A GENERAL THEORY OF ARBITRAGE PRICING: WHEN THE IDIOSYNCRATIC RISKS ARE DEPENDENT AND THEIR SECOND MOMENTS DO NOT EXIST

by

C. Jevons Lee
Taychang Wang

(29-88)

RODNEY L. WHITE CENTER FOR FINANCIAL RESEARCH
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6367

A General Theory of Arbitrage Pricing: When the Idiosyncratic Risks Are Dependent and Their Second Moments Do Not Exist

Taychang Wang and C. Jevons Lee*

A. B. Freeman School of Business
Tulane University
New Orleans, LA 70118-5698

Tel: (504) 865-5439

September, 1988

*Respectively, Associate Professor, National Taiwan University and Professor, Tulane University. Major portion of this project was completed when both authors were associated with the Wharton School, University of Pennsylvania. Helpful comments and discussions from Mark Grinblatt, Richard Kihlstrom, Andrew Lo, and Craig MacKinlay are appreciated. Any remaining error is ours. Financial support from the Yin's Foundation is acknowledged.

A General Theory of Arbitrage Pricing: When the Idiosyncratic Risks Are Dependent and Their Second Moments Do Not Exist

Abstract

In this paper, we generalize the Arbitrage Pricing Theory (APT) to incorporate the cases where the idiosyncratic risks of the factor model are dependent and/or the second central absolute moments (variances) of the assets returns do not exist. A bound on the pricing errors, similar to the one derived in Ross (1976) and Huberman (1982), is derived in our generalized framework. Specifically, it is shown that as long as the idiosyncratic risks are weakly dependent (or when the sequence of the idiosyncratic risks is a lacunary system), the approximate linear pricing relation holds in the absence of "arbitrage" in the sense of convergence in p^{th} mean (ACPM). It can be demonstrated that the models in Huberman (1982), Ingersoll (1984) and Chamberlain and Rothschild (1983) are all special cases of this version of the APT. It is also established that, under suitable assumptions on the linear factor structure, the approximate linear pricing relation implies the nonexistence of asymptotic arbitrage opportunities. Thus the no-asymptotic-arbitrage position is a necessary and sufficient condition for the approximate linear pricing relation.

I. Introduction

In financial economics, variance has almost always been used as a measure of risk and dispersion. It plays an obvious role in mean-**variance** analysis and in theories derived from it, such as the Sharpe-Lintner-Mossin-Treynor Capital Asset Pricing Model (CAPM). It is also used as a measure of risk in the Arbitrage Pricing Theory (APT) where a riskless arbitrage portfolio is defined to be a portfolio with no **variance** and the idiosyncratic risk is always assumed to have bounded **variance**. The literature has not closely examined the robustness of these theories when asset return distributions have undefined or infinite variances. There are strong empirical evidences showing that this is a relevant concern. For example, Fama (1965) found that the distribution of price changes conforms better to the stable Paretian distribution which is not normal. In the class of stable Paretian distributions, only the normal has a finite variance. In this paper, we extend the earlier research in asset pricing by employing different concepts of dispersion. As an instance, when variances are infinite, the mean absolute deviation (MAD) or the mean of some power of the absolute deviation (other than two) may exist. By using these alternative concepts of dispersion, asset pricing can be shown to extend to other classes of asset return distribution. In particular, our research focuses on the APT.

The APT has generated extensive research interests since it was first introduced by Ross in 1976. The theory was proposed as an alternative to the CAPM, a major analytic tool in financial literature at that time. It represents one of the major attempts to surmount the problems with testability and the anomalous empirical findings that have plagued other theories.[1] The APT is a one period model in which investors share the same belief that the stochastic properties of capital assets returns are consistent with a linear factor structure of which the idiosyncratic risks have finite variances.[2] If there are no asymptotic arbitrage opportunities, then the expected returns on these capital assets are approximately linearly related to the factor loadings. People may be curious about what happens to the APT when the second central absolute moments (variances) of the assets return do not exist.[3] This is solved in the context where the distributions of the idiosyncratic risks are stable Paretian with characteristic exponent α $(1 \le \alpha < 2)$ and the asymptotic arbitrage is defined in terms of convergence in p^{th} mean.[4]

Moreover, we generalize this theory even further by allowing the sequence of idiosyncratic risks to correlate under some constraints and by letting the

---

[1]. Following the early success of the empirical research on the CAPM, extensive evidences in recent years indicate the existence of anomalies. For example, market capitalizations, dividend yields, and price-earnings ratios of common stocks are found to be significantly associated with the asset returns after risks are adjusted according to the CAPM. To explain these anomalies, the literature explores the possible economic and financial variables that are omitted in the one-period CAPM. For example, theories that incorporates institutional characteristics such as taxation, skewness, and intertemporal changes in investment opportunities are suggested. None of them is successful in getting empirical supports. Furthermore, Roll (1977) wondered that the CAPM may not be a testable scientific theory.

[2] Even if all investors agree on the factor structure, however, there is still significant scope for disagreement on the underlying probability distributions. As long as all investors agree on the impacts of the factors on returns through factor loadings, they can hold a variety of views on the distributions of the factors without violating the approximate linear relation. Similarly, investors can also disagree on the distributions of the residual risks.

[3] Throughout this paper, we assume that the first moments of the asset returns always exist.

[4] When $p = \alpha = 2$, we have the usual convergence in quadratic mean. Note that $1 \le p < \alpha$ when $\alpha \neq 2$.

variances of the idiosyncratic risks to be infinite in some cases. We prove that if the idiosyncratic risks are weakly dependent (or, specifically, the sequence of the idiosyncratic risks is a lacunary system of order p for some p ≥ 1) and if the definition of arbitrage is defined in terms of convergence in p^{th} mean instead of being restricted to convergence in quadratic mean ($p = 2$), then the APT holds. The results concerning the asset pricing relation in Ross (1975, 1976), Huberman (1982), Ingersoll (1984) and Chamberlain and Rothschild (1983) are shown to be special cases of our model.[5]

We also demonstrate that, under suitable assumptions on the linear factor structure, the approximate linear pricing relation implies the nonexistence of asymptotic arbitrage opportunities. Thus, the no-asymptotic-arbitrage position is a necessary and sufficient condition for the approximate linear pricing relation. The empirical implications of this finding will be discussed.

In this paper, we assume the number of the assets traded in the economy are countably infinite. The price system is normalized by letting each asset cost one dollar. The assets are arranged in a sequence. We always examine the effect on various objects such as arbitrage opportunities and pricing relation as the number of assets, n, increases to infinity.

The rest of the paper is organized as follows. In Section II, we discuss the basic model and its assumptions. Although the basic model has been extensively explored in the literature, the scope of examination has been limited to the sufficient conditions of the APT. In this section, we will

---

[5] In their proofs of the APT, Ross (1976) and Huberman (1982) assumed the residual risks to be uncorrelated, Ingersoll (1984) assumed the variance-covariance matrix of the residual risks to be uniformly bounded, and Chamberlain and Rothschild (1983) and Chamberlain (1983) allowed only K eigenvalues of the returns variance-covariance matrix to be unbounded. All these theories of arbitrage pricing are special cases of our model.

derive the necessary condition as well. Section III discusses the APT in a specific framework where asset are stably distributed. Section IV provides a generalization of the results in Section III. We will introduce the concept of lacunary system to describe an interesting type of weakly dependent relations among the idiosyncratic risks. In Section V, applications of the generalized APT are discussed. Section VI concludes the paper.

II. The Basic Model

The basic model of the APT in the literature is analyzed in this section. Basically, the model is composed of three elements, namely, the linear factor structure, the concept of asymptotic arbitrage , and the approximate linear pricing relation. We will first discuss the assumptions on the linear factor structure of asset returns. Then we will define asymptotic arbitrage in terms of convergence in quadratic mean and the approximate linear pricing relation expressed in terms of Euclidean norm. The necessary and sufficient condition for the approximate linear pricing relation in this basic model will be derived.

II.A. The Linear Factor Structure

The first element of the APT is the linear factor structure of asset returns. Throughout this section we shall assume the following linear factor structure:

$$\mathbf{x} = \mathbf{a} + \mathbf{B}\,\mathbf{f} + \mathbf{e} \qquad (1a)^{6}$$

---

[6] Variables $\mathbf{x}$, $\mathbf{f}$, and $\mathbf{e}$ are random and the sequences $\mathbf{x}$, $\mathbf{a}$, $\mathbf{B}$, $\mathbf{f}$, and $\mathbf{e}$ all depend on n. To avoid messy notations, unless the omission can cause confusion, we do not label the randomness, neither do we index the order of

$E(\mathbf{e}) = \mathbf{0}$ (1b)

$E(f) = 0$ (1c)

$E(\mathbf{ff}') = \mathbf{I}_K$ (Identity matrix of rank K) (1d)

$E(\mathbf{ef}') = \mathbf{0}$ (n x K matrix) (1e)

$E(\mathbf{ee}') = \mathbf{D}$ ($\mathbf{D}$ is a diagonal matrix) (1f)

$\mathbf{1}_i'\mathbf{D}\mathbf{1}_i < \sigma^2 < \infty$ for all i, where $\mathbf{1}_i$ is the i^{th} column of an identity matrix with rank n. (1g)[7]

The vectors **x**, **a**, and **e** are each n x 1 in dimension and represent the realized returns, expected returns and (nonobservable) "residual" portions of the returns, respectively. The e_i, which is often called idiosyncratic risk or residual in the literature, measures the uncertainty unexplained by the common factors. The expected return of i^{th} asset, a_i, is assumed to be bounded for all i. **f** is a K x vector of nonobservable values of the common factors. The second moment of f_k is assumed to exist for all k. **B** is the n x K matrix of bounded factor loadings, i.e., $|b_{ik}| < \infty$ for all i and k. **D** is the n x n positive definite diagonal variance-covariance matrix of the idiosyncratic risks. The i^{th} diagonal element of **D** is denoted as σ_i^2.

The linearity assumption in (1a) is the backbone of the APT and will be kept intact throughout this paper. As is discussed in Ingersoll (1984), assumption (1b) is always possible through appropriate choice of expected returns, **a**. Assumptions (1c) and (1d), which can be met by subtracting any

---

sequence.

