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Abstract

ADVERSE RISK INCENTIVES AND THE DESIGN OF PERFORMANCE-BASED CONTRACTS

In this paper, option pricing theory is used to value and analyze many performance-based fee contracts that are currently in use. A potential problem with some of these contracts is that they may induce portfolio managers to adversely alter the risk of the portfolios they manage. This paper is prescriptive, in that it derives conditions for contract parameters that provide proper risk incentives for classes of investment strategies. For buy-and-hold and rebalancing strategies, adverse risk incentives are avoided when the penalties for poor performance outweigh the rewards for good performance.

In late 1985, the Securities and Exchange Commission approved the use of performance-based fees for portfolio managers. They were also approved by the Department of Labor in August 1986 for ERISA governed pension funds. Since then, performance-based contracts, which reward portfolio managers on the basis of their portfolio return relative to the performance of a benchmark portfolio, (typically the S&P 500), have become increasingly common. Currently, billions of dollars in pension fund assets are being managed under these contracts. Proposals that are currently being considered indicate that this amount will shortly grow to hundreds of billions of dollars. It is not surprising, then, that the proper design of these contracts is a major concern of the pension fund industry.[1]

The focus of this paper is on performance-based fee contracts that are currently in use. Hence, rather than solving for the optimal design of contracts in general, we analyze the selection of parameters for observed classes of contracts.[2] The analysis shows that if the parameters for these contracts are not properly chosen, the portfolio manager might face incentives to alter the risk of the managed portfolio to a level that would be detrimental to the welfare of his clients.

The risk-incentive problems are analyzed within an option pricing framework. We show that performance-based contracts provide portfolio managers with the opportunity to earn a fee that is equivalent to the payoff on a portfolio of European options. The options arise because these contracts have a floor on the manager's compensation, (the base fee), and sometimes a ceiling as well, (referred to as the cap).

The underlying asset of these options is typically the difference between the value of the managed portfolio and that of a comparison or benchmark portfolio. Since option values are sensitive to the risk of the underlying asset, the portfolio manager can control the present value of his fee by altering the risk of his portfolio. For fee contracts that lack a cap on the performance-based fee, the present value of the fee monotonically increases as the beta of the portfolio deviates from one and as the unsystematic risk of the portfolio increases. Hence, the contracts can provide an incentive to the manager to deviate from the level of systematic and unsystematic risk preferred by the client, regardless of the client's risk-preference. Depending on their design, contracts that include caps may also have these "adverse risk incentives." One contribution of this paper is that it delineates necessary and sufficient conditions for selecting contract parameters that induce proper risk incentives for classes of investment strategies.

These issues may also be relevant when the manager's current fee is independent of the fund's performance. We argue that for these cases, multiperiod reputational considerations may make the manager's current plus future fees a convex function of his performance in the current period. This convexity property, which induces the incentive to increase risk, will hold if the future benefits from performing well outweigh the losses from performing poorly.

The presumed motivation for offering performance-based fee contracts is to create incentives for the manager to use superior information for the benefit of the fund. However, the results in the paper apply both to portfolio managers with superior information and to portfolio managers with no better information than "the market." In particular, we show that for certain classes of portfolio strategies, informed portfolio managers select the same portfolio as uninformed portfolio managers if the realized fee can be hedged in the manager's personal portfolio. This result, which is a simple extension of Fisher separation, suggests that effective portfolio management contracts may require implicit or explicit restrictions on the manager's personal portfolio.

The adverse risk incentives are first illustrated for a simple performance-based contract in section I of the paper. Section II derives conditions for the construction of more complicated performance-based contracts that do not offer incentives to substantially alter the risk of the managed portfolio. Section III analyzes the robustness of our findings. In particular, it focuses on systematic risk, unsystematic risk, multiperiod reputation considerations, and dynamic portfolio strategies (some of which, in the absence of transaction costs, earn the maximum fee with probability one, irrespective of contract design). This section also discusses a similar risk incentive in corporate finance, the bondholder-stockholder conflict, and contrasts our contractual solution with those proposed for this other problem. In Section IV, we show that informed managers and uninformed managers place the same value on every portfolio strategy. The section also discusses contractual covenants that may induce an informed manager to use superior information in managing the portfolio. Section V briefly concludes the paper.

I. SIMPLE FEE CONTRACTS

The simplest performance-based contract has two features: a base fee and a bonus based on the degree to which the managed portfolio's return exceeds the return of some benchmark. The realized value of this contract, with units expressed as fractions of the net asset value of the fund at the beginning of the evaluation period, is represented by the expression

$$F = B + \max[0, m(\tilde{R}_p - \tilde{R}_s)], \text{ where} \tag{1}$$

B = base fee

m = fraction of the return of the fund in excess of the benchmark return awarded to the manager as a bonus for good performance,

