TAX SMOOTHING WITH FINANCIAL INSTRUMENTS

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Abstract

The paper explores how the structure of government debt affects the budget in a stochastic environment. In the theoretical part, I present two models that motivate why governments should care about the risk inherent in its choice of liabilities. The models are based on tax-smoothing and risk aversion of taxpayers, respectively. Debt should be structured to hedge against macroeconomic shocks that affect the government budget, in particular against shocks to aggregate output. The optimal structure of government liabilities generally includes some "risky" securities which are state contingent in real terms.

The empirical part studies state-contingencies implemented by some specific securities. I find that nominal debt and long-term debt have desirable properties as hedges. This may motivate the current practice of issuing non-indexed debt of various maturities. The argument justifying "risky" nominal and long-term debt suggests that the government may improve welfare by taking a short position in the stock market. This is strongly supported by the data. Finally, I find that issuing selected foreign currency bonds may be beneficial.

1. Introduction

The United States government issues Treasury bonds and bills of various maturities. They are considered risk-free in terms of default risk, though their real value may fluctuate considerably. This paper is concerned with questions of what may motivate such a debt structure and whether it is an optimal one.

The paper analyzes government policies that maximize welfare. The welfare maximizing approach of analyzing government debt policy was introduced by Barro (1979). He shows that, in a deterministic environment, optimal tax and debt policy should smooth tax rates over time. We show that this objective generalizes, in a stochastic environment, to a debt policy that smooths taxes over time and states of nature. The optimal policy calls for a structure of government debt that makes the value of debt state-contingent. The government should hedge against macroeconomic shocks that affect its budget. A similar state-contingent debt structure is obtained in a model with risk averse taxpayers.

Bonds denominated in fiat currency (U.S. dollars) and bonds with different maturities embody specific state-contingencies. In an optimal-policy context, this raises the question why securities with these characteristics--and only these--are being issued by the United States government (and most other governments).

The first question is about indexation, which has received considerable attention in the literature (see, e.g., Fischer (1983)). We show that the state-contingency implemented by relating the real value of government debt to inflation has indeed desirable hedging properties that are highly significant statistically and economically. This is consistent with the theoretical analysis in Bohn (1988b).

The second question is about the maturity structure of debt. In a discrete-time framework, the real value of debt with maturity greater than one period is contingent on a nominal (or real, if indexed) interest rate in the next period. We show that this type of contingency is also desirable for a welfare-maximizing government, though the evidence is weaker than that in favor of nominal debt.

Third, one may ask whether the government should consider other types of securities than those presently issued. From a theoretical perspective, tax rates could be stabilized perfectly, if debt were simply contingent on tax rates. However, such debt securities may be impractical for other reasons that are beyond the scope of this paper (see below). Alternatively, the government could hedge against any specific shock by making debt contingent on this shock. We identify uncertainty about business cycles as a key source of shocks, because tax revenues are cyclical. Thus, a hedging position could be created by creating and issuing a security with value contingent on output (GNP or some related measure of business conditions).

But even assuming that the government does not create and issue new securities tied directly to shocks or to tax rates, one may ask if the government can improve on the current practice of issuing only nominal debt securities. The answer is yes, which we demonstrate in the two cases of equity and foreign currency debt. Since stock prices are correlated with output, the government can hedge against cyclical fluctuations by taking a short position in the stock market (the S&P 500 index in our analysis). We also show that issuing foreign currency debt, which is a common practice in many countries, could be beneficial for the U.S. government.

Two issues are purposely omitted in this paper, namely asset pricing and incentives. In general, determining the optimal structure of debt is a portfolio question, which depends on risk and return differentials between securities. We focus on risk alone and totally exclude differentials in expected returns by assuming that there are always some risk-neutral investors.[1]

Recently, time consistency issues have received considerable attention in analyzing government policy (Kydland and Prescott (1977)). Time consistency is relevant in particular for questions of nominal government liabilities (Calvo (1978), Bohn (1988b)) and in the context of maturity choice (Lucas and Stokey (1983)). It may also explain why the government does not issue securities directly conditional on shocks or tax rates.[2] We omit any analysis of these issues, not to dispute their obvious relevance, but to concentrate on the portfolio problem. In the portfolio problem, many different securities can be treated in a similar and symmetric way even though they may raise very different incentive issues. For a balanced characterization of optimal policy, one would have to integrate the present analysis with a model of the incentive issues peculiar to the securities in question (see Bohn (1988a) and (1988b)).

The paper is organized as follows. Section 2 sets up two simple models (based on risk averse taxpayers and excess burden, respectively) that motivate why governments may want to hedge against economic uncertainty. Section 3 studies securities that may have a role in hedging. After outlining the methodology, we address questions of indexation and maturity choice, and finally turn to the stock market and foreign currency debt. Some extensions are presented in Section 4. Section 4.1. analyzes correlations of tax rates with security returns and Section 4.2 considers the hedging properties of the various securities against uncertain government spending. Section 5 summarizes the conclusions.

2. A Framework for Analysis

Why should a government be concerned about risk? We will provide two motivations. First, Barro (1979) has shown that, with distortionary taxes, the government should smooth tax rates over time. In a stochastic environment, Barro's approach generalizes to tax smoothing over states of nature. The government behaves as if it is averse to the risk of changing tax rates even if all individuals are identical and risk neutral.

Alternatively, governments are naturally concerned about risk, if their constituents, taxpayers, are risk averse. A second model motivates risk averse government behavior by assuming that some taxpayers are risk averse and have limited access to capital markets. We provide both motivations to demonstrate a common feature that we want to establish: Uncertainty of current and future aggregate output or income is a key source of risk. Therefore, the government should hedge against low realizations of aggregate output.[3]

2.1. Tax Smoothing

We consider a model similar to Barro (1979), except that we add risky securities. In period t, identical, infinitely lived individuals maximize

$$U_t = E_t \sum_{j\geq 0} \rho^j \cdot c_{t+j} , \qquad (1)$$

where $0 < \rho < 1$ is a discount factor and c_{t+j} is consumption in period $t + j$. They own a stream of endowments Y_{t+j} and may trade $K + 1$ assets. Let $A_{k,t}$ be the quantity of asset k ($k = 0, 1, \ldots, K$) purchased in period t, $p_{t,k}$ the price of asset k in terms of consumption goods (ex dividend), and $f_{t+j,k}$, $j \geq 1$, the stream of cash flows (interest payments or dividends) in future periods.

Individuals pay taxes on endowments at a rate τ_t. We assume that taxes are distortionary, e.g., because of wasteful efforts of evading or sheltering income. Following Barro (1979), the excess burden of taxation is summarized by a loss function $h(\tau_t)$, which indicates the fraction of endowment "wasted" when taxes are τ_t. Then the individual budget constraint is

$$c_t + \sum_k p_{t,k} \cdot A_{t,k} = Y_t \cdot \left(1 - \tau_t - h(\tau_t)\right) + \sum_k (p_{t,k} + f_{t,k}) \cdot A_{t-1,k} \qquad (2)$$

Individual optimization implies asset pricing equations

$$p_{t,k} = \rho \cdot E_t(p_{t+1,k} + f_{t+1,k}) \qquad (3)$$

for all k.

It is convenient to introduce several specific securities that we want to analyze. First, let $k = 0$ be a riskfree (in real terms, i.e. price level indexed) one-period security that has a price $p_{t,0} = 1$ for all t. It must yield $f_{t,0} = r = 1/\rho - 1 > 0$. Then we can define excess returns

$$\hat{r}_{t+1,k} = (p_{t+1,k} + f_{t+1,k} - E_t(p_{t+1,k} + f_{t+1,k}))/p_{t,k}$$

$$= (p_{t+1,k} + f_{t+1,k} - p_{t,k})/p_{t,k} - r$$

on assets $k \geq 1$. Individual optimization can be summarized by $E_t\hat{r}_{t+1,k} = 0$.

Second, we want to discuss assets with returns defined in terms of a nominal unit of account, money. But we do not want to focus on asset pricing issues specific to monetary models nor do we want to discuss optimal monetary policy. While both issues are important topics in themselves,[4] all we need here is a well-defined price level. Therefore, we will just assume that the price level P_t and the rate of inflation, $\pi_t = \log(P_t/P_{t-1})$, follow some stochastic processes. It could be motivated more rigorously as a limit of a cash-in-advance model with "small" monetary sector.[5]

Third, some securities may be denominated in a foreign currency. Given risk neutrality of domestic individuals, the market clearing conditions for the closed economy[6] are not essential for our arguments. Therefore, allowing the existence of "other" individuals abroad does not change the model significantly. Whenever necessary, we will assume that payoffs of some securities may depend on variables defined within a foreign economy.

The government uses taxes $T_t = \tau_t \cdot Y_t$ to finance government spending G_t and to service the government debt.[7] We assume that the government can issue arbitrary quantities $D_{k,t}$ of the securities k at the market price.[8] The government budget constraint is

$$T_t = \tau_t \cdot Y_t = G_t + \sum_k (p_{t,k} + f_{t,k}) \cdot D_{t-1,k} - \sum_k p_{t,k} \cdot D_{t,k} \tag{4}$$

Individual welfare can be written as a function of government policy by substituting (4) and (2) into the individual objective function (and dropping irrelevant terms)[9]

$$U_t = E_t \sum_{j\geq 0} \rho^j \cdot [Y_{t+j} \cdot (1 - h(\tau_{t+j}))] , \tag{5}$$

The government chooses tax rates and debt structure to maximize (5) subject to (4). In effect, the government objective is to minimize the expected present value of excess burden (Barro (1979)). The first order conditions for optimal policy are

$$E_t[h'(\tau_{t+1})] = h'(\tau_t) \qquad \text{for } k = 0 \tag{6a}$$

$$E_t[h'(\tau_{t+1}) \cdot \hat{r}_{t+1,k}] = 0 \qquad \text{for all } k > 0 . \tag{6b}$$

If we assume that excess burden is quadratic, i.e. $h(\tau_t) = h/2 \cdot \tau_t^2$ for some $h > 0$, equation (6a) implies tax smoothing over time, $E_t \tau_{t+1} = \tau_t$, as in Barro (1979). This also determines the path of total debt.

Under the same assumption, equations (6b) imply that the conditional covariances of tax rates and excess returns are zero, i.e.,

$$\mathrm{cov}_t(\tau_{t+1}, \hat{r}_{t+1,k}) = E_t[\tau_{t+1} \cdot \hat{r}_{t+1,k}] = 0 \tag{7}$$

That is, the government should stabilize taxes across possible states of nature. Since taxes are a function of debt policy through equation (4), these orthogonality conditions implicitly characterize the optimal debt structure. Our main objective is to compute this optimal debt structure. One can also use these conditions directly to test for optimality. This approach is pursued in Section 4.1.

