A MEAN-VARIANCE FRAMEWORK FOR TESTS OF ASSET PRICING MODELS

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Comments Welcome

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ABSTRACT

This paper presents a mean-variance framework for likelihood ratio tests of asset pricing models. A pricing model is tested by examining the position of one or more reference portfolios in sample mean-standard-deviation space. Included are tests of both single-beta and multiple-beta relations, with or without a riskless asset, using either a general or a specific alternative hypothesis. Tests with factors that are not portfolio returns are also included. The mean-variance framework is illustrated by testing the zero-beta CAPM, a two-beta pricing model, and the consumption-beta model.

1. Introduction and Summary

Many asset pricing models imply a linear relation between the expected return on an asset and covariances between that asset's return and one or more factors. The implications of such models can also be stated in terms of the mean-variance efficiency of a benchmark portfolio. In single-beta pricing relations, the benchmark portfolio can be identified specifically. For example, in the Capital Asset Pricing Model (CAPM) of Sharpe (1964), Lintner (1965), and Black (1972), it is well known that mean-beta linearity is equivalent to mean-variance efficiency of the market portfolio [Fama (1976), Roll (1977), and Ross (1977)]. Similarly, the consumption-beta model implies the mean-variance efficiency of the portfolio having maximal correlation with consumption [Breeden (1979)]. In multi-beta pricing relations, the benchmark portfolio generally cannot be identified specifically but instead is characterized as being some combination of a set of reference portfolios. For example, an exact K-factor arbitrage pricing relation is equivalent to the mean-variance efficiency of some portfolio that combines K factor-mimicking portfolios [Grinblatt and Titman (1987) and Huberman, Kandel, and Stambaugh (1987)].

Although the equivalence between linear pricing relations and mean-variance efficiency is well understood at a theoretical level, links between tests of the pricing models and a mean-variance framework are limited to a few special cases.[1] This study presents a complete framework for the characterization and investigation of likelihood ratio tests of the pricing restrictions in a mean-variance setting. Our treatment includes tests with either a single beta or multiple betas, with or without a riskless asset, using either a general or a specific alternative hypothesis. We also extend

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[1] See Jobson and Korkie (1982, 1988), Gibbons, Ross, and Shanken (1985),

the mean-variance framework to tests of the pricing relation with factors that are not portfolio returns. All of the tests considered should be viewed as tests of mean-variance efficiency defined in terms of unconditional distributions rather than tests of conditional mean-variance efficiency.

A major virtue of the mean-variance framework presented in this paper is that it allows the researcher to represent graphically in two familiar dimensions the outcome of a test of a multi-dimensional pricing restriction. A pricing model is tested by examining the position of one or more reference portfolios in sample mean-standard-deviation space. In this approach, the likelihood-ratio-test statistic can be viewed not simply as the outcome of a numerical procedure but also as a quantity with simple economic and statistical interpretations.

One case for which the mean-variance framework has been developed is where a pricing model that includes a riskless asset is tested against a general alternative hypothesis. The likelihood ratio test in this case can be characterized as comparing the position in sample mean-standard-deviation space of a benchmark portfolio, or a set of reference portfolios, to the position of the sample tangent portfolio [e.g., Jobson and Korkie (1982) and Gibbons, Ross, and Shanken (1985)]. The rejection region in sample mean-standard-deviation space is defined by a pair of lines.

We show that, in the absence of a riskless asset, the rejection region for the likelihood ratio test using a general alternative hypothesis is defined by a hyperbola in sample mean-standard-deviation space. As in the case where a riskless asset exists, the rejection region depends only on the estimated means and variance-covariance matrix for the observed universe of assets and does not depend on the specified benchmark or reference portfolios. It is not necessary to estimate a zero-beta expected return in order to conduct the test. With a single benchmark portfolio, the likelihood ratio test consists of asking whether the position of the benchmark portfolio in sample mean-standard-deviation space lies within the rejection region. With a collection of reference portfolios, the researcher first plots the sample minimum-standard-deviation boundary of all combinations of the reference portfolios. The test then consists of asking whether this entire boundary lies within the rejection region. We illustrate these procedures by testing a zero-beta CAPM and a two-beta pricing model.

The mean-variance framework is also used to investigate likelihood ratio tests of pricing models against specific alternative hypotheses. We consider tests of a K1-beta pricing model against a specific K2-beta pricing model. The null hypothesis identifies K1 reference portfolios to be used in explaining expected returns, and the specific alternative hypothesis identifies an additional set of K2-K1 reference portfolios. If a riskless asset exists, then a test of a K1-beta model against a K2-beta model is conducted by testing whether the tangent portfolio of the K1 portfolios is also the tangent portfolio of the larger set of K2 portfolios. The test is identical to the test of a K1-beta model against a general alternative, except that the set of K2 reference portfolios replaces the original universe of n assets. No other information about the other n-K2 assets is used.

When a riskless asset is not included, the specific alternative hypothesis is that some combination of the K2 portfolios is efficient with respect to the set of n assets. As in the case with a riskless asset, there is a close correspondence between the mean-variance representations of the tests against the general and specific alternatives, and the critical hyperbolas in sample mean-standard-deviation space are from the same class. Unlike the case with a riskless asset, however, the critical hyperbola in the case without a riskless asset depends on the returns of all n assets. Tests using a specific alternative are illustrated by testing a single-beta model against a two-beta model.

We extend the mean-variance framework to tests of a pricing relation with factors, such as consumption, that are not portfolio returns. In particular, we consider the role of reference portfolios with weights estimated, within the sample, to approximate those of the portfolios having maximal correlations with the factors. We show that, if a riskless asset exists, then the likelihood-ratio test of a single-beta pricing model, where betas are defined with respect to a factor, is similar to the test of a single-beta model using a reference portfolio with prespecified weights. In both cases, the position of the reference portfolio is compared to the position of the sample tangent portfolio of the observed universe of n assets. With a prespecified reference portfolio, the critical value for this comparison depends only on the sample means and variance-covariance matrix of the n assets. In the test with estimated weights, however, the critical value also depends on the sample correlation between the return on the estimated reference portfolio and the factor. We illustrate this procedure by testing the consumption-beta model.

The mean-variance framework offers directions for future research beyond the scope of the present study. For example, the mean-variance framework presented here, coupled with previously developed analysis, allows the researcher to investigate problems associated with measuring accurately the returns on relevant benchmark or reference portfolios. Kandel and Stambaugh (1987) conduct such an investigation for the Sharpe-Lintner form of the CAPM, where a riskless asset is included. They compute the maximum correlation between a given benchmark portfolio and a portfolio that gives a different between the benchmark and the ex ante tangent portfolio exceeds a given level. Their analysis combines a mean-variance framework for the likelihood ratio test with the results of Kandel and Stambaugh (1986), which derives the maximum correlation between a given portfolio and another portfolio with a given location in mean-variance space. Similar analyses can be conducted for other pricing models by combining the mean-variance framework for tests of these models with the results of Kandel and Stambaugh (1986).

The paper proceeds as follows. Section 2 defines terms and notation used in the paper. Section 3 analyzes likelihood ratio tests using a general alternative hypothesis, and section 4 presents tests using specific alternative hypotheses. As each test is discussed, we include an illustration using weekly returns on stock-market indexes and common-stock portfolios formed according to firm size. Section 5 extends the framework to models with factors that are not portfolio returns and provides an illustration using consumption data. Section 6 concludes the paper.

2. Definitions and Notation

We consider a set of n risky assets, which themselves are often portfolios formed from a larger universe of individual assets, and a set of K reference portfolios which comprise a subset of the n assets. If a riskless asset does not exist, then R_t denotes returns in period t on the K reference portfolios and r_t denotes returns in period t on the remaining n-K assets. If a riskless asset exists, then R_t and r_t denote excess returns on these assets, i.e., returns in excess of the riskless rate r_{Ft}.[2] It is assumed throughout

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[2]If the riskless rate is changing, the use of excess returns is problematic in terms of defining unconditional mean-variance efficiency. In that our primary goal is to provide a simple framework in which to interpret

the paper that the n-vector of returns $(r_t' \; R_t')'$ is distributed multivariate normal with a nonsingular variance-covariance matrix.[3] Define

T: the number of time-series observations in the sample of returns

E: the column vector of length n containing the sample means of $(r_t' \; R_t')'$.

V: the sample covariance matrix of $(r_t' \; R_t')'$.

$\mu(p)$: the sample mean return of portfolio p.

$\sigma^2(p)$: the sample variance of portfolio p.

The matrix summarizing the sample feasible set and its determinant (D):

$$\begin{bmatrix} L & M \\ M & N \end{bmatrix} \equiv \begin{bmatrix} \iota_n' V^{-1} \iota_n & \iota_n' V^{-1} E \\ \iota_n' V^{-1} E & E' V^{-1} E \end{bmatrix}, \qquad D \equiv L \cdot N - M^2. \tag{1}$$

$\bar{\sigma}^2(m)$: the minimum sample variance of any portfolio with sample mean return m that is constructed from the set of n assets.

$\bar{\sigma}_K^2(m)$: the minimum sample variance of any portfolio with sample mean return m that is constructed from the set of K assets with returns R_t.

The following are defined only for the case where a riskless asset exists:

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use of excess returns. This issue, along with more general questions about the appropriateness of testing unconditional relations, lie beyond the scope of this study.

[3]The simple partitioning of the set of n assets into sets of size K and n-K is for ease of discussion. Both the K reference portfolios and the other n-K assets can be combinations of the n "primitive" assets.

$S(p)$: the sample Sharpe measure of portfolio p, defined as the ratio of the mean excess return on p to the standard deviation of excess return on p. That is,

$$S(p) = \frac{\mu(p)}{\sigma(p)} \quad , \tag{2}$$

where excess returns are used in computing $\mu(p)$ and $\sigma(p)$.

p^* : the portfolio having the highest absolute value of the sample Sharpe measure of any portfolio constructed from the set of n assets.

p_K^* : the portfolio having the highest absolute value of the sample Sharpe measure of any portfolio constructed from the set of K assets with excess returns R_t.

3. Likelihood Ratio Tests Using A General Alternative Hypothesis

Numerous studies have developed and applied tests of asset pricing models against a general (unspecified) alternative hypothesis using the multivariate regression,

$$r_t = a + BR_t + u_t \quad . \tag{3}$$

A linear mean-beta pricing relation states that, for some scalar γ,

$$E(r_t) = \gamma\iota_{n-K} + B[E(R_t) - \gamma\iota_K] \tag{4}$$

where E( ) is the expectation operator and ι_{n-K} denotes an (n-K)-vector of ones. Furthermore, if a riskless asset exists (so that r_t and R_t are stated as excess returns) then $\gamma = 0$. The pricing relation in (4) implies the

$$a = \gamma(\iota_{n-K} - B\iota_K) \quad , \tag{5}$$

which simplifies to the restriction $a = 0$ when a riskless asset exists.[4]

This section presents a framework in sample mean-variance space for conducting likelihood ratio tests of the above pricing restrictions. We first summarize existing results for models with a riskless asset (section 3.1); we then state new results for models without a riskless asset (sections 3.2 and 3.3).