$\tilde{R}_p$ = return of the managed portfolio in the evaluation period, and

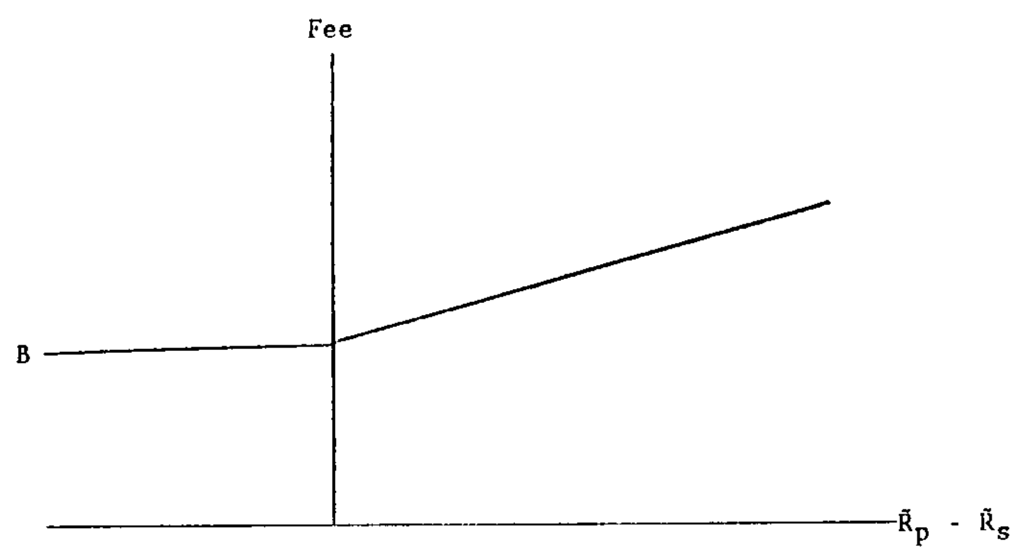
$\tilde{R}_s$ = return of the benchmark portfolio used in the evaluation period.

The bonus portion of the contract can be viewed as offering the manager a menu of options, each corresponding to a different portfolio that he might choose. Each item in the menu consists of m European options to exchange the manager's portfolio for the benchmark portfolio.[3] Diagram 1, which illustrates this, plots the contract's percentage reward to the manager as a function of the difference between the return of his portfolio and the return of the benchmark.

Since the value of an option is related to the volatility of the underlying security, the present value of the fee contract must be related to the volatility of the combination of a long position in the managed portfolio and a short position in the benchmark. This, in turn, is affected by the risk of the manager's portfolio. An analytic solution for the present value of the fee can be obtained if assumptions are made about the manager's investment strategy. Consider, for instance, a manager with a buy-and-hold strategy in the benchmark portfolio and a risk-free asset. Let β with $\beta > 1$ denote the fraction of the initial net asset value of the fund invested in the benchmark, (or equivalently, his portfolio beta when measured against the benchmark). The fraction $\beta - 1$ of the fund's net asset value is then borrowed at the risk-free rate to finance the risky equity purchase,

DIAGRAM 1

MANAGEMENT FEE AS A FUNCTION OF THE RETURN OF THE PORTFOLIO
IN EXCESS OF THE RETURN OF THE BENCHMARK PORTFOLIO



implying that

$$\tilde{R}_p = \beta\tilde{R}_s + (1 - \beta)R_f.$$

Substituting this into equation (1) yields

$$F = B + m(\beta - 1)\max(0, \tilde{R}_s - R_f). \qquad (2)$$

For $\beta < 1$, (i.e. the fund is long in the risk-free asset), it becomes

$$F = B + m(1 - \beta)\max(0, R_f - \tilde{R}_s). \qquad (3)$$

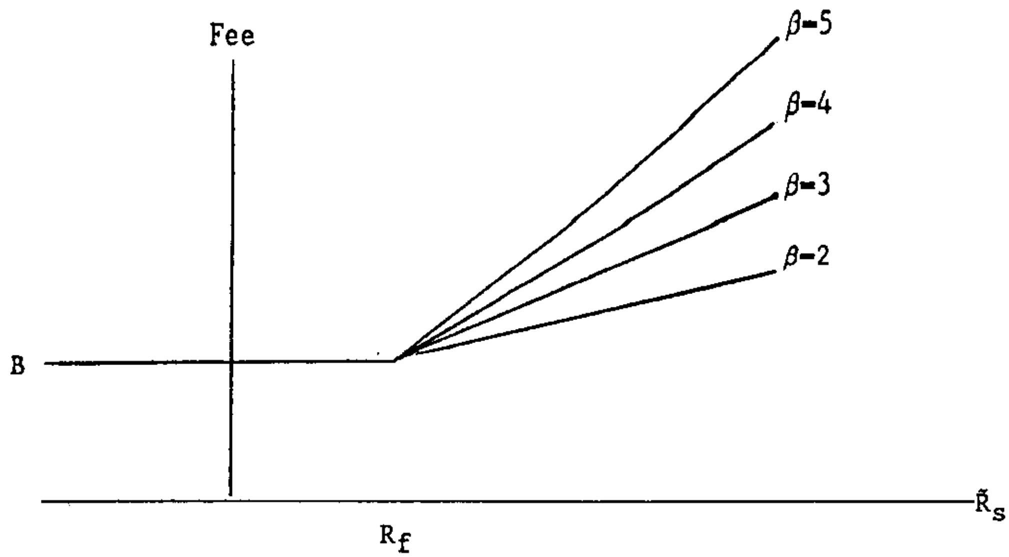
Equations (2) and (3) are plotted in Diagram 2 for various values of β. Increasing the leverage of the managed portfolio when β exceeds one (or increasing the long position in the risk-free investment for $\beta < 1$) implies that the fee structure has more call (put) options implicit in the payoff. Since call (put) options always have positive value, as the manager increases the leverage (the risk-free position) of his portfolio, he increases the present value of his fee contract.

If he desires, the portfolio manager can capture the value of these options risklessly by hedging in his personal portfolio. For a portfolio with a beta that is greater (less) than one, he merely needs to write call (put) options on a one dollar investment in the benchmark with striking prices equal to one plus the risk-free rate. The number of options he writes is the product of m, the absolute value of $\beta - 1$, and the initial net asset value of the fund. Since these should be European options, this hedge is only approximate if the manager uses currently traded American index options. However, he can perfectly hedge his fee with dynamic positions in these options or in index futures. (Note that this would be somewhat more difficult if options on the benchmark or some portfolio that is highly correlated with the benchmark are not traded, as would be the case for some less traditional benchmarks.)