An explicit solution for the optimal debt structure is derived in Appendix 1, using linear approximations. The basic idea is that tax rates are constrained by initial debt, the present value of government spending, and the path of endowments. Because of tax-smoothing, any innovation in current or future endowment or in government spending or any unexpected change in the value of debt forces the government to revise tax rates (unless it is perfectly hedged). If we define innovations in growth and government spending (relative to current output) as $\hat{y}_{t+1+j} = E_{t+1}y_{t+1+j} - E_t y_{t+1+j}$ and $\hat{g}_{t+1+j} = (E_{t+1}G_{t+1+j} - E_t G_{t+1+j})/Y_t$ and let $d_{t,k} = p_{t,k} \cdot D_{t,k}/Y_t$ be the ratio of security-k debt to output, the innovation in tax rates is

$$\hat{\tau}_{t+1} = \tau_{t+1} - E_t\tau_{t+1} = (1 - \psi)\cdot\exp(-\bar{y}) \cdot [\textstyle\sum_k \hat{r}_{t+1,k} d_{t,k} + \sum_{j\geq 0} \rho^j \cdot \hat{g}_{t+1+j}] - \tau_t \cdot \sum_{j\geq 0} \psi^j \cdot \hat{y}_{t+1+j} , \tag{8}$$

where $0 < \psi < 1$ is a discount factor defined in the appendix. As one would expect, taxes are increased, if the value of debt increases unexpectedly, if estimates of future government spending are revised upwards, or if expected output declines. Then the first order conditions (7) are equivalent to

$$\sum_s \mathrm{cov}_t(\hat{r}_{t+1,s}, \hat{r}_{t+1,k}) \cdot d_{t,s} + \sum_{j\geq 0} \rho^j \cdot \mathrm{cov}_t(\hat{r}_{t+1,k}, \hat{g}_{t+1+j})$$

$$- w \cdot \sum_{j\geq 0} \psi^j \cdot \mathrm{cov}_t(\hat{r}_{t+1,k}, \hat{y}_{t+1+j}) = 0$$

for all k, where $w = \exp(\bar{y})/(1 - \psi) \cdot \tau_t$ is the weight on output risk.

To write this equation more conveniently, define the innovation in the expected present value of output, $PV(\hat{y})_{t+1} = \sum_{j\geq 0}\psi^j \cdot \hat{y}_{t+1+j}$ and the innovation in the expected present value of government spending, $PV(\hat{g})_{t+1} = \sum_{j\geq 0} \rho^j \cdot \hat{g}_{t+1+j}$. Let Σ_r be the variance-covariance matrix of returns, assumed to be non-singular,[10] and let $\Sigma_{g,r}$ and $\Sigma_{y,r}$ be the vectors of covariances between returns and $PV(\hat{y})_{t+1}$ and $PV(\hat{g})_{t+1}$, respectively. Finally, let d_t be the vector of risky government debt securities $d_{t,s}$. Then the first order conditions are

$$\Sigma_s \mathrm{cov}_t(\hat{r}_{t+1,s}, \hat{r}_{t+1,k}) \cdot d_{t,s} + \mathrm{cov}_t(\hat{r}_{t+1,k}, PV(\hat{g})_{y+1})$$

$$- w \cdot \mathrm{cov}_t(\hat{r}_{t+1,k}, PV(\hat{y})_{t+1}) = 0$$

$$\Leftrightarrow \qquad \Sigma_r \cdot d_t + \Sigma_{g,r} - w \cdot \Sigma_{y,r} = 0$$

and the solution for the debt structure is

$$d_t = \Sigma_r^{-1} \cdot [w \cdot \Sigma_{y,r} - \Sigma_{g,r}] \ . \qquad (9)$$

Thus, general formula for optimal debt structure involves covariances of returns with innovations in output and government spending.

Output (or equivalently, aggregate income) matters in this model, because it forms the tax base and because high tax _rates_ cause distortions. Given desired _levels_ of revenues, tax rates must increase if output falls. Since tax rates are smoothed over time, any news about future output is also relevant. Consequently, permanent changes in output have much larger effects than temporary changes. Unexpectedly high government spending has the obvious effect on tax rates and creates additional demand for hedging.

2.2. Risk Averse Taxpayers

Now we turn to an alternative model of optimal debt structure. Risk averse government behavior may also be due to risk averse taxpayers. Suppose there are two types of individuals, 1 and 2. Type 1 individuals (fraction α of the population) are risk averse with a concave utility function $\bar{u}(c, G) = u(c + \beta G)$, $0 \leq \beta < 1$.[11] Type 2 individuals (fraction $1 - \alpha$ of the population) are risk neutral and maximize the objective function (1), which implies asset pricing equations (3).[12] Both types have per-capita endowments Y_t.

To motivate a governmental role in risk allocation, it is crucial that type 1 individuals have limited access to financial markets. We just assume that type-1 individuals face a binding liquidity constraint, i.e., they consume all of their current disposable income. Such a constraint may arise, e.g., if endowments Y_t are human capital so that various information problems (moral hazard) prevent borrowing against it; but a rigorous treatment of this issue would complicate the model too much.[13]

To make the problem interesting, assume that the government cannot distinguish the two types, i.e. it has to levy the same per-capita tax T_t on each. It maximizes welfare

$$U_t = E_t \sum_{j\geq 0} \rho^j \cdot [\alpha \cdot u(c_t^1 + \beta \cdot G_t) + (1 - \alpha) \cdot c_{t+j}^2]$$

$$= \alpha \cdot E_t \sum_{j\geq 0} \rho^j \cdot u(c_t^1 + \beta \cdot G_t) + (1 - \alpha) \cdot E_t \sum_{j\geq 0} \rho^j \cdot c_{t+j}^2 \quad (10)$$

Consumption of type 1 is $c_t^1 = Y_t - T_t$ and consumption of type 2 satisfies equation (2) with c_t replaced by c_t^2 and $h(\tau) = 0$. Consumption can be obtained as function of government policy by using equations (2) - (4), resulting in

$$c_t^1 = Y_t - G_t + \sum_{t,k} p_{t,k} \cdot D_{t,k} - \sum_k (p_{t,k} + f_{t,k}) \cdot D_{t-1,k} \qquad (11)$$

$$E_t \sum_{j \geq 0} \rho^j \cdot c_{t+j}^2 = E_t \sum_{j \geq 0} \rho^j \cdot (Y_{t+j} - G_{t+j}) + A^{net}{}_t \ ,$$

where $A^{net}{}_t = \Sigma_k (p_{t,k} + f_{t,k}) \cdot (A_{t-1,k} - D_{t-1,k})$ is the initial net wealth of type-2 individuals (possibly nonzero in an open economy).

First order conditions for optimal debt policy are

$$E_t[u'(c_{t+1}^1 + \beta \cdot G_{t+1})] = u'(c_t^1 + \beta \cdot G_t) \qquad (12a)$$

$$E_t[u'(c_{t+1}^1 + \beta \cdot G_{t+1}) \cdot \hat{r}_{t+1,k}] = 0 \qquad \text{for all } k > 0 \ . \qquad (12b)$$

Thus, policy smooths the path of consumption for the type-1 agents. Equation (12a) implies that expected marginal utility of consumption a martingale; this characterizes optimal taxes and the level of debt. Equations (12b) imply that the government should hedge against shocks that may force it to change type-1 consumption unexpectedly; this characterizes the optimal structure of government debt.

The optimal debt structure can be obtained, if we use a quadratic approximation of $u(\cdot)$ and approximate Y_{t+j} as in Section 2.1 (see Appendix 1). Using the linear approximation, condition (12b) implies

$$\Sigma_r \cdot d_t + (1 - \beta) \cdot \Sigma_{g,r} - \exp(\bar{y})/(1 - \psi) \cdot \Sigma_{y,r} \qquad (13)$$

which is equivalent to

$$d_t = \Sigma_r^{-1} \cdot [\exp(\bar{y})/(1 - \psi) \cdot \Sigma_{y,r} - (1 - \beta) \cdot \Sigma_{g,r}] \ . \qquad (14)$$

Notice how similar equation (14) is to equation (9) of the tax-smoothing model. The optimality conditions and the solutions have the same structure, with only the weights on output and government spending being different. Since $\tau_t < 1$ and $\beta \geq 0$, the consumption-smoothing model places lower weight on the present value of innovations in government spending than the tax smoothing model. Intuitively, the objective here is to "protect" the entire path of consumption against shocks, while in the tax-smoothing model the government "protects" only the government sector. Since the government sector absorbs a share τ_t of endowment, the weight on output in a tax smoothing model is only τ_t (versus 1 here). On the other hand, if government spending provides utility, only $(1 - \beta)$ of any shock to government spending must be offset in a consumption smoothing model, but all of it in the tax smoothing model.

3. Empirical Analysis

In this section we consider a number of securities that may have a role as hedges against shocks to the government budget: Nominal and indexed bonds, long- and short-term bonds, common stocks, and foreign currency bonds.

3.1. Methodology

The two theoretical models identify uncertain output and uncertain government spending as sources of risk. For most of the empirical analysis, we will focus on the role of cyclical risk. A priori, it seems likely that output variation is a quantitatively significant source of risk; the cyclical volatility of budget deficits is well documented. In contrast, there are several considerations that may limit the feasibility of hedging against uncertain government spending.[14] We will turn to uncertain government spending in Section 4.2 and assume known government spending ($\Sigma_{g,r} = 0$) in this section. Both models then imply that the optimal structure of debt is proportional to the vector $h = \Sigma_r^{-1} \cdot \Sigma_{y,r}$, which depends only on the variance-covariance matrix of innovations in output and security returns.[15]

Our strategy is to take different sets of securities k, to estimate the covariance of their returns with the expected present values of output and government spending, and thereby to determine which of them should be issued. The data are for the post-war United States. They are described in Appendix 2.

The main empirical problem is to identify the innovations, in particular the change in expectations of "far out" realizations of output growth needed in the expected present value term $PV(\hat{y})_{t+1}$. We will use vector-autoregression (VAR) techniques, which seem ideally suited for the task of extracting the covariance structure of a multivariate process.

Let $x_t = (x_{1t}, \ldots, x_{St})'$ be the vector of variables in a VAR-process (where the prime ' denotes the transpose). Specifically, assume $S \geq K + 1$ and let $x_{1t} = y_t$ and $x_{(k+1)t} = \hat{r}_{t,k}$ for $k = 1, \ldots, K$. If $S > K + 1$, variables x_{st} for $s > K + 1$ are additional variables in the information set; if $S = K + 1$, only output and returns are included. Let A_n be the coefficient matrix at lag n ($n = 1, \ldots, N$), A_0 the vector of intercepts, and u_t the vector or residuals. Then the VAR can be written as

$$x_t = A_0 + \sum_n A_n x_{t-n} + u_t, \quad \text{where} \quad E[u_t u_t'] = \Sigma_u . \qquad (15)$$

Appendix 3 shows that the elements of the covariance vector $\Sigma_{y,r}$ can be written in terms of the VAR as

$$c_k = \text{cov}_t\left(PV(\hat{y})_{t+1}, \hat{r}_{t+1,k}\right) = i_{k+1} \cdot \Sigma_u \cdot C_\psi \cdot i_1 , \qquad (16)$$

where $C_\psi' = (I_{S\times S} - \sum_n \psi^n A_n)^{-1}$ and i_s is the vector of zeros and ones that selects variable s. That is, the covariance is the element in row 1, column k + 1 of the matrix product $\Sigma_u \cdot C_\psi$.

