i(p)})| \le f\{i(2)-i(1),\ i(3)-i(2),\ \ldots\ ,i(p)-i(p-1)\}\pi_{j=1}^{p}[E(e_{i(j)}{}^{p})]^{1/p}$$

for all $1 \le i(1) < \ldots < i(p)$,

and for $\Sigma_{k=1}^{\infty}\Sigma_{j(1)=1}^{k} \ldots \Sigma_{j(p-2)=1}^{k} f\{j(1),\ \ldots,\ j(p-2),\ k\} < \infty$. ▪

Definition: A sequence of random variables $\{e_i\}$ satisfies **Condition B_p** (weakly multiplictive dependence restriction of type B_p) with respect to an even integer p, if $E(e_i{}^p) < \infty$, for all i, and there exists a symmetric function f of p/2 arguments such that

$$|E(e_{i(1)} \ldots E(e_{i(p)})| \le f\{i(2)-i(1),\ i(4)-i(3), \ldots ,i(p)-i(p-1)\}\pi_{j=1}^{p}(E(e_{i(j)}{}^{p})^{1/p}$$

for all $1 \le i(1) < \ldots < i(p)$,

and for $\Sigma_{k=1}^{\infty}\Sigma_{j(1)=1}^{k} \ldots \Sigma_{j(p/2\ -\ 1)=1}^{k} f\{j(1),\ \ldots,\ j(p/2 - 1),k\} < \infty$. ▪

Conditions A_p and B_p are two types of orthogonality-related dependence restrictions. They are exactly the same when p = 2. They can also be regarded as a simple relaxation of orthogonality. Longnecker and Serfling (1978) showed that the notion of quasi-orthogonality treated in Kac, Salem and Zygmund (1948) is a special case of Conditions A_2 and B_2.

Example 1: Suppose that $x_i = a_i + b_i f + e_i$, i = 1, 2, ... , where the e_i are "almost" uncorrelated: $cov(e_i,e_j) = 0$ if $|i-j| > 1$. Then e_i satisfy both conditions A_p and B_p when p = 2 (Chamberlain and Rothschild, 1983). ▪▪

Example 2: Given the same linear one factor structure as in Example 1, suppose the correlation matrix of the e, R_n, is

$$\begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\ \alpha & 1 & \alpha & \cdots & \alpha^{n-2} \\ \alpha^2 & \alpha & 1 & \cdots & \alpha^{n-3} \\ . & . & . & \cdots & . \\ \alpha^{n-1} & \alpha^{n-2} & \alpha^{n-3} & \cdots & 1 \end{bmatrix}$$

Then e_i also satisfy Conditions A_2 and B_2. ▪▪

Conditions A_p and B_p, when $p \geq 4$, are substantially more powerful than orthogonality. Two specialized forms of Condition B_p are presented in Appendix B.

The third weakly multiplictive dependence restriction is closely related to Condition A_p and B_p, but it deal with the fourth order product moment of a Guassian series. Anderson (1971:39) showed that if $e_{i(1)}$, $e_{i(2)}$, $e_{i(3)}$, and $E_{i(4)}\}$ are multivariate normally distributed with zero mean vector, the fourth order product moment is

$$E\{e_{i(1)}e_{i(2)}e_{i(3)}e_{i(4)}\}$$
$$= E\{e_{i(1)}e_{i(2)}\}E\{e_{i(3)}e_{i(4)}\} + E\{e_{i(1)}e_{i(3)}\}E\{e_{i(2)}e_{i(4)}\}$$
$$+ E\{e_{i(1)}e_{i(4)}\}E\{e_{i(2)}e_{i(3)}\}.$$

Moreover, if after rearranging the series of asset returns we can get $E(e_ie_j) = R(j-i))$, and if $|R(k)|$ is nonincreasing, then it can be shown (Longnecker and Serfling, 1978) that, for $i(1) \leq i(2) \leq i(3) \leq i(4)$,

$|E\{e_{i(1)}e_{i(2)}e_{i(3)}e_{i(4)}\}| \le$

$2|R(i(2)-i(1))R(i(4)-i(1))| + \min[|R(i(2)-i(1)|,|R(i(4)-i(3)|]|R(i(3)-i(2)|.$

The first term on the right hand side of the above inequality is of the form of Condition $B2_4$ (see Appendix B). The second term motivates the following definition.

