ASYMPTOTIC ARBITRAGE OPPORTUNITIES AND ASSET MARKET EQUILIBRIUM

by

C. Jevons Lee
Taychang Wang

(28-88)

RODNEY L. WHITE CENTER FOR FINANCIAL RESEARCH
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6367

Asymptotic Arbitrage Opportunities and Asset Market Equilibrium

C. Jevons Lee and Taychang Wang*

A. B. Freeman School of Business
Tulane University
New Orleans, LA 70118-5698

Tel: (504) 865-5439

September, 1988

* Respectively, Professor, Tulane University and Associate Professor, National Taiwan University. We appreciate the comments from Mark Grinblatt, Richard Kihlstrom, and Andrew Lo. Financial support from the Yin's Foundation is acknowledged.

Asymptotic Arbitrage Opportunities and Asset Market Equilibrium

Abstract

In this paper, we show the reason why the absence of asymptotic arbitrage opportunities in the sense of convergence in quadratic mean (ACQM) as defined in Huberman (1982) is only a necessary condition for an asset market equilibrium. For certain classes of risk-averting investors, a portfolio that is not an ACQM may sometimes provide infinitely blissful gratification. These investors would relentlessly explore such a portfolio and cause market disequilibrium. Consequently, the APT that is based on Huberman's concept of arbitrage is not a valid description of the no-arbitrage pricing relation as other types of asymptotic arbitrage opportunities may exist in the economy. To resolve this inconsistency, we replace Huberman's concept of asymptotic arbitrage (convergence in quadratic mean) with that of convergence in probability. We show that if the idiosyncratic risks of the linear K-factor structure are weakly dependent (or, more precisely, the sequence of the idiosyncratic risks is a lacunary system of order p for some $p \geq 1$), the absence of the arbitrage opportunities in the sense of convergence in probability (ACP) implies an approximate linear pricing relation which is consistent with an asset market equilibrium for a broader class of preferences. However, there are still circumstances in which the absence of the ACP is not compatible with the asset market equilibrium. Finally, we claim that the absence of "asymptotically exact" arbitrage opportunities is consistent with asset market equilibrium for all risk-averting investors.

I. INTRODUCTION

The arbitrage theory of capital asset pricing was developed by Ross (1975, 1976) as an alternative to the mean-variance capital asset pricing model (CAPM). The derivation of market equilibrium condition in the CAPM depends on specific assumptions about the investors' utility function. However, Huberman (1982) suggested a definition for "arbitrage" and proved the arbitrage pricing theory (APT) without referring to the investors' utility function.[1] Huberman's approach, the so-called "arbitrage derivation", became the commonly accepted framework for analyzing the APT in the literature. Since the APT depends only on the absence of arbitrage with this approach, it seems that the APT should also requires less structure on preference than the CAPM.[2] In this paper, we demonstrate the importance of considering investors' preferences in deriving the APT with "arbitrage" approach.

It is well known that the no-arbitrage pricing relation is only a necessary condition for an asset market equilibrium. The no-arbitrage condition in Huberman (1982) does not imply the nonexistence of arbitrage

---

[1] "Arbitrage" in this paper should always be interpreted as an asymptotic-arbitrage unless otherwise stated.

[2] Jarrow (1988) provided conditions on preferences and market structures that imply the no-arbitrage condition (Condition i in page 116) are sufficient to derive the APT. He showed that only mild restrictions on preferences and market structures are needed to obtain the APT. One natural question to ask is what would happen to his result if the no-arbitrage condition is defined in terms of other mode of convergence instead of convergence in quadratic mean. Jarrow's condition is stronger than the convergence in quadratic mean no-arbitrage condition. In this paper, we provide other types of no-arbitrage conditions which are also stronger than the convergence in quadratic mean no-arbitrage condition, but in a different way. While Jarrow examined the structure of preferences for a given no-arbitrage condition, our purpose is to find a definition for the no-arbitrage condition so that portfolios of assets satisfying this condition will provide no free lunches to any investors with increasing and concave utility functions. These two papers can be viewed as a set of complementary pieces.

opportunities in a broader sense, which, in turn, does not always indicate asset market equilibrium. In this paper, we will establish the relation between no-arbitrage condition and asset market equilibrium using two types of risk-averting investors who derive an infinitely large utility from a costless portfolio with a non-zero variance in the limit.[3] If these two kinds of investors exist, then the absence of arbitrage opportunities in the sense of convergence in quadratic mean does not imply an asset market equilibrium. Hence, the approximate linear pricing relation derived from the APT using Huberman's concept may not be consistent with market equilibrium.