To obtain the optimal debt structure, let Σ_u^* be the K×K-submatrix of Σ_u consisting of rows and columns 2, ..., k + 1. It is an estimate of the covariance matrix of excess returns Σ_r. Let Σ^X be the S×S matrix containing the elements of Σ_u^{*-1} in rows and columns 2 to K + 1 and zeros elsewhere. Then the elements in row 1, columns 2, ..., K + 1 of the matrix $\Sigma^X \cdot \Sigma_u \cdot C_\psi$ are estimates for the vector $h = \Sigma_r^{-1} \cdot \Sigma_{y,r}$. Since the vector of optimal debt d_t is proportional to h, the sign of the k-th element

$$h_k = i_{k+1} \cdot \Sigma^X \cdot \Sigma_u \cdot C_\psi \cdot i_1 \tag{17}$$

is sufficient to determine whether or not security k should be issued by the government.

The matrix products $\Sigma_u \cdot C_\psi$ and $\Sigma^X \cdot \Sigma_u \cdot C_\psi$ are consistently estimated as functions of the corresponding estimates of the VAR process. Appendix 3 derives their asymptotic covariance matrix and Wald tests for individual and joint significance of c_k's and h_k's. The derivation may be of independent interest, since it provides a joint asymptotic covariance matrix of the variances and covariances of VAR-innovations and estimated VAR-coefficients.

A simplified procedure can be applied, if only one risky security is considered and a bivariate VAR-process is estimated (S = 2, K = 1). Estimating such a process is useful--at least as an initial step--to find out whether a security has a role as hedge in the absence of other risky debt. Appendix 3 shows that $\Sigma_{y,r} > 0$ and $h_1 > 0$ if and only if the auxiliary equation

$$x_{1t} = \sum_{n=1}^{N} b_{1n} \cdot x_{1t-n} + \sum_{n=0}^{N} b_{2n} \cdot x_{2t-n} + vt \tag{18}$$

yields estimates satisfying $\sum_{n=0}^{N} \psi^n b_{2n} > 0$. The simple linear constraint

H_0: $\sum_{n=0}^{N} \psi^n b_{2n} = 0$ can be tested easily (conditional on ψ) and frequently appears more significant than Wald tests of $c_1 = 0$ or $h_1 = 0$.[16]

Results are reported in several tables which will be discussed below. In the tables, we provide at least 5 pieces of information:

(1) correlations ρ_k between each return series k with the present value of output

(2) covariances c_k between each return series k with the present value of output,

(3) their asymptotic standard errors (std-c_k),

(4) vectors of h_k's indicating the optimal debt structure, and

(5) their standard errors (std-h_k).

Correlations are provided in addition to covariances to improve readability, since the correlations are independent of the unit of measurement. All estimates were computed for parameter values $\psi = 0.99$ and $\psi = 1.0$;[17] but since they were very similar, only results for $\psi = 0.99$ are reported. Notice that, in alternative regressions, the c_k's vary with the set of variables in the information set while the h_k's vary in addition with the set of securities that is being considered.

For all securities, we start with the minimal bivariate VAR including only GNP-growth (for y_t) and the return series, using quarterly U.S.-data from 1954:2 to 1987:4 and including a constant and 4 lags. All macroeconomic variables are log-differences. As alternatives, the processes were re-estimated with 8 lags, in log-levels (with a time trend included), and for shorter sample periods of 1954:2-1972:4 and 1973:1-1987:4 (intended to capture potential breaks caused by exchange rate regimes), respectively. Next, VARs with several additional variables in the information set were estimated. Following Bernanke (1986), we include military spending, money supply M1, the monetary base, and the 3-month T-bill rate in one information set.[18] As an alternative, we replace military spending by import prices, which might be especially important for predicting inflation. Finally, we estimate processes that include returns on several financial assets simultaneously, with and without additional information variables. For easier reference, the list of VAR-specifications is shown in a separate table, Table 1.

3.2. Nominal and Indexed Debt

Currently, all U.S. government debt is denominated in U.S. dollars, with various maturities. From a theoretical perspective, such nominal debt may be preferable to indexed debt as a hedge, if inflation is uncertain and correlated with other macroeconomic variables of interest, as I have argued elsewhere (Bohn (1988b)).[19] We therefore start with two securities, an indexed bond ($k = 0$) and a one-period nominal bond ($k = 1$). Maturity choice is considered in the next section.

A nominal bond issued in period t pays a known nominal return of i_t between t and t + 1, i.e. yields $r_{t+1,1} = i_t - \pi_{t+1}$ in real terms. In equilibrium,

$$i_t = r + E_t\pi_{t+1}$$

and

$$\hat{r}_{t+1,1} = -(\pi_{t+1} - E_t\pi_{t+1}) = -\hat{\pi}_{t+1} .$$

The empirical results are displayed in Table 2, where the change in the GNP-deflator, P, is used as variable for inflation. For all VAR-specifications, the correlations of innovations in returns and the present value of output are highly positive, i.e., the correlations with inflation are negative.[20] Except for the split sample, all estimates of covariances c_1 and estimates of optimal debt values h_1 are significantly positive. The correlations in bivariate VARs, around 0.90, seem extraordinarily high and may reflect the omission of other variables that predict output and inflation. But even if monetary and fiscal variables are included, the correlations remain around 0.50. Thus, the optimal government debt portfolio should clearly contain nominal bonds.

To put the result in perspective, recall that we have simplified the analysis to focus on issues of risk. Most importantly, we left out issues related to asset pricing and incentives, which would have to be added for a definitive policy recommendation. Incentive issues may reduce the optimal amount of nominal debt, but cannot eliminate it, if nominal debt is desirable for hedging reasons (see Bohn (1988b)). If there were differences in expected returns on indexed and nominal debt, the optimal debt structure would be shifted towards the lower-cost security.

The results are not only statistically but also economically significant. Based on the point estimates, the U.S.-government should even hold indexed bonds and issue nominal debt in an amount far exceeding its total debt.[21] If these simultaneous large long and short positions are impractical, th government may operate at a corner solution of issuing only nominal bonds. Alternatively, it may be that the 3-month maturity of nominal bonds implicit in quarterly data is insufficient. Hence, we turn to longer term securities.

3.3. Term to Maturity

There are three sets of issues related to maturity: One can study the term structure of nominal interest rates, the term structure of real interest rates, and the comparison of nominal and indexed bonds of different maturities. We start with the third issue, which is related to the previous section, and then turn to the first issue.[22] With many different maturities, the potentially large number of return series is a problem for computing optimal policy (multicollinarity in Σ_r). Therefore, we confine ourselves to two period bonds (6-month T-bills) and perpetuities (approximated by the longest term Treasury-bonds).

To compare nominal and indexed perpetuities (k = 0, 1, respectively), note that the real perpetuity has real return r. The nominal perpetuity pays i units of money in every period, i.e., $f_{t+j,1} = i/P_{t+j}$. By assumption, its expected real returns is equal to r. Appendix 1 shows that its excess return is approximately

$$\hat{r}_{t+1,1} = -\Omega \cdot PV(\pi)_{t+1} = -\Omega \cdot \sum_{j\geq 1}\phi^{j} \cdot (E_{t+1}\pi_{t+s} - E_t\pi_{t+s}), \qquad (19)$$

where ϕ is a discount factor and Ω is a positive constant. It is a weighted average of news about future rates of inflation. As before, the government should issue this nominal perpetuity if and only if $\Sigma_{y,r} = \text{cov}_t(PV(\hat{y})_{t+1}, \hat{r}_{t+1,1}) > 0$. In the VAR framework, this is equivalent to

$$\Sigma_{y,\pi} = \text{cov}_t(PV(\hat{y})_{t+1}, PV(\hat{\pi})_{t+1}) = i_1 \cdot C_\psi \cdot \Sigma_u \cdot C_\phi' \cdot i_2 = -c_1 < 0, \qquad (20)$$

where C_ϕ is defined analogous to C_ψ above.

Table 3 displays correlations between $PV(\hat{y})_{t+1}$ and $PV(\hat{\pi})_{t+1}$ for parameters ψ = 1.0, 0.99 and ϕ = 1.0, 0.99, 0.95 computed from the VAR estimates. All correlations turn out to be highly negative, suggesting that long-term nominal bonds should be issued. Unfortunately, the standard errors of the covariance estimates are so high that the estimates are insignificant (and therefore not displayed). Since the innovation in $PV(\hat{\pi})_{t+1}$ is close to being a multiple of the innovation in current inflation, the result suggests that long-term nominal bonds can be used to make the real value of debt very sensitive to inflation without issuing a large volume of nominal debt.[23]

Next, consider the term structure of nominal interest rates. Let i_{1t}, i_{2t}, and i_{Lt} be the yields on nominal one-period bonds, two-period bonds, and perpetuities (securities k = 1, 2, 3). Their real returns are approximately $i_{1t} - \pi_{t+1}$, $i_{2t} - \pi_{t+1} - (i_{1t+1} - i_{2t})$, and $i_{Lt} - D \cdot (i_{Lt+1} - i_{Lt}) - \pi_{t+1} = r^N_{t+1,3} - \pi_{t+1}$, respectively, where D is the duration of the perpetuity and $r^N_{t+1,3}$ the nominal return. Since data are available for inflation, 3-month T-bills yields (i_1) and nominal returns on long-term bonds (as proxy for perpetuities), we write innovations in returns as

$$\hat{r}_{t+1,1} = -\hat{\pi}_{t+1}$$

$$\hat{r}_{t+1,2} = -\hat{\pi}_{t+1} - \Delta\hat{i}_{1,t+1} = \hat{r}_{t+1,1} - \Delta\hat{i}_{1,t+1}$$

$$\hat{r}_{t+1,3} = -\hat{\pi}_{t+1} + \hat{r}^N_{t+1,3} = \hat{r}_{t+1,1} + \hat{r}^N_{t+1,3}$$

and use series for $-\hat{\pi}_{t+1}$, $-\Delta\hat{i}_{1,t+1}$, and $\hat{r}^N_{t+1,3}$ in the VAR-estimation.[24] The empirical series are labeled P, DTB, and LRET, respectively, and the results are displayed in Tables 4 and 5.

In Table 4, all correlations between the present value of output with DTB and LRET except one[25] have the positive sign indicating that two-period or long-term nominal bonds should be issued. Unfortunatley, few covariances and h_k-values are significantly positive. But the correlations have an interesting pattern: Both return series are negatively correlated with innovations in current output (-0.33 and -0.27, respectively, for VAR-specifications #1). The correlations with the present value of output are only positive because the coefficients of lagged returns on output are strongly positive. Consistent with this pattern, significant results are obtained for the 8-lag VAR (#4) with DTB as return series.

Overall, the positive point estimates for the correlations and covariances suggest that government debt should include components that are sensitive to innovations in nominal interest rates. Since inflation and nominal yields are correlated, however, this conclusion is only preliminary: In a portfolio setting, the optimal supply of two-period or long-term bonds depends on the joint correlation structure of (innovations in the present value of) output, inflation, and nominal returns. Such variance-covariances matrices are computed in Table 5.

Panel A is based on a VAR with output, inflation, and the change in T-bill yields. Panel B is estimated with M1 and the monetary base in the information set, and Panel C includes the long-term return into the potential debt-portfolio. The point estimates for all securities are positive in all regressions, implying that the value of debt should be sensitive to inflation and changes in nominal interest rates and confirming the result of the bivariate VAR's. The results on P are highly significant, while the sampling error makes the conclusions concerning DTB and LRET somewhat tentative.