3.1 Tests of Models With a Riskless Asset

When a riskless asset exists, efficiency is defined with respect to the set of n risky assets plus the riskless asset. If the pricing model contains a single beta, i.e. the matrix B in (4) has one column, then a test of the pricing model is equivalent to a test of the mean-variance efficiency of the specified reference portfolio with return R_t. If the pricing model contains several betas, i.e. B has more than one column, then in general one cannot identify a specific benchmark portfolio that is implied by the pricing model to be mean-variance efficient. The linear pricing relation in (4) is equivalent to the statement that some portfolio of the K reference portfolios is mean-variance efficient [Jobson and Korkie (1985), Grinblatt and Titman (1987) and Huberman, Kandel, and Stambaugh (1987)].

The finite-sample distribution of the likelihood ratio test statistic for models with a riskless asset is given by Gibbons, Ross, and Shanken (1985).

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[4]In the absence of a riskless asset and when $K > 1$, a test of the restrictions $a = 0$ and $B\iota_K = \iota_{n-K}$ is equivalent to a test of "mean-variance spanning", i.e., that the mean-variance frontier of the K assets coincides

They show that a transformation of the likelihood ratio statistic for testing $a = 0$ in (3) (when r_t and R_t are stated in excess of the riskless rate) obeys an F distribution in finite samples.[5] The following proposition summarizes the sample mean-variance representation of this test provided by Jobson and Korkie (1982) and Gibbons, Ross, and Shanken (1985).[6]

<u>Proposition 1:</u> The likelihood ratio test with significance level α rejects the hypothesis that some portfolio of the K reference portfolios is efficient with respect to the set of n assets plus the riskless asset if and only if

$$|S(p_K^*)| < S_{CRIT},$$

where

$$S_{CRIT} = \left[\frac{S(p^*)^2 - \nu F_\alpha(n\text{-}K,\ T\text{-}n)}{1 + \nu F_\alpha(n\text{-}K,\ T\text{-}n)} \right]^{1/2} \tag{6}$$

if the bracketed quantity in (6) is positive, S_{CRIT} equals zero otherwise (in which case there is no rejection), $F_\alpha(n\text{-}K, T\text{-}n)$ is the critical value for significance level α of the F distribution with n-K and T-n degrees of freedom, and $\nu = (n\text{-}K)/(T\text{-}n)$.

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[5] Jobson and Korkie (1985) and MacKinlay (1987) also present the same result for the single-beta CAPM. A similar result is also presented by Jobson and Korkie (1982), except that they characterize what is in fact the finite-sample distribution as being valid only asymptotically, and they misstate the number of degrees of freedom.

[6] These results are also summarized in a recent paper by Jobson and Korkie

Proof. See appendix.

For a given sample of assets and returns, there may exist no specification of the reference portfolio(s) that results in a rejection of the pricing model. This situation, wherein the maximum squared sample Sharpe measure $S(p^{*})^2$ is less than $\nu F_{\alpha}(n\text{-}K, T\text{-}n)$ and thus the bracketed quantity in (6) is negative, is more likely to occur as the number of assets (n) grows large relative to the number of time-series observations (T).

As the above proposition states, in a test of a single beta model (K = 1) the efficiency of a portfolio can be tested by plotting its position in sample mean-standard-deviation space, where all returns are stated in excess of the riskless rate. The tested portfolio's position is compared to the location of the two critical lines with intercepts of zero and slopes with absolute values equal to S_{CRIT}.

Proposition 1 also indicates that in the test of a multi-beta model (K > 1) the portfolio tested is p_K^*, the sample tangent portfolio for the set of K assets. The position of portfolio p_K^* is compared to the two critical lines in sample mean-standard-deviation space in precisely the same manner as was the single reference portfolio in the case of K = 1. (The differences in S_{CRIT} between the two cases simply reflect different degrees of freedom.) Note that $|S(p_K^*)| < S_{CRIT}$, and thus the multi-beta model is rejected, if and only if the minimum-standard-deviation boundary of the K reference portfolios does not intersect either of the two critical lines.

We illustrate here a test of a two-beta pricing model (K = 2) with the weekly-returns data used by Kandel and Stambaugh (1987) in tests of the Sharpe-Linter version of the CAPM (K = 1). The set of twelve risky assets (n = 12) consists of two market proxies--the equally weighted and the value-

value-weighted portfolios of common stocks formed by ranking all firms on both exchanges by market value at the end of the previous year. The riskless rate is the return on a U.S. Treasury Bill with one week to maturity.[7] A two-beta model is tested here using the market proxies as the two reference portfolios. As can be seen in figure 1, which displays the test at a 5% significance level for the 324-week period from 10/8/75 to 12/31/81, the hyperbola representing combinations of the two reference portfolios does not intersect either critical line, and thus the two-beta model is rejected.[8]

3.2 Tests of Single-Beta Models Without a Riskless Asset

A likelihood ratio test of (5) with a single beta, where γ is an unknown zero-beta rate, was first proposed by Gibbons (1982). The hypothesis tested is equivalent to the mean-variance efficiency of the benchmark portfolio with respect to the n risky assets. The exact finite sample distribution of the likelihood ratio test statistic has not been obtained for this case, although a lower bound for the distribution is obtained by Shanken (1986).[9] Thus, selection of an appropriate critical value is more difficult than in the case where a riskless asset exists. Once a critical value is specified, however, we show that this test can be conducted in a mean-variance framework.

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[7]We thank Richard Rogalski for providing the Treasury Bill data.

[8]This is the third of three subperiods examined by Kandel and Stambaugh (1987). For both proposition 1 as well as the propositions to follow, we use this single subperiod simply to illustrate the testing framework rather than to conduct comprehensive new tests of asset pricing models.

[9]For discussions of finite-sample properties of the likelihood ratio statistic and other large-sample equivalents, see also Stambaugh (1982), Shanken (1985), and Amsler and Schmidt (1985). Shanken (1985) derives an upper bound on the finite-sample distribution of one alternative to the

Define

W(p) : a monotonic transformation of the likelihood ratio test statistic for testing the efficiency of a given portfolio p with respect to the set of n assets, where $T \cdot \ln[1 + W(p)]$ is asymptotically distributed as χ^2 with n-2 degrees of freedom if portfolio p is efficient.

W^* : the critical value for W(p) at the chosen significance level. That is, the efficiency of portfolio p is rejected if $W(p) > W^*$.

<u>Proposition 2.</u> The likelihood ratio test rejects the efficiency of portfolio p in the absence of a riskless asset, that is $W(p) > W^*$, if and only if

$$\sigma^2(p) \;>\; \delta_1(W^*) \;+\; \delta_2(W^*) \cdot \bar{\sigma}^2(\mu(p)) \tag{7}$$

where the functions $\delta_1(\cdot)$ and $\delta_2(\cdot)$ are given by

$$\delta_1(x) \;=\; \frac{x(x + 1)}{Lx - D} \qquad \text{and} \qquad \delta_2(x) \;=\; \frac{-D(x + 1)}{Lx - D} \qquad , \tag{8}$$

and where L and D are defined in (1).

Proof: See appendix.

Proposition 2 states that the likelihood ratio test of efficiency can be performed by first constructing a critical parabola in sample mean-variance (μ, σ^2) space given by the equation $\sigma^2 = \delta_1 + \delta_2 * \bar{\sigma}^2(\mu)$. Note that this critical parabola is a linear transformation of the sample minimum-variance constructed from the set of n assets). If the tested portfolio lies inside the convex region defined by this critical parabola, then the efficiency of that portfolio is rejected. The critical parabola becomes a critical hyperbola in sample mean-standard-deviation space, and we use the latter representation in the illustration below.

Using the same twelve assets and the same sample period as in the previous example, we test the zero-beta CAPM [Black (1972)] with each of the two indexes as the market proxy. Since this formulation of the model does not include a riskless asset, raw (not excess) returns are used. The critical value, W^*, is based on the result by Shanken (1986) that, under the null hypothesis, the lower bound on the distribution of $W(p)\cdot(T-K-1)$ is a T^2-variate with degrees of freedom n-K and T-K-1. Equivalently, the lower bound on the distribution of $W(p)\cdot(T-n)/(n-K)$ is central F with degrees of freedom n-K and T-n. Therefore, for a significance level of 5% and for the 324-week sample size the critical value is

$$W^* = F_{0.05}(11,\ 312) \cdot 11 / 312 = 0.0641 \quad .$$

Figure 2 displays the results of this test. Each of the two market proxies lies inside the rejection region defined by the critical hyperbola, and thus the the efficiency of each of the two indexes is rejected at a significance level of no more than 5%.

3.3 <u>Tests of Multiple-Beta Models Without a Riskless Asset</u>

Gibbons (1982) and Shanken (1985, 1986) discuss likelihood ratio tests of (5) in the absence of a riskless asset for cases where K > 1. This restriction [imposed by the pricing equation in (4)] is equivalent to the

Huberman, Kandel, and Stambaugh (1987)]. Proposition 2 states that the critical region for testing the efficiency of a given portfolio, in the absence of a riskless asset, is given by a linear transformation of the sample minimum-variance boundary. The following proposition establishes a similar result for the test of the efficiency of some combination of the K assets. Define

W_K : a monotonic transformation of the likelihood ratio test statistic for testing the hypothesis that some portfolio of the K assets is mean-variance efficient with respect to the set of n assets, where $T \cdot \ln[1 + W_K]$ is asymptotically distributed as χ^2 with n-K-1 degrees of freedom under the null hypothesis.

W_K^* : the critical value for W_K at the chosen significance level. That is, the null hypothesis is rejected if $W_K > W_K^*$.

Since an exact small-sample distribution for the likelihood ratio statistic has not been obtained, choosing the critical value W_K^* is again more difficult than in the cases where a riskless asset exists. One could use, for example, the lower bound on the distribution obtained by Shanken (1986). Once the critical value is chosen, however, the test can be conducted in sample mean-variance space as shown by the following proposition.

<u>Proposition 3.</u> The likelihood ratio test rejects the hypothesis that some portfolio of the K assets is efficient with respect to the n risky assets, that is $W_K > W_K^*$, if and only if

$$\bar{\sigma}_K^2(m) \quad > \quad \delta_1(W_K^*) \quad + \quad \delta_2(W_K^*) \cdot \bar{\sigma}^2(m) \qquad \text{for all m.} \tag{9}$$

Proof: See appendix.

Note the similarity between propositions 2 and 3. In both cases, the rejection region is defined by a critical parabola that is simply a linear transformation of the sample minimum-variance boundary. (In fact, the definitions of the parabolas are identical except for the possibly different critical values, W^* and W_K^*.) A given portfolio's efficiency is rejected if it lies inside the convex rejection region enclosed by the critical parabola. The efficiency of any combination of the K assets is rejected if the entire feasible set of portfolios of those K assets lies within that rejection region.

We illustrate this test with the same data used to construct the previous two examples. As in the first example, we test a two-beta model where the value-weighted and equally weighted market proxies are specified as the two reference portfolios. In this case, however, there is no riskless asset. The critical value W_K^* is computed with the same method used in the previous example, using Shanken's (1986) lower bound, except that K2 = 2, so

$$W_K^* = F_{0.05}(10,\ 312) \cdot 10 \ / \ 312 \ = \ 0.0596 \quad .$$

Figure 3 illustrates the results of this test. All combinations of the two reference assets lie within the rejection region defined by the critical hyperbola, and thus the two-beta model is rejected.