We also tentatively investigate the no-arbitrage condition that avoids the above-mentioned situation for certain investors (Type I) by making the no-arbitrage condition somewhat more restrictive. Moreover, we are able to, under this stronger no-arbitrage condition, relax the assumptions on the idiosyncratic risks in the proof of the APT. However, the problem of inconsistency with market equilibrium persists when we consider the other type of investors (Type II). Finally, we suggest a no-arbitrage condition that conforms to market equilibrium for all risk-averting investors. The validity of the APT in such a setting is apparent since a stronger assumption is used.

In this paper, we consider a market in which a countable number of assets are traded. The price system is normalized by assuming each asset costs one dollar. The assets are arranged in a sequence. We shall look at what happens to various objects as n increases to infinity. We also assume that preferences can be characterized by an expected utility function.

---

[3] Obviously, this portfolio does not satisfy the condition of arbitrage defined by Huberman.

The rest of paper is organized into three sections. In Section II, we use two examples to illustrate that the relation between the no-arbitrage condition and market equilibrium depends on the utility function of the investors. In Section III, we relax the assumptions on the linear factor structure, but tighten the restrictions on the no-arbitrage condition.[4] We then establish the APT under this new setting. Moreover, conditions for the consistence between no-arbitrage and market equilibrium will be provided. Section IV concludes this paper.

II. ARBITRAGE OPPORTUNITIES AND ASSET MARKET EQUILIBRIUM

II.1 The Basic Model

Basically, the APT is composed of three elements: the linear K-factor structure, the no-arbitrage condition, and the approximate linear asset pricing relation. The linear K-factor structure of asset return, $\mathbf{x}$, is described as follows,

$$\mathbf{x} = \mathbf{a} + \mathbf{B}\,\mathbf{f} + \mathbf{e} \qquad (1a)^{5}$$

$$E(\mathbf{e}) = \mathbf{0} \qquad (1b)$$

$$E(\mathbf{f}) = \mathbf{0} \qquad (1c)$$

$$E(\mathbf{ff}') = \mathbf{I}_K \quad \text{(identity matrix of rank K)} \qquad (1d)$$

$$E(\mathbf{ef}') = \mathbf{0} \quad \text{(n x K matrix)} \qquad (1e)$$

$$E(\mathbf{ee}') = \mathbf{D} \quad \text{(}\mathbf{D}\text{ is a diagonal matrix)} \qquad (1f)$$

$\mathbf{1}_i'\mathbf{D}\mathbf{1}_i < \sigma^2 < \infty$ for all i, where $\mathbf{1}_i$ is the i^{th} column of

[4] For example, the variances of idiosyncratic risks may not be finite.

[5] In this paper, **x**, **f**, and **e** are random. Moreover, **x**, **a**, **B**, **f**, and **e** all depend on n. To simplify notations, we omit the labels of randomness and the subscripts for sequential indices.

an identity matrix with rank n. (1g)

The vectors **x**, **a**, and **e** are each n x 1 in dimension and represent the realized returns, expected returns and (nonobservable) "residual" portions of the returns, respectively. The e_i, which is often called idiosyncratic risk or residual in the literature, measures the uncertainty unexplained by the common factors. The expected return of i^{th} asset, a_i, is assumed to be bounded for all i. **f** is a K x 1 vector of nonobservable values of the common factors. The second moment of f_k is assumed to exist for all k. **B** is the n x K matrix of bounded factor loadings, i.e., $|b_{ik}| < \infty$ for all i and k. **D** is the n x n positive definite diagonal variance-covariance matrix of the idiosyncratic risks. The i^{th} diagonal element of **D** is denoted as σ_i^2.

Following Huberman (1982) and Ingersoll (1984), asymptotic arbitrage is defined as follows:

Definition: Arbitrage in the sense of convergence in quadratic mean (hereafter, **ACQM**) is the existence of a subsequence $\tilde{n}$ of arbitrage portfolios, $\mathbf{w}(\tilde{n})$, $\tilde{n} = 1, 2, \ldots$, whose returns $z(\mathbf{w}(\tilde{n}))$ satisfy

$$\mathbf{w}(\tilde{n})'\mathbf{1}(\tilde{n}) = 0, \quad (2a)$$

$$E\, z(\mathbf{w}(\tilde{n})) = \mathbf{w}(\tilde{n})'\mathbf{a}(\tilde{n}) \geq m > 0, \quad (2b)$$

$$\text{Var}\, z(\mathbf{w}(\tilde{n})) = \mathbf{w}(\tilde{n})'\Sigma(\tilde{n})\mathbf{w}(\tilde{n}) \longrightarrow 0 \quad (2c)$$

where $\Sigma(\tilde{n}) = \mathbf{B}(\tilde{n})'\mathbf{B}(\tilde{n}) + \mathbf{D}(\tilde{n})$, and **1** is a column of 1s. ▪

Condition (2a) indicates a zero-cost portfolio, condition (2b) means that the portfolio has a positive expected return, and condition (2c) signifies that the portfolio has a zero variance random return in the limit.