[7] The validity of (1g) can be tested by using an interesting result in Laha and Rohatgi (1979:62, E24). It can be shown that
$E(e_i)=0$ and $E(e_i^2)=\sigma_i^2 < \infty$ imply that
$$P\{e_i>x\} \le \sigma_i^2/(\sigma_i^2+x^2) \text{ if } x > 0$$
$$\text{and } P\{e_i>x\} \le x^2/(\sigma_i^2+x^2) \text{ if } x < 0.$$
Hence, a researcher can show the empirical validity of (1g) by examining the histogram of the sample data.

factor means and then orthognoalizing and rescaling the parameters, are innocuous. Also note that it is assumed that **B** is an n x K matrix with rank K.[a] Assumption (1e) is, however, rather troublesome. Ingersoll (1987) argued that (1e) could be achieved through appropriate choices for **a** and **B**. As to be shown in Appendix A, that statement is not quite true. Note that (1e) is not crucial to the derivation of the APT. However, it is required in proving that the no-asymptotic-arbitrage position is a necessary condition of the approximate linear pricing relation.

Assumption (1f) indicates zero covariance of the idiosyncratic risks and assumption (1g) indicates the finiteness of the variances of the idiosyncratic risks. Conditions (1d) and (1g) together imply that variances of x_i are bounded. All these assumptions will be relaxed in our generalized model of the APT.

---

[a] The matrix **B** is assumed to be a n x K matrix with rank K. If the rank of **B** is less than K, say K_1, let $B = (B_0 \ \ B_1)$ where B_0 is an n by K_1 matrix. Suppose $B_1 = B_0 P$, we have

$$(B_0 \ B_1) \begin{bmatrix} I & -P \\ 0 & I \end{bmatrix} = (B_0).$$

Since $\mathbf{B}\mathbf{f} = \mathbf{B}\mathbf{Q}^{-1}\mathbf{Q}\mathbf{f}$, take

$$Q = \begin{bmatrix} I & -P \\ 0 & I \end{bmatrix},$$

the rank problem is then solved by letting $B_{(new)} = \mathbf{B}\mathbf{Q}^{-1}$ and $f_{(new)} = \mathbf{Q}^{-1}f$. Now we can not say that $E(ff') = I_k$. But, this is not of great importance. We are more concerned about the fact that $E[(\mathbf{BF})(\mathbf{BF})'] = \mathbf{BB}'$.

II.B. Asymptotic Arbitrage

The second basic component of the APT is the nonexistence of the asymptotic arbitrage opportunities. Huberman (1982) and Ingersoll (1984) suggested a definition of asymptotic arbitrage , which is defined in terms of convergence in quadratic mean.

Definition: Arbitrage in the sense of convergence in quadratic mean (ACQM) is the existence of a subsequence $\hat{n}$ of arbitrage portfolios, $\mathbf{w}(\hat{n})$, $\hat{n} = 1, 2, \ldots$, whose returns $z(\mathbf{w}(\hat{n}))$ satisfy

$$\mathbf{w}(\hat{n})'\mathbf{1}(\hat{n}) = 0, \qquad (2a)$$

$$\mathrm{Var}\ z(\mathbf{w}(\hat{n})) = \mathbf{w}(\hat{n})'\Sigma(\hat{n})\mathbf{w}(\hat{n}) \dashrightarrow 0 \qquad (2b)$$

$$E\ z(\mathbf{w}(\hat{n})) = \mathbf{w}(\hat{n})'\mathbf{a}(\hat{n}) \geq m > 0, \qquad (2c)$$

where $\Sigma(\hat{n}) = \mathbf{B}(\hat{n})'\mathbf{B}(\hat{n}) + \mathbf{D}(\hat{n})$. ∎

The less stringent condition (2c) replaces Huberman's (1982) requirement of $Ez(\mathbf{w}(\hat{n})) \dashrightarrow \infty$ (Ingersoll, 1984).[9]

The approximate linear pricing relation in the APT is derived from the

---

[9] According to Ingersoll (1984), the scale of asymptotic arbitrage is arbitrary. Hence, condition (2c) should serve the purpose. Although the scale of quadratic mean arbitrage can be magnified, yet the choice of the scalar is by no means arbitrary. Given a subsequence $\hat{n}$ of arbitrage portfolio; $\mathbf{w}(\hat{n})$, $c(\hat{n})\mathbf{w}(\hat{n})$ can be also an arbitrage portfolio for some $c(\hat{n}) > 0$. We can not take an arbitrary $c(\hat{n})$, otherwise condition (2c) might be violated. However, if $c(\hat{n}) = (\mathbf{w}(\hat{n})'\Sigma(\hat{n})\mathbf{w}(\hat{n}))^{p}$, where $-1/2 < p < 0$, then $\hat{\mathbf{w}}(\hat{n}) = c(\hat{n})\mathbf{w}(\hat{n})$ is a quadratic mean arbitrage opportunity with an infinite profit in the limit. The reason is given as follows.

Suppose $\mathbf{w}(\hat{n})'\Sigma(\hat{n})\mathbf{w}(\hat{n}) \dashrightarrow 0$, we have

$$\hat{\mathbf{w}}(\hat{n})'\mathbf{1}(\hat{n}) = c(\hat{n})\mathbf{w}(\hat{n})'\mathbf{1}(\hat{n}) = 0,$$

$$\hat{\mathbf{w}}(\hat{n})'\mathbf{a}(\hat{n}) = c(\hat{n})\mathbf{w}(\hat{n})'\mathbf{a}(\hat{n}) \geq mc(\hat{n}) \dashrightarrow \infty \qquad m > 0, \text{ and}$$

$$\hat{\mathbf{w}}(\hat{n})'\Sigma(\hat{n})\ \hat{\mathbf{w}}(\hat{n}) = c(\hat{n})^{2}\mathbf{w}(\hat{n})'\Sigma(\hat{n})\mathbf{w}(\hat{n}) = (\mathbf{w}(\hat{n})'\Sigma(\hat{n})\mathbf{w}(\hat{n}))^{1+2p}$$

$$\dashrightarrow 0 \qquad \text{for } -1/2 < p < 0.$$

absence of asymptotic arbitrage opportunities. The no-arbitrage condition is implied by the definition of arbitrage through a simple logic:

not(A and B) ≡ (A ==> not B).

Hence, we define the nonexistence of ACQM, or **NACQM**, as follows.

Definition: The **NACQM** condition is the situation that when there is a portfolio with zero cost and zero variance in the limit, its expected return must also converge to zero, i.e., for all $\mathbf{w} \in \mathbf{R}^n$,

$\mathbf{w}'\mathbf{1} \rightarrow 0$ and $Var(\mathbf{w}'\mathbf{x}) \rightarrow 0 \Rightarrow \mathbf{w}'\mathbf{a} \rightarrow 0.$ (2')[10]

Whenever possible, the indices in the seqence of the arbitrage portfolio are omitted for brevity.

II.C. The Approximate Linear Pricing Relation (ALPR)

The third basic component of the APT is the approximate linear pricing relation, which gives expected returns with a mean squared error of zero. The nature of the approximate linear pricing relation depends on the specification of the error bound. The linear pricing relation that is usually examined in the literature is described in terms of the finiteness of the Euclidean norm of the pricing error vector.

Definition: The **approximate linear pricing relation in terms of Euclidean norm (ALPR)** indicates that the expected returns, **a**, is a linear combination of the

[10] Chamberlain and Rothschild (1983) made another no-arbitrage assumption: If $\mathbf{w}'\mathbf{1} \rightarrow 1$, $Var(\mathbf{w}'\mathbf{x}) \rightarrow 0$, and $E(\mathbf{w}'\mathbf{x}) \rightarrow \alpha$ (Condition Aii), then $\alpha > 0$. Jarrow (1988) showed that this condition implies (2'). However, to prove the APT, the weaker one (2') will be sufficient.

factor loadings, $b_{.k}$, plus an error term, $\mathbf{v}$. And it gives expected returns a with a mean squared error of zero. In other words, for n = 1, 2, ... , there exist $\mathbf{c}' = (c_0, c_1, c_2, \ldots , c_K)$ such that

$$\mathbf{a} = \mathbf{B}^* \mathbf{c} + \mathbf{v}, \text{ and} \tag{3a}$$

$$\lim_{n \to \infty} (1/n) \|\mathbf{v}\|^2 = 0, \text{ or somewhat more strongly,} \tag{3b}$$

$$\lim_{n \to \infty} \|\mathbf{v}\| < \infty, \tag{3c}$$

where $\mathbf{B}^* = (\mathbf{1} \;\; \mathbf{B})$, $\mathbf{1}$ is a vector of 1s; all $\mathbf{a}$, $\mathbf{B}$, and $\mathbf{c}$ depend on n, and $\| \; \|$ is the Euclidean norm, i.e., $\|\mathbf{v}\| \equiv (\Sigma_{i=1}^{n} |v_i|^2)^{1/2}$. ∎

This paper will discuss several types of approximate linear pricing relation. However, the ALPR in terms of Euclidean norm appears most often in the text. Unless otherwise specified, the "ALPR" in the rest of the paper indicates the "approximate linear pricing relation in terms of Euclidean norm".

Two interesting remarks about the asymptotic linear pricing relation can be made here. First, the literature (e.g., Ingersoll (1987)) interprets the asymptotic linear pricing model, $\mathbf{a} \approx \mathbf{B}^* \mathbf{c}$, as that the market evaluates asset prices "correctly" for almost all assets, and it can be extremely bad at pricing a finite number of assets. This interpretation is rather ambiguous and sometimes misleading. For example, if we let the pricing error, v_i, be $\alpha/2^{i/2}$ for any α such that $0 < \alpha < \infty$, then the norm of $\mathbf{v}$ is a finite α. Here, every v_i is greater than zero and $v_{i+1} = v_i/2^{1/2}$. Hence, it is misleading in saying that the model prices assets correctly for <u>almost all</u> assets. Also, the definition in (3) does not entitle us to make assertion on individual pricing error.

Second, the literature (e.g., Ingersoll, 1987) asserts that the theory prices all of the assets together with a negligible mean squared error, i.e.,

$\|v\|^2/n \to 0$; this condition for asymptotic linear pricing relation is unnecessarily weak. In fact, it is also true that $\|v\|/\log(n) \to 0$. The strongest condition we can get is $\lim_{n\to\infty}\|v\| < \infty$.[11] This accuracy bound was first derived by Huberman(1982) for uncorrelated idiosyncratic risks. Chamberlain and Rothschild (1983), Stambaugh (1983), and Ingersoll (1984) have independently proved this same result for correlated idiosyncratic risks.

II.D. Necessary and Sufficient Conditions for The APT

Now, we are ready to show the "if and only if" relation between the nonexistence of the ACQM and the approximate linear pricing result in the basic model. The necessary and sufficient condition for the ALPR in the basic model is formally stated and proved in Proposition 1.

Proposition 1:

(i) Given the linear factor structure (1), the NACQM described in (2') implies the ALPR stated in (3c).