4. Likelihood Ratio Tests Using Specific Alternative Hypotheses

This section examines likelihood ratio tests of the pricing restriction in (4), where the restriction is tested against a specific alternative. The specific alternative is one in which a multi-beta model of higher dimension portfolios is tested as the null hypothesis against the alternative hypothesis that a model with K2 reference portfolios holds (K2 > K1), where the latter set includes the original K1 reference portfolios. The total set of n risky assets is held fixed, so the alternative hypothesis simply identifies a larger number of the n assets as reference portfolios to be used in explaining expected returns on the other assets.

4.1 Tests of Models With a Riskless Asset

We first consider the case where a riskless asset exists. In this case, the null hypothesis is equivalent to the statement that the tangent portfolio of the K1 reference portfolios is the tangent portfolio of the n assets. The alternative hypothesis is that the tangent portfolio of the set of K2 reference portfolios is also the tangent portfolio of the set of n assets. As before, p^*_{K1} and p^*_{K2} denote the portfolios from the sets of K1 and K2 assets having the highest absolute Sharpe measures. Let H0 denote the hypothesis that a K1-beta model holds in the presence of a riskless asset, and let HA denote the hypothesis that a K2-beta model holds.

Proposition 4. The likelihood ratio test with significance level α rejects H0 against HA if and only if

$$|S(p^*_{K1})| < S_{CK2} \quad , \tag{10}$$

where

$$S_{CK2} \equiv \left[\frac{S(p^*_{K2})^2 - \nu F_\alpha(K2\text{-}K1,\ T\text{-}K2)}{1 + \nu F_\alpha(K2\text{-}K1,\ T\text{-}K2)} \right]^{1/2} \tag{11}$$

if the bracketed quantity in (11) is positive, S_{CK2} equals zero otherwise (in which case there is no rejection), $F_{\alpha}(K2-K1, T-K2)$ is the critical value for significance level α of the F distribution with K2-K1 and T-K2 degrees of freedom, and $\nu = (K2-K1)/(T-K2)$.

Proof: See appendix.

A comparison of propositions 1 and 4 reveals that a test of a K1-beta model against a K2-beta model, when a riskless asset exists, is conducted by testing whether some combination of the K1 reference portfolios is the tangent portfolio of the set of K2 reference portfolios. Observe that the test defined in the above proposition is identical to the test defined in proposition 1, except that the critical Sharpe measure S_{CRIT} is replaced by S_{CK2} and the degrees of freedom are changed. Therefore, only information about the K2 reference portfolios, and the subset of K1 portfolios, is used to conduct the test against the specific alternative. No other information about the original n assets is used.

Using the same data as in the previous examples (twelve assets and the period 10/8/75-12/23/81), we illustrate proposition 4 by testing the Sharpe-Lintner model (K1 = 1) against the specific alternative of a two-beta pricing model (K2 = 2) in which the two NYSE-AMEX indexes serve as the reference portfolios. In other words, the specific alternative states that some combination of the value-weighted and equally weighted NYSE-AMEX portfolios is the tangent portfolio. (As in proposition 1, excess returns are used in this test, and tangency is defined with respect to the origin.) The Sharpe-Lintner model is tested with each of the two portfolios as the market index.

Figure 4 displays lines corresponding to the critical Sharpe measures for tests using both a general alternative and the specific alternative

alternative includes the rejection region for the test using the specific alternative. (This issue is analyzed below.) Given the positions of both index portfolios, the Sharpe-Lintner model is rejected for either specification of the market index using either the general alternative or the specific two-beta alternative.

When the maximum squared sample Sharpe measure of combinations of the K2 reference portfolios is sufficiently low relative to the appropriate F statistic, there can be samples in which no choice of the K1 reference portfolios (from among combinations of the K2 portfolios) produces a rejection using the specific alternative hypothesis. Recall that the test using the general alternative can encounter a similar situation (cf. the discussion immediately following proposition 1). For the values of n and T used in the example illustrated here, (11) implies that such a situation occurs when the maximum squared Sharpe measure of combinations of the two reference portfolios is less than $(1/322)\cdot F_{0.05}(1, 322)$, and we find this to be the case in a different period (7/2/69 to 10/1/75). In that period, the Sharpe-Lintner model is rejected using a general alternative for any market index that combines the equally weighted and value-weighted NYSE-AMEX portfolios, but no such combination rejects the model using the specific alternative.

As noted earlier, the test in proposition 1 is equivalent to testing whether a = 0 in (3) when r_t and R_t are stated as excess returns. A similar equivalence holds for the test in proposition 4. Partition the vector of excess returns on the n assets as $(r_t' \;\; R_{1t}' \;\; R_{2t}')'$, where R_{1t} contains returns on K1 assets and R_{2t} contains returns on K2-K1 assets. A test of a K1-beta pricing model against a general alternative, as in proposition 1, is

$$\begin{bmatrix} r_t \\ R_{2t} \end{bmatrix} = \begin{bmatrix} a \\ a_2 \end{bmatrix} + \begin{bmatrix} B \\ B_2 \end{bmatrix} R_{1t} + \begin{bmatrix} u_t \\ u_{2t} \end{bmatrix} , \qquad (12)$$

where all returns are stated in excess of the riskless rate. Likewise, a test of a K2 pricing model against a general alternative is equivalent to testing whether $a^* = 0$ in the regression

$$r_t = a^* + B^* \begin{bmatrix} R_{1t} \\ R_{2t} \end{bmatrix} + \varepsilon_t . \qquad (13)$$

Given proposition 4 and the same type of equivalence to a regression test, a test of a K1-beta model against the specific alternative of a K2-beta model is equivalent to testing whether $a_2 = 0$ in the regression

$$R_{2t} = a_2 + B_2 R_{1t} + u_{2t} . \qquad (14)$$

As noted above, the returns on the other assets, r_t, are not used in this test.

An interesting special case occurs when K1 = 1 and K2 = 2, i.e. the tangency of a given portfolio with excess return R_{1t} is tested against the alternative that some combination of this portfolio and a second portfolio with excess return R_{2t} is the tangent portfolio of the n assets. Given the above discussion, this test is equivalent to regressing R_{2t} on R_{1t} and testing whether the scalar intercept is equal to zero.

4.2 Tests of Models Without a Riskless Asset

When a riskless asset is not included, the null hypothesis to be tested is that some combination of the K1 reference portfolios is efficient with respect to the set of n risky assets. The alternative hypothesis is that some combination of the K2 reference portfolios is efficient with respect to the n assets. The following proposition gives a mean-variance representation of the likelihood ratio test in this case. As in the above case with a riskless asset, there is a close correspondence between this result and the mean-variance representation of the test against a general alternative (proposition 3). Let H0 denote the hypothesis that a K1-beta model holds with no riskless asset, and let HA denote the hypothesis that a K2-beta model holds, where K2 > K1. Define

$W_{K1,K2}$: a monotonic transformation of the likelihood ratio test statistic for testing H0 against HA, where $T \cdot \ln[1 + W_{K1,K2}]$ is asymptotically distributed as χ^2 with K2-K1 degrees of freedom under H0.

$W^*_{K1,K2}$: the critical value for $W_{K1,K2}$ at the chosen significance level. That is, H0 is rejected in favor of HA if $W_{K1,K2} > W^*_{K1,K2}$.

Proposition 5. The likelihood ratio test rejects H0 against HA, that is $W_{K1,K2} > W^*_{K1,K2}$, if and only if

$$\bar{\sigma}_K^{-2}(m) \quad > \quad \delta_1(X_{K2}) \quad + \quad \delta_2(X_{K2}) \cdot \bar{\sigma}^{-2}(m) \qquad \text{for all m,} \tag{15}$$

where δ_1 and δ_2 are defined in (8), W_K is defined in section 3.3, and

Proof: See appendix.

The critical parabola in sample mean-variance space defined by proposition 5 is from the same class of parabolas defined in proposition 3, which addresses the test against a general alternative. Different parabolas corresponding to different critical values for the likelihood ratio test are obtained by varying W_K^* in proposition 3 and by varying $W_{K1,K2}^*$ (and thus X_{K2}) in proposition 5. If $W_K^* = X_{K2}$, then the critical parabolas for the two tests coincide.

Unlike the test in proposition 4, the test in proposition 5 uses information about all n assets. The critical parabola in (15) depends on a number of quantities that require returns on each of the n assets, such as the functions $\delta_1(\cdot)$ and $\delta_2(\cdot)$ and the quantity W_{K2}, a transformation of the likelihood ratio statistic for testing a K2-beta model against a general alternative. Therefore, the ability to use only the K2 reference portfolios in testing a K1-beta model against a K2-beta model is limited to the case where a riskless asset is included. We suggest that the intuition for this asymmetry lies in the fact that, in the absence of a riskless asset, the specific K2-beta alternative still requires information about all of the n assets in order to identify the expected zero-beta return.[10] This is not true, of course, when a riskless (zero-beta) rate is observed.

We illustrate proposition 5 by testing the zero-beta CAPM (K1 = 1) against the specific alternative of a two-beta pricing model (K2 = 2) in which the two NYSE-AMEX indexes serve as the reference portfolios. The data are the

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same as in the previous examples (twelve assets and the period 10/8/75-12/23/81). Figure 5 displays the critical hyperbola for the test using the specific alternative as well as the critical hyperbola for the test using a general alternative. The latter test was presented earlier in figure 2 for a slightly different choice of the critical value W^*. In order to compare directly the tests against both types of alternative hypotheses, we choose the critical values for both tests using the asymptotic χ^2 distributions for the likelihood ratio test statistics. Therefore, again using a 5% significance level, the critical value for the test using the general alternative (proposition 2) is

$$W^* = e^{\frac{1}{T} \chi^2_{0.05}(10)} = 1.05813 \quad ,$$

and the critical value for the test using the specific two-beta alternative is

$$W^*_{1,2} = e^{\frac{1}{T} \chi^2_{0.05}(1)} = 1.01927 \quad ,$$

where $\chi^2_\alpha(\nu)$ is the critical value for significance level α of the χ^2 distribution with ν degrees of freedom.

As noted earlier, the zero-beta CAPM is rejected with either portfolio as the market index when a general alternative hypothesis is used. When the specific alternative is used, however, the zero-beta CAPM is accepted if the value-weighted NYSE-AMEX portfolio is specified as the market index. In this case, the rejection region for the test using the general alternative is larger than the rejection region for the test using the specific alternative.

not in the latter.

4.3 Comparisons Between Tests Using General and Specific Alternatives

As illustrated in figures 4 and 5, the mean-variance framework allows tests using both general and specific alternatives to be represented on the same graph. It may be, as in the last example, that a given pricing model is rejected against one form of alternative hypothesis (specific or general) but is not rejected against another form of alternative.