The Arbitrage Pricing Theory asserts that if the returns on the risky assets are given by the linear factor structure described in (1), and there are no asymptotic arbitrage opportunities as defined in (2), then there exists an approximate linear pricing relation which gives expected returns with finite sum of squared pricing errors. In other words, for n = 1, 2, ... , there exist $c_0, c_1, c_2, \ldots, c_K$ such that

$$\lim_{n \to \infty} \| \mathbf{a} - \mathbf{B}^* \mathbf{c} \|^2 < \infty,$$

where $\mathbf{c}' = (c_0 \; c_1 \; c_2 \; \ldots \; c_K)$, $\mathbf{B}^* = (\mathbf{1} \;\; \mathbf{B})$, $\mathbf{1}$ is a column vector of 1s, and $\mathbf{a}$, $\mathbf{B}$, and $\mathbf{c}$ all depend on n.

In next subsection, we show that absence of the arbitrage opportunities defined by Huberman (1982), ACQM, does not always indicate nonexistence of other types of arbitrage opportunities.

II.2 Some Examples of Market Disequilibrium

Before showing the inconsistency between the absence of ACQM and the asset market equilibrium, it is worthwhile to mention that Huberman (1982) noticed that one needs to make assumption on agents' preferences in order to relate existence of equilibria to absence of arbitrage. However, his statement that "... a result of the type 'no arbitrage implies a certain behavior of returns' should involve no consideration of preference structure of the agents involved" (Huberman 1982:P.190) is rather ambiguous. He did not fully explore this issue. Huberman gave two examples showing that arbitrage portfolios which satisfy (2) are not desirable for some expected utility maximizers. In other words, (2) does not suffice to assert that $\lim_{n \to \infty} E\, u(z(\mathbf{w}(\hat{n}))) = U(+\infty)$ for all monotone utility functions u. In the first example, he assumed a CRRA-like utility function: $u(x) = -1/x$ for $x > 0$ and $U(x) = -\infty$ for $x \leq 0$, whereas in the second example he assumed a CARA utility function (of exponential type). When these two types of investors spot an ACQM, they are so repelled by it that they would rather die (if death is the ultimate sufferance) than touch it. General conditions which asserts that (2) implies $u(z(\mathbf{w}(\hat{n}))) = U(+\infty)$ are not known. Jarrow (1988) provided sufficient conditions on preferences and market structure for the APT pricing relation to hold. Jarrow also showed that his Proposition 3 includes as a special case the conditions on preferences contained in Ross (1976).[8]

On the other hand, if there is a subsequence n^* of arbitrage portfolios, $\mathbf{w}(n^*)$, $n^* = 1, 2, \ldots$, whose return $z(\mathbf{w}(n^*))$ does not satisfy (2), does this mean that no one is going to have infinite utility levels? In this subsection, we show that the portfolio satisfying the Huberman's condition of no-arbitrage may actually give rise to an infinite level of utility for some risk-averting investors.

Example 1 : Suppose we have a sequence of random arbitrage portfolio returns z_n with outcomes { $1-n$, 1, $1+n$ } and with probabilities { n^{-2}, $1-2n^{-2}$, n^{-2} }. This portfolio does **not** provide an ACQM because

$E(z_n) = 1 > 0,$

$Var(z_n) \rightarrow 2 \neq 0.$ ▪▪

---

[8] Ross (1976) showed that if the investor's utility function is strictly increasing, concave and bounded below or uniformly integrable, then the utility function will imply $u(z(\mathbf{w})) = U(+\infty)$. However, Ross' argument applies only to the random variables converging to zero in quadratic mean. He did not generalize his results in Appendix 2 to other modes of convergence.