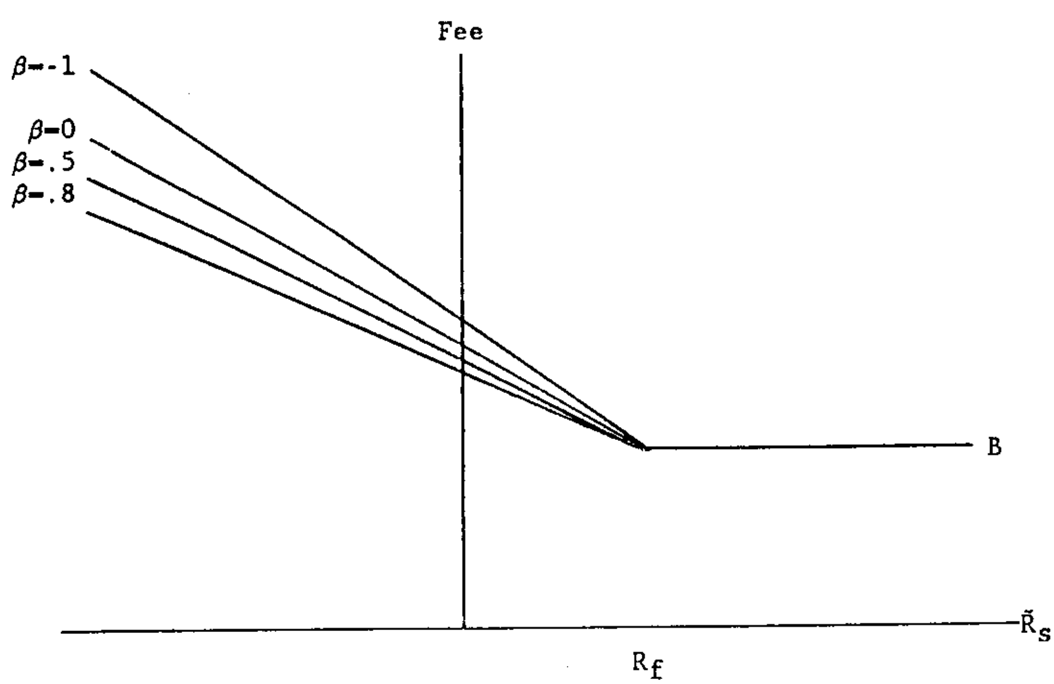
DIAGRAM 2

MANAGEMENT FEE FOR VARIOUS PORTFOLIOS OF THE RISK-FREE ASSET AND THE BENCHMARK AS A FUNCTION OF THE RETURN OF THE BENCHMARK

Panel A: Leverage Positions in the Benchmark



Panel B: Portfolios with Long Positions in the Risk-Free Asset



II. COMPLEX FEE CONTRACTS

Most performance-based fees are more complex than those examined in the previous section. They often include caps on the maximum fee and bonuses that are triggered when the portfolio return exceeds the benchmark return by a hurdle amount H, which can be either positive or negative. For example, if H is 2%, the manager earns the base fee unless his returns are more than 2% greater than the benchmark return. The manager then keeps the fraction m of the returns in excess of the 2% hurdle up to the cap, if one exists. Contracts with a negative H can be thought of as having a penalty for performance below the benchmark return. -H is then the minimum difference between the benchmark return and the portfolio return that results in the maximum penalty.

This more general contract is represented by the formula

$$F = B + m\,\{\min\,[\max(0,\ \tilde{R}_p - \tilde{R}_s - H),\ (C - B)/m)]\},$$

$$= B + m\,[\max(0,\ \tilde{R}_p - \tilde{R}_s - H) - \max(0,\ \tilde{R}_p - \tilde{R}_s - H - (C - B)/m)], \quad (4)$$

where

H = hurdle point that triggers the bonus and

C = the cap on the performance fee, which exceeds B.

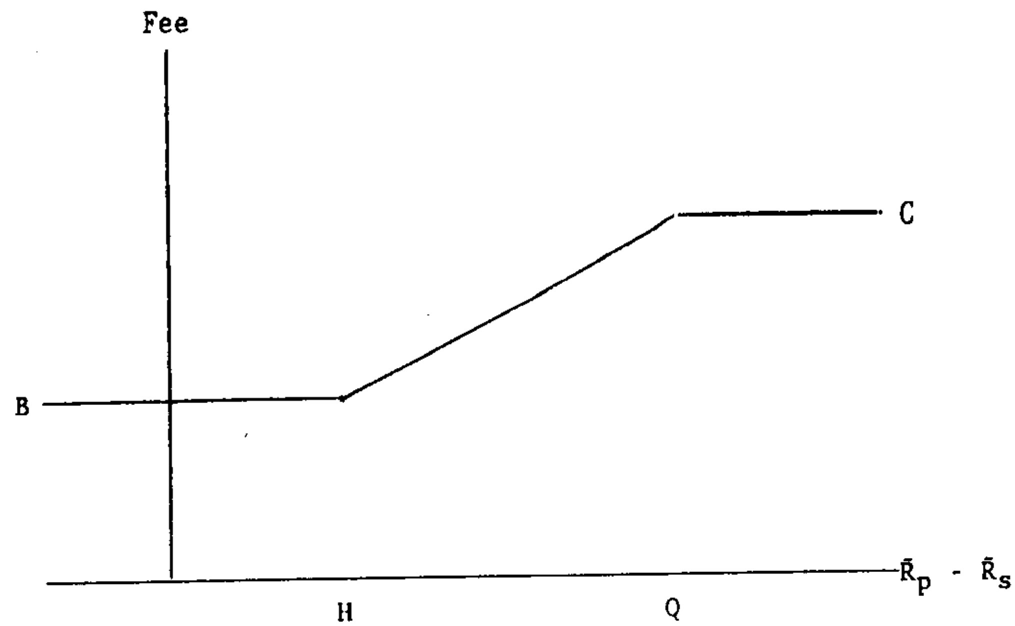
This contract is plotted for both positive and negative hurdles in Diagram 3. Note that the contract without the cap is a limiting case in which the cap, C, approaches infinity.[4] Note also that H + (C - B)/m, which we henceforth denote as Q, should exceed zero. Otherwise, this is a trivial problem: The manager earns the cap by purchasing the benchmark portfolio. Q is the minimum amount by which the return of the managed portfolio must exceed the return of the benchmark in order for the cap to be binding.