Given that the rejection regions in sample mean-variance space are of the same form for tests using either general or specific alternatives, the rejection region for one type of test will, ex post, include the rejection region of the other. Whether the test using the general alternative has the larger or smaller rejection region, however, depends on the specific sample. The relative size of the rejection region for the test using the specific alternative depends on the degree to which the specific alternative is satisfied in the sample.

When a riskless asset exists, the rejection regions are defined by critical Sharpe measures. For a given significance level α, the critical Sharpe measure for the test of a K1-beta model against a general alternative, S_{CRIT}, obeys the following relation to the critical Sharpe measure for testing the same hypothesis against the specific alternative, S_{CK2} (provided neither measure is zero):

$$\frac{1 + S^2_{CRIT}}{1 + S^2_{CK2}} = \left[\frac{1 + S(p^*)^2}{1 + S(p^*_{K2})^2} \right] \cdot \left[\frac{1 + \frac{(K2-K1)}{(T-K2)} F_\alpha(K2-K1,\ T-K2)}{1 + \frac{(n-K1)}{(T-n)} F_\alpha(n-K1,\ T-n)} \right] . \qquad (17)$$

The second bracketed expr[illegible]

monotonic transformation of the statistic for testing a K2-beta pricing model against a general alternative (cf. proposition 1). If the K2-beta model performs sufficiently well against a general alternative, so that the first bracketed expression in (17) is sufficiently close to unity, then the entire expression on the right-hand side of (17) is less than unity, and $S_{CK2} > S_{CRIT}$. In this case, the rejection region for the test against the specific alternative includes the rejection region for the test against the general alternative. If, on the other hand, the first bracketed expression is sufficiently large, i.e. if the K2-beta model performs less well, then the rejection region in the general-alternative test is larger.

A similar analysis is possible for the tests in the absence of a riskless asset. Propositions 3 and 5 imply that the rejection region for the test using the specific alternative includes the rejection region for the test against the general alternative if $X_{K2} < W^*_{K1}$. For the purposes of this discussion we assume that the critical values W^*_{K1} and $W^*_{K1,K2}$ are chosen, as in the last example, based on the large-sample distributions for the likelihood ratio test statistics. Thus, for a given significance level α, $T\cdot[\ln(1 + W^*_{K1})] = \chi^2_\alpha(n\text{-}K1\text{-}1)$ and $T\cdot[\ln(1 + W^*_{K1,K2})] = \chi^2_\alpha(K2\text{-}K1)$, where $\chi^2_\alpha(\nu)$ denotes the critical value for significance level α of the χ^2 distribution with ν degrees of freedom. Given these specifications, the relation between X_{K2} and W^*_{K1} is given by

$$\frac{1 + X_{K2}}{1 + W^*_{K1}} = \left[1 + W_{K2} \right] \cdot \left[e^{\frac{1}{T}(\chi^2_\alpha(K2\text{-}K1) \; - \; \chi^2_\alpha(n\text{-}K1\text{-}1))} \right] . \qquad (18)$$

As in (17), the second bracketed term is less than unity and does not depend transformation of the likelihood ratio statistic for testing a K2-beta model against a general alternative. If W_{K2} is sufficiently small, i.e. if the K2-beta model performs sufficiently well in the sample, then the right-hand side of (18) is less than unity, $X_{K2} < W^*_{K1}$, and the rejection region for the test using the specific alternative includes the rejection region for the test using the general alternative.

In the previous example (figure 5), the rejection region of the test using the general alternative includes the rejection region of the test using the specific alternative. As explained earlier, this example, as the others, is based on data from the period from 10/8/75 to 12/23/81. If we choose an earlier period, form 1/2/63 to 6/25/69, then the rejection region of the specific-alternative test is the larger one (and the zero-beta CAPM is rejected against either alternative using either market index). In that earlier period, the two-beta alternative is supported better by the data. Recall from figure 3 that the two-beta model is rejected in the later period used in the examples. The same two-beta model is not rejected in the earlier period, however.[11]

5. Tests of Models With Factors That Are Not Portfolio Returns

The tests considered up to this point in the paper apply to models in which the betas [B in (3) and (4)] are defined with respect to returns on a prespecified set of reference portfolios. In other cases the researcher may wish to test a pricing model in which the betas are computed with respect to a

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[11] In that case, some feasible portfolios of the two reference assets lie outside the rejection region. Those portfolios, however, have mean returns less than the mean return of the global-minimum-variance portfolio. The latter case illustrates a shortcoming common to all of these tests: there is

set of factors that are not portfolio returns. This section examines likelihood ratio tests of such models in a mean-variance framework. In particular, we consider the role of reference portfolios that are constructed within the sample, i.e. portfolios with weights that depend on sample parameter estimates.

5.1 A Mean-Variance Framework for the Likelihood Ratio Test

It is assumed throughout this section that a riskless asset exists. Consider the multivariate regression

$$r_t = a + Bf_t + u_t \quad , \tag{19}$$

where r_t contains excess returns on n assets and f_t is a K-vector of factors. The asset pricing model states that, for some set of factor premiums contained in the K-vector ζ,

$$E(r_t) = B\zeta \quad . \tag{20}$$

The pricing model in (20) implies the following restriction on the parameters in the regression in (19),

$$\begin{aligned} a &= B[\zeta - E(f)] \\ &= B\phi \quad . \end{aligned} \tag{21}$$

We consider in detail the case of a single factor, so that B is an n×1 vector and ϕ is a scalar. Define

$\hat{p}$: the reference portfolio that is constructed within the sample, i.e. the portfolio of the n assets having maximum sample correlation with

R_t: the return on portfolio $\hat{p}$.

$\hat{\rho}$: the sample correlation between R_t and f_t.

W_{f1}: a monotonic transformation of the likelihood ratio statistic for testing the restriction in (21) against a general alternative, where $T \cdot \ln(1 + W_{f1})$ is asymptotically distributed as χ^2 with n-1 degrees of freedom under the null hypothesis.

W^*_{f1}: the critical value for W_{f1} at the chosen significance level. That is, the restriction in (21) is rejected if $W_{f1} > W^*_{f1}$.

As before, p^* denotes the portfolio of the n assets having the highest absolute Sharpe measure.

Unlike the tests discussed in sections 3 and 4, where the composition of the reference portfolio(s) is specified ex ante, the weights in the reference portfolio $\hat{p}$ are estimated within the sample. When only the factor f_t is specified, the true weights in the reference portfolio of the n assets are unobservable. Thus, one cannot simply test the ex ante tangency of $\hat{p}$ using the tests presented earlier, since the pricing theory does not require that the estimated reference portfolio $\hat{p}$ be ex ante mean-variance efficient. If such a procedure were followed, the pricing model would tend to be rejected incorrectly more often than the nominal rejection frequency (size) of the test. The following proposition allows us to compare the test using a factor to the earlier test using a prespecified reference portfolio.

Proposition 6. The sample statistics $S(p^*)$, $S(\hat{p})$, and $\hat{\rho}$ are sufficient to compute the likelihood ratio statistic for testing (21) with a single factor. Specifically, the likelihood ratio test rejects (21) if and only if where

$$S_{Cf1} = \left[1 - \frac{W^*_{f1}(1 - \hat{\rho}^2)}{\hat{\rho}^2} \right]^{\frac{1}{2}} \cdot \left[\frac{S(\hat{p}^*)^2 - W^*_{f1}}{1 + W^*_{f1}} \right]^{\frac{1}{2}} \tag{22}$$

if the bracketed quantities in (22) are positive and S_{Cf1} equals zero otherwise (in which case there is no rejection).

Proof: See appendix.

Proposition 6 reveals that the test of the single-beta pricing model using a factor is similar to the test of a single-beta model using a prespecified reference portfolio (proposition 1). In both cases, the absolute Sharpe measure of a reference portfolio is compared to the maximum absolute Sharpe measure of the n assets. In the test presented here, however, this comparison depends on $\hat{\rho}$, the sample correlation between the return on the reference portfolio and the return on the factor.[12]

In general (with probability 1), $\hat{\rho}$ is less than unity and the weights in the reference portfolio $\hat{p}$ contain measurement error. The inclusion of $\hat{\rho}$ in the test statistic W_{f1} can be viewed as an adjustment for this measurement error. For a given $\hat{\rho}$, a critical value for the test statistic W_{f1} determines a critical Sharpe measure S_{Cf1} against which the absolute value of $S(\hat{p})$ is compared.[13] Thus, for a given location in sample mean-variance space of the reference portfolio $\hat{p}$, the null hypothesis could be rejected for one value of

[12] Recall that in proposition 1, the only sample quantity affecting the critical Sharpe measure S_{CRIT} is $S(p^*)$, the maximum absolute Sharpe measure of the n assets.

[13] This critical Sharpe

$\hat{\rho}$ but accepted for some lower value of $\hat{\rho}$. As $\hat{\rho}$ decreases, or as the variance of the measurement error in the weights in $\hat{p}$ increases, the reference portfolio $\hat{p}$ must lie farther from the sample tangent portfolio (in terms of absolute Sharpe measures) in order to reject the null hypothesis.

When $\hat{\rho} = 1$ the test statistic W_{f1} is undefined, which can be seen by observing that the covariance matrix of u_t in (19) is singular if some linear combination of the elements of r_t yields f_t.[14] However, it can be shown that

$$\lim_{\hat{\rho} \to 1} W_{f1} = \frac{S(p^*)^2 - S(\hat{p})^2}{1 + S(\hat{p})^2} . \tag{23}$$

The limiting value in (23) is in the form of the test statistic underlying proposition 1. Also note that if $\hat{\rho} = 1$, then the critical Sharpe measure S_{Cf1} in (22) is of the same form as S_{CRIT} in (6) [with W^*_{f1} in place of $\nu F_\alpha(\cdot)$]. In other words, in the special case $\hat{\rho} = 1$, the weights in the reference portfolio are estimated without error and the correct test of the single-beta model is to test the ex ante tangency of portfolio $\hat{p}$ using the test in proposition 1. One would replace f_t by R_t in (19), eliminate one asset from the multivariate regression (leaving n-1 equations), and test whether $a = 0$.

5.2 <u>An Illustration Using Consumption Data</u>

The consumption-beta model of Breeden (1979) is tested here using the mean-variance framework developed above. Breeden, Gibbons, and Litzenberger (1986) conduct tests of this model using a series of estimated quarterly consumption growth rates. Their study includes tests of restrictions on parameters in a multivariate regression as in (19), in which quarterly stock

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and bond returns (in r_t) are regressed on quarterly consumption growth (f_t).[15] We also conduct a test of the consumption-beta model using quarterly data, covering the period beginning in the second quarter of 1929 and ending in the third quarter of 1978.[16] Our set of twelve assets ($n = 12$) consist of (i) a portfolio of long-term U.S. Government Bonds, (ii) a portfolio of bonds rated below Baa by Moody's, and (iii) ten value-weighted portfolios of common stocks formed by sorting firms according to size (with approximately the same number of firms in each portfolio). Excess returns are used, where the riskless rate is the yield on the shortest-maturity U.S. Government security with a maturity of at least three months.