Definition: A sequence $\{e_i\}$ satisfies **Condition C_p** (weakly multiplictive dependence restriction of type C_p) with respect to an even integer p if $E(e_i^p) < \infty$, for all i, and there exists a function f of p/2 - 1 arguments such that

$|E(e_{i(1)}\ldots e_{i(p)})| \le$

$\min\{f(i(2)-i(1), f(i(p)-i(p-1)\} g\{i(3)-i(2), i(5)-i(4), \ldots, i(p-1)-i(p-2)\}\cdot \pi_{j=1}^{p}(E(e_{i(j)}^p)^{1/p}$

for all $1 \le i(1) < \ldots < i(p)$, if $\Sigma_{j=1}^{\infty} f(j) < \infty$, and if $\Sigma_c\, g\{j(1), \ldots, j(p/2-1)\} < \infty$, where the subscript $_c$ denotes the set of all (p/2 - 1)-tuples (j(1), ..., j(p/2-1)) with $1 \le j(v) \le j(m)$ for $v \ne m$, $1 \le j(m) < \infty$, and m = 1, ..., p/2 - 1. ∎

Lemma 3: Let the sequence $\{e_i\}$ satisfy, for an even integer p > 2, either Condition A_p, Condition B_p, or Condition C_p. Then the condition $\Sigma_{i=1}^{\infty} b_i^2 d_i^2 < \infty$ implies the almost sure convergence of $\Sigma_{i=1}^{\infty} d_i e_i$, where $b_i \equiv E(e_i^p)$.

Proof: See Theorem 5.2 of Longnecker and Serfling (1978). Q.E.D.

Remark: This lemma is an extension of the result of Komlós (1972) for multiplictive sequences to weakly multiplictive sequences (see Wang and Lee, 1988).

Definition: The linear factor structure is a **W-type factor structure** if the asset return vector, $\mathbf{x}$, satisfies the requirement of a basic linear factor structure, i.e, condition (1); and $\{e_i\}$ satisfies, for an even integer $p \geq 2$, either Condition A_p, Condition B_p, or Condition C_p, (Note that $E|e_i|^p < \infty$ for all i). ∎

Definition: The no-asymptotic-arbitrage condition defined in terms of almost sure convergence (**NAASC**) satisfies the requirement that

when $\mathbf{w}'\mathbf{1} \dashrightarrow 0$, and $\mathbf{w}'\mathbf{x} - \mathbf{w}'\mathbf{a} \dashrightarrow 0$ a.s.,

then $\mathbf{w}'\mathbf{a} \dashrightarrow 0$,

where --> 0 a.s. denotes almost sure convergence to zero. ∎

Proposition 5: Given a W-type factor structure, the NAASC implies the ALPR.

Proof: Projecting the vector **a** onto the space spanned by **B** and the vector **1**, we have:[12]

$$\mathbf{a} = \mathbf{B}^*\mathbf{c} + \mathbf{v}, \text{ where, } \mathbf{c} \in R^{k+1} \text{ and } \mathbf{B}^{*\prime}\mathbf{v} = \mathbf{0}.$$

Consider an arbitrage portfolio, $\mathbf{w}$, such that

$$\mathbf{w} = \frac{\mathbf{v}}{\|\mathbf{v}\|^r},$$

where, $1 < r \leq 2$. The return of portfolio $\mathbf{w}$ is

$$\frac{\mathbf{v}'\mathbf{x}}{\|\mathbf{v}\|^r} = \frac{1}{\|\mathbf{v}\|^r}(\mathbf{v}'\mathbf{a} + \mathbf{v}'\mathbf{B}\mathbf{f} + \mathbf{v}'\mathbf{e}) = \frac{}{\|\mathbf{v}\|^r}(\mathbf{v}'\mathbf{a} + \mathbf{v}'\mathbf{e}).$$