Various subsets of the following assumptions will now be made to allow a

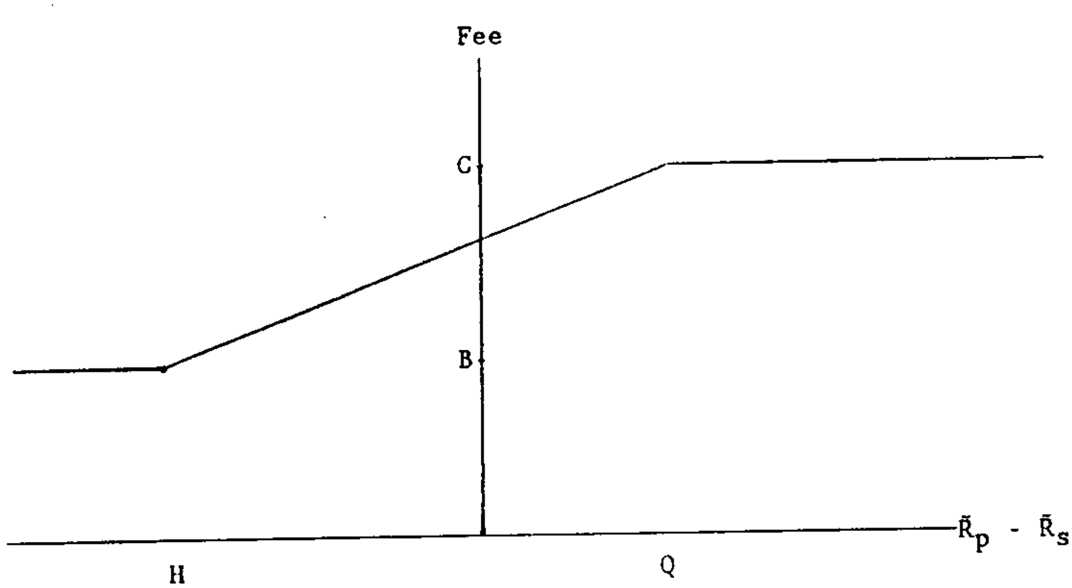
DIAGRAM 3

MANAGEMENT FEES WITH CAPS AS A FUNCTION OF THE RETURN OF THE PORTFOLIO IN EXCESS OF THE RETURN OF THE BENCHMARK PORTFOLIO

Panel A: Positive Bonus Trigger Point



Panel B: Negative Bonus Trigger Point



tractable analysis of the adverse risk incentives that are associated with more complicated performance-based fee contracts. Later sections of the paper examine the robustness of the results to relaxations of these assumptions.

Assumption 1: The manager follows a buy-and-hold strategy for the duration of the evaluation period. His portfolio has the fraction β invested in the benchmark portfolio and the fraction $1-\beta$ invested in the risk-free asset, where the risk parameter β is selected by the portfolio manager.[5]

In a later section, we employ an alternative assumption.

Assumption 1': The value of the managed portfolio is generated by a stationary geometric brownian motion process, (i.e. the portfolio is continuously rebalanced) and can be continuously traded in perfect markets in the manager's personal portfolio.

Assumption 2: The manager's horizon is one evaluation period.

Assumption 3: There are no arbitrage opportunities, the benchmark portfolio can be continuously traded in perfect markets in the manager's personal portfolio, and the value of the benchmark portfolio is generated by a diffusion process with known variance parameter(s).

Assumption 4: The benchmark portfolio's net return is bounded below by negative 100%.

For certain closed form solutions, a stronger assumption may be employed:

Assumption 4': The benchmark portfolio's gross return is generated by a stationary geometric brownian motion process (and is therefore lognormally distributed).

Note that Assumption 3 implies that without loss of generality, risk-neutral valuation techniques are applicable to any contingent claim on the benchmark portfolio. (See Cox and Ross (1976).) When combined with Assumption 4', it implies that the Black and Scholes (1973) option pricing formula can be used to value the performance-based fee contracts.

Assumption 1 implies that the realized fee from equation (4) can be expressed as

$$F = B + m\,[\max(0,\ b(\tilde{R}_s - R_f) - H) - \max(0,\ b(\tilde{R}_s - R_f) - Q],\qquad (5)$$

where $b = \beta - 1$, $Q = H + (C - B)/m > 0$, and $R_f = 1 +$ risk-free rate. The bracketed term in (5), denoted F^*, can be rewritten for $b > 0$ as

$$F^* = b[\max(0, \tilde{R}_s - R_f - H/b) - \max(0, \tilde{R}_s - R_f - Q/b)] \qquad (6a)$$

and for $b < 0$ as

$$F^* = -b[\max(0, H/b + R_f - \tilde{R}_s) - \max(0, Q/b + R_f - \tilde{R}_s)]. \qquad (6b)$$

The expression in (6a) ((6b)) represents b ($-b$) times the difference between the value of two call (put) options on the benchmark portfolio with striking prices respectively equal to $R_f + H/b$ and $R_f + Q/b$. The first option is an option associated with the bonus, which is triggered by performance in excess of $\tilde{R}_s + H$. The second option is associated with the cap, which becomes effective when performance exceeds the benchmark return by Q. Since these options differ only in their striking prices and Q/b is positive for the calls and negative for the puts, the difference in their values must always be nonnegative.