Since DTB and LRET are nominal returns, one has to be careful in interpreting the optimal debt vector h. In a portfolio of P and DTB (and possibly LRET), the h_k-value for P indicates the total exposure to inflation risk. To compute the optimal supply of one-period bonds in a portfolio of one-period and two-period (and possibly long-term bonds) the difference between the h_k-value associated with P and the sum of the other h_k-values (indicating the optimal supply of longer term bonds) must be computed. This difference is positive in all regressions, implying that the value of debt should be sensitive to inflation and changes in nominal interest rates in a way that can be implemented by positive amounts of one-period, two-period, and long-term nominal bonds.

As in Section 3.2, desired nominal debt/GNP ratios (a multiple of the h_k-values for P) still far exceed the current total debt/GNP ratio for the United States. Thus, an optimal debt portfolio does not include indexed bonds at all, but would instead call for holding indexed debt. If this is not practically possible, nominal bonds are again a corner solution and indexed bonds should not be issued at all.

3.4. Other Risky Securities

Nothing in the preceding analysis suggests that governments should restrict their liabilities to nominal or indexed debt securities. We will only consider two other classes of securities, stocks and foreign currency debt.

If economists were asked to name a financial market variable closely correlated with economic activity, many would probably pick a stock market index. This makes common stocks a natural candidate for hedging output risk. Since the correlation is likely positive, our analysis suggests that the government should take a short position in stocks, or perhaps more practically, in index futures. To test this intuition, we consider the correlation between output and stock returns as measured by the Standard and Poors 500 index.

The results appear in Tables 6 and 7. Looking at stocks as the only risky liability (Table 6), correlations are almost all positive and highly significant, suggesting that the government would indeed benefit from taking a short position in the stock market.[26]

Given that nominal bonds are also issued, we compute the joint covariance structure of nominal bonds and stocks (P and STOCK) in Table 7, Panel A. Panels B - D add two-period bonds as return series and/or M1 and the monetary base as information variables. In all estimates, the optimal debt structure includes a short position in stocks. The optimal supply of one or two period nominal bonds remains positive. We see in Panel C that both nominal bonds and stocks appear highly significant in the optimal debt structure. (Panel C should be considered the most interesting one since it omits DTB, which is insignificant in itself, and adds information variables that are clearly important to predict P.)

Next, consider foreign currency debt. Many countries issue such debt, which may be quite risky in terms of the domestic currency. For the United States, the issue of foreign currency denominated debt has recently come up in the context of stabilizing exchange rates. We concentrate on two major currencies, Japanese yen and German mark, to evaluate whether foreign currency debt may be desirable for hedging reasons.[27]

As in the domestic context, we consider one-period nominal bonds, two-period-nominal bonds, and nominal perpetuities.[28] Using the duration formulas, the relevant excess returns are $\hat{e}_{t+1} - \hat{\pi}_{t+1}$, $\hat{e}_{t+1} - \hat{\pi}_{t+1} - \Delta\hat{i}^*_{1t+1}$, and $\hat{e}_{t+1} - \hat{\pi}_{t+1} - D \cdot \Delta\hat{i}^*_{Lt+1}$, respectively, where e_t denotes the rate of depreciation of the U.S.-dollar relative to the foreign currency, a * indicates foreign variables, and D denotes the duration of the foreign perpetuities.

Since the excess returns are linear combinations of several series, it is more instructive to analyze the components. That is, we compute the covariances of the present value of innovations in output with $\hat{e}_{t+1}$, $-\Delta\hat{i}^*_{1t+1}$, and $-\Delta\hat{i}^*_{Lt+1}$ to see whether or not the government could hedge output risk by issuing a security contingent on the corresponding variable. Notice that securities with real returns contingent on $\hat{e}_{t+1}$, $-\Delta\hat{i}^*_{1t+1}$, or $-\Delta\hat{i}^*_{Lt+1}$ can be interpreted as forward contracts. They would have to be repackaged to interpret them as optimal supplies of bonds. An optimal debt portfolio of, say, h_1 of a security contingent on $-\hat{\pi}_{t+1}$, h_2 of a security contingent on $\hat{e}_{t+1}$, and h_3 of a security contingent on $-\Delta\hat{i}^*_{Lt+1}$, would imply an optimal supply of long-term foreign bonds of h_3/D, an optimal supply of short-term foreign bonds of $h_2 - h_3/D$, and an optimal supply of domestic nominal bonds of $h_1 - h_2$.

Results based on bivariate VARs are displayed in Table 8 and the multivariate results are shown in Table 9, both for the period of flexible exchange rates 1973:1 - 1987:4. The return series are the rates of dollar-depreciation relative to mark and yen (EG and EJ, respectively) and minus one times the change in short- and long-term German and Japanese interest rates (SG and LG for Germany, SJ and LJ for Japan, respectively).

Since we are mainly interested in the question of whether there is an incremental benefit in issuing German or Japanese currency bonds in a setting with domestic nominal bonds, we concentrate on the multivariate framework (the bivariate results are similar).[29] In Panel A of Table 9, nominal domestic bonds (with real returns contingent on inflation P) are considered jointly with German and Japanese bonds (with nominal returns contingent on EG and EJ). The correlations between innovations in the present value of output and both exchange rates become positive, though insignificant. The point estimates suggest that it may be beneficial to supplement domestic nominal debt by German bonds, but not by Japanese bonds. Panel B adds the mark-dollar rate and a short-term German yield series to the domestic inflation series. Panel C does the same for Japanese data, and Panels D and E replace short-term by long-term yield data.

The results with yield series are remarkably uniform and significant: Exposure to changes in all foreign interest rates is desirable.[30] The strongest result is obtained in Panel C for long-term German bonds, where both

EG and LG appear significant at the 5% level. Notice, however, that optimal exposure to yield change exceeds the optimal total exposure to exchange rate risk in all cases (for plausible values of durations of long-term bonds). Again, exposure to German exchange rates appears more desirable than exposure to Japanese exchange rates. Thus, the optimal risk exposure can only be implemented by bond portfolios that include simultaneous large supplies of bonds (domestic and long-term foreign) and bond holdings (short-term foreign).

Overall, exposure to selected foreign interest rates appears to be very desirable. Though a more comprehensive analysis of foreign currency debt is beyond the scope of this article, there seems to be some potential for improvements in United States debt policy in this direction.

4. Extensions

4.1. Testing for Optimal Policy

The first order conditions (6a,b) should be satisfied, if the government chooses an optimal debt structure in a tax-smoothing model. These conditions can be used to test for such a policy.

Equation (6a) implies that tax rates follow a random walk, which is not rejected in standard regression tests.[31] Equation (6b) implies that innovations in tax rates should be uncorrelated with all innovations in returns. Results for bivariate VAR's with tax rates and our return series are displayed in Table 10.

Covariances with all domestic return series (Panel A) are significantly negative. This suggests that tax smoothing was imperfect. The government should have issued longer term bonds, i.e. taken more inflation and interest rate risk to hedge against shocks.[32] Also, tax rates could have been stabilized better, if the government had taken a short position in the stock market, confirming the result from Section 3.3.[33]

Concerning international data (Panel B), the covariances with all yield series are positive, most significantly with long-term German bond yields. This suggests that the U.S.-government might benefit from issuing such bonds, again confirming the result from Section 3.3.

Notice that I simply use the ratio of federal tax revenue to GNP as measure of tax rates instead of, say, marginal federal income tax rates. The reason is that although marginal tax rates are the theoretically relevant determinant of excess burden, it is unclear what "the" marginal tax rate is on aggregate. Taxes are levied not just on income but also on other activities, rates differ across individuals, and tax laws contain a multitude of exemptions and exceptions. On aggregate, however, all taxes must be paid out of the available economic resources, GNP. A higher revenue to GNP ratio implies higher tax rates somewhere, no matter how tax laws are structured in detail. Therefore we identify changes of the revenue to GNP ratio with changes in tax rates (or more precisely, with the marginal excess burden $h'(\tau)$).

Also notice that these tests of optimal policy require the strong assumption that tax policy is revised quarterly in light of news about all relevant variables. If, e.g., taxes are set ahead for a full fiscal year, significant correlations in quarterly data should not be interpreted as rejections of optimality (subject to the constraint of policies formulated by fiscal year) but rather as a suggestion that tax smoothing could be improved by more frequent policy adjustments.

4.2. Uncertain Government Spending

If uncertainty in government spending constitutes a significant risk, hedging against changes in the present value of government spending, $PV(\hat{g})$, may be appropriate. The analysis proceeds similar to the analysis for output risk.[34] Results are displayed in Table 11, Panel A.

While the covariances with inflation and bond series are insignificant, the covariance of the present value of spending and stock prices is significantly negative (though the correlation is smaller than that with output). This makes the optimal short position in stocks suggested by the analysis of output risk even more desirable and increases its optimal size.[35]

However, if the government is better informed about its own spending than the public, the VAR-estimates of future government spending may not be meaningful. If this is a serious concern, military spending may be a better measure of shocks on the spending side of the budget, assuming that it is equally unpredictable and exogenous for the public and the government. Panel B of Table 11 displays correlations of military spending with return series. All of them turn out to be small and insignificant. Overall, there seems to be little scope for hedging on the spending side of the government budget.[36]

5. Conclusions

We have analyzed the optimal structure of government debt on the government budget in a stochastic environment. We have two models that motivate why governments should care about the risk inherent in their choice of liabilities, based on tax-smoothing and risk aversion of taxpayers, respectively.

In the empirical analysis, we study how issuing nominal versus index debt, maturity choice, and the practice of the issuing debt securities affects the macroeconomic risks borne by the government. In an uncertain macroeconomic environment, an optimal structure of debt will generally include some "risky" debt (which is state contingent in real terms). We show that the contingencies represented by nominal debt and long-term debt have desirable properties as hedges against shocks affecting the government budget. This may motivate the current practice of non-indexing.

The hedging argument suggests further that the government may improve welfare by taking a short position in the stock market or issues some, newly created securities contingent on economic activity (e.g. on GNP). This is supported by the data. Finally, we find that it may be beneficial to issue some foreign currency debt, German mark denominated bonds in particular.

Footnotes

[1]The government's maximization problem could be solved even if there were differences in expected returns. The optimal debt structure would be similar to our solution but with a bias towards issuing securities with relatively low expected returns. The problem is how to motivate such return differentials. Adding risk aversion would complicate the model tremendously without providing a sufficient explanation for return differentials (see Mehra and Prescott (1985)). Providing an alternative explanation is beyond the scope of this paper.

[2]Securities with payoffs contingent on tax rates or government spending (a potential source of shocks) would severely distort government incentives for choosing the size of the public sector.

[3]Notice that either non-lump-sum taxes or heterogeneity must be assumed. If all taxpayers were identical and if taxes were lump sum, the optimal structure of government debt would be indeterminate as would be the level of debt, see Barro (1974).

[4]See, e.g., Lucas (1984) on asset pricing with money and Calvo (1978), Lucas and Stokey (1983), and Bohn (1988b) on incentive issues in monetary and fiscal policy.