(ii) With an additional assumption that **D** is asymptotically positive definite, (2') is implied by (3c).

Proof:

(i): Projecting the vector **a** onto the space spanned by **B** and the vector **1**, we have:[12]

---

[11] In the proof of part (i) of Proposition 1, we can set the arbitrage portfolio **w** to be $v/(\log n * \|v\|)$. Its expected return is $\|v\|/\log n$ and its variance is less than $\sigma^2/(\log n)^2$. From these we can assert that

$\|v\|/\log n \to 0$ as $n \to \infty$,

which is directly from the finiteness of $\|v\|$.

[12] This is due to the <u>projection theorem</u>. If S is a closed linear subspace of a complete inner product space (Hilbert space) L, then every $l \in L$ has a unique decomposition as $l = l_1 + l_2$ where $l_1 \in S$ and $l_2 \in S^{\perp}$ (i.e., the

$a = B^{*}c + v$, where, $c \in R^{k+1}$ and $B^{*}v = 0$.

Consider an arbitrage portfolio, **w**, such that

$$w = \frac{v}{\|v\|^{r}} ,$$

where, $1 < r \leq 2$. The return of portfolio **w** is

$$\frac{v'x}{\|v\|^{r}} = \frac{1}{\|v\|^{r}} (v'a + v'Bf + v'e) = \frac{1}{\|v\|^{r}} (v'a + v'e).$$

Therefore, the expected return of portfolio **w** is

$$\frac{v'a}{\|v\|^{r}} = \frac{1}{\|v\|^{r}} (v'B^{*}c + v'v) = \frac{v'v}{\|v\|^{r}} = \|v\|^{2-r} .$$

The variance is

$$\frac{v'Dv}{\|v\|^{2r}} = \frac{1}{\|v\|^{2r}} \Sigma_{i=1}^{n} v_i^2 \sigma_i^2 \leq \sigma^2 \|v\|^{2-2r}.$$

If $\lim_{n \to \infty} \|v\|$ is not finite, then the expected return remains a positive number while the variance goes to zero as $n \to \infty$ which violates the assumption of no arbitrage. Therefore, $\lim_{n \to \infty} \|v\|^2 < \infty$.

(ii): We need to show that if $\lim_{n \to \infty} \|a - B^{*}c\| < \infty$, then

$w'1 = 0$ and $Var(w'x) \to 0$ imply $w'a \to 0$.[13]

Consider an arbitrage portfolio **w** with the following properties:

$w'1 = 0$ and $Var(w'x) \to 0$.

Since $Var(w'x) = w'BB'w + w'Dw \to 0$,

we have $w'B \to 0'$ and $\Sigma_{i=1}^{n} w_i^2 \sigma_i^2 \to 0$ by (1f).

Due to that fact $\Sigma_{i=1}^{n} w_i^2 \sigma_i^2 \geq \beta \Sigma_{i=1}^{n} w_i^2$, $\Sigma_{i=1}^{n} w_i^2 \to 0$, where $\beta = \min_i \sigma_i^2$.

---

inner product $(l_2, l_1) = 0$ for every $l_1 \in S$).

[13] The logical structure of the second part of Theorem 1 is that the assumption A:[ALPR] implies that if C:[$w'1 \to 0$ and $Var(w'x) \to 0$] is true, then E:[$w'a \to 0$] is true. In logic symbols, this is written

A ==> [C ==> E].

The symbolic logic statement can also be written [A and C] ==> E.

By Schwarz inequality we have

$$\|\mathbf{w}\| \ \|\mathbf{a} - \mathbf{B}^{*}\mathbf{c}\| \geq |\mathbf{w}'(\mathbf{a} - \mathbf{B}^{*}\mathbf{c})| = |\mathbf{w}'\mathbf{a}|.$$

Since $\|\mathbf{w}\| \to 0$ and $\lim_{n \to \infty}\|\mathbf{a} - \mathbf{B}^{*}\mathbf{c}\| < \infty$, $\mathbf{w}'\mathbf{a}$ must converge to zero, i.e., there are no asymptotic arbitrage opportunities. Q.E.D.

If we tighten up the constraint on the pricing error bound, we can relax the constraint on the variance-covariance matrix of the idiosyncratic risks. Corollary 1 shows the trade off between these two constraints.

Corollary 1 Given the linear factor structure, if **D** is asymptotically positive semi-definite and if $\lim_{n \to \infty}\|\mathbf{a} - \mathbf{B}^{*}\mathbf{c}\| = 0$, then the ACQM does not exist, i.e., $\mathbf{w}'\mathbf{1} \to 0$, and $\mathrm{Var}(\mathbf{w}'\mathbf{x}) \to 0$ imply that $\mathbf{w}'\mathbf{a} \to 0$.

Proof: Using the fact that among assets with zero idiosyncratic risk the pricing must be exact (Ingersoll, 1988: Chapter 7, Theorem 2). Then apply Proposition 1. Q.E.D.

The proof of sufficiency (part (i)) in Proposition 1 is similar to Theorem 1 of Huberman (1982). The arbitrage condition is defined in terms of variance, and the error bound in the ALPR is defined in terms of squared errors. Later on we will derive the more general results by relaxing the assumptions (1f) and (1g) on the idiosyncratic risks in the linear factor structure, and we will use alternative definitions for asymptotic-arbitrage. The proof of necessity (part (ii)) is new and will also be extended.

III. Nonexistence of the Second Moments in Asset Returns: The Case of Stable Paretian Distribution

III.A. Empirical Evidences on Stock Returns

The stochastic properties of asset returns plays an important role in constructing financial theories. Most financial models use variance as a measure of risk and dispersion. However, empirical evidences indicate that the second moments of asset returns may not exist.

Fama (1965) tests the normality hypothesis on the daily returns of the Dow Jones Industrial stocks. The results reveal more kurtosis (fatter tails) than that predicted from a sample of independent and identically distributed normal variates. Fama thus concludes that the distribution of price changes conforms better to the stable Paretian distribution with characteristic exponent less than two.[14]

A frequently proposed alternative to the stable Paretian model is the mixtures of normal distributions hypothesis, which suggests that stock returns are represented by combinations of normal distributions with different variances and possibly different means (e.g., Clark ,1973; Hsu et al., 1974; Westerfield, 1977; and Hagerman, 1978). Both the stable Paretian and mixtures of normal distributions models are capable of describing the higher frequencies

---

[14] The family of probability distributions known as "stable Paretian" is probably one of the most popular models to describe the stochastic properties of daily common stock returns (Fama, 1965; Fama and Miller, 1972; Mandelbrot, 1963, 1971; Blume, 1970; and Roll, 1970). Blattberg and Gonedes (1974) showed that the stochastic properties of monthly returns are better described by the normal distribution. Although many empirical researchers in finance continue to use monthly data in their studies, most of recent empirical works employs daily data in order to segregate information events (e.g. Aharony and Swary, 1980) or to take advantage of a much larger sample size (e.g., Roll and Ross, 1980). Therefore, it is increasingly important to face the issue of stable Paretian distribution in the asset pricing theory.

of stock returns observed near the mean and in the tail areas when compared with a normal distribution.

Fielitz and Rozelle (1983) found evidence showing that although the majority of stock return distributions are consistent with a mixture-of-distribution hypothesis, it is difficult to differentiate whether the mixtures of distributions are normal with changing variance, or nonnormal stable with changing scale parameter (dispersion). In this section, we will discuss the APT when the returns have no second moments, or more specifically, when the distributions of the returns are symmetric stable Paretian.

III.B. Stable Paretian Distribution

Definition: A **stable Paretian distribution** has the following log characteristic function:

$$\log \Phi_x(t) = \log E(e^{ixt})$$

$$= i\delta t - \sigma|t|^{\alpha}[\, 1 + i\beta(t/|t|)w(t,\alpha)], \qquad (4)^{15}$$

where x is the random variable, t is some real number, and

$$0 < \alpha \le 2;\ -1 \le \beta \le 1;\ -\infty < \delta < \infty;\ 0 < \sigma. \qquad \blacksquare$$

The parameter α is the characteristic exponent, a main parameter of the stable law; β is the degree of asymmetry of a stable distribution, where $\beta = 0$ indicates a symmetric distribution; δ is the location parameter; and σ is the scale parameter of the stable Paretian distribution. When $\alpha < 2$, the variance of the stable distribution is infinite. However, there is a finite parameter σ which defines the scale of the distribution. Suppose that $\alpha = 1$, $\beta = 0$ (Cauchy

---

[15] There are other ways to represent a stable Paretian distribution. See Zolotatarev (1986).

distribution), σ is the semi-interquartile range. In the remaining part of this section, we will consider only the case where $\alpha > 1$, $\beta = 0$ so that the mean value exists and the density is symmetric.

By definition, a stable Paretian distribution is any distribution that is stable or invariant under addition.[16] In other words, the distribution of the sum of i.i.d. stable Paretian variables is itself stable Paretian and has the same distribution as the summands (but, values of parameters are different). More generally, stable random variables are stable or invariant under "weighted" addition.[17]

Some properties of the stable random variables, which are useful in deriving a generalized version of the APT are summarized in Lemma 1.

Lemma 1:

(i) For each $p \in (0,\alpha)$ and each $q \in (0,\alpha)$, there exist constants A and B such that for each $n \geq 1$ and any $\{d_i\} \in \mathbf{R}^n$,

$$A(\Sigma_{i=1}^n |d_i|^p)^{1/p} \leq (E(\Sigma_{i=1}^n |d_i e_i|^p)^{\alpha/p})^{1/\alpha} \leq B(\Sigma_{i=1}^n |d_i|^p)^{1/p}.$$

---

[16] It is well known that the class of stable distributions provides a generalization of the normal central limit theorem; i.e., if a weighted sum of random variables has a limiting distribution, the limiting distribution is a member of the stable class. The stable nonnormal distributions generalize the central limit theorem to the case in which the variances of the underlying variables do not exist.

[17] For example, let $x_1, x_2, \ldots, x_n$ be independent symmetric stable variables whose distributions have the same characteristic exponent α, but different (possibly) location and scale parameters a_i and σ_i. Then $z_n = \Sigma_{i=1}^n d_i x_i$, is symmetric stable with characteristic exponent α and with location and scale parameters as follows:

$$E(z_n) = \Sigma_{i=1}^n d_i a_i,$$

$$\sigma(z_n) = [\Sigma_{i=1}^n \sigma_i |d_i|^\alpha], \text{ for all constants } d_i.$$

The location parameter of asset i's return stable distribution is the expected asset return a_i if $\alpha > 1$. The notation δ is used in the statistics literature. For notation consistency, we will replace δ_i by a_i for all i. Throughout the discussion, we will assume $\alpha > 1$ so that the first moment exists.