Therefore, the expected return of portfolio $\mathbf{w}$ is

$$\frac{\mathbf{v}'\mathbf{a}}{\|\mathbf{v}\|^r} = \frac{1}{\|\mathbf{v}\|^r}(\mathbf{v}'\mathbf{B}^*\mathbf{c} + \mathbf{v}'\mathbf{v}) = \frac{\mathbf{v}'\mathbf{v}}{\|\mathbf{v}\|^r} = \|\mathbf{v}\|^{2-r}.$$

Consider $u \equiv v/\|v\|$, then $\Sigma_{i=1}^n b_i^2 u_i^2 = \|v\|^{-2}\Sigma_{i=1}^n b_i^2 v_i^2 < \beta < \infty$,

---

[12] This is due to the projection theorem. If S is a closed linear subspace of a complete inner product space (Hilbert space) L, then every $l \in L$ has a unique decomposition as $l = l_1 + l_2$ where $l_1 \in S$ and $l_2 \in S^\perp$ (i.e., the inner product $(l_2, l_1) = 0$ for every $l_1 \in S$).

where $\beta = \max_i b_i$.

According to Lemma 3, $\Sigma_{i=1}^{n} v_i e_i / \|v\|$ converges almost surely.

If $\lim_{n \to \infty} \|v\|$ is not finite, then the expected return of portfolio **w** remains a positive number while its random return becomes

$$\mathbf{w}'\mathbf{x} = \mathbf{w}'\mathbf{e} = (\Sigma_{i=1}^{n} v_i e_i / \|v\|) / \|v\|^{r-1}$$

which goes to zero almost surely as $n \to \infty$ which violates the NAASC condition. Therefore, $\lim_{n \to \infty} \|v\|^2 < \infty$. Q.E.D.

IV.B. The C-Type Factor Structure

Now, we will introduce the concept of the **convergence system** (Gaposhkin, 1966) which is a suitable structure for the idiosyncratic risks as we deal with the NAASC. Specifically, the convergence system is the weakest restriction on the idiosyncratic risks one can get to derive the ALPR using the concept of NAASC.

Definition: Suppose that $\{e_i\}$ is a sequence of random variables satisfying the following condition:

$\Sigma_{i=1}^{\infty} d_i e_i$ converges almost surely for all real sequences $\{d_i\}$ such that $\Sigma_{i=1}^{\infty} d_i^2 < \infty$ (i.e., $\{d_i\} \in l_2$).

Then it is called a **convergence system**. If any rearrangement of the system $\{e_i\}$ is a convergence system, then $\{e_i\}$ is called an **unconditional convergence system**. *

Example 3: If e_i are i.i.d., $E(e_i) = 0$, $E(e_i^2) = \sigma^2 < \infty$ for all i, then $\{e_i\}$ is a convergence system. **

Example 4: If $E(e_{i+1}|e_1, \ldots, e_i) = 0$ for all $i \geq 1$ and $\sup_i E(e_i^2) < \infty$, then, by the martingale convergence theorem (see Chung, 1974), $\{e_i\}$ is a convergence system. ▪▪

Example 5: Let $\{e_i\}$ be a generalized linear process generated by an orthogonal S_p system $\{u_h\}$ with $p > 2$ and if ess $\sup_{0 \leq \theta \leq 2\pi} f(\theta) < \infty$, where f is the spectral density of $\{e_i\}$. Then $\{e_i\}$ is a convergence system (Lai and Wei, 1984). ▪▪

Other structures which are convergence systems include stationary Gaussian sequences with absolutely summable correlations and certain types of weakly multiplictive sequences.