Case 1, Type I Investor: Consider an investor with a concave utility function,

$$u(z) = \begin{cases} .8z & \text{if } z \geq 0 \\ z & \text{if } z < 0 \end{cases} \quad . \tag{3}$$

The expected utility of Type I investor is

$$E\ u(z_n) = (1-n)*1/n^2 + .8*(1-2/n^2) + .8*(1+n)*1/n^2$$

$$= .8 - 1.8/n + 1.8/n^2 \text{ --> } .8.$$

The arbitrage portfolio is of zero cost and can be scaled up to any level. If we scale up the above portfolio by $n^{1/2}$, then

$$E\ u(z_n') = .8*n^{1/2} - .2/n^{1/2} + .2/n^{1.5} \text{ --> } \infty \quad \text{as } n \text{ --> } \infty.$$

Hence, a portfolio that does not offer an ACQM turns out to be a golden chance for Type I investor. ▪▪

Case 2, Type II Investor: Consider another class of investors whose utility function is described as

$$u(z) = \begin{cases} z^{2/3} - 4/27 & \text{if } 8/27 < z \\ z & \text{if } -4/9 < z \leq 8/27 \\ -(-z)^{3/2} -4/27 & \text{if } z \leq -4/9. \end{cases} \tag{4}$$

This utility function is concave. In fact, it is strictly concave except for the segment between -4/9 and 8/27. The values of flexing points are assumed for convenience and are not crucial to the result here. Nevertheless, we have

$$E\ u(z_n) = [(1+n)^{2/3}-4/27]*n^{-2} + 23/27*(1-2n^{-2}) - [(n-1)^{3/2}+4/27]*n^{-2}$$

$$\longrightarrow 23/27 > 0.$$

If we scale up the portfolio by $n^{1/4}$, then

$$E\ u(z_n') = [n^{1/6}(1+n)^{2/3}-4/27]*n^{-2} + 23/27*n^{1/4}*(1-2n^{-2})$$

$$- [n^{1/6}(n-1)^{3/2}+4/27]*n^{-2}$$

$$\longrightarrow \infty \quad \text{as } n \longrightarrow \infty.$$

Here again, the portfolio that is not an asymptotic arbitrage opportunity in Huberman's sense is a fantasy-comes-true to Type II investor. ∎

The existence of Type I and Type II investors in an economy poses a serious problem to the consistency of the APT derived from Huberman's approach with market equilibrium. If the above-mentioned portfolio is an enrapturing joy to Type I and Type II investors, they will seize and explore this "opportunity". Since this portfolio requires zero investment, nothing can hold off these investors' relentless pursue of happiness. Hence, absence of ACQM does not imply asset market equilibrium condition. Consequently, the APT that is based on Huberman's concept of arbitrage may not be a valid description for the equilibrium asset pricing behavior if there are Type I and Type II investors in the economy. In next section, we will provide alternative definitions of arbitrage to see if they can accommodate these two kinds of investors.

III. ALTERNATIVE DEFINITIONS OF THE ARBITRAGE OPPORTUNITIES

In this section, we first provide a new definition of arbitrage and show that the APT holds with respect to this more general arbitrage condition.[7] Then, we will examine the robustness of this condition with respect to Type I and Type II investors. Finally, a no-arbitrage condition that would be consistent with asset market equilibrium with respect to all risk-averting investors is given.

III.1. The Setup

Definition: Arbitrage in the sense of convergence in probability (ACP) is the existence of a subsequence $\hat{n}$ of arbitrage portfolios, $\mathbf{w}(\hat{n})$, $\hat{n} = 1, 2, \ldots$, whose returns $z(\mathbf{w}(\hat{n}))$ satisfy

$$w(\hat{n})'1(\hat{n}) = 0 \tag{5a}$$

$$E\, z(\mathbf{w}(\hat{n})) = w(\hat{n})'\mathbf{a}(\hat{n}) \geq m > 0 \tag{5b}$$

$$z(\mathbf{w}(\hat{n})) \text{ -p-> } \mathbf{w}(\hat{n})'\mathbf{a}(\hat{n}), \text{ i.e., } \operatorname{plim} z(\mathbf{w}(\hat{n})) = \mathbf{w}(\hat{n})'\mathbf{a}(\hat{n}). \tag{5c}$$ [8]

where -p-> denotes convergence in probability. ∎

While Huberman's arbitrage is defined in terms of convergence in quadratic mean, condition (5c) is defined in terms of convergence in probability. Note that convergence in quadratic mean always implies convergence in probability, but not vice versa; condition (5c) is weaker than condition (2c). Thus the "no-arbitrage" condition based on the concept of ACP is stronger than the "no-

---

[7] The more general arbitrage condition implies the more restrictive no-arbitrage condition in proving the APT. However, the assumptions about the linear factor structure can be relaxed as shown in next subsection.