[5]Assume individuals pay by check for all purchases. Also, there is no cash and reserve requirements are 100%, so that money supply, monetary base, and deposits are equal. A fraction ε of all checks fail to clear before the end of the period so that the seller is forced to keep $\varepsilon \cdot Y_t \cdot P_t$ on account. Assuming reserves are non-interest bearing, money reduces individuals' real cash flow by $\varepsilon \cdot (Y_t - P_{t-1}/P_t \cdot Y_{t-1})$, which would have to be subtracted on the right hand side of (2) and added to the government's resources in (4). Assuming nominal money supply $M_t(\varepsilon)$ follows some exogenous

process for given ε, prices are determined by $M_t(\varepsilon) = \varepsilon \cdot Y_t \cdot P_t$ or $P_t = (M_t(\varepsilon)/\varepsilon)/Y_t$. If ε is small, the effect of money on the model is negligible. To be precise, constraints (2) and (4) and a well-defined price level P_t are obtained in the limit as $\varepsilon \to 0$, provided $M_t(\varepsilon)/\varepsilon$ has a finite, positive limit M_t.

[6]That is, the requirement that net holdings of individuals and the government add up to zero for each security.

[7]For the empirical analysis, we use government spending in a wide sense as including all federal expenditures except interest on the federal debt. The reason is that all expenditures must be financed by distortionary taxes.

[8]As explained in the introduction, we leave out issues of time-consistency, that may arise if the government can manipulate payoffs of some securities. We also leave out non-negativity constraints or other limits on debt portfolio that may be a concern in practice.

[9]To be exact, $\sum_k(p_{t,k} + f_{t,k}) \cdot (A_{t-1,k} - D_{t-1,k}) - E_t \sum_{j\geq 0} \rho^j \cdot G_{t+j}$ should be added to the right hand side of (5). But since this is an additive, exogenous term, it does not affect decisions.

[10]Otherwise, redundant securities can be dropped from the analysis without loss of generality.

[11]We could have defined all other utility functions over consumption and government spending, too, but it is only critical for risk averse individuals. This utility function implies that one unit of per capita government spending G provides as much utility as β units of consumption.

[12]Risk neutrality is imposed to simplify asset pricing and to obtain a tractable welfare function for the government.

[13]Alternatively, one may assume that each type-1 agent represents a sequence of individuals who each live for only one period.

[14]First, there may be a severe problem of measuring innovations in spending, if the government knows more about future spending than the econometrician. Second, we exclude incentive problems in this paper. These problems may be particularly severe for debt contingent on government spending, because spending itself is a government choice variable and, again, because of potential asymmetric information. Third, the weight on spending is small in the consumption model, if β is close to one. If we interpret the liquidity constrained type-1 consumers as relatively "poor" individuals, it seems likely that they benefit significantly from government spending, i.e. that β is high.

[15]The factor of proportionality differs across models and depends critically on the discount rate applied to future output (see equations (9) and (14)). For quarterly real growth of $\bar{y}$ = 0.75% (the sample average), τ_t = 0.2 (ratio of federal revenue to GNP was 951.6/4604.0 = 0.206 in 1987), and a real discount rate of 1% per quarter, the proportionality factor is $w = \exp(\bar{y})/(1 - \psi) \cdot \tau_t \cong 20$ in the tax smoothing model and $\exp(\bar{y})/(1 - \psi) \cong 100$ in the model with risk-averse taxpayers. But if a discount rate of 0.5% per quarter is assumed, the factors double.

[16]In contrast to h_1, the test statistic g does not involve estimated variances. Since uncertainty in the estimated covariance matrix contributes to the standard error of h_1, it is not surprising that it is more difficult to reject $h_1 = 0$ than $g = 0$.

[17]GNP growth over our sample period is close to 3% annual. It is difficult to find the appropriate real discount factor, but fortunately, the choice of ψ is not critical. The value $\psi = 0.99$ corresponds to a, rather high, 7% real rate on an annual basis, while $\psi = 1.0$ is the theoretical upper limit for ψ.

[18]We use the change in the T-bill rate, while Bernanke uses the level, because the change is the relevant return series in this study. The choice of change or level in the information does not affect predictions in any significant way.

[19]This is true even though the type and transmission of shocks is different here. In that paper, a crucial macro-variable is the real interest rate, which is set constant in this study.

[20]Notice that nominal debt is desirable as a hedge, if innovations in the expected present value of output are <u>negatively</u> correlated with inflation, i.e., if $\mathrm{cov}_t(PV(\hat{y})_{t+1}, \hat{\pi}_{t+1}) = -c_1 < 0$. The optimal amount of nominal debt is proportional to $h_1 = c_1/\mathrm{var}(\hat{\pi}_{t+1})$. It may be interesting to note that the correlation of current output and inflation is rather small (below 0.1 in absolute value for most VARs), which is consistent with the literature (see Bernanke (1986)). The negative correlation of the present value of output with inflation is due to the fact that shocks that raise current inflation have a net negative effect on future output.

[21]Taking the lowest estimate of $h_1 = 1.62$ and a proportionality factor of $w = 20$ (computed in a previous footnote), the ratio of nominal debt (with one quarter maturity) to GNP should be about $d_1 = 32$, as opposed to the current total debt/GNP ratio of 0.366 (based on GNP of $4598 and federal debt of $1685 billion at the end of 1987).

[22]We do not attempt an empirical analysis of the term structure of indexed bonds since our assumption of risk neutrality would make an interpretation impossible.

[23]For Var #1 and parameters of column 1 of Table 3, for example, $PV(\hat{\pi})_{t+1} = 7.50 \cdot \hat{\pi}_{t+1} + 0.74 \cdot \hat{y}_{t+1}$, so that any innovation in inflation, which affects $PV(\hat{y})_{t+1}$ negatively, raises $PV(\hat{\pi})_{t+1}$ by a factor of 7.5. This

[24]Using innovations in inflation and nominal returns instead of a set of real returns facilitates the comparison with Section 3.2 by focussing on the new series. Yields on 6-month (two-period) T-bills are only available from 1959 on. Using 3-month holding period returns on such bills instead of DTB for 1960:2-1987:4 made little difference in the estimation. Also, using changes in long-term yields instead of returns lead to similar results.

[25]The exception is process #5 (in GNP-levels) for DTB. But notice that one should be cautious in interpreting specification #5 since GNP may contain a unit root.

[26]The small positive value for stocks in the VAR with import prices (#12) is somewhat puzzling. Further exploration shows that, over our sample period, import prices have been a powerful predictor of future output (especially 2-6 period ahead) and that they had some predictive power for future stock returns (9% significance level in a 4-lag VAR with GNP, stock returns and import prices). This may explain the reduced role for stock prices, though the predictability of stock prices is difficult to believe. Taken seriously, it suggests that the government should make its liabilities contingent on import prices.

[27]The current discussion is largely about yen denominated debt. Mark denominated bonds are interesting because such bonds were actually issued in 1978. The discussion about using foreign currency debt as a tool for exchange rate stabilization is apparently motivated by the potential incentive effects of such debt. As before, we do not consider incentive arguments, though they could complement our analysis; this is left for future research.

[28]Bonds indexed to foreign prices could also be considered, but they would raise a number of issues related to modeling real exchange rat movements that are beyond the scope of the paper.

[29]In the bivariate VAR's displayed in Table 8, the covariances with all return series except the short-term German yields are positive and highly significant, while the covariances of the present value output with exchange rates are insignificant.

[30]Short-term German rates, SG, are insignificant by themselves; but SG and EG are jointly significant at the 5% level.

[31]For example, in a four-lag autoregression for the full sample period 1954:2-1987:4, the relevant F(4,130)-statistic had a value of 0.91, which is a marginal significance level of 46%.

[32]If this was prevented by legal restrictions on the amount of long term Treasury bonds, this analysis implies that such constraints have been welfare-reducing.

[33]Results for the 73-87 sample period were similar; they were therefore omitted.

[34]Provided the share of government spending in GNP, $\bar{g}$, does not vary too much, the expected present value of government spending relative to current GNP, $PV(\hat{g})_{t+1} = \sum_{j\geq 0} \rho^j \cdot (E_{t+1}G_{t+1+j} - E_tG_{t+1+j})/Y_t = \sum_{j\geq 0} \rho^j \cdot (E_{t+1}G_{t+1+j} - E_tG_{t+1+j})/E_tY_{t+1+j}$, can be approximated by $\bar{g} \cdot \sum_{j\geq 0} \rho^j \cdot \dot{g}_{t+1+j} = \bar{g} \cdot PV(\dot{g})_{t+1}$, where $\dot{g}$ is the innovation in the expected growth rate of real government spending. Results in Table 11 are for the covariances of $PV(\dot{g})$ and returns.

[35]But notice that output risk still has a much larger weight in determining optimal debt than uncertainty in government spending, which is largely due to the factor $1/(1 - \psi)$ in the optimal debt equations (9) and (14).

[36]In addition, such hedging would be unnecessary, if spending increases utility (in terms of the model of Section 2.2., if β is close to 1).

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Appendix 1: Approximations

In the tax smoothing model of Section 2.1, optimal taxes can be approximated as follows.

Let y_t be the growth in endowments and let $\bar{y}$ be it mean, where we assume $\bar{y} < r$. We have $Y_{t+j+1} = Y_t \cdot \exp\left[\sum_{s=0}^{j}(y_{t+1+s} - \bar{y})\right]$. Using the approximation $\exp(x) \cong 1 + x$ for small x, we can write

$$Y_{t+j+1} \cong Y_t \cdot \exp[(j+1)\cdot\bar{y}] \cdot \left[1 + \sum_{s=0}^{j}(y_{t+1+s} - \bar{y})\right]$$

Notice that $E[Y_{t+j+1}] \cong Y_t \cdot \exp[(j+1)\cdot\bar{y}]$. Taking a Taylor expansion of τ_{t+j+1} around $T_t \cdot \exp[(j+1)\cdot\bar{y}]$ and $E[Y_{t+j+1}]$, we get

$$\begin{aligned}\tau_{t+j+1} &= T_{t+1+j}/Y_{t+1+j}\\ &\cong \tau_t + 1/Y_t \cdot \left[T_{t+1+j} \cdot \exp[-(j+1)\cdot\bar{y}] - T_t\right]\\ &\qquad - \tau_t/Y_t \cdot \left[Y_{t+1+j} \cdot \exp[-(j+1)\cdot\bar{y}] - Y_t\right]\\ &\cong 1/Y_t \cdot \exp[-(j+1)\cdot\bar{y}] \cdot T_{t+1+j} - \tau_t \cdot \left[\sum_{s=0}^{j} y_{t+1+s} - \bar{y})\right]\end{aligned}$$

We solve this for T_{t+1+j}, take conditional expectations and use (6a) to obtain

$$E_{t+1}T_{t+j+1} \cong Y_t \cdot \exp[(j+1)\cdot\bar{y}] \cdot \left[\tau_{t+1} + \tau_t \cdot \sum_{s=0}^{j}(E_{t+1}y_{t+1+s} - \bar{y})\right]$$