Although the condition of the convergence system covers many idiosyncratic risk structures, it does not necessarily hold if the $\{e_i\}$ are only assumed to be uncorrelated with $E(e_i) = 0$, and $E(e_i^2) = \sigma_i^2 \leq \sigma^2 < \infty$ for all i,[13] or when the $\{e_i\}$ are independent with zero means and $\sup_i E|e_i|^p < \infty$ for some $0 \leq p < 2$.[14]

Definition: The linear factor structure is a **C-type factor structure** if the asset return vector, **x**, satisfies the requirement of a basic linear factor structure (condition (1)) and $\{e_i\}$ is a convergence system. ▪

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[13] According to Chen, Lai, and Wei (1981), by Tandori's Theorem, there exists a sequence $\{e_i\}$ of orthogonal random variables such that $E(e_i^2) = 1$ for all i and $\Sigma_{i=1}^{\infty}(c_i e_i/s_i)$ is everywhere divergent which also implies that $(\Sigma_{i=1}^{n} c_i e_i)/s_n$ diverges almost surely by Lemma 3 (p.327) where $s_n \equiv \Sigma_{i=1}^{n} c_i^2$.

[14] This assertion is illustrated by the following example. Assume that $\{e_i\}$ are i.i.d random variables and $P\{e_n = n\} = P\{e_n = -n\} = 1/n$, then $P\{e_n = n \text{ i.o.}\} = 1$. Thus if $c_n = 1/n$, $\lim_{n \to \infty} P\{a_n e_n \to 0\} = 0$. Therefore, $\Sigma_{i=1}^{n} c_i e_i$ does not converge almost surely.

Proposition 6: Given a C-type factor structure, the NAASC implies the ALPR.

Proof: We have $\mathbf{a} = \mathbf{B}^*c + v$. Let $u = v/\|v\|$. Since $\Sigma_{i=1}^{n} u_i^2 = 1 < \infty$ for all n, $\Sigma_{i=1}^{n} u_i e_i$ converges almost surely. The rest of the proof is similar to the one given in Proposition 5. Q.E.D.

If $\{e_i\}$ is an lacunary system of order p ($p > 2$), a proposition similar to Proposition 5 can be proved using the following lemma.

Lemma 4: Let $\{e_i\}$ be an S_p system for some $p > 2$, and $\{d_i\}$ be a sequence of real constants. Suppose that $\Sigma_{i=1}^{n} d_i^2 < \infty$. Then the series $\Sigma_{i=1}^{n} d_i e_i$ unconditionally converges almost surely (i.e., every rearrangement of the series converges almost surely).

Proof: This is established by Gaposhkin (1966). Q.E.D.

Corollary 1: Given a lacunary factor structure (for some $p > 2$), then the NAASC implies the ALPR.

Proof: Use Lemma 4. Q.E.D.

IV.C. S-Type Factor Structure

In Lemma 4, it is assumed that the second moments of asset returns exist. The literature suggests that stochastic properties of asset returns might well be characterized by a stable Paretian distribution in which the second moment does not exist.[15] We now generalize our model to the case where the idiosyncratic risks do not have finite second moments. Before discussing the ALPR, we need the follow results from statistics literature.

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[15] For extensive references on this issue, see Wang and Lee (1988).

Definition: For a normed space E we define constants $S_n^p(E)$, $0 \le p \le 2$, n = 1,2,... as follows

$S_n^p(E) \equiv \inf\{ s \in \mathbf{R}^+$: for all $d_1, d_2, \ldots, d_n \in E$, and

$$(E\|\Sigma_{i=1}^n d_i e_i\|^{p/2})^{2/p} \le s(\Sigma_{i=1}^n \|d_i\|^p)^{1/p}\},$$

where e_i are independent real stable random variables.[16] We shall say that a normed space E is of **stable type p** (in short, $E \in$ s-type p) if there exists a constant $\tau > 0$ such that for all $n \in \mathbf{N}$, $S_n^p(E) \le \tau < \infty$. ∎

It can be shown (Woyczynski, 1978) that $E \in$ s-type p if and only if there exists a constant $\tau > 0$ such that, for all α, $0 \le \alpha \le p^*$, and for all finite $d^i \in E$ ($p^* = p$ if $p < 2$, $p^* = \infty$ if $p = 2$), $(E\|\Sigma_{i=1}^n d_i e_i\|^\alpha)^{1/p} \le \tau\ (\Sigma_{i=1}^n \|d_i\|^p)^{1/p}$.