We first construct the reference portfolio $\hat{p}$, which is the portfolio of the twelve assets having the maximum sample correlation with consumption. In this example, the maximum correlation ($\hat{\rho}$) equals 0.55. Figure 6 displays the minimum-standard-deviation boundary of the twelve assets as well as lines corresponding to two critical Sharpe measures, both for significance levels of 0.10. The first of these corresponds to S_{CRIT} (proposition 1), which would be appropriate for testing the ex ante efficiency of $\hat{p}$. The second line corresponds to S_{Cf1}, the critical Sharpe measure given in proposition 6.[17] Note that the pricing model would be rejected if the position of the reference portfolio $\hat{p}$ were compared to the line corresponding to S_{CRIT}. On the other hand, $\hat{p}$ lies essentially on the line corresponding to the appropriate critical Sharpe measure S_{Cf1}, so the pricing model is not rejected at a ten percent

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[15] Those authors test a zero-beta version of the model, so the restriction in (21) becomes $a = \iota_n \phi_1 + B\phi_2$, for some scalars ϕ_1 and ϕ_2.

[16] We are grateful to Mike Gibbons for providing us with the consumption data.

significance level (the p-value is approximately 0.10).

An interesting outcome occurs if the above example is modified so that the significance level is 0.05 or so that the set of assets consists of only the common stock portfolios (n = 10). In these cases, the right-most bracketed expression in (22) is negative, and, as explained earlier, this expression corresponds to S_{CRIT} in proposition 1. A negative value indicates that even the largest squared sample Sharpe measure is too low, relative to the critical value of the statistic's distribution, to allow any portfolio to be inferred inefficient (cf. the discussion following proposition 1). For this sample, there are no portfolios of the ten assets that would be inferred inefficient at the 0.10 significance level, and there are no portfolios of the twelve assets considered above that would be inferred inefficient at the 0.05 significance level. Given the definition of S_{Cf1}, this also means that a one-factor pricing model would not be rejected in such cases for any realizations of the factor. We see that a necessary condition for measurement of the factor (e.g. consumption) to have any relevance in a given sample is that there be some portfolios that would be inferred to be ex ante inefficient. If no such portfolios exist in the sample, then the researcher need not be concerned with measuring the factor, because a rejection of the pricing model could not occur in any event. In the example displayed in figure 6, the two bond portfolios are included in order to construct an example in which the specification of the factor can affect the outcome of the test at a 0.10 significance level.

6. Conclusions

Likelihood ratio tests of many asset pricing models, including multi-beta models, can be conducted in a mean-variance framework.

standard-deviation space. When a riskless asset exists, a rejection region in sample mean-standard-deviation space is defined by a pair of lines determined by a critical Sharpe measure. When no riskless asset exists, the rejection region is defined by a critical hyperbola. Single-beta pricing models are rejected if a given reference portfolio lies within the rejection region. Multi-beta pricing models are rejected if all combinations of a number of reference portfolios lie within the rejection region.

The mean-variance framework developed here allows likelihood ratio tests to be conducted using either a general alternative hypothesis or a specific alternative of a higher-dimensional linear pricing model. The rejection regions are of the same form in both cases, and the rejection region for one test will include the rejection region of the other. If the specific alternative is satisfied sufficiently well within the sample, then the rejection region for the test using that alternative will be larger than the rejection region for the test using a general alternative.

When a factor such as consumption is used to test a single-beta pricing model, the likelihood ratio test can be conducted by examining the position in sample mean-standard-deviation space of the portfolio of a given set of assets having the maximum sample correlation with the factor. The Sharpe measure defining the critical region is, however, less than the Sharpe measure appropriate for testing a single-beta model where this portfolio is specified ex ante as the reference portfolio.

A necessary condition for the rejection of a single-beta model using a factor is that there exist some portfolios within the sample that would be inferred to be ex ante inefficient. If such portfolios do not exist, then no realization of the factor could produce a rejection of the pricing theory.

APPENDIX

This appendix describes the analysis leading to propositions 1 through 6. The analysis uses a general $n \times \ell$ matrix A to denote a set of ℓ portfolios that are combinations of the n assets. Matrix A has full column rank with $A'\iota_n = \iota_\ell$, where $\ell \le n$. The set of K reference portfolios with returns R_t is represented by a specific choice of A with K columns. It is often convenient during the analysis to regard K as fixed but to consider other sets of ℓ portfolios (sometimes containing only one member), which are represented by different specifications of A. For example, R_t can be replaced in (3) and (4) by an ℓ-vector of returns R_t^a, defined by another choice of A, and r_t is then replaced by an $(n-\ell)$ vector of returns r_t^a, where r_t^a and R_t^a are linearly independent. In that case, the regression in (3) becomes

$$r_t^a = a + BR_t^a + u_t , \tag{A.1}$$

where a and B are redefined accordingly.

For the matrix A, define

$$\begin{bmatrix} L(A) & M(A) \\ M(A) & N(A) \end{bmatrix} \equiv \begin{bmatrix} \iota_\ell'(A'VA)^{-1}\iota_\ell & \iota_\ell'(A'VA)^{-1}A'E \\ \iota_\ell'(A'VA)^{-1}A'E & E'A(A'VA)^{-1}A'E \end{bmatrix} , \tag{A.2}$$

$$Q(A, \gamma) \equiv \frac{N - N(A) - 2\gamma[M - M(A)] + \gamma^2[L - L(A)]}{1 + N(A) - 2\gamma M(A) + \gamma^2 L(A)} \tag{A.3}$$

$$\bar{Q}(A) \equiv \min_{\gamma} Q(A, \gamma) \tag{A.4}$$

$$\hat{\gamma}(A) \equiv \operatorname{argmin} Q(A, \gamma) \tag{A.5}$$

$$S^*(\gamma) \equiv \max_p \left| \frac{p'E - \gamma}{(p'Vp)^{1/2}} \right| \qquad \text{s.t. } p'\iota = 1 \tag{A.6}$$

$$s^*(A, \gamma) \equiv \max_w \left| \frac{w'A'E - \gamma}{(w'A'VAw)^{1/2}} \right| \qquad \text{s.t. } w'\iota_\ell = 1 \tag{A.7}$$

$p^*(A, \gamma)$: an (n+K)-vector containing the weights in the tangent portfolio of the set A with respect to the intercept γ. That is, $p^*(A, \gamma) = Aw^*$, where

$$w^* \equiv \text{argmax} \left| \frac{w'A'E - \gamma}{(w'A'VAw)^{1/2}} \right| \qquad \text{s.t. } w'\iota_\ell = 1 \tag{A.8}$$

Note that when returns are stated in excess of a riskless rate r_{Ft}, then $S^*(0)$ is the maximum absolute Sharpe measure of the set of n assets, and $s^*(A, 0)$ is the maximum absolute Sharpe measure of the set of portfolios in A.

The function $Q(A, \gamma)$ provides the basis for the likelihood ratio tests discussed in section 3. Relations between various forms of $Q(A, \gamma)$ and likelihood ratio tests are established in studies by Kandel (1984), Shanken (1985, 1986), and Gibbons, Ross, and Shanken (1985). In discussing these relations, it is useful to recognize two alternative expressions for $Q(A, \gamma)$. It is straightforward to show that

$$[s^*(A, \gamma)]^2 = N(A) - 2\gamma M(A) + \gamma^2 L(A) \quad , \tag{A.9}$$

and an analogous expression holds for $S^*(\gamma)$ in terms of N, M, and L.

$$Q(A,\ \gamma) = \frac{[S^{*}(\gamma)]^2 - [s^{*}(A,\ \gamma)]^2}{1 + [s^{*}(A,\ \gamma)]^2} \tag{A.10}$$

The second alternative expression for $Q(A,\ \gamma)$ involves parameters estimated from a multivariate regression [see, for example, Shanken (1986)]. Let $\hat{a}$ and $\hat{B}$ denote ordinary least squares estimates of the parameters in (A.1) For a given γ, define $\alpha(\gamma) = \hat{a} - \gamma[\iota_{(n-\ell)} - \hat{B}\iota_{\ell}]$, and let Σ denote the estimated variance-covariance matrix of the u_t's (using T as a divisor). Through straightforward (but somewhat tedious) algebra it can be shown that

$$Q(A,\ \gamma) = \frac{\alpha(\gamma)'\Sigma^{-1}\alpha(\gamma)}{1 + (E_A - \gamma\iota_{\ell})'V_A^{-1}(E_A - \gamma\iota_{\ell})}\ , \tag{A.11}$$

where E_A and V_A denote the sample mean vector and variance-covariance matrix of R_t^a.[18]

Gibbons, Ross, and Shanken (1985) show that Q(A, 0), where Q(•) is in the form of (A.11) and where all returns are stated in excess of a riskless rate r_{Ft}, is a monotonic transformation of the likelihood ratio statistic for testing the efficiency of some combination of the portfolios in A in the presence of a riskless asset. These authors also use the form of Q(•) in (A.10) with $\ell = 1$ to provide the geometric interpretation of the test of the efficiency of a given portfolio (proposition 1 with K=1).

Gibbons, Ross, and Shanken show that $[(T-n)/(n-\ell)]Q(A,\ 0)$ is distributed as F with degrees of freedom $(n-\ell)$ and $(T-n)$ under the null hypothesis. The

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[18]The function denoted Q(·) in Shanken (1986) ... (A.11) ...

general result of Proposition 1 ($K \geq 1$) then follows directly from (A.10) and the observations that $Q(A, 0)$ is decreasing in $[s^*(A, 0)]^2$ and that $s^*(A, 0) = s^*(p^*[A, 0], 0)$.

Kandel (1984) analyzes the likelihood ratio test of the efficiency of a given portfolio, with weights given by the n-vector p, in the absence of a riskless asset. He shows that the likelihood ratio test statistic is a monotonic transformation of $\bar{Q}(p)$; specifically, $T \cdot \ln[1 + \bar{Q}(p)]$ is distributed asymptotically as χ^2 with n-2 degrees of freedom under the null hypothesis. [The expression for $Q(\cdot)$ derived by Kandel corresponds to the special case of (A.3) for $\ell = 1$.]

Shanken (1985, 1986) shows that $\bar{Q}(A)$, where $Q(A, \gamma)$ is stated in the form of (A.11), is a monotonic transformation of the likelihood ratio statistic for testing the hypothesis that some combination of the portfolios in A ($\ell > 1$) is mean-variance efficient in the absence of a riskless asset. Specifically, $T \cdot \ln[1 + \bar{Q}(A)]$ is distributed asymptotically as χ^2 with $n-\ell-1$ degrees of freedom under the null hypothesis.

Propositions 2 and 3 in section 3 are obtained by combining the results of the above studies with a further investigation of the properties of the function $Q(A, \gamma)$.

<u>Lemma 1.</u> For a single portfolio with weights denoted by the n-vector p,

$$\sigma^2(p) = \delta_1 + \delta_2 \bar{\sigma}^2(\mu(p)) \quad , \tag{A.12}$$

where

$$\delta_1 = \frac{\bar{Q}(p)[\bar{Q}(p) + 1]}{I\bar{Q}(p) - D} \quad \text{and} \quad \delta_2 = \frac{-D[\bar{Q}(p) + 1]}{I\bar{Q}(p) - D} \quad . \tag{A.13}$$

Proof: Kandel (1984) derives $\hat{\gamma}(p)$, the MLE for the zero-beta rate. Substituting $\hat{\gamma}(p)$ for γ in (A.3) and simplifying (with tedious but straightforward algebra) gives the above result.