[8] In this paper, we will state the no-asymptotic-arbitrage condition as follows. The nonexistence of ACP means that when there is a portfolio with zero cost and its random return converges to zero in probability, its expected return must also converges to zero, i.e., for all $\mathbf{w}_n \in \mathbf{R}^n$, $\mathbf{w}_n'1_n \to 0$ and $z(\mathbf{w}_n)$ -p-> $\mathbf{w}_n'\mathbf{a}_n \Rightarrow \mathbf{w}_n'\mathbf{a}_n \to 0$. This is true by simple logic: not(A and B) ≡ (A ==> not B).

arbitrage" condition on that of ACQM. It is not difficult to envision that using (5) can accommodate a more general class of risk-averting investors. Although a stronger structure is imposed on the no-arbitrage condition, yet we may relax the assumption about the idiosyncratic risks in the linear factor structure by applying the concept of weakly dependent relations among the random variables. Specifically, we will assume that the sequence of idiosyncratic risks is a lacunary system of order p with $p \geq 1$.

Definition: Given $p > 0$, a sequence of real-valued random variables $\{e_i\}$ is called a **lacunary system** of order p, if there exists a positive constant K_p such that for any sequence of real constant d_i,

$$E|\Sigma_{i=m}^{n} d_i e_i|^p \leq K_p(\Sigma_{i=m}^{n} d_i^2)^{p/2} \qquad \text{for all } n \geq m.$$

If the system $\{e_i\}$ is an S_p system for every $p > 2$, then it is called an S_∞ system.

The lacunary system has properties that are, in some sense, similar to properties of systems of independent variables. Probably we may call it a weakly dependent system. If $\{e_i\}$ is a lacunary system, then $\Sigma_{i=1}^{\infty} d_i e_i$ is called a lacunary series. This concept is discussed in detail in Gaposhkin (1966) and Lai and Wei (1983). It can be shown that both sequences of i.i.d. normal random variables and the i.i.d. Bernoulli random variables are examples of lacunary system, and the assumptions on the idiosyncratic risks in the linear factor structure used by Chamberlain and Rothschild (1983) and Ingersoll (1984) are all special cases of a lacunary system. Wang and Lee (1988) provided extensive discussion on a general theory of arbitrage pricing when the sequence of idiosyncratic risks is a lacunary system of order p for some $p \geq 1$.

Proposition 1 : If the returns on the risky assets are given by a K factor linear structure, where the sequence of residual risks, $\{e_i\}$, is a lacunary system for some $p \geq 1$, then the absence of ACP implies an approximate linear pricing relation, i.e., there exist a column vector $\mathbf{c}' = (c_0 \; c_1 \; c_2 \; \ldots \; c_K)$ such that $\lim_{n \to \infty} \| \mathbf{a} - \mathbf{B}^* \mathbf{c} \|^2 < \infty$, where $\| \; \|$ denotes the Euclidean norm.[9]

Proof:

Projecting the vector **a** into the space spanned by the matrix **B** and vector **1**, we have

$$\mathbf{a} = \mathbf{B}^* \mathbf{c} + \mathbf{v}, \; \mathbf{c} \in \mathbf{R}^{k+1}, \text{ and } \; \mathbf{B}^* \mathbf{v} = \mathbf{0}.$$

Consider an arbitrage portfolio **w**, such that

$$\mathbf{w} = \frac{\mathbf{v}}{\|\mathbf{v}\|^r}, \text{ where } r \in (1, 2],$$

whose return, $z(\mathbf{w})$, is

$$z(\mathbf{w}) = \frac{\mathbf{v}'\mathbf{x}}{\|\mathbf{v}\|^r} = \frac{1}{\|\mathbf{v}\|^r} (\mathbf{v}'\mathbf{a} + \mathbf{v}'\mathbf{e}).$$

Hence, the expected return, $E\, z(\mathbf{w})$ is

$$E\, z(\mathbf{w}) = \frac{\mathbf{v}'\mathbf{a}}{\|\mathbf{v}\|^r} = \|\mathbf{v}\|^{2-r}.$$

By Markov inequality,

$$P(\, |z(\mathbf{w}) - \mathbf{w}'\mathbf{a}| \geq d \,)$$

$$\leq \frac{E\, (|z(\mathbf{w}) - \mathbf{w}'\mathbf{a}|^p)}{d^p}, \quad \text{for some } d > 0 \text{ and } 0 < p < \infty.$$

$$\leq \frac{K_p (\mathbf{w}'\mathbf{w})^{p/2}}{d^p} = (K_p/d^p)(\|\mathbf{v}\|^{2-2r})^{p/2} \dashrightarrow 0 \quad \text{for some } K_p < \infty.$$

The last inequality follows from the assumption that $\{e_i\}$ is a lacunary system of order p.