Letting PV( ) denote the present value function, the values of the expressions in (6a) and (6b) are respectively

$$PV(F^*) = b[c(R_f + H/b) - c(R_f + Q/b)] \text{ for } b > 0 \text{ and} \qquad (7a)$$

$$PV(F^*) = -b[p(R_f + H/b) - p(R_f + Q/b)] \text{ for } b < 0, \text{ where} \qquad (7b)$$

$c(K)$ = value of a European call option on a one dollar investment in the benchmark portfolio with a striking price equal to K and

$p(K)$ = value of a put option with the same features as $c(K)$.

Note that if K is zero or negative, $c(K)$ equals the absolute value of the striking price plus the value of the underlying asset and $p(K)$ equals zero. If K is positive and if assumption (4′) holds, $c(K)$ and $p(K)$ equal their Black-Scholes (1973) values, respectively

$$c(K) = N\{-[\log(K/R_f)]/\sigma + \sigma/2\} - (K/R_f)N\{-[\log(K/R_f)]/\sigma - \sigma/2\}, \text{ and}$$

$p(K) = c(K) + 1 - K/R_f$, where

σ = annualized standard deviation of $\tilde{R}_s$.

We can now describe how deviations of beta from one (i.e. from holding the benchmark portfolio) affect the value of the contract.

Proposition 1: If the fee is described by equation (4) and if Assumptions (1), (2), (3), and (4) hold, then

(i) In the absence of a cap (i.e. infinite Q), the value of the performance-based fee contract is monotonically increasing in the absolute deviation of the β of the managed portfolio from one.

(ii) If H is negative and there is a cap, the value of the performance-based fee contract is decreased by small increases in β above $\beta = 1$ and increased for small decreases in β below one.

Proof: See the Appendix.

Part (i) of Proposition 1 derives from the fact that options are increasing functions of the volatility of the underlying asset. Part (ii) of the proposition follows from the truncation of the distribution of the benchmark portfolio return. For small increases in β above one, the bonus portion of the option is always in the money. That is, even in the worst possible scenario, where the benchmark return is -100%, one still receives more than the base fee. Because of this truncation, small changes in the volatility of the underlying asset, brought about by increasing beta slightly above one, have no effect on the bonus option. However, since they increase the value of the cap option, they reduce the value of the fee. For small decreases in β below one, the cap is never reached, even in the best possible scenario, where the benchmark return is -100%. As a consequence, volatility increases that are brought about by small decreases in beta below one increase the value of the bonus option, but have no effect on the value of the cap option. This results in an increase in the present value of the fee.

Proposition 1 suggests that any cap and any negative hurdle will deter sufficiently small increases from a beta of one. It also indicates that performance-based fee contracts cannot be designed to provide incentives for the portfolio manager to buy-and-hold the benchmark portfolio, (unless the maximum fee is awarded for zero excess performance, i.e. $Q \leq 0$). This is not a serious problem if a contract can be designed that induces the portfolio manager to choose a finite beta and no unsystematic risk (relative to the benchmark portfolio), where the optimal beta is independent of the volatility of the benchmark. In this case, one could alter the risk of the benchmark with positions in the risk-free asset to induce incentives for any desired risk level in the managed portfolio. For example, if the S&P 500 benchmark induces a value maximizing portfolio beta of .9, (relative to the S&P 500), a value maximizing beta of one is induced by a benchmark with a leveraged position in the S&P 500: 10/9 dollars invested in the S&P 500 with 1/9 of a dollar borrowed at the risk-free rate.

Improperly designed contracts, however, may create an incentive for the manager to deviate from any finite risk level (beta) that is targeted for the fund. With such contracts, it is impossible to induce the portfolio manager to select an appropriate beta by changing the composition of the benchmark. The following proposition delineates the features of contracts that create incentives for the manager who follows a buy-and-hold strategy to seek a finite risk level.

Proposition 2: If the fee contract is represented by eq. (4) and assumptions (1), (2), (3), and (4′) hold, then

(i) If there is a finite cap and if H is nonnegative, the value of the performance-based contract is monotonically increasing in the deviation of the β of the managed portfolio from one.

(ii) If H is negative and smaller in absolute value than Q, the value of the performance-based contract is monotonically increasing in the absolute deviation of the portfolio beta from β^*, where

$\beta^* = 1 - HQ/[R_f(H + Q)]$.

(iii) If H is negative and larger in absolute value than Q, the value of the performance-based contract is monotonically decreasing in the absolute deviation of the portfolio beta from β^*, which implies that for these parameter values, the fee-maximizing beta is β^*.[6] It is necessarily less than one.

Proof: See the Appendix.

To understand the intuition of this proposition, consider the effect of volatility on options with different striking prices. When underlying asset values are generated with a buy-and-hold strategy in a risky and a risk-free asset, volatility increases in the underlying asset have a greater effect on the value of an option, the closer that option is to being at-the-money. Part (i) of Proposition 2 follows because the bonus option is closer to being at-the-money than the cap option. In part (ii), the bonus option is also closer to being at-the-money than the cap option, however there is another effect that dominates the volatility effect for small changes in beta. This is the effect of the truncated distribution of the benchmark portfolio return, which is discussed in Proposition 1. In part (iii), the cap option is closer to being at-the-money than the bonus option, however the truncation effect again dominates for betas close to one, which shifts the optimal beta to the left of one.

Note that $\beta*$ in Proposition 2 is a monotonic function of H, holding Q constant, and a monotonic function of Q, holding H constant. Hence, as the penalty increases, $\beta*$ increases towards one; as the bonus increases (i.e. the cap is increased), $\beta*$ decreases towards minus infinity. An infinitely restrictive penalty is required to induce the portfolio manager to hold the benchmark portfolio. However, one does not have to be too restrictive to be close to a beta of one. For instance, if -H is 300 basis points and Q is 200 basis points, the contract maximizing beta is close to .95.