Lemma 5: Let $1 \le p < 2$. The following properties of a Banach space E are equivalent:

(i) $E \in$ s-type p;

(ii) For any $\{d_i\} \in E$ with $\Sigma\|d_i\|^p < \infty$ and i.i.d. stable random variables e_i of characteristic exponent p, the series $\Sigma_{i=1}^n d_i e_i$ converges almost surely (and also in $L_q(E)$ if $q < p$).

Proof: See Woyczynski (1978). Q.E.D.

Definition: The linear factor structure is a **S-type factor structure** if

i. the asset return vector, **x**, satisfies the requirement of a basic linear factor structure, i.e, condition (1);

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[16] In other words, $\{e_i\}$ have common distribution and their characteristic function has the property that $E \exp(ite_i) = \exp(-\|t\|^p)$. $t \in \mathbf{R}$.

ii. $\{e_i\}$ is a sequence of i.i.d. symmetric stable random variables with characteristic exponent $p \in (1,2)$; and

iii. the space of the portfolio weights is of s-type p. ∎

Corollary 2: Given a S-type factor, the NAASC implies the ALPR.

Proof: Use Lemma 5. Q.E.D.

V. Conclusion

This paper explores the necessary and sufficient conditions for the APT by systematically examining the associations among the linear factor structure in the economy, the lack of asymptotic arbitrage opportunities in the assets market, and the approximate linear pricing relation. A complete articulation of these relationships can enhance our understanding about how we can perturb the assumptions on the linear factor structure and on the no arbitrage condition without ruining the ALPR result.

This study allows us, once gathering empirical results on two of the three elements of the APT, to infer about the nature of the third elements. Tables 2 and 3 summarize the relations among the three elements of the APT. Table 2 shows the combinations of various factor structures and assorted no-asymptotic-arbitrage conditions that imply the ALPR.[17] Table 3 indicates various kinds of factor structure under which the ALPR would imply the lack of asymptotic arbitrage opportunities.

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[17] The word "use" in Tables 2 and 3 indicates that the given relationship among the three elements of the APT has not been formally proved. Using the relevant theorem discussed in this paper, one can easily prove the given relationship.

Our results in Tables 2 and 3 demonstrate that the APT is consistent with many mixtures of linear factor structures and definitions of no-asymptotic-arbitrage . Moreover, this paper, by explicitly deriving the relations among the three elements of the APT, may provide the empirical researchers some guidance on studying the arbitrage mechanism, the stochastic nature of the asset returns, and asset pricing models in the real world.