<u>Proof of proposition 2</u>: Note that the right-hand sides of (7) and (A.12) are the same, except that the (A.12) replaces W^* by $\bar{Q}(p)$. Also, given the discussion above, $W(p) = \bar{Q}(p)$. The right-hand side of (7) is increasing in W^*, which can be shown using the condition that $\bar{\sigma}^2(\mu(p)) \geq 1/L$ (since $1/L$ is the global minimum sample variance of the set of n assets). Therefore, if $W(p) > W^*$, the inequality in (7) follows from lemma 1.

<u>Lemma 2.</u> For all γ, $Q(A, \gamma) = Q(p^*(A, \gamma), \gamma)$.

Proof: Use (A.10) and the observation that $s^*(A, \gamma) = s^*(p^*[A, \gamma], \gamma)$.

<u>Lemma 3.</u> For all A, γ, and w such that $w'\iota_\ell = 1$,

$$Q(Aw, \gamma) \geq Q(p^*[A, \gamma], \gamma).$$

Proof: By definition, $[s^*(p^*[A, \gamma], \gamma)]^2 \geq [s^*(Aw, \gamma)]^2$, and by (A.10), $Q(A, \gamma)$ is decreasing in $[s^*(A, \gamma)]^2$.

<u>Lemma 4.</u> For all A and w such that $w'\iota_\ell = 1$, $\bar{Q}(Aw) \geq \bar{Q}(A)$.

Proof: Using, in order, (A.5), lemma 3, lemma 2, and (A.4), observe that $\bar{Q}(Aw) = Q(Aw, \hat{\gamma}(Aw)) \geq Q(p^*[A, \hat{\gamma}(Aw)], \hat{\gamma}(Aw)) = Q(A, \hat{\gamma}(Aw)) \geq \bar{Q}(A)$.

<u>Proof of proposition 3</u>: Let W_ℓ^* denote the critical value for the likelihood ratio test of the hypothesis that some combination of the portfolios in A is efficient, i.e., this null hypothesis is rejected if $\bar{Q}(A) > W_\ell^*$. Note that if $\bar{Q}(A) > W_\ell^*$, then by lemma 4, $\bar{Q}(Aw) > W_\ell^*$ for all w. Therefore, using lemma 1 and the same argument in the proof of proposition 2, the inequality in (7) must hold for any $p = Aw$ (where W_ℓ^* replaces W^*). A

for all p's on the minimum-variance boundary of the set A, and this is equivalent to (9).

We now turn to the tests against specific alternatives, which are addressed in propositions 4 and 5. Let E^* and V^* denote the true (population) mean vector and variance-covariance matrix of the n risky assets. The pricing restriction in (4) can be viewed as a set of restrictions on E^* and V^*. For a given number of K reference portfolios, partition V^* as

$$V^* \equiv \operatorname{cov}\begin{bmatrix} r_t \\ R_t \end{bmatrix} \equiv \begin{bmatrix} V_{11}^* & V_{12}^* \\ V_{21}^* & V_{22}^* \end{bmatrix} , \qquad \text{(A.14)}$$

and similarly partition E^* as $E_1^* \equiv E(r_t)$ and $E_2^* \equiv E(R_t)$. It is easily shown that the restriction in (4) can be written as

$$E_1^* = \gamma\iota_{n-K} + (V_{22}^*)^{-1}V_{12}[E_2^* - \gamma\iota_K] \qquad \text{(A.15)}$$

Let the parameter vector θ contain the elements of E^* and V^*, and let Ω denote the entire parameter space (wherein E^* can be any real-valued vector and V^* can be any symmetric positive-definite matrix). The restriction in (A.14) is represented as $\theta \in \omega(K)$, where $\omega(K)$ denotes the region of Ω defined by the restriction. The notation "$\omega(K)$" is chosen to emphasize the fact that this region depends on the choice of the K reference portfolios. Let X denote the sample of T observations of $(r_t' \; R_t')$, and let $f(\theta; X)$ denote the likelihood function (given by the multivariate normal distribution).

The likelihood ratio for testing a K-beta pricing model against a general

$$\lambda(K) \equiv \frac{\max\limits_{\theta \in \Omega} f(\theta;\, X)}{\max\limits_{\theta \in \omega(K)} f(\theta;\, X)} , \tag{A.16}$$

and the likelihood ratio for testing a K_1-beta pricing model against a specific alternative of a K_2-beta model is given by

$$\lambda(K_1,\, K_2) \equiv \frac{\max\limits_{\theta \in \omega(K_2)} f(\theta;\, X)}{\max\limits_{\theta \in \omega(K_1)} f(\theta;\, X)} . \tag{A.17}$$

From (A.16) and (A.17), observe

$$\lambda(K_1,\, K_2) = \frac{\lambda(K_1)}{\lambda(K_2)} \tag{A.18}$$

Since the tests discussed previously involve the computation of $\lambda(K)$, the computation of $\lambda(K_1,\, K_2)$ is straightforward using (A.18).

Let A_2 denote the set of K2 reference portfolios, and let the submatrix A_1 denote the set of K1 reference portfolios. When a riskless asset exists, Gibbons, Ross, and Shanken (1985) show that

$$\begin{aligned} \lambda(K_1) &= [1 + Q(A_1,\, 0)]^{T/2} \qquad \text{and} \\ \lambda(K_2) &= [1 + Q(A_2,\, 0)]^{T/2} \qquad , \end{aligned} \tag{A.19}$$

where excess returns are used in the computations. Using (A.18) and (A.10),

$$\frac{[s^*(A_2,\ 0)]^2 - [s^*(A_1,\ 0)]^2}{1 + [s^*(A_1,\ 0)]^2} = \lambda(K1,\ K2)^{2/T} - 1 \quad , \tag{A.20}$$

so that the left-hand side of (A.20) is a monotonic transformation of the likelihood ratio test statistic $\lambda(K1,\ K2)$. Now observe that the left-hand side of (A.20) is equal to $Q(A_1,\ 0)$ but where the set of K2 reference portfolios replaces the original set of n assets in the definition of $Q(\cdot)$. Therefore, proposition 4 follows as a straightforward relabeling of proposition 1, where the set of K2 reference portfolios takes the role of the set of n assets and the set of K1 reference portfolios takes the role of the set of K assets.

In the tests where a riskless asset does not exist, use (A.18) and the definitions of W_{K1}, W_{K2}, and $W_{K1,K2}$ to obtain

$$1 + W_{K1,K2} = \frac{1 + W_{K1}}{1 + W_{K2}} \quad . \tag{A.21}$$

From (A.21), $W_{K1,K2} > W^*_{K1,K2}$ if and only if $W_{K1} > X_{K2}$ [defined in (16)]. Proposition 5 then follows directly from proposition 3 with X_{K2} in place of W^*_K.

Proposition 6 returns to a test against a general alternative, but a factor is used instead of the return on a reference portfolio. The starting point for this analysis is a result of Shanken (1985). Let b denote the ordinary least squares estimate of B in (19), and let Σ denote the sample variance-covariance matrix of the residuals from that regression. Shanken

$$W_{f1} = \min_{\phi} \frac{e' \Sigma^{-1} e}{1 + \frac{\phi^2}{\sigma_f^2}} \qquad \text{(A.22)}$$

where $e = E - \phi b$ and σ_f^2 is the sample variance of the factor f_t. Through straightforward algebra, it can be verified that the solution to the above minimization yields

$$W_{f1} = h\sigma_f^2 \left[1 - \frac{2\,\frac{g^2}{h}}{[(z - h\sigma_f^2)^2 + 4g^2\sigma_f^2]^{1/2} + z - h\sigma_f^2} \right] , \qquad \text{(A.23)}$$

where $z \equiv E'\Sigma^{-1}E$, $h \equiv b'\Sigma^{-1}b$, and $g \equiv b'\Sigma^{-1}E$. Using the facts that the weights in p^* are proportional to $V^{-1}E$ and that the weights in $\hat{p}$ are proportional to $V^{-1}b$, along with the relation

$$V^{-1} = \Sigma^{-1} - \frac{\sigma_f^2 \Sigma^{-1} bb' \Sigma^{-1}}{1 + \sigma_f^2 h} , \qquad \text{(A.24)}$$

it is easily shown that

$$\hat{\rho}^2 = \sigma_f^2 b' V^{-1} b = \frac{\sigma_f^2 h}{1 + \sigma_f^2 h} , \qquad \text{(A.25)}$$

$$S(p^*)^2 = E' V^{-1} E = z - \frac{\sigma_f^2 g^2}{\quad} \quad \text{and} \qquad \text{(A.26)}$$

$$S(\hat{p})^2 = \frac{(b'V^{-1}E)^2}{b'V^{-1}b} = \frac{g^2}{h} \quad . \tag{A.27}$$

Combining (A.23) with the three relations above yields

$$W_{f1} = \frac{\hat{\rho}^2}{1-\hat{\rho}^2}\left[1 - \frac{2S(\hat{p})^2}{[Y^2 + 4\hat{\rho}^2 S(\hat{p})^2]^{1/2} + Y}\right] \quad , \tag{A.28}$$

where

$$Y = S(p^*)^2 - \hat{\rho}^2[1 + S(p^*)^2 - S(\hat{p})^2] \quad . \tag{A.29}$$

Proposition 6 follows by substituting W^*_{f1} for W_{f1} and S_{Cf1} for $S(\hat{p})$ and then solving for S_{Cf1}.