(ii) For each $q \in (0,\alpha)$, and each $p \in (\alpha,\infty)$, there exist constants A and B such that for each $n \geq 1$ and any $\{d_i\} \in \mathbf{R}^n$,

$$A(\Sigma_{i=1}^n |d_i|^\alpha)^{1/\alpha} \leq (E(\Sigma_{i=1}^n |d_i e_i|^p)^{q/p})^{1/q} \leq B(\Sigma |d_i|^\alpha)^{1/\alpha}.$$

Proof: See Schwartz (1969/70) and Woyczynski (1978). Q.E.D.

III.C. The APT with Stably Distributed Asset Returns

As discussed in the basic model, the APT in the literature is derived assuming finite variance. When the second moments of asset returns do not exist, we need to use an alternative concept of asymptotic arbitrage called NACPM and a new concept of approximate linear pricing relation called ALPR-p. These concepts are defined as follows.

Definition: The no-asymptotic-arbitrage condition defined in terms of convergence in p^{th} mean (**NACPM**) satisfies the requirement that

when $\mathbf{w}'\mathbf{1} \dashrightarrow 0$, and $\mathbf{w}'\mathbf{x} - \mathbf{w}'\mathbf{a} \xrightarrow{L_p} 0$, for some $p \geq 1$,

then $\mathbf{w}'\mathbf{a} \dashrightarrow 0$,

where $\xrightarrow{L_p} 0$ denotes the convergence to zero in p^{th} mean. ∎

Definition: An approximate linear pricing relation expressed in term of norm defined by $\|\mathbf{x}\| \equiv (\Sigma |x_i|^p)^{1/p}$ (**ALPR-p**) can be stated as

$\mathbf{a} = \mathbf{B}^*\mathbf{c} + \mathbf{v}$, and $\lim_{n\to\infty} \|\mathbf{v}\|_p < \infty$,

where $\|\mathbf{v}\|_p \equiv (\Sigma_{i=1}^n |v_i|^p)^{1/p})$ and $p \geq 1$. ∎

Note that $\lim_{n\to\infty} \|\mathbf{v}\|_p < \infty$ is stronger than $\lim_{n\to\infty} \|\mathbf{v}\|_2 < \infty$, since $\|\mathbf{v}\|_p > \|\mathbf{v}\|_2$ if $p < 2$.

Proposition 2: Given the linear factor structure, (1), with the exception that the idiosyncratic risks, e_i, are i.i.d. symmetric stable random variables with characteristic exponent $\alpha \in (1,2)$, the NACPM (for $p \in (1,\alpha)$) implies ALPR-p.

Proof: As in Proposition 1, by orthogonal projection,

$$\mathbf{a} = \mathbf{B}^{*}\mathbf{c} + \mathbf{v}, \text{ and } \mathbf{B}^{*}\mathbf{v} = \mathbf{0}.$$

Then, the proposition can be proved by contradiction.

Suppose that $\|\mathbf{v}\|_p \to \infty$ as $n \to \infty$.

Let $\mathbf{w} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|_p^p}$,

then $\mathbf{w}$ is a zero-cost portfolio.

$$z = \mathbf{w}'\mathbf{x} = \frac{1}{\|\mathbf{v}\|_p^p}(\mathbf{v}'\mathbf{a} + \mathbf{v}'\mathbf{e}).$$

$$E(z) = \mathbf{w}'\mathbf{a} = \frac{\|\mathbf{v}\|_2^2}{\|\mathbf{v}\|_p^p} \geq \min(1, \delta\min_i\{v_i \mid v_i \neq 0\}) > 0, \text{ for some } \delta > 0.$$

$$E|z - E(z)|^p = E|\mathbf{w}'\mathbf{x} - \mathbf{w}'\mathbf{a}|^p = E|\mathbf{w}'\mathbf{e}|^p = E|\Sigma_{i=1}^n w_i e_i|^p$$

$$\leq B(\Sigma_{i=1}^n |w_i|^p) = B\|\mathbf{v}\|_p^{(1-p)p}.$$

The last inequality follows from (ii) of Lemma 1.

Now $\lim_{n\to\infty} E(z_n) > 0$ and $\lim_{n\to\infty} E|z_n - E(z_n)|^p = 0$ which violate the NACPM, thus $\lim_{n\to\infty}\|\mathbf{v}\|_p < \infty$. Q.E.D.

Note that, in Proposition 2, we did not make any assumption on the stochastic properties of the factor, f_k. We only assume that the idiosyncratic risks, e_i, are stably distributed.

To obtain the necessary condition result in Proposition 3, we assume that all interrelationships among the returns on individual assets are generated by the linear K-factor model:

$$x_i = a_i + \Sigma_{k=1}^K b_{ik} f_k + e_i, \qquad i = 1, 2, \ldots, n.$$

where f_k are assumed to be mutually independent, independent of the e_i and both

e_i and f_k are stably distributed the same value of α. The market model and the market-industry model in Fama and Miller (1972) are special cases of this model. It can be shown that x_i are also stably distributed with characteristic exponent α.

The following lemma will be used in the proof of Proposition 3 which simply says no-arbitrage condition is a necessary condition for the linear pricing relation.

Lemma 2: If $E|X_n + Y_n|^p \dashrightarrow 0$ for some $p > 0$ and X_n and Y_n are independent for all n, then $E|X_n|^p \dashrightarrow 0$ and $E|Y_n|^p \dashrightarrow 0$.

Proof: See Appendix B. Q.E.D.

Proposition 3: Suppose the linear factor structure (1) holds, except that the idiosyncratic risks, e_i, and the factors, f_k, are now assumed to be i.i.d. symmetric stable random variables with characteristic exponent $\alpha \in (1,2)$, then:

the approximate linear pricing relation expressed in term of norm defined by $\|\mathbf{x}\| \equiv (\Sigma|x_i|^q)^{1/q}$ **(ALPR-q)** implies the NACPM, where $p \in (1,\alpha)$, p is such that $1/p + 1/q = 1$, $p \le q < \infty$.

Proof: $\mathbf{w'x} - \mathbf{w'a} \xrightarrow{L_p} 0 \iff E|\mathbf{w'Bf} + \mathbf{w'e}|^p \dashrightarrow 0$

$$\implies E|\mathbf{w'Bf}|^p \dashrightarrow 0 \quad \text{and}$$

$$E|\mathbf{w'e}|^p \dashrightarrow 0. \qquad (5)$$

(5) follows from Lemma 2.

According to **(ii)** of Lemma 1,

$$E|\Sigma_{k=1}^{K} u_{nk} f_k|^p \ge A_K \Sigma_{k=1}^{K} |u_{nk}|^p, \text{ where } u_{nk} = \Sigma_{i=1}^{n} w_i b_{ik}.$$

The fact that $E|\Sigma_{k=1}^{K} u_{nk} f_k|^p \dashrightarrow 0$ leads to

u_{nk} --> 0 for all k. (6)

From (ii) of Lemma 1,

$A_n \Sigma_{i=1}^{n} |w_i|^p \leq E|\Sigma_{i=1}^{n} w_i e_i|^p$ --> 0

==> $\Sigma_{i=1}^{n} |w_i|^p$ --> 0 (7)

Let $c_1' = (c_1, c_2, \ldots, c_k)$.

By (i), $\mathbf{w'a} = \mathbf{w'B^{-}c} + \mathbf{w'v}$

$= \mathbf{w'}1c_0 + \mathbf{w'Bc_1} + \mathbf{w'v}$

--> $\mathbf{w'v}$

since $\Sigma_{k=1}^{K} u_{nk} c_k$ --> 0 by (6) and K is finite.

$\lim_{n \to \infty} \|\mathbf{v}\|_q < \infty$ and (7) imply that $\mathbf{w'v}$ --> 0 by Holder's inequality.

Therefore $\mathbf{w'a}$ --> 0. Q.E.D.

Corollary 2: Suppose the linear factor structure (1) holds, except that the idiosyncratic risks, e_i, and the factors, f_k, are assumed to be i.i.d. symmetric stable random variables with characteristic exponent $\alpha \in (1,2)$, then:

[$\mathbf{w'1}$ --> 0, and $\mathbf{w'x} - \mathbf{w'a}$ $-L_p$-> 0 for $p \in (1,\alpha)$ ==> $\mathbf{w'a}$ --> 0]

<==> $\mathbf{a} = \mathbf{B^{-}c} + \mathbf{v}$, and $\lim_{n \to \infty} \|\mathbf{v}\|_q < \infty$,

where $\|\mathbf{v}\|_q \equiv (\Sigma_{i=1}^{n} |v_i|^q)^{1/q})$ and p is such that $1/p + 1/q = 1$.

Proof: See Appendix C. Q.E.D.

IV. A Generalized Arbitrage Pricing Theory

In the previous section, we prove the APT under the assumption of i.i.d. stably distributed idiosyncratic risks, which is unnecessarily strong. In this section, we will prove the APT under a more general structure which includes the cases of infinite second moments and weak dependence among idiosyncratic risks. The notion of weak-dependence relation (lacunary system) is introduced first. The APT is then derived under this more general setting.

IV.A. The Lacunary System

Definition: Given $p > 0$, a sequence of real-valued random variables $\{e_i\}$ is called a **lacunary** system of order p, or an S_p system, if there exists a positive constant K_p such that for any sequence of real constants d_i,

$$E|\Sigma_{i=m}^{n} d_i e_i|^p \le K_p(\Sigma_{i=m}^{n} d_i^2)^{p/2} , \quad \text{for all } n \ge m.$$

If the system $(e_i\}$ is an S_p system for every $p > 2$, then it is called an S_∞ system (Gaposhkin, 1966). ∎

The lacunary system has properties that are, in some sense, similar to properties of systems of independent variables. In fact, we may call it a weakly dependent system. Here, the given property must be "hereditary" for the system $\{e_i\}$, that is, it must remain valid for every subsystem of the given system. If $\{e_i\}$ is a lacunary system, then $\Sigma_{i=1}^{\infty} d_i e_i$ is called a lacunary series. A few examples may help to clarify the nature of the lacunary systems.