III. EXTENSIONS

1. Adverse Risk Incentives in Corporate Finance

The tendency to alter the risk of the managed portfolio is similar to the incentives that equityholders have to expropriate wealth from existing bondholders by increasing leverage, investing in riskier projects, and paying dividends. Green (1984) and others have suggested that convertible debt or warrants can be used to eliminate this incentive. This solution to the bondholder-stockholder incentive problem is equivalent to implementing a performance-fee contract where the fee increases at points greater than Q, but at a slope m′ less than m. This type of contract adds m′ call options on $\tilde{R}_p - \tilde{R}_s$ with striking prices of Q to the old fee. In this case, the adverse risk incentives are eliminated only when H is negative and larger in absolute value than Q + z, where z is some positive number that is increasing in m'. (The proof is available on request.)

By analogy, our solution suggests that the bondholder-stockholder conflict cannot be eliminated with any type of warrant or convertible debt issue. Only when the striking price of these option securities is sufficiently close to the current stock price will the incentive be eliminated. Moreover, the fewer the number of option securities issued, the closer the striking price must be to the current stock price.

Bondholders, aware of the adverse incentives of value-maximizing equity holders, also write covenants into their bonds that limit these forms of expropriation. Funds could similarly restrict the behavior of their performance-compensated portfolio managers. For instance, it is possible to contractually prohibit the manager from borrowing or holding short or long positions in a risk-free asset or equivalent positions in futures or option contracts. The manager could, however, still achieve higher fees by choosing to hold securities with either very high betas or very low betas. One could try to counter this behavior by rewarding a measure of performance that is adjusted for beta (e.g. the Jensen (1968) Measure), but there is a vast literature in finance on classes of securities that outperform beta-based benchmarks as traditionally computed. Moreover, in this case, the manager can gain by choosing stocks with large amounts of unsystematic risk. This could be even worse from the fund's perspective, since it might increase the fund's riskiness without increasing its expected return. It should also be noted that contractual limitations on the manager could be counterproductive if it limits his flexibility, and hence, his ability to achieve abnormal returns.

2. Unsystematic Risk and Rebalancing Strategies

In the last section, we showed that a manager who holds no unsystematic risk will maximize the value of the performance contract with a finite beta only if the maximum penalty under the contract (relative to holding the benchmark portfolio) exceeds the maximum allowed benefit (i.e. the cap minus the fee from holding the benchmark). The inclusion of unsystematic risk makes this problem difficult to analyze with the techniques used in the previous section. In particular, with the buy-and-hold strategy, the underlying assets in the fee options no longer have distributions that are easily analyzed with a Black-Scholes approach.

Despite this, most of the intuition developed for the simpler case should apply in more general settings. For instance, the main point of Propositions 1 and 2, that contracts with large caps or small penalties create incentives to adversely alter the risk of the managed portfolio, is very general.

We cannot demonstrate this, except for special cases,[7] unless the contract and the investment strategy are slightly modified, so that tractable solutions for the relevant option values can be derived in this more general setting. Modifying the investment strategy alone is not sufficient. For instance, if the values of the managed portfolio and the benchmark portfolio are generated by stationary lognormal diffusion processes, risk neutral valuation techniques can be used to value the two implicit call options with boundary conditions given by equation (4). In this case, the value of a call option on $\tilde{R}_p - \tilde{R}_s$ with a striking price of K is given by

$$(1/R_f)\int_{-\infty}^{\infty}\int_{y+K}^{\infty}(x - y - K)f(x,y)dxdy,$$

has used this technique to value options when the underlying asset's value is generated by more than one state variable. Unfortunately, the valuation formula obtained from the above expression is very complex and is unlikely to be sufficiently tractable to generate analytic conditions for the design of contracts without adverse risk incentives. For this reason, the contract must be slightly modified as well.

We will assume that the fee contract is of the form

$$F = B + m[\max(0, \log(\tilde{R}_p) - \log(\tilde{R}_s) - H) - \max(0, \log(\tilde{R}_p) - \log(\tilde{R}_s) - Q] \quad (8)$$

This contract is based on the difference between the instantaneous rates of return of the portfolio and the benchmark, rather than the cumulated returns. These returns are normally distributed, rather than lognormally distributed.[8] We also assume that the value of the managed portfolio is generated by a stationary geometric Brownian motion process. If asset returns are generated by such stationary processes, then this assumption implicitly assumes a rebalancing strategy for the managed portfolio rather than a buy and hold strategy.

The performance fee given by equation (8) can be hedged with dynamic positions in the benchmark portfolio and the assets in the managed portfolio if assumptions (1′) and (3′) hold. This is because the fee is a function only of the state variables that generate $\tilde{R}_p$ and $\tilde{R}_s$. As a consequence, the fee contract can be valued as a special case of the risk neutral valuation formula in Brennan (1979, eq.(39)) for call options on normally distributed variables. In this case, the current value of the underlying asset is set equal to zero, implying that the value of a call option with striking price K is

$$c(K) = (-K/R_f)N(-K/\sigma) + (1/R_f)\sigma n(-K/\sigma), \text{ where} \quad (9)$$

$n(x) = N'(x) =$ standard normal density function, and

$\sigma =$ annualized standard deviation of $\log(\tilde{R}_p) - \log(\tilde{R}_s)$.[9]

The relevant striking prices for the bonus and cap options are respectively H and Q. With this modification of the fee, we can confirm the intuition given above, as the following proposition demonstrates.

<u>Proposition 3</u>: If the fee is represented by equation (8), and if assumptions (1'), (2), (3), and (4') hold, then

(i) The instantaneous variance of the $\log(\tilde{R}_p/\tilde{R}_s)$, (or, equivalently, $\tilde{R}_p/\tilde{R}_s$), denoted σ^2, can be achieved with any combination of systematic and unsystematic variance that satisfies

$$\sigma^2 = (\beta - 1)^2\sigma_s^2 + \sigma_\epsilon^2,$$

where β is the regression coefficient that relates $\log(\tilde{R}_p)$ to $\log(\tilde{R}_s)$, σ_ϵ^2 is the variance of the residual from the regression, and σ_s^2 is $\text{var}(\tilde{R}_s)$.