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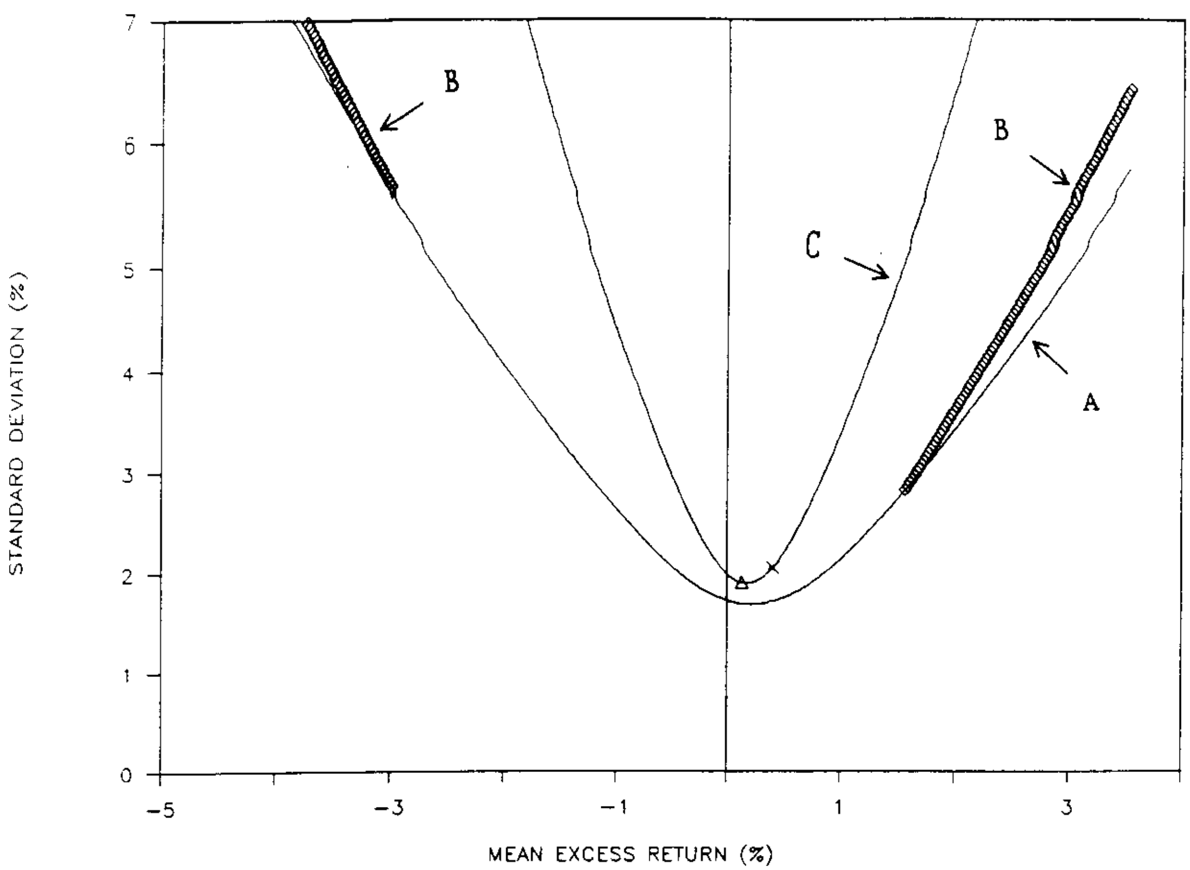

Figure 1. Likelihood ratio tests of a two-beta pricing model in the presence of a riskless asset. The tests are based on weekly returns in excess of a riskless rate. The two reference portfolios are the value-weighted NYSE (△) and the equally weighted NYSE (x). Hyperbola "A" is the sample minimum-standard-deviation boundary of twelve assets: ten size-based portfolios plus the two market proxies. Lines "B" represent the critical Sharpe measure at a 5% significance level. Hyperbola "C" is the sample minimum-standard-deviation boundary of the two reference portfolios.

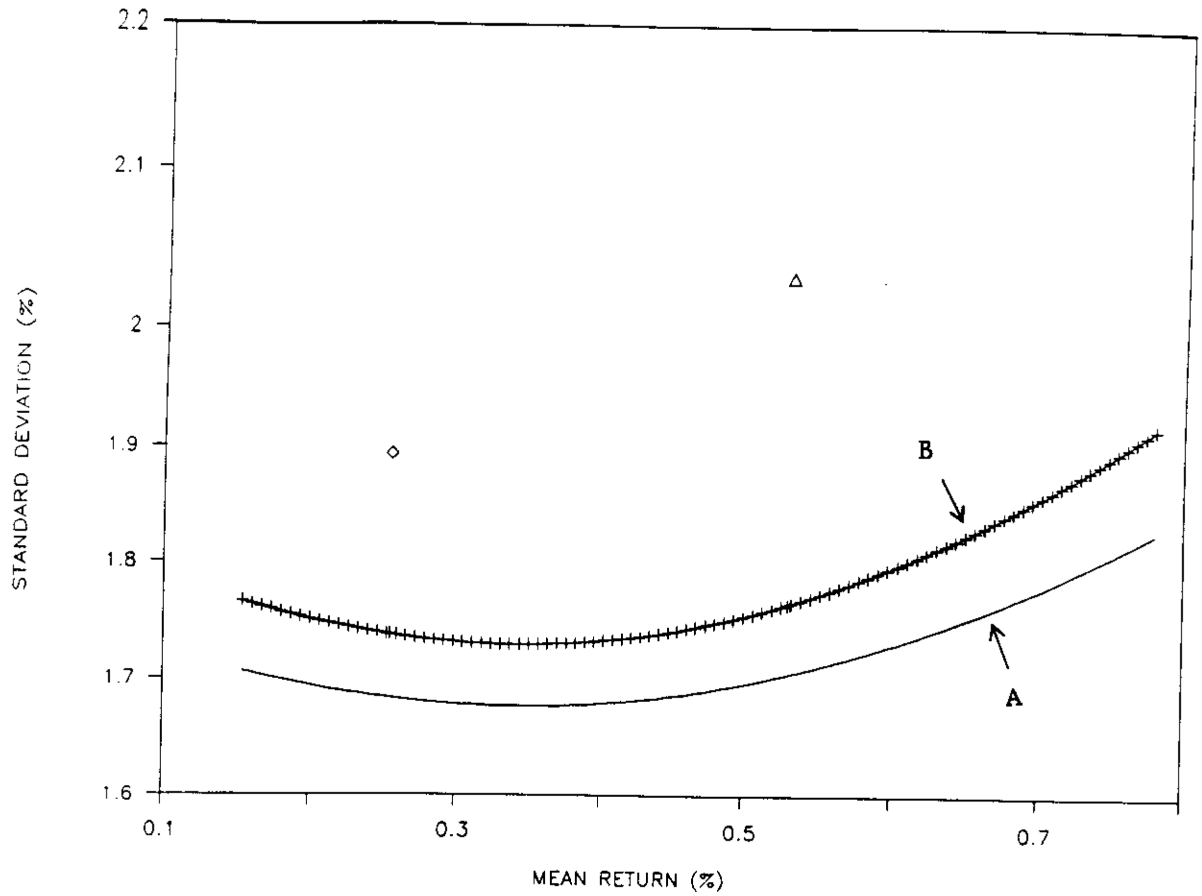

Figure 2. Likelihood ratio tests of the zero-beta CAPM (without a riskless asset). The tests are based on weekly returns. The market proxies are the value-weighted NYSE (◇) and the equally weighted NYSE (△). Hyperbola "A" is the sample minimum-standard-deviation boundary of twelve assets: ten size-based portfolios plus the two market proxies. Hyperbola "B" is the critical hyperbola at a 5% significance level.

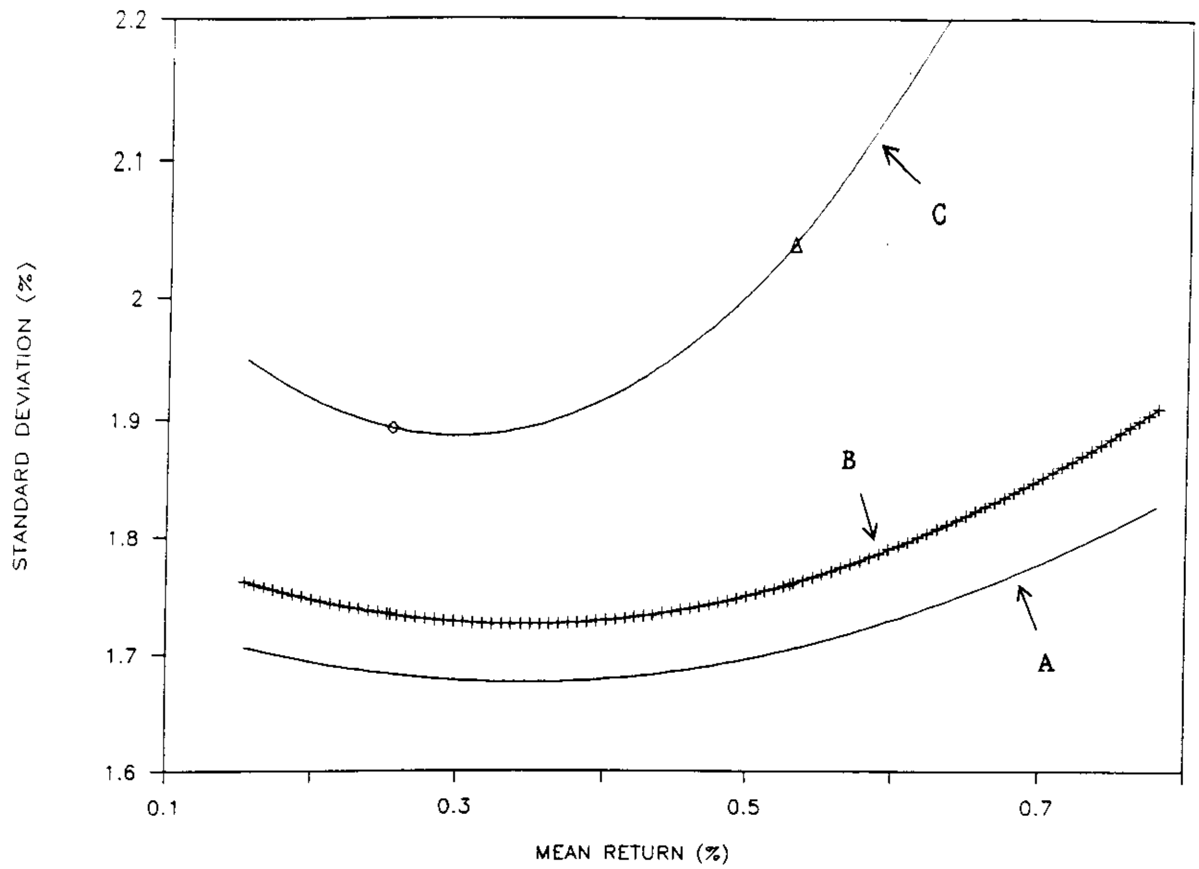

Figure 3. Likelihood ratio tests of a two-beta pricing model in the absence of a riskless asset. The tests are based on weekly returns. The two reference portfolios are the value-weighted NYSE (◇) and the equally weighted NYSE (△). Hyperbola "A" is the sample minimum-standard-deviation boundary of twelve assets: ten size-based portfolios plus the two market proxies. Hyperbola "B" is the critical hyperbola at a 5% significance level. Hyperbola "C" is the sample minimum-standard-deviation boundary of the two reference portfolios.