Example 1: If $\{e_i\}$ are independent random variables such that $E(e_i) = 0$ for all i, and $\sup_i E|e_i|^p < \infty$ for some $p \ge 2$. Then $\{e_i\}$ is an S_p system.∎∎

Example 2: Let $\{e_i\}$ be i.i.d. standard normal random variables, then $\{e_i\}$ is an S_p system since $E|\Sigma_{i=1}^{n} d_i e_i|^p = K_p(\Sigma_{i=1}^{n} d_i^2)^{p/2}$, where $K_p \equiv E|N(0,1)|^p$.∎∎

Example 3: If $\{e_i\}$ are i.i.d. Bernoulli random variables and $P\{e_i=1\} = P\{e_i=-1\} = 1/2$. Then for every $p > 0$ there exists a positive constant K_p such that $E|\Sigma_{i=1}^{n} d_i e_i|^p \le K_p(\Sigma_{i=1}^{n} d_i^2)^{p/2}$ for all $n \ge 1$ and all d_i. Thus $\{e_i\}$ is an S_∞ system. See Khintchine (1924) for a proof. ▪▪

Example 4: Suppose that $\{e_i\}$ are i.i.d. with a symmetric stable distribution and that the characteristic exponent α is between 1 and 2 ($1 < \alpha < 2$), then $\{e_i\}$ is a lacunary system. Note that expected values exist only when $\alpha > 1$.

Proof: According to Feller (1971: 171),

$$E|\Sigma_{i=m}^{n} d_i e_i|^p = (\Sigma_{i=m}^{n} |d_i|^p) E|e_i|^p,$$

for all $d_i \in R^{n-m+1}$, $m \le n$, and $1 < p < \alpha$.

Since there exists a positive constant θ such that

$$(\Sigma_{i=m}^{n} |d_i|^p)^{1/p} \le \theta(\Sigma_{i=m}^{n} |d_i|^2)^{1/2},$$ [18]

therefore, $E|\Sigma_{i=m}^{n} d_i e_i|^p \le K_p(\Sigma_{i=m}^{n} d_i^2)^{p/2}$, where $K_p = \theta^p E|e_i|^p$.

Thus, $\{e_i\}$ is a lacunary system of order p. ▪▪

Example 5: Suppose that $E(e_i) = 0$, $Var(e_i) = 1$ for all i, and

$$E(e_i e_j) = Cov(e_i, e_j) = \pi > 0 \text{ for } i \ne j,$$

then $\{e_i\}$ is not an S_2 system.

Note that the largest eigenvalues of the variance-covariance matrix of $\{e_i\}$, $\mathbf{\Omega}$, is $1 + (n-1)\pi$ which diverges to infinity as $n \to \infty$.[19] Thus $\{e_i\}$ does not meet Chamberlain and Rothschild's (1983) criteria. However, it can be shown that it is a lacunary system of order 1.[20] ▪▪

---

[18] As $n \to \infty$, it is assumed $\{d_i\}$ belongs to $\mathbf{R}^\infty$, an infinite dimensional complete linear space. Since $\|\mathbf{d}\|_2 < \|d\|_p$, by equivalent norm theorem (Royden, 1968), there exist a $\theta > 0$ such that $\theta\|\mathbf{d}\|_2 \ge \|\mathbf{d}\|_p$, where $\mathbf{d} = (d_m \ldots d_n)$, $m \le n$.

[19] The other eigenvalues are all equal to $1-\pi$. See, for example, Anderson (1971:289).

[20] $E\|\Sigma d_i e_i\| \le E(\Sigma\|d_i e_i\|) \le K_0 \Sigma\|d_i\| \le K_1(\Sigma\|d_i\|^2)^{1/2}$, where $K_0 \equiv \max_i E\|e_i\| \le Var(e_i\|e_i = \arg\{\max_i\|e_i\|\}) = 1 < \infty$; $K_1 \equiv \theta K_0$. The last inequality is due to the argument in Example 4.

IV.B. Lacunary System of Idiosyncratic Risks and the APT: Sufficient Conditions

In this section, we will provide sufficient conditions for the ALPR. The assumptions (1f) and (1g) in the basic model are relaxed substantially. The L-type factor structure replaces the linear factor structure [equation set (1)] of the basic model. We will show that the APT holds in such a more general factor structure.

Definition: In a **L-type factor structure**, we have the following linear K-factor model:

$$x = \mathbf{a} + \mathbf{Bf} + \mathbf{e} \ ,$$

where $\{e_i\}$ is a lacunary system of order p for some $p \geq 1$, and without loss of generality, **a** and **B** are normalized such that $E(\mathbf{e}) = \mathbf{0}$, and $E(\mathbf{f}) = \mathbf{0}$. ∎

Theorem 1: Given the L-type factor structure, the NACPM implies the ALPR stated in (3c).

Proof: Setting the portfolio weights, **w**, as:

$$\mathbf{w} = \frac{\mathbf{v}}{\|\mathbf{v}\|^r} \ , \text{ where } r \in (1,2] \qquad \text{and}$$

using the definition of the lacunary system, we can prove this theorem in the same fashion (by contradiction) as Proposition 2. Q.E.D.

The assumptions used in Theorem 1, $E(\mathbf{f}) = \mathbf{0}$ and $E(\mathbf{e}) = \mathbf{0}$, are rather weak and innocuous. We do not have to impose restrictions on the form of the multivariate distribution of (f,e) beyond the requirement that $\{e_i\}$ is a lacunary system. In particular, neither need f_k be jointly independent, nor need they be independent of the e_i. They need not possess variances. None of the random variables need be normally distributed. In the early APT literature (Ross, 1976; Huberman, 1982), the assumption of uncorrelated idiosyncratic risks is quite strong. Chamberlain and Rothschild (1983) introduced the concept of an "approximate factor structure" to prove the APT with weaker qualification on the asset return generating process.

However, Grinblatt and Titman (1985) demonstrated that the notion of an approximate factor structure is not a significantly weaker restriction on the return generating process than that in Ross (1976) and Huberman (1982). They argued that "Any economy that satisfies the Chamberlain-Rothschild approximate factor structure can be transformed, in a manner that does not alter the characteristics of investor portfolios, into an economy that satisfies the Ross exact factor structure. This insight also applies to Ingersoll (1984) economy that yields the equal-weighted pricing bound, ..." (Grinblatt and Titman, 1985, 1369-68). Using a different definition of no-asymptotic-arbitrage condition, we generalize the APT to allow for the nonexistence of the second moments of asset returns. In the general model, for example, when p is less than two, variances need not exist. We can simply model investors' preference as a function of mean and "dispersion". The latter (dispersion) can be measured in terms of mean absolute deviation (MAD) or mean of some power (less than two) of the absolute deviation, instead of variance. The nature of investors' utility functions is a yet to be resolved empirical issue. Our generalization of the APT from a restrictive mean-variance framework could be regarded as a real extension of the APT.

Furthermore, applying Theorem 1, we can easily get Corollary 3, a slightly more general interpretation of the APT.

Corollary 3: Given the L-type factor structure, the no-asymptotic-arbitrage condition:

[$\mathbf{w}'\mathbf{1} \longrightarrow 0$ and $\mathbf{w}'\mathbf{x} - \mathbf{w}'\mathbf{a} \xrightarrow{L_r} 0 \Longrightarrow \mathbf{w}'\mathbf{a} \longrightarrow 0$ with $1 \le r \le p$]

implies the ALPR in (3c).

Theorem 1 shows that the L-type factor structure and the NACPM are the sufficient conditions for an ALPR. The L-type factor structure is rather general, though. Result in Lemma 3 will be used in showing this point later. Lemma 3 also summaries some important properties of a lacunary system.

Lemma 3:

(i) Let $0 < q < p$. If $\{e_i\}$ is an S_p system, then it is also an S_q system.

(ii) Let $\{e_i\}$ be a Gaussian system with zero mean. If $\{e_i\}$ is an S_q system for some $q > 0$, then it is an S_∞ system.

(iii) Let $\{e_i\}$ be an S_p system for some $p \ge 2$. Let $\{d_i\}$ be a sequence of real constants. Suppose that $\Sigma_{i=m}^{n} d_i^2 < \infty$. Then the series $\Sigma_{i=m}^{n} d_i e_i$ converges in L_p norm. Moreover, $E(\sup_{n\ge m}|\Sigma_{i=m}^{n} d_i e_i|^p) < \infty$.

Proof:

(i) From Hölder's inequality, $E|x|^q \le (E|x|^p)^{q/p}$ for $0 < q < p$. The result follows immediately.

(ii) See Lai and Wei (1983).

(iii) See Gaposhkin (1966). Q.E.D.

Thus if we restrict the distribution of $\{e_i\}$ to be Gaussian and assume that $\{e_i\}$ is an S_q system for some $q > 0$. Then we can relax the assumption of no arbitrage in proving the APT by invoking (ii) of Lemma 3.

Corollary 4: Given the linear factor structure (1) except letting $\{e_i\}$ be a Gaussian system with zero mean and be a S_q system for some $q > 0$,[21] then the no-asymptotic-arbitrage condition: for all $\mathbf{w} \in \mathbf{R}^n$,

$$\mathbf{w}'1 \to 0 \quad \text{and} \quad \mathbf{w}'\mathbf{x} - \mathbf{w}'\mathbf{a} \xrightarrow{L_r} 0 \quad \text{for all } 1 \le r < \infty$$

$$\Longrightarrow \quad \mathbf{w}'\mathbf{a} \to 0,$$

implies the ALPR in (3c).

Unlike Corollary 3, the value of r in Corollary 4 can be any value that is not less than 1.

The lacunary system may not be a familiar concept to many of the readers. In Lee and Wang (1988), another structure which is used quite intensively in probability and statistics literature is introduced. It can be shown that if $\{e_i\}$ is an L_2 bounded martingale difference system such that $E(e_i|e_j) = 0$ for all $j < i$ and $\sup_i E|e_i|^2 < \infty$, and if there is NACPM, then the ALPR holds. However, the assumption of martingale difference system is unnecessarily restrictive. Here, we propose a general structure, the "generalized linear process", for the idiosyncratic risks, and show that the ALPR still holds in this setting.

Definition: If $\{u_h\}$ is any orthonormal sequence ($E(u_h) = 1$ and $E(u_h u_j) = 0$ for all $h \neq j$), the sequence

$$e_i = \text{l.i.m.}_{N \to \infty} \Sigma_{h=-N}^{N} \mathbf{m}_{ih} u_h, \tag{8}$$

[21] To satisfy this condition, one can just assume e_i to be i.i.d. standard normal random variables.

is called a **generalized linear process** generated by $\{u_h\}$ where the m_{ih} are constants such that $\Sigma_{i=1}^{\infty} m_{ih}^2 < \infty$.[22] ▪

The notation, l.i.m., means limit in quadratic mean (which exists by the assumption of orthonormality and $m_{ih} \in l_2$). According to Doob (1953), every wide-sense stationary sequence $\{e_i\}$ can be represented in the form of (8) if and only if its spectral density is absolutely continuous. If the orthonormal u_h are i.i.d., the convergence in (8) also holds almost surely and the sequence $\{e_i\}$ defined by (8) is called a linear process. We can apply the concept of generalized linear process to describe the stochastic process of asset returns as a G-type factor structure.