---

[9] When $p < 2$, the condition used here is weaker than those used by Ingersoll (1984) and Chamberlain and Rothschild (1983).

If $\lim_{n \to \infty} \|v\|$ is not finite, then the expected return remains a positive number while $z(\mathbf{w})$ converges to a positive number in probability which violates the assumption of no-arbitrage. Therefore, $\lim_{n \to \infty} \|v\|^2 < \infty$. Q.E.D.

III.2. Some Investors' Poison, Others' Meat

It is obvious that the random portfolio return in Example 1 converges to 1 in probability. Hence, although the portfolio is not an ACQM, it is an ACP. Consequently, by ruling out the ACP, we may be able to restore consistency in some cases.

Even though the new definition of arbitrage eliminates the inconsistency between the no-arbitrage condition and market equilibrium condition demonstrated in Example 1, the problem is not completely resolved. Example 2 shows that a portfolio that is not an ACP can nevertheless make Type I investors ecstatic if its random return has some special stochastic structure.

Example 2: Let $\{z_n\}$ be a sequence of random arbitrage portfolio returns with outcomes
$\{(1-n)/n, 1/n, (1+n)/n\}$ with probabilities $\{1/4 - 1/n, 1/4 + 2/n, 1/2 - 1/n\}$.
The mean and variance of this portfolio are:

$E(z_n) = (1 + n/4)/n \to 1/4,$

$Var(z_n) = (11/16 * n^2 - 2n) \to 11/16 \neq 0.$ ▪▪

The portfolio returns, z^n, does not converge to $E(z_n)$ in probability. Hence, this portfolio is **not** an ACP.

Case 1, Type I Investor: Let

$$u(z) = \begin{cases} .8z & \text{if } z \geq 0 \\ z & \text{if } z < 0 \end{cases}$$

$$\begin{aligned} E\,u(z_n) &= \{(1-n)(1/4-1/n) + .8(1/4+2/n) + .8(1+n)(1/2-1/n)\}/n \\ &= \{3n/20 + 37/20 - 9/(5n)\}/n \\ &\to 3/20. \end{aligned}$$

If we scale up the above portfolio by n, then $E\,u(z_n) \to \infty$. Thus this portfolio is in fact a fantastic opportunity to Type I investor. ▪▪

Case 2, Type II Investor:

$$u(z) = \begin{cases} z^{2/3} - 4/27 & \text{if } 8/27 \leq z \\ z & \text{if } -4/9 \leq z < 8/27 \\ -(-z)^{3/2} - 4/27 & \text{if } z < -4/9 \end{cases}$$

$$\begin{aligned} E\,u(z_n) &= [(\{1+n\}/n)^{2/3}-4/27](1/4-1/n) + (1/n)(1/4+2/n) \\ &\quad - [(\{n-1\}/n)^{3/2}+4/27](1/2-1/n) \quad \text{for } n > 4, \\ &\to -13/36. \end{aligned}$$

If we scale up the above portfolio by n, then $E\,u(z_n) \to -\infty$. This portfolio is indeed an awful choice to Type II investor. ▪▪

Example 2 articulates that a portfolio which provides no ACP might be some investors' poison, but the others' meat. The ruthless wealth pursuit of the Type I investors would make the absence of ACP inconsistent with asset market equilibrium. Next, we attempt to provide a definition of "no-arbitrage" that is consistent with market equilibrium for **all** risk-averting investors and for all asset return distributions. We will also discuss the validity of the APT in such a setting. In fact, a stronger pricing relation is obtained.

The no-arbitrage condition that is consistent with market equilibrium for all investors must satisfy the following requirement:

> For any sequence of asset returns $\{x_i\}_{i=1}^{n}$ and for any increasing and concave utility functions, u_j, when $\mathbf{w}'\mathbf{1} \to 0$ as n goes to infinity, $E\{u_j(I + \mathbf{w}_n'\mathbf{x}_n)\} \to \alpha \leq u_j(I)$ for all j, where I denotes the nonrandom initial wealth.

The absence of the asymptotically exact arbitrage opportunities defined below satisfies this requirement.

Definition: The **asymptotically exact no-arbitrage condition** requires that whenever a zero-cost portfolio has no systematic risks in the limit, its expected returns must also converge to zero.