Denote $\psi = \rho \cdot \exp(\bar{y})$, the expected present value of taxes is therefore

$$\begin{aligned}PV(T)_{t+1} &= E_{t+1}\sum_{j\geq 0}\rho^j \cdot T_{t+j+1}\\ &= Y_t \cdot \exp(\bar{y})/(1-\psi) \cdot \tau_{t+1}\\ &\qquad + Y_t \cdot \exp(\bar{y}) \cdot \tau_t \cdot \sum_{j\geq 0}\psi^j\left[\sum_{s=0}^{j}(E_{t+1}y_{t+1+s} - \bar{y})\right]\\ &= Y_t \cdot \exp(\bar{y})/(1-\psi) \cdot \left[\tau_{t+1} + \tau_t \cdot \sum_{j\geq 0}\psi^j(E_{t+1}y_{t+1+s} - \bar{y})\right],\end{aligned}$$

which, by equation (3), is also equal to

$$PV(T)_{t+1} = \sum_{j\geq 0} \rho^j \cdot E_{t+1}G_{t+1+j} + \sum_k (p_{t+1,k} + f_{t+1,k}) \cdot D_{t,k} .$$

Hence, we obtain the tax rate in period t + 1 as

$$\tau_{t+1} = (1 - \psi)/Y_t \cdot \exp(-\bar{y})\left[\sum_{j\geq 0} \rho^j \cdot E_{t+1}G_{t+1+j} + \sum_k (p_{t+1,k}+f_{t+1,k}) \cdot D_{t,k}\right]$$

$$-\tau_t \cdot \sum_{j\geq 0}\psi^j (E_{t+1}y_{t+1+j} - \bar{y}) .$$

Define innovations in growth and government spending (relative to current output) as $\hat{y}_{t+1+j} = E_{t+1}y_{t+1+j} - E_t y_{t+1+j}$ and $\hat{g}_{g+1+j} = (E_{t+1}G_{t+1+j} - E_t G_{t+1+j})/Y_t$ and let $d_{t,k} = p_{t,k} \cdot D_{t,k}/Y_t$ be the ratio of security-k debt in output. Then the innovation in tax rates is

$$\hat{\tau}_{t+1} = \tau_{t+1} - E_t\tau_{t+1} = (1 - \psi) \cdot \exp(-\bar{y}) \cdot \left[\sum_k \hat{r}_{t+1,k} \cdot d_{t,k} + \right.$$

$$\left. \sum_{j\geq 0} p^j \cdot \hat{g}_{t+1+j}\right] - \tau_t \cdot \sum_{j\geq 0} \psi^j \cdot \hat{y}_{t+1+j} . \tag{8}$$

In the consumption smoothing model of Section 2.2, equations (12a,b) imply

$$E_t(c^1_{t+1} + \beta \cdot G_{t+1}) = (c^1_t + \beta \cdot G_t)$$

and

$$E_t\left[(c^1_{t+} + \beta \cdot G_{t+1}) \cdot \hat{r}_{t+1,k}\right] = 0 ,$$

if u(·) is approximated by a quadratic function. Then equation (11) implies

$$E_t\sum_{j\geq 0} \rho^j \cdot (c^1_{t+j+} \beta \cdot G_{t+j}) = 1/(1 - \rho) \cdot (c^1_t + \beta \cdot G_t)$$

$$= E_t\sum_{j\geq 0} \rho^j \cdot \left(Y_{t+j} - (1 - \beta) \cdot G_{t+j}\right) - \sum_k (p_{t,k} + f_{t,k}) \cdot D_{t-1,k}$$

$$\cong Y_{t-1} \cdot \exp(\bar{y})/(1 - \psi) \cdot E_t \sum_{j\geq 0} \psi^j \cdot (E_t y_{t+j} - \bar{y})$$

$$- (1 - \beta) \cdot E_t \sum_{j\geq 0} \rho^j \cdot G_{t+j} - \sum_k (p_{t,k} + f_{t,k}) \cdot D_{t-1,k}$$

Looking at innovations in period t + 1, we therefore have

$$(c + \beta G)_{t+1} - E_t(c + \beta G)_{t+1} = -(1 - \rho) \cdot Y_{t-1} \cdot [\textstyle\sum_k \hat{r}_{t+1,k} \cdot d_{t,k}$$

$$+ (1 - \beta) \cdot \textstyle\sum_{j\geq 0} \rho^j \cdot \hat{g}_{t+1+j} - \exp(-\bar{y})/(1 - \psi) \cdot \sum_{j\geq 0} \psi^j \cdot \hat{y}_{t+1+j}] .$$

so that $\mathrm{cov}_t\big((c + \beta G)_{t+1}, \hat{r}_{t+1,k}\big) = 0$ implies

$$\textstyle\sum_s \mathrm{cov}_t(\hat{r}_{t+1,s}, \hat{r}_{t+1,k}) \cdot d_{t,s} + (1 - \beta) \cdot \sum_{j\geq 0} \rho^j \cdot \mathrm{cov}_t(r_{t+1,k}, \hat{g}_{t+1+j})$$

$$- \exp(\bar{y})/(1 - \psi) \cdot \textstyle\sum_{j\geq 0} \psi^j \cdot \mathrm{cov}_t(\hat{r}_{t+1,k}, \hat{y}_{t+1+j}),$$

for all k. This is equivalent to equation (13).

Equation (20) is obtained by noting that the nominal bond is valued at

$$p_{t,1} = i \cdot E_t \textstyle\sum_{j\geq 1} \rho^j/P_{t+j} = i/P_t \cdot E_t \sum_{j\geq 1} \exp(-j \cdot r - \sum_{s=1}^{j} \pi_{t+s})$$

and that (with $1/\rho = \exp(r) \tilde{=} 1 + r$)

$$p_{t+1,1} + f_{t+1,1} = i/P_t \cdot E_{t+1} \textstyle\sum_{j\geq 0} \exp[-j \cdot r - \sum_{s=0}^{j} \pi_{t+1+s}]$$

$$= i/P_t \cdot (1 + r) \cdot E_{t+1} \textstyle\sum_{j\geq 1} \exp[-j \cdot r - \sum_{s=1}^{j} \pi_{t+s}] .$$

The excess return $\hat{r}_{t+1} = \big(p_{t+1,1} + f_{t+1,1} - (1 + r) \cdot p_{t,1}\big)/p_{t,1}$ can be approximated by

$$\hat{r}_{t+1,1} = (1 + r)/p_{t,1} \cdot \textstyle\sum_{j\geq 1} [E_{t+1} \exp[-j \cdot r - \sum_{s=1}^{j} \pi_{t+s}]$$

$$-E_t \exp[-j \cdot r - \textstyle\sum_{s=1}^{j} \pi_{t+s}]]$$

$$= -(1 + r)/p_{t,1} \cdot \textstyle\sum_{j\geq 1} \exp[-j \cdot (r + \bar{\pi})]$$

$$\cdot [E_{t+1} \exp[\textstyle\sum_{s=1}^{j} (\pi_{t+s} - \bar{\pi})] - E_t \exp[\sum_{s=1}^{j} (\pi_{t+s} - \bar{\pi})]]$$

$$\tilde{=} - (1 + r)/p_{t,1} \cdot \textstyle\sum_{j\geq 1} \exp[-j \cdot (r + \bar{\pi})] \cdot \Sigma_{s=1}^{j} (E_{t+1}\pi_{t+s} - E_t \pi_{t+s})$$

where $\bar{\pi}$ is mean inflation. Thus, the excess return $\hat{r}_{t+1}$ is a weighted average of innovations in expected future inflation, with geometrically declining weights. That is,

$$\hat{r}_{t+1,1} = -\Omega \cdot PV(\pi)_{t+1} = -\Omega \cdot \sum_{j\geq 1} \phi^{j} \cdot (E_{t+1}\pi_{t+s} - E_{t}\pi_{t+s}) ,$$

where $\phi = \rho \cdot \exp(-\bar{\pi})$ and Ω is the factor of proportionality in the previous equation.

Appendix 2: Data

All data are from the WEFA (Wharton Econometric Forecasting Associates) database if not otherwise noted. They are defined as follows:

output y_t - real GNP, quarterly, from National Income and Product Accounts (NIA)

prices P_t - GNP deflator, quarterly, from NIA

government spending G_t - nominal federal expenditures excluding net interest payments divided by the GNP-deflator, quarterly, from NIA

military spending - nominal federal defense spending divided by the GNP-deflator, quarterly, from NIA

tax revenues T_t - nominal federal receipts divided by the GNP-deflator, quarterly, from NIA

P - inflation π_t - the log-growth rate in P_t

DTB - change in log-yields on the 3-month Treasury-bill between the last day of the current quarter and the last day of the previous quarter, from the CRSP file (following Fama's (1984) definitions)

LRET - log-return on long-term Treasury bonds over the quarter, from Ibbotsen Associates

STOCK - log-return on the S&P 500 stock index, from Ibbotsen Associates

M1 - growth rate in the M1-money supply, computed as log-change between the last months of the quarters. Federal Reserve Board data for 1959-1987 are spliced to a 1953-1958 series from Banking and Monetary Statistics.

money base - log-growth rate in the money base, computed as log-change between the last weekly observations of the quarters. Federal Reserve Board data for 1959-1987 are spliced to a series from the St. Louis Federal Reserve Bank for 1953-1958.

import prices - log-growth rate in the quarterly U.S.-import price deflator, from NIA.

Ex - log-growth of the U.S.-dollar exchange rate relative to the country indicated by "x" between the last days of the current quarter and the day month of the previous quarter (IFS series OOAE).

Sx - change in money market yields in country "x" between the last month of the current quarter and the last month of the previous quarter (IFS series 60B).

Lx - change in long-term government bond yields in country "x" between the last month of the current quarter and the last month of the previous quarter (IFS series 61).

where x = G - Germany

x = J - Japan

Appendix 3: VAR-Based Projection and Testing

Equation (16) is derived as follows. Let i_s be the vector of zeros and ones that select variable s and let

$$\tilde{A} = \begin{matrix} A_1 & A_2 & \cdot\cdot & A_n \\ I & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & I & 0 \end{matrix}, \quad \tilde{x}_t = \begin{matrix} x_t \\ x_{t-1} \\ \cdot \\ x_{t-n+1} \end{matrix}, \quad \tilde{A}_0 = \begin{matrix} A_0 \\ 0 \\ \cdot \\ 0 \end{matrix}, \quad \text{and } \tilde{u}_t = \begin{matrix} u_t \\ 0 \\ \cdot \\ 0 \end{matrix} .$$

Then we have $E_t\tilde{x}_{t+j} = \tilde{A}_0 + \tilde{A}_j \cdot \tilde{x}_t$, hence $E_t\tilde{x}_{t+j} - E_{t-1}\tilde{x}_{t+j} = A^j \cdot \tilde{u}_t$, $\hat{y}_{t+j} = i_1 \cdot \tilde{A}_j \cdot \tilde{u}_t$, and

$$PV(\hat{y})_{t+1} + \sum_{j\geq 0} \psi^j \cdot \hat{y}_{t+1+j} = i_1 \cdot \sum_{j\geq 0} \psi^j \cdot \tilde{A}^j \cdot \tilde{u}_{t+1} =$$

$$i_1 \cdot (I - \psi - \tilde{A})^{-1} \cdot \tilde{u}_{t+1} .$$

It is straightforward to verify that the first S×S elements of $(I - \psi \cdot \tilde{A})^{-1}$ are $C' = (I - \sum_n \psi^n A_n)^{-1}$, so that $PV(\hat{y})_{t+1} = i_1 \cdot C' \cdot u_{t+1}$. We obtain (16) by using this formula in the covariance (recall that return k has position k + 1 in the vector x_t).