Definition: In a **G-type factor structure**, we have the following linear K-factor model:

$$\mathbf{x} = \mathbf{a} + \mathbf{Bf} + \mathbf{e} \ ,$$

where $\{e_i\}$ is a generalized linear process generated by an orthonormal S_p system $\{u_h\}$ with $p \geq 2$. Moreover, it is assumed that ess $\sup_{0 \leq \theta \leq 2\pi} f(\theta) < \infty$ (ess sup ≡ essential supremium), where f is the spectral density of $\{e_i\}$. Without loss of generality, **a** and **B** are normalized such that $E(\mathbf{e}) = \mathbf{0}$, and $E(\mathbf{f}) = 0$.[23] ▪

---

[22] From this definition, it follows that
$E(\Sigma_{i=1}^{n} e_i)^2 = \Sigma_{h=-\infty}^{\infty} (\Sigma_{i=1}^{n} m_{ih}^2) = o(n^2)$
and therefore $\{e_i\}$ satisfies the weak law of large numbers: $n^{-1}\Sigma_{i=1}^{n} e_i \xrightarrow{p} 0$. The definition in (8) also provides an important stochastic model in the engineering literature, where the sequence $\{u_h\}$ is a white noise sequence and {ei} is the output sequence obtained by passing $\{u_h\}$ through a linear filter defined by $\{m_{ih}\}$ (cf. Kailath, 1974).

[23] The sequence $\{e_i\}$ is thus wide-sense stationary and has a spectral density f.

Corollary 5: Given the G-type factor structure, the NACPM implies the ALPR in (3c).

Proof: It can be proved that $\{e_i\}$ is an S_p system with $p \geq 2$ (Lai and Wei, 1983, p.189). Then, the corollary can be derived by applying Theorem 1.

Q.E.D.

IV.C. Pricing Error Bound and Dependence Structure of Idiosyncratic Risks

In the above subsection, we examine the trade-off between the constraint on the factor structure and the no-asymptotic-arbitrage condition in deriving an approximate pricing relation. From Theorem 1, however, we can conjecture that the bound on errors in the linear pricing relation depends crucially on the assumption about the dependence structure of the idiosyncratic risks. The strength of constraint on the idiosyncratic risks seems to affect the tightness on the pricing error bound.

To study the relation between the pricing error bound and the dependence structure of idiosyncratic risks, we need to introduce another concept, the **S_{pq} system**.

Definition: The sequence of idiosyncratic risks, $\{e_i\}$, is an S_{pq} system if there exists a positive constant K_p for any sequence of real constants d_i such that

$$E|\Sigma_{i=1}^{n} d_i e_i|^p \leq K_p(\Sigma_{i=1}^{n}|d_i|^q)^{p/q} \qquad \text{for all } n \geq 1,$$

where $0 < q < \infty$. ∎

Remark: When $q = 2$, the S_{pq} system is a lacunary system of order p.

Example 6: Suppose that $\{e_i\}$ are i.i.d. standard normal random variables. Since

$$E|\Sigma_{i=1}^{n} d_i e_i|^p = (\Sigma_{i=1}^{n} d_i^2)^{p/2} E|N(0,1)|^p \le E|N(0,1)|^p (\Sigma_{i=1}^{n} d_i^{\alpha})^{p/\alpha} ,$$

for some $\alpha \ge 2$. Thus, $\{e_i\}$ is an $S_{p\alpha}$ system. ∎∎

Example 7: Suppose that $\{e_i\}$ is a sequence of i.i.d. symmetric stable random variables with characteristic exponent $\alpha \in (1,2)$, then according to **(iii)** of Lemma 1 (with $q = 1$), there exists a positive real constant H_p such that

$$E(\Sigma_{i=1}^{n} |d_i e_i|^p) \le H_p (\Sigma_{i=1}^{n} |d_i|^{\alpha})^{p/\alpha} \text{ for } p \in (\alpha, \infty).$$

Thus, $\{e_i\}$ is an $S_{p\alpha}$ system. ∎∎

Corollary 6: If the sequence of idiosyncratic risks is a S_{pq} system ($p \ge 2$; $0 < q < \infty$), then NACPM implies the following approximate linear pricing relation,

$$\mathbf{a} = \mathbf{B}_n \mathbf{c} + \mathbf{v}, \text{ and } \lim_{n \to \infty} \|\mathbf{v}\|_q < \infty,$$

where $\|\mathbf{v}\|_q \equiv (\Sigma_{i=1}^{n} |v_i|^q)^{1/q}$.

Proof: Apply Theorem 1. Q.E.D.

From Corollary 6, we can see that if the sequence of idiosyncratic risks is a lacunary system ($q = 2$), the bound is expressed in terms of Euclidean norm, which is the approximate linear pricing relation usually studied in the literature. If the assumption we make about the idiosyncratic risks is stronger, a tighter bound is obtained, and vice versa. Theorem 1 is a special case of Corollary 6.

IV.D. Necessary and Sufficient Conditions for a General APT

So far we have provided, in several kinds of factor structure, a variety of sufficient conditions for the ALPR. Now let's turn to the issue of necessary condition. The derivation of the necessary as well as the sufficient condition for the general APT has important empirical research implications.[24]
First, if the linear pricing model is not supported by the data, then we could assert that arbitrage opportunities do exist or that the assumption of linear K-factor structure is not tenable according to the data used in testing the APT. If it is the "arbitrage opportunities" that ruins the ALPR, we might infer that transaction costs are so large the utility-maximizer are deterred from taking these advantages. Secondly, suppose that the ALPR holds, from the if-and-only-if relation we can say there are no arbitrage opportunities in the economy. In addition, the ALPR can be tested by examining the existence of arbitrage opportunities. Given the data available, if we can find just one arbitrage portfolio with zero cost and zero absolute central pth moment (e.g. variance) has a positive mean, then the hypothesis of linear pricing may be rejected.

Before stating the formal theorem, we first introduce the concepts of Banach system and Bessel inequality which are needed in deriving the necessary condition.

Definition: A sequence of real-valued random variables $\{e_i\}$ is called a **Banach system** if there exists a positive constant H such that, for every sequence of real constants d_i,

---

[24] Testability of the APT has always been a controversial issue in the literature (Shanken, 1982; Dybvig and Ross, 1985). Strictly speaking, it is not

$$E|\Sigma_{i=m}^{n} d_i e_i| \geq H(\Sigma_{i=m}^{n} d_i^2)^{1/2} \qquad \text{for all } n \geq m .$$ ∎

Clearly, if $\{e_i\}$ is a Banach system, then for every $p \geq 1$, there exists a $H_p > 0$ such that

$$E|\Sigma_{i=m}^{n} d_i e_i|^p \geq H_p(\Sigma_{i=m}^{n} d_i^2)^{p/2} \qquad \text{for all } n \geq m$$

holds for all constant d_i (Banach, 1930). A few examples may help the reader to gain some insight about the nature of the Banach system. They are given as follows.

Example 8: Let $\{e_i\}$ be i.i.d. standard normal random variables. Then $\{e_i\}$ is a Banach system since

$$E|\Sigma_{i=1}^{n} d_i e_i|^p = H_p(\Sigma_{i=1}^{n} d_i^2)^{p/2}$$

where $H_p \equiv E|N(0,1)|^p$. ∎∎

Example 9: If $\{e_i\}$ are i.i.d. Bernoulli random variables and

$$P\{e_i=1\} = P\{e_i=-1\} = 1/2.$$

Then, for every $p > 0$, there exists a positive constant H_p such that

$$H_p(\Sigma_{i=1}^{n} d_i^2)^{p/2} \leq E|\Sigma_{i=1}^{n} d_i e_i|^p , \quad \text{for all } n \geq 1 \text{ and all } d_i.$$

Thus, $\{e_i\}$ is a Banach system. See Khintchine (1924) for a proof. ∎∎

Example 10: Let $\{e_i\}$ be an orthonormal sequence. Then $\{e_i\}$ is an S_r system for all $0 < r \leq 2$. Furthermore, if $\{e_i\}$ is an S_p system for some $p > 2$, then, as shown by Gaposhkin (1966), $\{e_i\}$ is a Banach system. ∎∎

Definition: A sequence of random variables $\{e_i\}$ is said to satisfy the **Bessel inequality** if there exists $M > 0$ such that for all constants d_i

$$E|\Sigma_{i=m}^{n} d_i e_i|^2 \geq M\, \Sigma_{i=m}^{n} d_i^2 , \qquad \text{for all } n \geq m.$$ ∎

Lemma 4: If $\{e_i\}$ is an lacunary system of order p for some $p \geq 2$, and if it also satisfies the Bessel inequality, then $\{e_i\}$ is a Banach system (Banach, 1930).

Proof: The lemma can be proved by using Schwarz inequality, Lyapounov's inequality and Hölder's inequality. For a complete proof, see Gaposhkin (1966). Q.E.D.

Definition: The linear factor structure is a **B-type factor structure** if the asset return vector, **x**, satisfies the requirement of a basic linear factor structure, i.e, condition (1); $E(\mathbf{ff}') = \mathbf{I}_K$; e and **f** are independent; and $\{e_i\}$ is a Banach system. ∎

Theorem 2: Given a B-type factor structure, the NACPM and ALPR are equivalent.

Proof: Same as in the proof of Proposition 3, by Lemma 2, we have

$$E|\mathbf{w}'\mathbf{B}\mathbf{f}|^p \dashrightarrow 0 \text{ and } E|\mathbf{w}'e|^p \dashrightarrow 0.$$

Since $p \geq 2$, $E|\Sigma_{k=1}^{K} u_{nk} f_k|^p \geq (E|\Sigma_{k=1}^{K} u_{nk} f_k|^2)^{p/2} = (\Sigma_{k=1}^{K} u_{nk}^2)^{p/2}$,

where $u_{nk} = \Sigma_{i=1}^{n} w_i b_{ik}$. The first inequality is due to Lyapounov.

The fact that $E|\Sigma_{k=1}^{K} u_{nk} f_k|^p \dashrightarrow 0$ leads to $\Sigma_{k=1}^{K} u_{nk}^2 \dashrightarrow 0$.