Table 2

Sufficient Condition

	Linear Factor Structure					
No Arbitrage Condition	BLFS & $E(e_i e_j)=0$ $Var(e_i)<\infty$	BLFS & $E(ee')=\Omega$ $\lVert\Omega\rVert<\infty$	M-type factor structure	W-type factor structure	L-type factor structure	C-type factor structure
$w'1 \to 0$ $Var(w'x) \to 0 \Rightarrow w'a \to 0$	$a \approx B^* c$ Huberman (1982)	$a \approx B^* c$ I(1984) C-R (1983)	$a \approx B^* c$ use Prop.1 ($p \geq 2$)	$a \approx B^* c$ use Prop.3 ($p \geq 2$)	$a \approx B^* c$ use Prop.3 ($p \geq 2$)	$a \approx B^* c$ Prop.C1 [1]
$w'1 \to 0$ $w'x - w'a \xrightarrow{L_p} 0 \Rightarrow w'a \to 0$	$a \approx B^* c$ use Prop.3 ($p \leq 2$)	$a \approx B^* c$ use Prop.3 ($p \leq 2$)	$a \approx B^* c$ use Prop.1	$a \approx B^* c$ use Prop.3	$a \approx B^* c$ use Prop.3	$a \approx B^* c$ Prop.C2 [1]
$w'1 \to 0$ $w'x - w'a \to 0$ a.s. $\Rightarrow w'a \to 0$	$a \approx B^* c$ use Cor.1	$a \approx B^* c$ use Cor.1	$a \approx B^* c$ use Cor.1	$a \approx B^* c$ Prop.5	$a \approx B^* c$ Cor.1 ($p>2$)	$a \approx B^* c$ Prop.6
$w'1 \to 0$ $w'x - w'a \xrightarrow{p} 0$ [2] $\Rightarrow w'a \to 0$	$a \approx B^* c$ use Lemma 1 L-W(1988)	$a \approx B^* c$ use Lemma 1 L-W(1988	$a \approx B^* c$ use Lemma 1 L-W(1988)	$a \approx B^* c$ use Lemma 1 L-W(1988)	$a \approx B^* c$ Lemma 1 L-W (1988)	$a \approx B^* c$ use Lemma 1 L-W(1988)

Notes:

1. The sufficient conditions include additional assumptions that x_i are independent and $|x_i| < \infty$.
2. -p-> means "converge in probability". We can replace it by -l-> (converge in law) here. Let c be a constant, x_n -l-> c <==> x_n -p->c.
3. I(1984) stands for Ingersoll (1984), C-R(1983) for Chamberlain-Rothschild (1983), and L-W(1988) for Lee and Wang (1988).

Table 3

Necessary Condition

	Linear Factor Structure					
No Arbitrage Condition	BLFS & $E(e_i e_j)=0$ $\min_i Var(e_i)>0$	BLFS & $E(ee')=\Omega$ $0<\lVert\Omega\rVert$	M'-type factor structure	W-type factor structure[1]	B-type factor structure	C-type factor structure[1]
w'1-->0 Var(**w'x**) -->0 ==> **w'a**-->0	$a \approx B^* c$ Prop.1 in W-L(1988)	$a \approx B^* c$ use Prop.4	$a \approx B^* c$ use Prop.2 (p=2)	$a \approx B^* c$ use Prop.4 (p=2)	$a \approx B^* c$ use Prop.4 (p=2)	$a \approx B^* c$ use Prop.4
w'1-->0 **w'x**-**w'a** $-L_p$->0 ==>**w'a**->0	$a \approx B^* c$ use Prop.4	$a \approx B^* c$ use Prop.4	$a \approx B^* c$ Prop.2	$a \approx B^* c$ use Prop.4	$a \approx B^* c$ Prop.4	$a \approx B^* c$ use Prop.4
w'1-->0 **w'x**-**w'a** -p->0 ==>**w'a**->0	$a \approx B^* c$ use Prop.C3 @[#] [2]	$a \approx B^* c$ use Prop.C3 @[#]	$a \approx B^* c$ use Prop.C3 @[#]	$a \approx B^* c$ use Prop.C3 @[#]	$a \approx B^* c$ use Prop.C3 @[#]	$a \approx B^* c$ use Prop.C3 @[#]
w'1-->0 **w'x**-**w'a** -->0 a.s. ==>**w'a**->0	$a \approx B^* c$ use Prop.C3 @[#] [3]	$a \approx B^* c$ use Prop.C3 @[#]	$a \approx B^* c$ use Prop.C3 @[#]	$a \approx B^* c$ use Prop.C3 @[#]	$a \approx B^* c$ use Prop.C3 @[#]	$a \approx B^* c$ use Prop.C3 @[#]

Notes:

1. $\{e_i\}$ is assumed to satisfy the Bessel inequality.
2. [#] indicates that the necessary conditions include the assumptions that e_i and f_i are all independent and $|e_i|<\infty$, $|f_k|<\infty$, or the assumption that **e** and **f** are independent. @ means other additional assumptions are included in the necessary conditions.
3. Use the fact that [**w'x** - **w'a** --> 0 a.s.] ==> [**w'x** - **w'a** -p-> 0].
4. W-L(1988) stands for Wang and Lee (1988)