To derive equation (18), let $a_{ij}(n)$ be the elements of A_n and σ_{ij} be the elements of Σ_u (i, j = 1, 2). For bivariate processes, we have $\Sigma_r > 0$ (a real number) and

$$\Sigma_{y,r} = i_2 \cdot \Sigma_u \cdot C_\psi \cdot i_1 = |C_\psi| \cdot \left[\left(I - \Sigma_n \psi^n a_{22}(n)\right) \cdot \sigma_{12} + \Sigma_n \psi^n a_{12}(n)\right) \cdot \sigma_{22}\right] .$$

If we define $v_t = u_{1t} - \sigma_{21}/\sigma_{22} \cdot u_{2t}$, the residual in the regression

$$x_{1t} = a_{10} + \Sigma_n\, a_{11}(n) \cdot x_{1t-n} + \Sigma_n a_{12}(n) \cdot x_{2t-n} + u_{1t}$$

can be replaced to obtain

$$x_{1t} = a_{10} + \Sigma_n a_{11}(n) \cdot x_{1t-n} + \Sigma_n a_{12}(n) \cdot x_{2t-n} + v_t$$

$$+ \sigma_{21}/\sigma_{22} \cdot \left[x_{2t} - a_{20} - \Sigma_n a_{21}(n) \cdot x_{1t-n} = \Sigma_n a_{22}(n) \cdot x_{2t-n}\right]$$

$$= b_0 + \Sigma_n b_{1n} \cdot x_{1t-n} + \Sigma_n b_{2n} \cdot x_{2t-n} + b_{20} \cdot x_{2t} + v_t$$

where $b_0 = a_{10} - \sigma_{21}/\sigma_{22} \cdot a_{20}$, $b_{in} = a_{1i}(n) - \sigma_{21}/\sigma_{22} \cdot a_{2i}(n)$ for $n = 1,\ldots, N$ and $i = 1, 2$, and $b_{20} = \sigma_{21}/\sigma_{22}$. But notice that

$$\Sigma_{y,r} = |C_\psi| \cdot \sigma_{22} \cdot \left[\sigma_{21}/\sigma_{22} + \Sigma_n \psi^n b_2(n)\right] ,$$

which is positive if and only if $b_{20} + \Sigma_n \psi^n b_{2n} > 0$.

Asymptotic standard errors of the estimates c_k and h_k (defined in equations (16) and (17)) are derived as follows, using Maximum Likelihood (ML) theory (see Amemiya (1985), Ch. 4). We assume normal i.i.d. errors and the absence of unit root problems, which guarantees standard properties of the coefficient estimates (see Schmidt (1976), Section 4.3 and p. 259). The two main problems are deriving the joint covariance matrix of the estimated coefficients and covariances and finding the derivatives of c_k and h_k with respect to coefficients and covariances.

To set up the estimation problem, let $Y = (x_1, \ldots, x_T)'$ and $X = \left(\binom{1}{\tilde{x}_1}, \ldots, \binom{1}{\tilde{x}_T}\right)'$, be the data matrices, $A = (A_0, A_1, \ldots, A_N)'$ the matrix of coefficients, and $u = (u_1, \ldots, u_T)'$ the matrix of residuals. Also define the vectors of stacked coefficients $a = \mathrm{vec}(A)$ and $s = \mathrm{vec}(\Sigma_u)$, where $\mathrm{vec}(\cdot)$ denotes stacking by columns. Notice that s contains S^2 elements $s_{ij} = \sigma_{1j}$, $(i, j = 1, \ldots, S)$, i.e. does not exploit the symmetry of Σ_u. Let $S = (s_{ij})$ be the $S \times S$ matrix of these elements. In contrast, define the $S^+ = S \cdot (S + 1)/2$-element vector of covariances

$$\sigma = (\sigma_{11}, \sigma_{21}, \sigma_{22}, \ldots, \sigma_{S1}, \ldots, \sigma_{SS})'$$

which stacks the elements of Σ_u recognizing the symmetry. Finally, let P be the $S^2 \times S^+$ matrix of zeros and ones that maps σ into s, $s = P \cdot \sigma$ and $\sigma = P' \cdot s$.

Assuming $u = Y - X \cdot A$ has a multivariate normal distribution, (Y, X) has a joint normal distribution conditional on $(x_0, x_{-1}, \ldots, x_{-n+1})$ with log-likelihood function

$$L(Y, X; a, \sigma) = - T/2 \cdot \log |\Sigma_u| - \tfrac{1}{2} \cdot \mathrm{tr}(\Sigma_u^{-1} \cdot u'u) ,$$

where deterministic terms have been omitted. For taking derivatives, it turns out to be easier to write this in terms of the general matrix S and to add the restriction $s = P \cdot \sigma$ in a second step. That is, let

$$\ell(Y, X; a, s) = - T/2 \cdot \log|S| - \tfrac{1}{2} \cdot \mathrm{tr}(S^{-1} \cdot u'u) ,$$

then $L(Y, X; a, \sigma) = \ell(Y, X; a, P \cdot s)$, $dL/d\sigma = P' \cdot d\ell/ds$, and $d\ell^2/(d\sigma d\sigma') = P' \cdot d\ell^2/(ds ds') \cdot P$. We have $dL/da = S^{-1} \otimes X'u$, where $\otimes$ indicates the Kronecker product and

$$dL/dS = -T/2 \cdot S^{-1} + \tfrac{1}{2} \cdot S^{-1} \cdot u'u \cdot S^{-1} = \tfrac{1}{2} \cdot S^{-1} \cdot Z \cdot S^{-1}$$

where $Z = u'u - T \cdot S = (z_{ij})$ is the $S \times S$ matrix with elements $Z_{ij} = \sum_t u_{it} \cdot u_{jt} - \sigma_{ij}$. Notice that $dL/ds = \tfrac{1}{2} \cdot \left((S^{-1})' \otimes S^{-1}\right) \cdot \mathrm{vec}(Z)$ (see Amemiya (1985) for all matrix formulas). This implies the familiar ML-estimators $\hat{A} = (X'X)^{-1} \cdot X'Y$ and $\hat{S} = \hat{\Sigma}_u = 1/T \cdot u'u$. To obtain their standard errors, we compute

$$dL^2(dada') = - S^{-1} \otimes X'X$$

$$dL^2/(dsda') = \left((S^{-1})' \otimes S^{-1}\right) \cdot vec(X'u)$$

$$dL^2/(dsds') = -T/2 \cdot \left((S^{-1})' \cdot \otimes S^{-1}\right)$$

$$dL^2/(d\sigma d\sigma') = -T/2 \cdot P'\left((S^{-1})' \otimes S^{-1}\right)P$$

Since $plim(X'u/T) = 0$, we have $plim\left(dL^2/(dsda')\right) = 0$ so that the information matrix is block-diagonal. The assumptions on u imply that $z_{ij}/\sqrt{T}$ converges in distribution to $N(0, 2\sigma_{ij}^2)$; linear combinations of different z_{ij}'s are also asymptotically normal with variances as indicated in Anderson (1958), p. 39. The other regularity assumptions necessary for Theorems 4.24 in Amemiya (1985), p. 121 can also be verified. Hence, $\sqrt{T} \cdot (\hat{a}, \hat{\sigma})'$ converges in distribution to

$$N\left[\sqrt{T} \cdot \begin{pmatrix} a \\ \sigma \end{pmatrix}, \begin{pmatrix} \Sigma_a & 0 \\ 0 & \Sigma_\sigma \end{pmatrix}\right].$$

where $\Sigma_a = \Sigma_u \otimes Q^{-1}$, $Q = plim(X'X/T)$, and $\Sigma_\sigma = 2 \cdot \left(P'(\Sigma_u^{-1} \otimes \Sigma_u^{-1})P\right)^{-1}$.

Let c be the vector of c_k's, which depends on a, s, and σ. The ML-estimator of c, $\hat{c} = c(\hat{a}, \hat{\sigma})$, has an asymptotical covariance matrix of

$$\Sigma_c = 1/T \cdot \left[(dc/da)' \, \Sigma_a (dc/da) + (dc/d\sigma)' \, \Sigma_\sigma (dc/d\sigma)\right],$$

where $dc/d\sigma = P' \cdot (dc/ds)$. Hence, the test statistic $c' \cdot \Sigma_c^{-1} \cdot c$ has a $\chi^2(K)$-distribution under $H_0 : c = 0$ (see Amemiya (1985), p. 142). This Wald-test is chosen because it does not require parameter estimates under the null hypothesis. The covariance matrix of $\hat{h}$, the vector of $\hat{h}_k$'s, is determined analogously. Similar Wald-statistics can be used to test subsets of c and h. Finally, straightforward differentiation yields

$$dc_k/da_{ij}(n) = \psi^n \cdot (C)_{i1} \cdot (\Sigma_u \cdot C)_{k+1,j}$$

$$dh_k/da_{ij}(n) = \psi^n \cdot (C)_{i1} \cdot (\Sigma^x \cdot \Sigma_u \cdot C)_{k+1},j$$

$$dc_k/ds_{ij} = (C)_{i1} \quad , \quad \text{if } j \neq k$$

$$= 0 \quad , \quad \text{if } j \neq k$$

$$dh_k/ds_{ij} = \left[(C)_{i1} - (\Sigma^x \cdot \Sigma_u \cdot C)_{i1}\right] \cdot (\Sigma^x)_{k+1,j}$$

where the notation $(M)_{rs}$ indicates on element in row r, column s of a matrix M.

Table 1: VAR-specifications and Notation

The subsequent tables provide information on the following variables:

ρ_k: the correlation between return series k and the present value of output,

c_k: the covariance between return series k and the present value of output,

std-c_k: the asymptotic standard error of c_k,

h_k: the indicator of optimal supply of security k defined in equation (17), and

std-h_k: the asymptotic standard error of h_k.

All subsequent tables use the following VAR-specifications:

#1: sample 1954:2-1987:4, 4 lags, bivariate (as described in text).

#2: sample 1973:1-1987:4, otherwise as #1.

#3: sample 1954:2-1972:4, otherwise as #1.

#4: 8 lag, otherwise as #1.

#5: using log-levels of GNP and P instead of their growth rates, including time as regressor, otherwise as #1.

#6: sample 1973:1-1987:4, otherwise as #5.

#7: with M1, money base, and DTB as information variables, otherwise as #1.

#8: with M1, money base, military spending, and DTB as information variables, otherwise as #1.

#9: with M1, money base, import prices, and DTB as information variables, otherwise as #1.

#10: with M1, money base, import prices, military spending, and DTB as information variables, otherwise as #1.

#11: with M1, money base, military spending, and inflation as information variables, otherwise as #1.

#12: with M1, money base, import prices, and inflation as information variables, otherwise as #1.

#13: with M1, money base, DTB, and inflation as information variables, otherwise as #1.

The significance of results is indicated by the following symbols:

*, **, ***: 10%, 5%, 1% significance in Wald tests, respectively.