Thus, $u_{nk} \dashrightarrow 0$, for all k. (9)

From Lemma 4,

$$A_p(\Sigma_{i=1}^{n} w_i^2)^{p/2} \leq E|\Sigma_{i=1}^{n} w_i e_i|^p \dashrightarrow 0 \implies \Sigma_{i=1}^{n} w_i^2 \dashrightarrow 0 . \quad (10)$$

From (9), we have $\mathbf{w}'\mathbf{a} \dashrightarrow \mathbf{w}'\mathbf{v}$.

Hence, $\lim_{n \to \infty} \|\mathbf{v}\| < \infty$ and (10) imply that $\mathbf{w}'\mathbf{v} \dashrightarrow 0$ by Schwarz inequality.

Therefore, $\mathbf{w}'\mathbf{a} \dashrightarrow 0$. Q.E.D.

Remark: In Theorem 2, we do not have to assume that $E(\mathbf{ff}') = \mathbf{I}_K$. When

$1 \leq p < 2$, a weaker condition that $\{f_k\}$ is a lacunary system of order p would be sufficient.

V. A Synthesis

In this section, we synthesize the APT literature and demonstrate that most of the results regarding the ALPR are special cases of Theorem 1, which is based on the L-type factor structure. Owen and Rabinovitch (1983) proposed a class of elliptical distributions to describe asset returns. We establish the validity of the APT when the asset return random variable are elliptically (spherically) distributed. The proof is a simple application of Theorem 1.

V.A. The APT Literature in Light of the General APT

To prove the APT with correlated idiosyncratic risks (Chamberlain and Rothschild, 1983; Ingersoll, 1984), we can just apply Theorem 1 with the assumption that $\{e_i\}$ is a lacunary system with order 2. Note that $\{e_i\}$ do not have to satisfy the Bessel inequality here. The reason is as follow. Because the condition that $\{e_i\}$ is a S_2 system is equivalent to that of

$$E(\Sigma_{i=1}^{n} w_i e_i)^2 \leq K_2(\Sigma_{i=1}^{n} w_i^2) \text{ for all } w.$$ [25]

[25] A system $\{e_i\}$ such that the quadratic form $\Sigma\Sigma\sigma_{ij} d_i d_j$ is bounded is called "quasi-orthogonal" (Kac, Salem and Zygmund, 1948). Given any sequence $\{e_i\}$ of functions belonging to L^2 in (a,b), the necessary and sufficient condition for quasi-orthogonality is that $\Sigma d_i e_i$ should converge in the mean with order 2 for any sequence $\{d_i\}$ with $\Sigma d_i^2 < \infty$. The necessity follows directly from Bessel inequality and the sufficiency from the fact that under the hypothesis of convergence in the mean with order 2, the integral

$$\int (\Sigma d_i e_i)^2 dx$$

is bounded for every sequence $\{d_i\}$ with $\Sigma d_i^2 < \infty$.

This is also equivalent to the condition that $\|\Omega\|$ is bounded (Ω is the variance-covariance of the idiosyncratic risks and $\|\Omega\| \equiv \sup_w w'\Omega w/w'w$).[26] Chamberlain and Rothschild (1983) assume that only K eigenvalues of the returns' variance-covariance matrix become unbounded. Stambaugh (1983) assumes that it is possible to decompose the returns' variance-covariance matrix as $\Sigma = \mathbf{BB'} + \mathrm{D} - \mathbf{A}$ where $\mathbf{D}$ is a diagonal matrix with bounded elements and $\mathbf{A}$ is nonnegative definite. Stambaugh (1983, footnote 11) has demonstrated that these two conditions are equivalent. The assumption of $\|\Omega\| \le \alpha < \infty$ (Ingersoll, 1984, Theorem 2) for some $\alpha > 0$ is likewise equivalent. This is shown in Ingersoll's footnote 8 (1984). We, thus, have the following corollary.

Corollary 7 (Chamberlain and Rothschild, 1983): Given the L-type factor structure with p = 2, (i.e., the idiosyncratic risks' variance-covariance matrix, Ω, is positive semi-definite and its largest eigenvalue is finite), then the NACQM described in (2') implies the ALPR stated in (3c).

Definition: Let p be a positive even integer. A sequence of random variables $\{e_i\}$ is said to be **multiplicative of order p** if $E(e_{i(1)} \cdot e_{i(2)} \cdot \ldots \cdot e_{i(p)}) = 0$ for all $i(1) < i(2) < \ldots < i(p)$ (Kolmós, 1972).

Remark: When p = 2, the multiplicative sequence of order p is reduced to the case of orthogonal random variables.

---

[26] Suppose $\mathbf{x}$ and $\mathbf{y} \in \mathbf{M}$ where $\mathbf{M}$ is the space of the nonnegative definite symmetric matrix space. It can be proved that $\| \ \|$ is indeed a proper norm;
(i) $\mathbf{x} = 0$ if and only if $\|\mathbf{x}\| = 0$.
(ii) $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$ since $g_1(\mathbf{x}+\mathbf{y}) \le g_1(\mathbf{x}) + g_1(\mathbf{y})$, where g_1 is the largest eigenvalue of the matrix (Rao, 1973: 68).
(iii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$, α is a real number.

Corollary 7 (Huberman, 1982): Let the sequence of idiosyncratic risks, $\{e_i\}$, be a multiplicative sequence of order 2 such that $\sup_i E(e_i^2) < \infty$, then the NACQM in (2') implies the ALPR in (3c).

Proof: The assumption implies that the sequence of random variables $\{e_i\}$ is an S_2 system. Use Theorem 1. Q.E.D.

Remark: Assume that $\{e_i\}$ is a multiplictive sequence of order p ($p \geq 4$ and is an even integer) such that $\sup_i E(|e_i|^p) < \infty$. Kolmós (1972) shows that $\{e_i\}$ is a lacunary system. Furthermore, if $\inf_i E(e_i^2) > 0$ then $\{e_i\}$ satisfies the Bessel inequality which implies that $\{e_i\}$ is a Banach system by Lemma 4.

Longnecker and Serfling (1978) show that if p is even and $\{e_i\}$ is a weakly multiplictive sequence of type A_p or B_p or C_p such that $\sup_i E|e_i|^p < \infty$, then $\{e_i\}$ is an S_p system. Therefore, L_p-bounded multiplictive sequences of even order p are S_p systems. A more extensive treatment of this issue is given in Lee and Wang (1988).

V.B. Non-normality: Elliptical Distributions

Owen and Rabinovitch (1983) proposed to use the class of elliptical (spherical) distributions to describe the stochastic properties of asset returns. They show the validity of Tobin's (1958) separation theorem, Bawa's (1975) rules of ordering uncertain prospects, Ross' (1978) mutual fund separation theorem and the CAPM in this class of distributions which are neither always normal nor necessarily stable.[27] They generalized the mean-covariance matrix framework to a mean-characteristic matrix framework in which the characteristic matrix is the basis for a spread or risk measure. Similarly, in the following corollary, the framework of convergence in mean and variance is generalized to one that is convergent in mean and dispersion in which dispersion is measured by the characteristic matrix.

Corollary 9: Consider the linear factor structure, $\mathbf{x} = \mathbf{a} + \mathbf{Bf} + \mathbf{e}$. Let $\mathbf{e}$ and $\mathbf{Bf}$ be elliptically distributed with mean $\mathbf{0}$, $\mathbf{0}$ (assuming its existence) and characteristic matrix $\mathbf{\Omega}$, $\mathbf{BB}'$ respectively. Suppose that $\|\Omega\| < \infty$ where $\|\Omega\| \equiv \sup_{\mathbf{w}} \mathbf{w}'\Omega\mathbf{w}/\mathbf{w}'\mathbf{w}$. Then the no-arbitrage condition:

$$\{\mathbf{w}'\mathbf{1} \dashrightarrow 0,\ \mathbf{w}'\Sigma\mathbf{w} \dashrightarrow 0 \implies \mathbf{w}'\mathbf{a} \dashrightarrow 0\}$$

implies the ALPR (3c), where $\Sigma = \mathbf{BB}' + \mathbf{\Omega}$.

Proof: Since $\mathbf{e}$ is a member of $ED_n(0,\Omega)$, $\mathbf{Bf}$ a member of $ED_n(0,\mathbf{BB}')$, then $\mathbf{x}$ is a member of $ED_n(\mathbf{a},\Sigma)$ (Kelker, 1970). The rest of the proof is similar to Theorem 1. Q.E.D.

The class of elliptical distributions contains the multivariate normal distribution as a special case; as well as many non-normal multivariate distribution such as multivariate Cauchy, the multivariate exponential, the symmetric stable distribution and non-normal variance mixture of multinormal distributions. Owen and Rabinovitch (1983) emphasize the potential usefulness of elliptical distribution in modeling the empirical distribution of

---

[27] With μ an n-component vector and Σ a positive definite n x n matrix, we say that the vector $\mathbf{x}$ is a member of the class of elliptical distributions (spherical distributions), $ED_n(\mu,\Sigma)$, if and only if the characteristic function of x is of the form $C_x(\mathbf{t}) = \Phi(t'\Sigma t)\exp(it'\mu)$ (Kelker, 1970).

speculative prices (returns). Since the shape of elliptical densities is flexible and allows for fat tails, this class provides a variety of possible multivariate models for speculative prices (returns).

VI. Conclusion

In this paper, we examined the robustness of the APT with respect to the assumptions on the factor structure of asset returns. Particularly, we established the generality of the APT when the idiosyncratic risks are weakly dependent and/or their second moments do not exist.

We showed that, under the suitable assumptions on the linear factor structure, if (i) the idiosyncratic risks are independent of the factors, (ii) the second moments of the factors exist, and (iii) the sequence of the idiosyncratic risks is a lacunary system of order p for some $p \geq 2$ and if it also satisfies the Bessel inequality, then the no-asymptotic-arbitrage condition and the approximate linear pricing relation are equivalent. However, only (iii) is needed in showing the nonexistence of arbitrage opportunities as a sufficient condition for the approximate linear pricing relation.[28] The models of APT in Ross (1978), Huberman (1982), Chamberlain and Rothschild (1983) and Ingersoll (1984) can be demonstrated as special cases of our general framework.

---

[28] p can be less than two here ($1 \leq p < \infty$).

Appendices

Appendix A

Ingersoll (1987) argued that (1e) in the text could be achieved through appropriate choices for **a** and **B**. However, it is not quite true by the following example.