APPENDIX A

Lemma A1: Let $1 < p < \infty$. There are positive real constants δ_p such that if $E(e_i|e_j) = 0$ for all $j < i$, then $(E|\Sigma_{i=1}^n e_i|^p)^{1/p} \le \delta_p (E|(\Sigma_{i=1}^n e_i^2)^{1/2}|^p)^{1/p}$.

Proof: See Burkholder (1973: 22). Q.E.D.

Proof of Lemma 1: The lemma can be proved by using Lemma A1 and Minkowski inequality. For a complete proof, see Lai and Wei (1983).

Q.E.D.

Lemma A2: Let $\{e_i\}$ be a sequence such that $E(e_i|e_j) = 0$ for all $j < i$. Suppose $\inf_i |e_i| > 0$, there exist an A such that

$$E|\Sigma_{i=1}^n d_i e_i| \ge A(E|\Sigma_{i=1}^n d_i e_i|^2)^{1/2} \quad \text{for all } n \ge 1.$$

Proof: See Lemma 4 of Burkholder (1968). Q.E.D.

Proof of Lemma 2:

According to Lemma A2 in the Appendix A, we have

$E|\Sigma_{i=m}^n d_i e_i| \ge \alpha(E|\Sigma_{i=m}^n d_i e_i|^2)^{1/2}$, for all $n \ge m$.

$(E|\Sigma_{i=m}^n d_i e_i|^p)^{1/p} \ge E|\Sigma_{i=m}^n d_i e_i|$, by Hölder's inequality.

Thus

$$E|\Sigma_{i=m}^n d_i e_i|^p \ge H_p(E|\Sigma_{i=m}^n d_i e_i|^2)^{p/2}, \text{ where } H_p = \alpha^p.$$ Q.E.D.

APPENDIX B

Two specialized forms of Condition B_p are presented here. For more discussion, see Longnecker and Serfling (1978).

Definition: A sequence $\{e_i\}$ satisfies **Condition B1$_p$** (weakly multiplictive of type B1$_p$) with respect to an even integer p if $E(e_i^p) < \infty$, for all i, and there exists a function f(j) such that

$$|E\{e_{i(1)}\dots e_{i(p)}\}| \le \min\{f[i(2)-i(1)], f[i(4)-i(3)], \dots, f[i(p)-i(p-1)]\}\cdot\pi_{j=1}^{p}(E(X_{i(j)}{}^p))^{1/p}$$

for all $1 \le i(1) < \dots < i(p)$, and if $\Sigma_{j=1}^{\infty} j^{p/2-1} f(j) < \infty$.

With $g(j_1,\dots j_{p/2}) = \min\{f(j_1),\dots,f(j_{p/2})\}$, both condition B and B1 have the same first part.

Definition: A sequence $\{e_i\}$ satisfies **Condition B2$_p$** (weakly multiplictive of type B2$_p$) with respect to an even integer p if $E(e_i^p) < \infty$, for all i, and there exists a function f(j) such that

$$|E\{e_{i(1)}\dots e_{i(p)}\}| \le f[i(2)-i(1)]f[i(4)-i(3)]\dots f[i(p)-i(p-1)]\ \pi_{j=1}^{p}(E(e_{i(j)}{}^p))^{1/p}$$

for all $1 \le i(1) < \dots < i(p)$, and if $\Sigma_{j=1}^{\infty} f(j) < \infty$.

APPENDIX C

Lemma C1 (Improved convergence lemma): For series of independent random variables, if the summands are uniformly bounded, and centered at expectations, then almost sure convergence and convergence in quadratic mean are equivalent.

Proof: See Loève (1977: 261). Q.E.D.