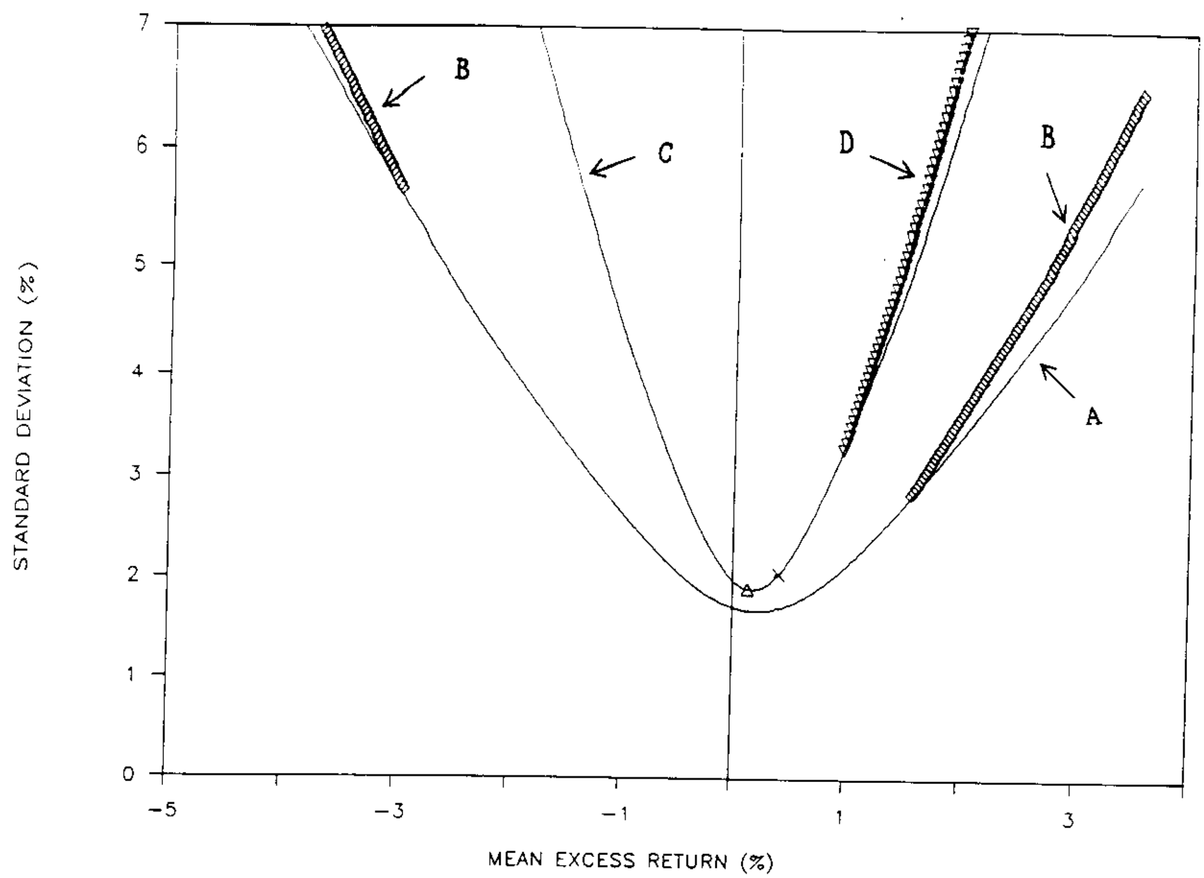

Figure 4. Likelihood ratio tests of a single-beta pricing model against general and specific alternative hypotheses in the presence of a riskless asset. The null hypothesis is a single-beta pricing model with a market proxy as the reference portfolio. The specific alternative hypothesis is a two-beta pricing model with two market proxies as the reference portfolios. The tests are based on weekly returns in excess of a riskless rate. The market proxies are the value-weighted NYSE (Δ) and the equally weighted NYSE (x). Hyperbola "A" is the sample minimum-standard-deviation boundary of twelve assets: ten size-based portfolios plus the two market proxies. Lines "B" represent the critical Sharpe measure at a 5% significance level against the general alternative hypothesis. Hyperbola "C" is the sample minimum-standard-deviation boundary of the two reference portfolios. Lines "D" represent the critical Sharpe measure at a 5% significance level against the specific alternative hypothesis.

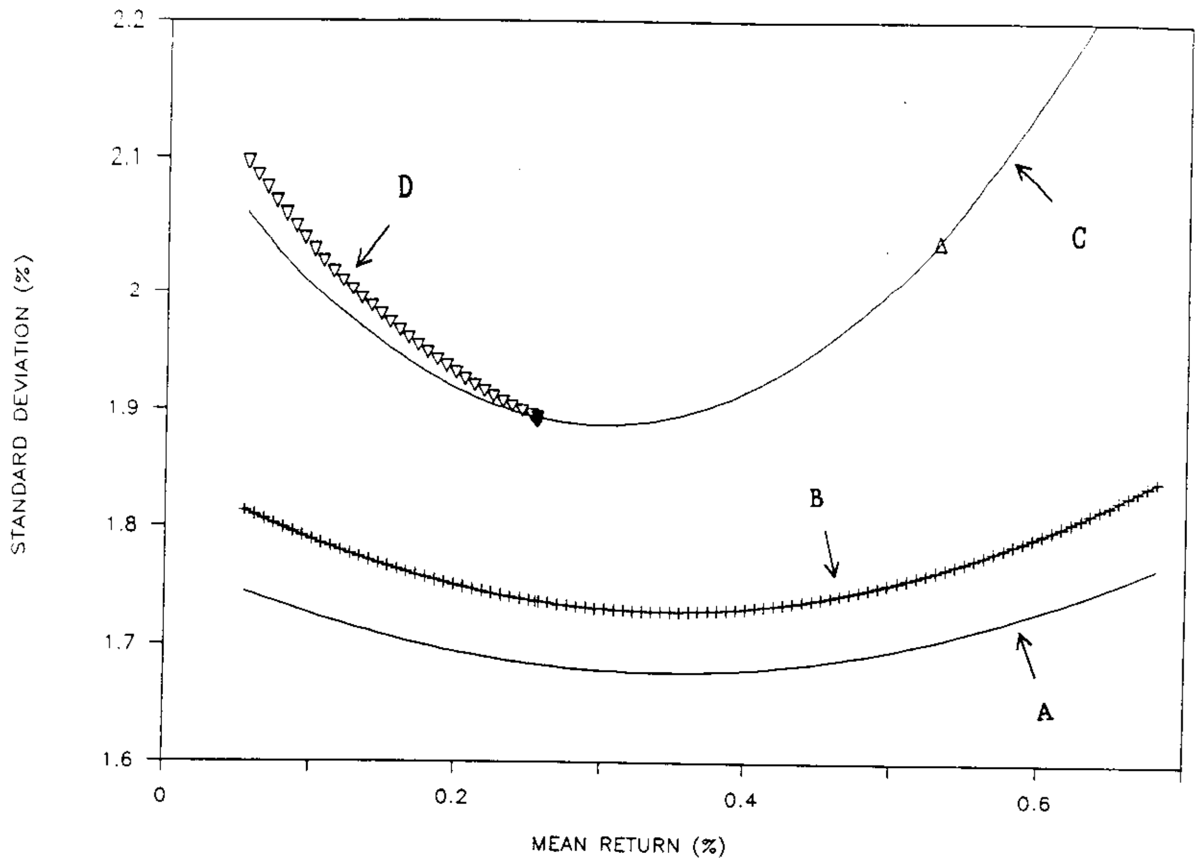

Figure 5. Likelihood ratio tests of a single-beta pricing model against general and specific alternative hypotheses in the absence of a riskless asset. The null hypothesis is a single-beta pricing model with a market proxy as the reference portfolio. The specific alternative hypothesis is a two-beta pricing model with two market proxies as the reference portfolios. The tests are based on weekly returns. The market proxies are the value-weighted NYSE (◆) and the equally weighted NYSE (△). Hyperbola "A" is the sample minimum-standard-deviation boundary of twelve assets: ten size-based portfolios plus the two market proxies. Hyperbola "B" is the critical hyperbola at a 5% significance level against the general alternative hypothesis. Hyperbola "C" is the sample minimum-standard-deviation boundary of the two reference portfolios. "D" is the relevant part of the critical hyperbola at a 5% significance level against the specific alternative hypothesis.

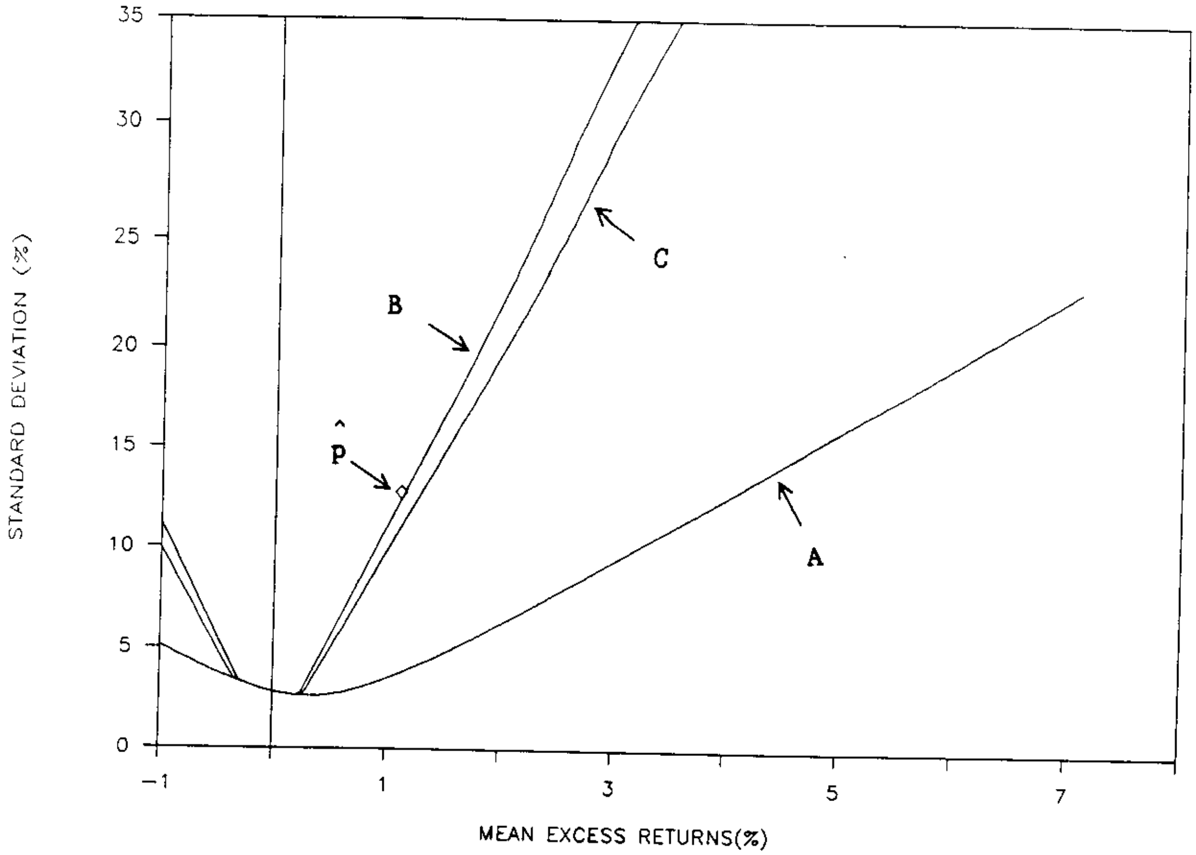

Figure 6. Likelihood ratio tests of the consumption-beta model. The tests are based on quarterly returns, in excess of a Treasury bill rate, and an index of quarterly consumption growth. Hyperbola "A" is the sample minimum-standard-deviation boundary of twelve assets: ten size-based portfolios plus a portfolio of long-term U.S. Government Bonds and a portfolio of bonds rated below Baa by Moody's. The portfolio $\hat{p}$ has maximum sample correlation with the consumption index. Lines "B" represent the critical Sharpe measure for testing the consumption-beta model at a 10% significance level. Lines "C" represent the critical Sharpe measure for testing the ex ante tangency of $\hat{p}$.