Suppose there are four states (s_1, s_2, s_3, s_4) in the economy. K = 1 and n = 2. The possible outcomes and probabilities are given as follows.

	s_1	s_2	s_3	s_4
e_1	1	0	0	-1
e_2	1	-2	0	1
f	1	-1	-1	1
Prob.	¼	¼	¼	¼

Thus, $E(e_1) = E(e_2) = E(f) = 0$, $E(e_1 f) = 0$, and $E(e_2 f) = 1$.

It is obvious that it is impossible to make $E(e_2 f) = 0$ by choosing appropriate **B** (**B** is 2 x 1 in dimension in this case). We can show that, in general, Ingersoll's statement does not hold. The proof is as follows.

Suppose $E(\mathbf{e}) = \mathbf{0}$, $E(\mathbf{f}) = \mathbf{0}$, then **a** is fixed. The only choice left to achieve (1e) is **B**. Equation (1a) can be rewritten as

$\mathbf{x} = \mathbf{a} + \mathbf{B}\mathbf{Q}^{-1}\mathbf{Q}\mathbf{f} + \mathbf{e} = \mathbf{a} + \mathbf{B}^{o}\mathbf{f}^{o} + \mathbf{e}$,

where $B^{o} \equiv \mathbf{B}\mathbf{Q}^{-1}$, $\mathbf{f}^{o} \equiv \mathbf{Q}\mathbf{f}$, and $Q \neq 0$.

$E(\mathbf{e}\mathbf{f}^{o\prime}) = E(\mathbf{e}\mathbf{f}'\mathbf{Q}') = E(\mathbf{e}\mathbf{f}')\mathbf{Q}'$.

We want $E(\mathbf{e}\mathbf{f}^{o\prime})$ to be 0. Now, there are nK equations for K^2 variables and K << n, The solution will be inconsistent in general. Q.E.D.

Appendix B

Proof of Lemma 2:

Without loss of generality assume that the median of Y_n is 0.

$$P\{|X_n+Y_n|>\alpha\} = P\{X_n+Y_n>\alpha\} + P\{X_n+Y_n<-\alpha\}$$

$$\geq P\{X_n+Y_n>\alpha|Y_n>0\}P\{Y_n>0\} + P\{X_n+Y_n<-\alpha|Y_n<0\}P\{Y_n<0\}$$

$$\geq P\{X_n+Y_n>\alpha|Y_n>0\}/2 + P\{X_n+Y_n<-\alpha|Y_n<0\}/2$$

$$\geq P\{X_n>\alpha\}/2 + P\{X_n<-\alpha\}/2$$

$$= P\{|X_n|>\alpha\}/2.$$

The last inequality is due to the assumption of independence.

$$E|X_n+Y_n|^p = \int_0^\infty P\{|X_n+Y_n|^p > \alpha\}\, d\alpha$$

$$= p\int_0^\infty \alpha^{p-1}P\{|X_n+Y_n|>\alpha\}\, d\alpha$$

$$\geq p/2\int_0^\infty \alpha^{p-1}P\{|X_n|>\alpha\}\, d\alpha$$

$$= E|X_n|^p/2.$$

We use Chung (1974:49, E17) to get the first equality and Rohatgi (1976: 86) the second equality. Q.E.D.

Appendix C

Proof of Corollary 2: It suffices to show that $\|v\|_p < \infty \Longrightarrow \|v\|_q < \infty$ for $0 < p < q < \infty$.

$$\frac{\|v\|_q}{\|v\|_p} = \left(\Sigma\left(\frac{|v|_i^p}{\|v\|_p^p}\right)^{q/p}\right)^{1/q} < \left(\Sigma\frac{|v|_i^p}{\|v\|_p^p}\right)^{1/q} = 1.$$

Q.E.D.

References

Aharony, Joseph and Itzhak Swary. "Quarterly Dividend and Earnings Announcements and Stockholders' Returns: An Empirical Analysis." Journal of Finance 35: 1-12.

Anderson, T. W. 1971. An Introduction to Multivariate Statistical Analysis. Wiley. New York.

Banach, S. 1930. "Über einige Eigenschaften der lacunaren trigonometrische Reiben." Studia Math. 2: 207-220.

Bawa, V.S. 1975. "Optimal Rules for Ordering Uncertain Prospects." Journal of Financial Economics 2: 95-121.

Blattberg, Robert and Nicholas Gonedes. 1974. "A Comparison of the Stable and Student Distributions as Statistical Models for Stock Prices." Journal of Business 47: 518-537.

Blume, Marshall. 1970. "Portfolio Theory: A Step Toward its Practical Applications." Journal of Business 43: 152-173.

Chamberlain, Gary. 1983. "Funds, Factors, and Diversification in Arbitrage Pricing Models." Econometrica 51: 1305-1323.

Chamberlain, Gary, and Michael Rothschild. 1983. "Arbitrage, Factor Structure, and Mean-Variance Analysis in Large Assets Markets." Econometrica 51: 1281-1304.

Chung, Kai Lai. 1974. A Course in Probability Theory. Academic Press. New York.

Clark, Peter. 1973. "A Subordinated Stochastic Process Model with Finite Variances for Speculative Prices." Econometrica 41: 135-155.

Doob, Joseph L. 1953. Stochastic Processes. Wiley. New York.

Fama, Eugene. 1965. "The Behavior of Stock Market Prices." Journal of Business 38: 34-105.

———. 1965. "Portfolio Analysis in a Stable Paretian Market." Management Science B 2: 409-419.

———, and Merton Miller. 1972. The Theory of Finance. Holt, Rinehart, and Winston. New York.

Feller, William. 1971. An Introduction to Probability Theory and its Applications. Vol. 2. 2nd ed. Wiley. New York.

Fielitz, Bruce D. and James P. Rozelle. 1983. "Stable Distributions and the Mixture of Distributions Hypotheses for Common Stock Returns." Journal of American Statistical Association 78: 28-36.

Gaposhkin, V.F., 1966. "Lacunary Series and Independent Functions." Russian Mathematical Surveys 21(6): 1-82.

Grinblatt, Mark and Sheridan Titman. 1985. "Approximate Factor Structures: Interpretations and Implications for Empirical Tests." Journal of Finance 40: 1367-1373.

Hagerman, Robert. 1978. "More Evidence on the Distribution of Security Returns." Journal of Finance 33: 1213-1221.

Hsu, Der-Ann, Robert Miller, and Dean Wichern. 1974. "On the Stable Paretian Behavior of Stock Market Prices." Journal of the American Statistical Association 69: 108-113.

Huberman, Gur. 1982. "A Simple Approach to Arbitrage Pricing Theory." Journal of Economic Theory 28: 183-191.

Ingersoll, Jonathan E., Jr. 1984. "Some Results in the Theory of Arbitrage Pricing." Journal of Finance 39: 1021-1039.

Ingersoll, Jonathan E., Jr. 1987. Theory of Financial Decision Making. Rowman. New Jersey.

Jarrow, Robert. 1988. "Preference, Continuity, and the Arbitrage Pricing Theory." Review of Financial Studies. Forthcoming.

Kac, M., R. Salem and A. Zygmund. 1948 "A Gap Theorem." Transactions American Mathematical Society 63: 235-248.

Kailath, T. 1974. "A View of Three Decades of Linear Filtering Theory." IEEE Trans. Inform. Theory IT-20: 146-181.

Kelker, Douglas. 1970. "Distribution Theory of Spherical Distributions and a Location-Scale Parameter Generalizations." Sankhya A32: 419-430.

Khintchine, A. 1924. "Über einen Satz der Wahrscheinlichkeitsrechnung." Fund. Math. 6: 9-20.

Komlós, J. 1972. "On the Series $\Sigma c_k \emptyset_k$." Studia Scientiarum Mathematicarum Hungarica 7: 451-458.

Laha, R. G. and V. K. Rohatgi. 1979. Probability Theory. Wiley. New York.

Lai, Tze Leung and Ching Zong Wei. 1983. "Lacunary systems and Generalized Linear Processes." Stochastic Processes and their Applications 14: 187-199.

Lee, C. Jevons and Taychang Wang. 1988. "Asymptotic Arbitrage Opportunities in

Various Modes of Stochastic Convergence and the Approximate Linear Pricing Relation." Working Paper. The Wharton School. University of Pennsylvania.

Longnecker, M., and R.J. Serfling. 1978. "Moment Inequalities for S_n under General Dependence Restrictions with Applications." Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete 43: 1-21.

Mandelbrot, Benoit. 1963. "The Variation of Certain Speculative Prices." Journal of Business 39: 242-255.

Owen, Joel and Ramon Rabinovitch. 1983. "On the Class of Elliptical Distributions and their Applications to the Theory of Portfolio Choice." Journal of Finance 38: 745-752.

Rohatgi, V. K.. 1976. An Introduction to Probability Theory and Mathematical Statistics. Wiley. New York.

Roll, Richard. 1970. The Behavior of Interest Rates: The Applications of the Efficient Market Model to U.S. Treasury Bills. Basic Books, Inc. New York.

———. 1977. "A Critique of the Asset Pricing Theory's Tests: Part I: On Past and Potential Testability of the Theory." Journal of Financial Economics 4: 129-179.

———, and Stephen A. Ross. 1980. "An Empirical Investigation of the Arbitrage Pricing Theory." Journal of Finance 35: 1073-1103.

Ross, Stephen A. 1975. "Return, Risk and Arbitrage." In Irwin Friend and J. Bicksler, eds., Studies in Risk and Return, Cambridge, Mass.: Ballinger Publishing Co.

———. 1976. "The Arbitrage Theory of Capital Asset Pricing." Journal of Economic Theory 13: 341-360.

———. 1978. "Mutual Fund Separation in Financial Theory-The Separating Distributions." Journal of Economic Theory 17: 254-286.

Royden, H.I. 1968. Real Analysis. 2nd ed. Macmillan Publishing Co. New York.

Schwartz, L. 1969/70. "Les applications o-radonifiantes dans les espaces de suites." Seminaire Schwartz. Exp. XXVI.

Stambaugh, Robert. 1983. "Arbitrage Pricing with Information." Journal of Financial Economics 12: 357-69.

Tobin, James. 1958. "Liquidity Preference as Behavior Towards Risk." Review of Economic Studies 25: 65-86.

Westerfield, Randolph. 1977."The Distribution of Common Stock Price Changes: An Application of Transactions Time and Subordinated Stochastic Models." Journal of Financial and Quantitative Analysis 11: 743-765.

Woyczynski, Wojbor A. 1978. "Geometry and Martingales in Banach spaces Part II: Independent Increments." in James Kuelbs, Probability on Banach Spaces. Marcel Dekker, Inc., New York.

Zolotatarev, V. M. 1986. One-Dimensional Stable Distribution. American Mathematical Society, Providence.