(ii) If H is less than or equal to -Q, the present value of the fee is monotonically decreasing in the variance of $\tilde{R}_p/\tilde{R}_s$, and hence maximized when $\tilde{R}_p = \tilde{R}_s$ at a fee value of B - mH.

(iii) If H is greater than -Q, the present value of the fee is monotonically increasing in the variance of $\tilde{R}_p/\tilde{R}_s$.

Proof: See Appendix.

If the results in Proposition 3 apply to the fee represented by equation (4), then adverse risk incentives are eliminated only when portfolio managers face penalties for poor performance that exceed the rewards for good performance. This result is very similar to the result in Proposition 2 for the buy-and-hold strategy. In contrast to the results in Proposition

2, however, the results in Proposition 3 are symmetric about $\beta = 1$. This derives both from the symmetry of the distributions of the underlying assets and because constant variance rebalancing strategies are not subject to the truncation effect, (discussed after Proposition 1), although either would be sufficient to induce symmetry.

One might also note that the results in Proposition 3 are identical to the results that would be obtained with the fee represented by equation (4), assuming that $\tilde{R}_p$ and $\tilde{R}_s$ are normally distributed. One merely substitutes $\tilde{R}_p$ and $\tilde{R}_s$ for the logs of these values. This result is not realistic, however, because it allows the value of the managed portfolio to be negative.

Proposition 3 also suggests that the particular benchmark can affect the investment strategy of the manager. If the contract's maximum penalty exceeds its maximum reward, the manager desires to hold the benchmark with no unsystematic risk. Hence, a client should tailor the benchmark to suit his risk preferences. These results extend to the lognormal return non-log fee of Proposition 2 with the slight modification discussed in the proposition.

3. Buy-and-hold vs. Rebalancing vs. More Complex Strategies.

By altering the performance-based contract, we were able to analyze stationary portfolio strategies that are continually rebalanced. Unfortunately, this strategy cannot be directly compared to the buy-and-hold strategy within our framework. However, we conjecture that, given lognormally distributed returns, buy-and-hold strategies result in higher expected fees than rebalancing strategies. This is because the risk of the buy-and-hold strategy changes as the value of the benchmark changes in a manner that increases the value of these types of fee contracts.

For the fee contracts examined here, it is better to decrease the volatility of the difference between the portfolio return and the benchmark return when near the cap and increase this volatility when further from the cap. If the initial β of a buy-and-hold strategy exceeds one, increases in the value of the benchmark place the manager closer to the cap and decrease volatility. If the initial β is less than one, increases in the benchmark increase volatility and place the manager farther from the cap.

Strategies that more aggressively increase or decrease the beta of the managed portfolio can generate even higher expected performance fees than the buy-and-hold strategies. In fact, in a perfect market, dynamic strategies exist that earn the maximum fee, irrespective of the design of the contract. Consider, for instance, a simplified setting where the benchmark return can earn either 1% more than the risk-free rate (the good state) or 1% less than the risk-free rate (the bad state) in a given time interval, and assume that there are T of these time intervals in one evaluation period. If one earns the maximum fee, C, by beating the benchmark by 2%, a doubling strategy will almost certainly earn the maximum fee if T is sufficiently large. In the first time interval, a beta of 3 ($b = 2$) followed by a beta of one in all subsequent intervals will earn the maximum fee if the good state occurs in the first interval. If it does not occur, a beta of 5, followed by betas of one will earn the maximum fee if the good state occurs in the second interval. If this does not happen, select a beta of 9 in the third interval, etc. Only if the bad state occurs in each of the T intervals would such a strategy fail. The probability of this goes to zero as T becomes large, which would be the case if continuous trading was possible.

Obviously, transaction costs and reputational considerations may limit the extent to which these dynamic strategies can be used. Hence, it is unlikely that portfolio managers will earn the maximum fee with certainty. Nevertheless, the example illustrates that even the contract designs in Section 2 that deter gaming with buy-and-hold strategies will not entirely eliminate adverse risk incentives unless one monitors the investment strategy of the manager very closely.

Portfolio managers are aware of the incentives to alter their volatility in response to the current performance of their portfolios. A recent Wall Street Journal (Dorfman (1986)) article reported the recollections of a partner in an investment consulting firm. He recalled "a couple of managers who stated their objectives as investing in blue-chip stocks with market capitalizations over $500 million. But when the managers' performance numbers sagged, they began to 'stretch for performance by going for takeover candidates and high-flying over-the-counter stocks.' "

We also know that portfolio turnover happens to be greater towards the end of an evaluation period than at the beginning. Traditional explanations for this include tax considerations and a desire to avoid listing "losers" on year-end and quarterly reports to clients. Our incentive fee model adds another explanation to this list. Managers who have done well will rebalance their portfolios to reduce risk, while managers who have done poorly will rebalance their portfolios to increase risk.

Similar arguments may also explain the observation that pension funds generally sell a target firm's stock subsequent to a tender offer announcement. These announcements generally result in large increases in the stock price moving the portfolio manager closer to the cap on his fee. Since these stocks also become riskier, an expected fee-maximizing portfolio manager would like to sell the securities rather than wait for the outcome of the offer.

4. Multiperiod Reputation

The model described above simplifies the rewards and penalties faced by actual portfolio managers. Most importantly, it assumes a one period horizon for managers. In reality, these managers may be more concerned about the renewal of lucrative contracts and about their long term reputations than about the additional amounts they can earn by altering risk in the manner outlined above.