+, ++, +++: 10%, 5%, 1% significance in regression tests (equation (18)), respectively.

Table 2: Returns on Nominal Bonds

VAR	ρ_k	c_k	std-c_k	h_k	std-h_k
#1	0.853	0.511	0.254**++	3.28	1.59**
#2	0.955	0.658	0.444+	4.25	2.76
#3	0.628	0.262	0.217	2.11	1.17
#4	0.923	0.377	0.162**++	2.56	1.06**
#5	0.957	3.585	1.182***	23.4	7.19***
#7	0.534	0.240	0.102**	1.95	0.80**
#8	0.540	0.237	0.096**	2.00	0.77***
#9	0.461	0.166	0.075**	1.63	0.70**
#10	0.388	0.133	0.069*	1.34	0.68**

Legend: See Table 1 for notation. The columns of c_k and std-c_k have been multiplied by 10^4 to improve readability.

Table 3: Correlation between the Present Values of Output and Inflation

VAR	ψ=0.99		ψ=1.0		
	ϕ=0.95	ϕ=0.99	ϕ=0.95	ϕ=0.99	ϕ=1.0
#1	-0.726	-0.717	-0.792	-0.784	-0.782
#2	-0.803	-0.775	-0.856	-0.832	-0.825
#3	-0.547	-0.562	-0.599	-0.613	-0.617
#4	-0.711	-0.678	-0.797	-0.768	-0.761
#5	-0.915	-0.861	-0.975	-0.941	-0.930
#7	-0.642	-0.571	-0.735	-0.689	-0.665
#8	-0.658	-0.576	-0.754	-0.698	-0.671
#9	-0.584	-0.521	-0.669	-0.618	-0.596

Legend: See Table 1 for VAR-specifications.

Table 4: Maturity Choice

Panel A: Changes in T-Bill Rates

VAR	ρ_k	c_k	std-c_k	h_k	std-h_k
#1	0.084	0.268	0.590	0.38	0.83
#2	0.128	0.608	1.339	0.49	1.08
#3	0.048	0.577	2.817	0.31	1.51
#4	0.568	1.553	0.747**+++	2.41	1.12**
#5	-0.042	-2.013	8.245	-2.92	12.0
#11	0.218	0.655	0.742	1.19	1.34

Panel B: Returns on Long-Term Bonds

VAR	ρ_k	c_k	std-c_k	h_k	std-h_k
#1	0.155	0.70	1.507	0.039	0.060
#2	0.330	3.089	3.535	0.073	0.083
#3	0.266	0.834	0.870	0.089	0.092
#4	0.334	1.796	1.397	0.074	0.057
#5	0.014	1.238	18.45	0.052	0.768
#11	0.239	1.605	1.815	0.068	0.076

Legend: See Table 1 for notation. The columns of c_k and std-c_k have been multiplied by 10^5 in Panel A and by 10^4 in Panel B to improve readability.

Table 5: Optimal Portfolios of Nominal and Long-Term Bonds

Panel A: Nominal and Two-Period Bonds†

RET	ρ_k	c_k	std-c_k	h_k	std-h_k
P	0.758	4.085	2.090*	2.739	1.351**
DTB	0.260	0.945	0.846	1.115	1.114

Panel B: Nominal and Two-Period Bonds†

RET	ρ_k	c_k	std-c_k	h_k	std-h_k
P	0.534	2.396**	1.023*	1.905	0.771**
DTB	0.214	0.646	0.779	0.984	1.343

Panel C: Nominal, Two-Period, and Long-Term Bonds†

RET	ρ_k	c_k	std-c_k	h_k	std-h_k
P	0.676	3.598	2.004*	2.371	1.307**
DTB	0.324	1.157	0.909	0.275	1.118
LRET	0.401	27.54	18.65	0.087	0.069

Legend:

† VAR with GNP and all returns listed under RET, otherwise VAR specification #1; see Table 1 for notation.

†† VAR with GNP, all returns listed under RET, and additional variables M1 and money base, otherwise VAR specification #1; see Table 1 for notation.

The columns of c_k and std-c_k have been multiplied by 10^5 to improve readability.

Table 6: Stock Returns

VAR	ρ_k	c_k	std-c_k	h_k	std-h_k
#1	0.614	0.576	0.178***++	0.096	0.027***
#2	0.693	0.832	0.348**+++	0.104	0.039***
#3	0.553	0.374	0.167**++	0.087	0.036**
#4	0.403	0.383	0.205*+	0.067	0.035*
#5	0.509	7.647	4.462*+++	1.317	0.751*
#11	0.546	0.522	0.192***	0.105	0.037***
#12	0.262	0.231	0.163	0.045	0.031
#13	0.422	0.367	0.163**	0.072	0.031**

Legend: See Table 1 for notation. The columns of c_k and std-c_k have been multiplied by 10^3 to improve readability.

Table 7: Optimal Portfolios of Nominal Bonds, Long-Term Bonds, and Stocks

Panel A: Nominal Bonds and Stocks†

RET	ρ_k	c_k	std-c_k	h_k	std-h_k
P	0.847	0.491	0.253**	2.932	1.652**
STOCKS	0.451	5.073	2.023**	0.050	0.034

Panel B: Nominal and Two-Period Bonds, and Stocks†

RET	ρ_k	c_k	std-c_k	h_k	std-h_k
P	0.760	0.397	0.211*	2.555	1.469**
DTB	0.319	0.107	0.072	1.372	1.093
STOCK	0.339	3.471	1.722**	0.028	0.035

Panel C: Nominal Bonds and Stocks††

RET	ρ_k	c_k	std-c_k	h_k	std-h_k
P	0.660	0.334	0.150**	2.031	1.035**
STOCK	0.530	5.198	1.876***	0.074	0.033**

Panel D: Nominal Bonds, Two-Period Bonds and Stocks††

RET	ρ_k	c_k	std-c_k	h_k	std-h_k
P	0.534	0.274	0.100**	1.6174	0.803**
DTB	0.306	0.084	0.067	1.242	1.320
STOCK	0.422	3.670	1.630**	0.052	0.034

Legend:

† VAR with GNP and all returns listed under RET, otherwise VAR specification #1; see Table 1 for notation.

†† VAR with GNP, all returns listed under RET, and additional variables M1 and money base, otherwise VAR specification #1; see Table 1 for notation.

The columns of c_k and std-c_k have been multiplied by 10^4 to improve readability.

Table 8: Foreign Securities

RET	VAR	ρk	c_k	std-c_k	h_k	std-h_k
EG	#2	-0.080	-0.078	0.318	-0.02	0.087
EG	#6	-0.478	-1.771	1.287	-0.48	0.337
EJ	#2	-0.410	-0.372	0.299	-0.11	0.084
EJ	#6	-0.564	-2.125	1.312	-0.63	0.373*
SG	#2	0.361	1.009	0.731	1.58	1.101
SG	#6	-0.056	-0.955	0.464	-1.46	7.082
LG	#2	0.701	1.324	0.709*++	7.27	3.657**
LG	#6	0.200	1.272	1.854++	7.65	11.07
SJ	#2	0.652	1.831	0.803**++	4.37	1.743**
SJ	#6	0.316	1.841	3.764++	9.80	9.429
LJ	#2	0.657	1.079	0.527**++	6.32	2.865**
LJ	#6	0.276	2.066	2.258++	13.0	14.05

Legend: RET indicates the return series; see Table 1 for other notation. The columns of c_k and std-c_k have been multiplied by 10^3 to improve readability.

Table 9: Optimal Portfolios with Foreign Currency Denominated Securities

Panel A: One-Period Dollar, Mark, and Yen Bonds†

RET	ρ_k	c_k	std-c_k	h_k	std-h_k
P	0.893	0.520	0.323	3.661	2.168*
EG	0.372	3.315	2.406	0.125	0.080
SG	0.169	1.422	2.400	-0.066	0.089

Panel B: One-Period U.S.$-Bonds and 1&2-Period Mark Bonds†

RET	ρ_k	c_k	std-c_k	h_k	std-h_k
P	0.796	0.387	0.285	3.264	2.239**
EG	0.324	2.373	2.154	0.079	0.064
EJ	0.251	0.076	0.075	2.606	1.587

Panel C: One-Period U.S.$ and Mark Bonds and Long-Term Mark Bonds†

RET	ρ_k	c_k	std-c_k	h_k	std-h_k
P	0.361	0.154	0.250	0.626	1.927
EG	0.340	2.286	1.844	0.133	0.960**
G	0.544	0.081	0.051	7.258	3.215**

Panel D: One-Period U.S.$-Bonds and 1&2-Period Yen Bonds†

RET	ρ_k	c_k	std-c_k	h_k	std-h_k
P	0.699	0.291	0.184	2.333	1.515
EJ	-0.191	-1.312	2.094	-0.036	0.063
SJ	0.795	0.188	0.075**	4.657	1.629***

Panel E: One-Period U.S.$ and Yen Bonds and Long-Term Yen Bonds†

RET	ρ_k	c_k	std-c_k	h_k	std-h_k
P	0.540	0.226	0.177	1.97	1.364
EJ	0.057	0.355	1.896	0.070	0.076
LJ	0.598	0.091	0.048*	7.528	3.526**

Legend: Same as for Table 7

Table 10: Correlations with Tax Rates

Panel A: 1954:2-1987:4

RET	ρk	c_k	std-c_k	h_k	std-h_k
P	-0.294	-0.051	0.016***+++	0.324	0.0906***
DTB	-0.201	-0.243	0.106**++	0.325	0.1360**
LRET	-0.281	-0.627	0.206***+++	0.025	0.0073***
STOCK	-0.196	-0.646	0.289**++	0.011	0.0057**

Panel B: 1973:1-1987:4

RET	ρ_k	c_k	std-c_k	h_k	std-h_k
EG	0.091	0.276	0.393	-0.0075	0.011
EJ	0.019	0.056	0.377	-0.0016	0.011
SG	-0.223	-0.029	0.017*	0.394	0.222*
LG	-0.431	-0.031	0.010***+++	1.468	0.396***+++
SJ	-0.255	-0.027	0.014*+	9.636	0.312**
LJ	-0.141	-0.096	0.088	0.548	0.497

Legend: RET indicates the return series. VAR's use specification #1 of Table 1 in Panel A and #2 in Panel B. The columns of c_k and std-c_k have been multiplied by 10^4 to improve readability.

Table 11: Correlations with Government Spending

Panel A: Total Federal Expenditures

RET	ρ_k	c_k	std-c_k	h_k	std-h_k
P	-0.088	-0.097	0.506	0.62	3.17
DTB	0.194	0.153	0.146	-2.06	1.95
LRET	-0.011	-0.159	3.268	0.006	0.13
STOCK	-0.404	-9.171	4.311**++	0.15	0.066**

Panel B: Military Spending

RET	ρ_k	c_k	std-c_k	h_k	std-h_k
P	-0.112	-0.224	0.861	1.43	5.51
DTB	0.017	0.024	0.268	-0.32	3.54
LRET	0.050	1.326	5.896	-0.054	0.24
STOCK	0.088	3.513	7.282	-0.061	0.13

Legend: RET indicates the return series, all VAR's use specification #1 of Table 1. The columns of c_k and std-c_k have been multiplied by 10^4 to improve readability.