Formally, for all $\mathbf{w}_n \in \mathbf{R}^n$,

$$\mathbf{w}_n'\mathbf{1}_n = 0 \quad \text{and} \quad \mathbf{w}_n'\mathbf{B}_n \to 0 \text{ imply that } \mathbf{w}_n'\mathbf{a}_n \to 0 \qquad (6)$$

*

Since for all concave utility functions, condition (6) implies $E\{u(I + \mathbf{w}_n'\mathbf{x}_n)\} \leq u(I + E\{\mathbf{w}_n'\mathbf{x}_n\}) \to u(I)$, the portfolio that satisfies the asymptotically exact no-arbitrage condition should not get any risk-averting investor excited.

Proposition 2 examines the validity of the APT under the asymptotically exact no-arbitrage condition.

Proposition 2: Assuming that (1a) through (1e) holds. The asymptotically exact no-arbitrage condition in equation (6) is a sufficient condition for the

= (c_0 c_1 c_2 ... c_K) such that $\lim_{n \to \infty} \|\mathbf{a} - \mathbf{B}^*\mathbf{c}\| = 0$, where $\| \ \|$ is the Euclidean norm.

Proof: Project **a** into the space spanned by **B** and **1**. Thus

$$\mathbf{a} = \mathbf{B}^*\mathbf{c} + \mathbf{v}, \text{ and } \mathbf{B}^*\mathbf{v} = \mathbf{0}, \text{ where } \mathbf{B}^* = (\mathbf{1}\ \mathbf{B}).$$

Consider a portfolio $\mathbf{w} = \mathbf{v}$. The expected return is $\mathbf{v}'\mathbf{a} = \mathbf{v}'\mathbf{v}$ which must converge to zero.

Q.E.D.

In fact, with a stronger "exact no-arbitrage" condition that, for all $\mathbf{w}_n \in R^n$,

$$\mathbf{w}_n'\mathbf{1}_n = 0 \quad \text{and} \quad \mathbf{w}_n'\mathbf{B}_n = 0 \text{ imply that } \mathbf{w}_n'\mathbf{a}_n = 0, \tag{7}$$

it is easy to see that this no-arbitrage condition is a necessary and sufficient condition for the "exact arbitrage pricing", i.e., there exists a column vector $\mathbf{c}_n$ such that $\mathbf{a}_n = \mathbf{B}_n^*\mathbf{c}_n$.

From the above discussion, we know that if we adopt (6) as our definition of no-asymptotic-arbitrage, no investor with increasing and concave utility functions are able to choose any zero-cost investment opportunity to increase their utilities.[10] Hence, under the condition of nonexistence of asymptotically exact arbitrage opportunities, there is no free lunch for any risk-averting investor in the limit. Proposition 2 says that, in this case, we can get an asymptotically exact linear pricing relation, a much stronger result. The linear pricing relation based on this no arbitrage condition is consistent with

[10] Condition like $\mathbf{w}_n'\mathbf{1}_n \to 0 \implies \mathbf{w}_n'\mathbf{a}_n \to 0$ will serve the same purpose. However, the above condition implies that the exact linear pricing relation holds asymptotically with risk premium vector being equal to **0**.

the market equilibrium, where the market is composed of all kinds of risk-averting investors.

Up to now, we have discussed the APT derived from the "arbitrage" approach. This method was adopted by Huberman (1982), Chamberlain and Rothschild (1983), Stambaugh (1983) and Ingersoll (1984). There is another approach -- the "equilibrium" derivation of the APT -- which is also widely adopted in the literature (Connor, 1984; Dybvig, 1983; Grinblatt and Titman, 1983; Chen and Ingersoll, 1983, and Cragg and Malkiel, 1982; Chamberlain, 1983). For example, Connor (1984) demonstrated that if the market portfolio is well diversified, every investor would then hold a well diversified portfolio. This and the first order condition of any investor imply asymptotically exact arbitrage pricing in a competitive equilibrium. Chamberlain (1983) argued that the asymptotically exact arbitrage pricing obtains if and only if there is a well diversified portfolio on the mean-variance frontier. Also, Chen and Ingersoll (1983) showed that in a finite economy if a well diversified portfolio exists and it is the optimal portfolio of some utility maximizing investor, then the first order conditions of that investor imply exact arbitrage pricing. Of course, the equilibrium derivation of the APT won't encounter the problem discussed in this paper. Thus, if we intend to use the arbitrage approach to derive the APT, at least, we can make the assumption employed by Connor, Chamberlain and Rothschild, and Chen and Ingersoll. Ideally, we had better find conditions on preferences that are consistent with the no arbitrage condition in terms of each mode of convergence in proving the APT. This is, however, beyond the scope of this paper.