Proposition C1: Given a linear factor structure. Suppose that x_i are independent, $|x_i| \le m < \infty$ for all i and $\{e_i\}$ is a convergence system. Then [$\mathbf{w}'1 \to 0$ and $\mathbf{w}'\mathbf{x} - \mathbf{w}'\mathbf{a} \xrightarrow{L_2} 0 \Rightarrow \mathbf{w}'\mathbf{a} \to 0$] $\Rightarrow$ [$\mathbf{a} \approx \mathbf{B}^*\mathbf{c}$].

Proof: Consider a portfolio $u = v/\|v\|$ where $v = \mathbf{a} - \mathbf{B}^*\mathbf{c}$. Since $\mathbf{u}'\mathbf{u} = 1 < \infty$, $\mathbf{u}'\mathbf{x} - \mathbf{u}'\mathbf{a}$ converges almost surely. We have $\mathbf{u}'\mathbf{x} - \mathbf{u}'\mathbf{a}$ converges in quadratic mean by Lemma C1. Let $\mathbf{w} = u/\|v\|^{r-1}$. Then apply Proposition 4 for p = 2. Q.E.D.

Lemma C2: Under the same condition as in Lemma C1, the convergence a.s. and convergence in pth mean are equivalent.

Proof: Use Lemma C1 and Basic inequality (Loève, 1977: 159). Q.E.D.

Proposition C2: Given a linear factor structure, suppose that x_i are independent, $|x_i| \le m < \infty$ for all i and $\{e_i\}$ is a convergence system. Then [$\mathbf{w}'1 \to 0$ and $\mathbf{w}'\mathbf{x} - \mathbf{w}'\mathbf{a} \xrightarrow{L_p} 0$ (for some $p > 0$) $\Rightarrow \mathbf{w}'\mathbf{a} \to 0$] $\Rightarrow$ [$\mathbf{a} \approx \mathbf{B}^*\mathbf{c}$].

Proof: Use Lemma C2. Q.E.D.

Proposition C3: Given a linear factor structure, suppose that $\{e_i\}$ is an Banach system, $\mathbf{e}$ and $\mathbf{f}$ are independent, e_i and f_k are all independent, $|e_i| < \infty$, and $|f_k| < \infty$. Then [$\mathbf{a} \approx \mathbf{B}^*\mathbf{c}$] $\Rightarrow$ [$\mathbf{w}'1 \to 0$ and $\mathbf{w}'\mathbf{x} - \mathbf{w}'\mathbf{a} \xrightarrow{p}$ (or $\xrightarrow{1}$) $0 \Rightarrow \mathbf{w}'\mathbf{a} \to 0$].

Proof: $\mathbf{w}'\mathbf{x} - \mathbf{w}'\mathbf{a} \xrightarrow{p} 0$

$\Leftrightarrow P\{|\mathbf{w}'\mathbf{x} - \mathbf{w}'\mathbf{a}| > \epsilon\} \to 0$

$\Leftrightarrow P\{|\mathbf{w}'\mathbf{B}\mathbf{f} + \mathbf{w}'\mathbf{e}| > \epsilon\} \to 0$

Without loss of generality, assume the median of **w'Bf** is 0, then, for some $\epsilon > 0$, $P\{|\mathbf{w'Bf}+\mathbf{w'e}| > \epsilon\} \geq P\{|\mathbf{w'e}| > \epsilon\}/2$ since **e** and **f** are independent by Lemma 2 in Wang and Lee (1988).

Similarly, $P\{|\mathbf{w'e}|>\epsilon\}$ --> 0 implies **w'e** $-L_2->$ 0 which means that w'w --> 0 since $\{e_i\}$ is a Banach system.

$P\{|w'Bf|>\epsilon\}$ --> 0 implies $E(|w'Bf|^2)$ --> 0 since $|f_k| < \infty$ and f_k are independent. Thus **w'BB'w** --> 0, i.e., $\Sigma_{i=1}^{n} w_i b_{ik}$ --> 0. Then apply Proposition 2 for the rest of the proof. Q.E.D.

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