Dybvig and Ross (1985), in a state preference framework, show the equivalence between absence of "arbitrage" and equilibrium (existence of an optimal demand for some agent who prefers more to less). In the same paper, they claimed that there is no substance in the distinction between the equilibrium derivation of the APT and the arbitrage derivation. It seems that their result and ours are contradictory. This discrepancy comes from the fact that Dybvig and Ross assumed a state preference framework with finite number of assets and states and finite-valued outcomes. This paper suggests that their result may not carry over to an economy with infinite number of assets and states and infinite-valued outcome.

IV. Conclusion

For certain classes of risk-averting investors, a portfolio that offers no ACQM may sometimes provide infinitely blissful gratification to some investors. These investors would explore such a wonderful opportunity and cause market disequilibrium. Consequently, the APT based on the Huberman's concept of arbitrage may not be a valid description of equilibrium pricing behavior in some circumstances.

To resolve this inconsistency, we employ a new definition of arbitrage. Instead of defining the arbitrage in terms of convergence in quadratic mean, we define it in terms of convergence in probability. If the sequence of idiosyncratic risks is a lacunary system of order p for some $p \geq 1$, the absence of ACP implies an approximate linear asset pricing relation. Although this new definition of arbitrage can accommodate a larger class of risk-averting investors, there are still circumstances in which the concept of ACP is inconsistent with the asset market equilibrium.

Finally, we defined an asymptotically exact no-arbitrage condition which is consistent with asset market equilibrium for all risk-averting investors. Under such a stronger condition of no-arbitrage, the "asymptotically exact" linear pricing relation holds. However, the asymptotically exact no-arbitrage condition may be too strong to describe the asset market behavior in the real world.

In this paper, we used several examples to examine the usefulness of alternative definitions of arbitrage. Unfortunately, we did not fully characterize the class of utility functions that is consistent with each definition of arbitrage. This should be an exciting challenge for future researchers.

REFERENCES

Chamberlain, Gary. 1983. "Funds, Factors, and Diversification in Arbitrage Pricing Models." Econometrica 51: 1305-1323.

Chamberlain, Gary, and Michael Rothschild. 1983. "Arbitrage, Factor Structure, and Mean-Variance Analysis in Large Assets Markets." Econometrica 51: 1281-1304.

Chen, Nai-fu and Jonathan E. Ingersoll, Jr. 1983. "Exact Pricing in Linear Factor Models with Finitely Many Assets," Journal of Finance, 38: 985-988.

Connor, Gregory. 1984. "A Unified Beta Pricing Theory." Journal of Economic Theory 34: 13-31.

Cragg, J. and B. Malkiel. 1982. Expectations and the Structure of Share Prices. University of Chicago Press.

Dybvig, Philip H. 1983. "An Explicit Bound on Individual Asset's Deviations from APT Pricing in a Finite Economy." Journal of Financial Economics 12: 483-496.

———, and Stephen A. Ross. 1985. Arbitrage. Unpublished Manuscript. Yale University.

Gaposhkin, V.F., 1966. "Lacunary Series and Independent Functions," Russian Mathematical Survey, 21(6): 1-82.

Grinblatt, Mark, and Sheridan Titman. 1983. "Factor Pricing in a Finite Economy." Journal of Financial Economics 12: 497-507.

Huberman, Gur. 1982. "A Simple Approach to Arbitrage Pricing Theory." Journal of Economic Theory 28: 183- 191.

Ingersoll, Jonathan E., Jr. 1984. "Some Results in the Theory of Arbitrage Pricing." Journal of Finance 39: 1021-1039.

Ingersoll, Jonathan E., Jr. 1987. Theory of Financial Decision Making. Rowman. New Jersey.

Jarrow, Robert. 1988. "Preference, Continuity, and the Arbitrage Pricing Theory." Review of Financial Studies. 1: 110-136.

Lai, Tze Leung and Ching Zong Wei, 1983. "Lacunary Systems and Generalized Linear Processes," Stochastic Processes and Their Applications 14: 187-199.

Ross, Stephen A. 1975. "Return, Risk and Arbitrage." In Irwin Friend and J. Bicksler, eds., Studies in Risk and Return. Cambridge, Mass.: Ballinger Publishing Co.

————. 1976. "The Arbitrage Theory of Capital Asset Pricing." Journal of Economic Theory 13: 341-360.

Stambaugh, Robert. 1983. "Arbitrage Pricing with Information." Journal of Financial Economics 12: 357-69.

Wang, Taychang and C. Jevons Lee. 1988. "A General Theory of Arbitrage Pricing: When the Residual Risks Are Dependent and Their Second Moments Do Not Exist." Working Paper, Rodney White Research Center, Wharton School, University of Pennsylvania.