If the risk of the portfolio is easily observed by the pension fund officers, these reputation considerations may mitigate the adverse incentive effects discussed here. However, since the risk of an actively managed portfolio is difficult to measure, our simple model may capture incentive problems that are present in more complicated settings.[10] Moreover, multiperiod considerations may reinforce our arguments about adverse risk incentives.

Regardless of the type of compensation contract, there is a maximum amount that the portfolio manager can lose by performing very poorly. He cannot do worse than lose all of his present and future business. However, the upside potential associated with significantly outperforming the benchmark is considerable, particularly if the manager is a newcomer to the profession who is managing small amounts of money. This suggests that, even for a manager with a fixed-percentage (i.e. flat) fee contract, the long run payout may be a convex function of the managed portfolio return minus the benchmark return in the current period. To offset the increased adverse risk incentives created by this additional convexity in the payoff function, it may be necessary to increase the penalties or decrease the cap on the performance contract from the levels specified in Propositions 2 and 3.

On the other hand, an established manager with an illustrious track record may already be managing a great deal of money. For such a manager, the potential loss from performing poorly may outweigh the gains from performing well. In this case, reputational considerations may add concavity to the payoff function, and it may be possible to decrease the penalties or increase the cap from the levels specified in section 2 without increasing the incentives to adversely alter the risk of the managed portfolio. These results can be formally analyzed within the compound option framework developed by Geske (1979). However, because the solution is analytically intractable, and would require numerical simulations, it is beyond the scope of this paper.

The ability to cancel the manager's contract without notice also falls within this framework. Dismissal of the manager can be thought of as a penalty that occurs at low levels of performance. If the manager's current performance level makes the potential reward from future portfolio gains exceed the potential penalties from future losses of the portfolio, adverse risk incentives may still occur. However, when the portfolio's value is close to the dismissal point, the concavity of the fee function around that

IV. SUPERIOR INFORMATION

An analysis of performance-based fees is necessarily incomplete without a discussion of superior information, since the primary motivation for these fees is the desire to obtain superior performance. However, the analysis in the previous sections made no assumptions about the manager's abilities or information. This implies that the existence of superior abilities or information does not alter our conclusions.

To understand this, consider the case where one manager has better information about the mean of the benchmark return, but has the same estimate of the instantaneous volatility of the benchmark return as a less informed manager. His continuous-time hedge of the fee for a given portfolio strategy, and hence his valuation of the performance-based fee, would then be identical to that of the less informed manager. This implies the Fisher separation result, which is summarized by Proposition 4. The result extends to models with discrete trading as long as the performance-based fees can be hedged with traded European options.

<u>Proposition 4</u>: Denote a portfolio strategy by α and the present value of the fee from that portfolio strategy by $PV_I(F(\alpha))$ for an informed portfolio manager and by $PV_U(F(\alpha))$ for an uninformed portfolio manager. If Assumption (3) holds,

$$PV_I(F(\alpha)) = PV_U(F(\alpha))$$

Proof: If options that are perfectly correlated with the fee can be directly traded by the informed and uninformed portfolio managers in their personal accounts, each will trade them until the values they place on every option are identical to their market prices. Hence, the value of the options to the portfolio managers will be independent of the manager's information.

QED

One implication of Proposition 4 is that uninformed and informed portfolio managers who are restricted to buy-and-hold or constant rebalancing strategies will maximize their fees by choosing identical portfolio strategies. This is a strong result, in part because the continuous-time trading environments, assumed for Proposition 4, place restrictions on heterogeneous information. For instance, it is not possible to model superior information about volatilities in this context. Moreover, a commonly used model of superior information in incomplete markets, where an informed investor has a finer partition of the state space than uninformed investors, would lead to arbitrage opportunities here because markets are effectively complete for uninformed managers in our framework.

Although these strong implicit restrictions on superior information should be noted, there is still the possibility that the manager's ability to trade for his personal account may defeat the purpose of incentive fees. A manager with superior information could rationally use the information for his personal account, and manage the portfolio under contract as if he were uninformed. Measures should thus be taken to prevent hedging in the manager's personal portfolio. Covenants in the fee contract, (explicit or implicit), that effectively prohibit hedging in the manager's personal account can cause informed and uninformed managers to value the options differently and will induce managers to use any private information they have when trading for the portfolios they manage. In the extreme case where the portfolio manager is required to place all of his personal wealth in the risk-free asset, it is easy to construct examples with performance-based fee contracts where the more optimistic portfolio manager chooses a higher beta than the less optimistic manager. These cases require us to specify the manager's utility function and thus cannot be examined within the option pricing framework developed here.

V. CONCLUSION

This paper has demonstrated that improperly designed performance-based fee contracts provide incentives for the portfolio manager to game the contract at the expense of the fund's beneficiaries by altering the risk of the fund. To mitigate the adverse risk incentives associated with performance-based fees, contracts should be designed with caps and should have penalties for performance below the benchmark. The penalties for poor performance should be at least as severe as the rewards for good performance. For the buy-and-hold and rebalancing strategies analyzed in this paper, contracts with these properties induce portfolio managers to choose portfolios with appropriate levels of risk.

The analysis also indicates that it is impossible to design a performance-based contract that will deter gaming when the class of dynamic strategies are not limited. This suggests that performance-based contracts should include covenants that specify allowable portfolio strategies.

Finally, we argued that the assumptions that allow us to value the performance fees, in particular the assumption that the fee can be perfectly hedged, also imply that the contracts will not induce portfolio managers to use their private information to construct the managed portfolios. The informed portfolio manager will instead choose the identical portfolio as his uninformed counterparts, and use his information to trade on his personal account. In order to induce a manager to use superior information, he must be prevented from hedging the contract in his personal portfolio. Hence, performance-based contracts should also include covenants that restrict the holdings in the manager's personal portfolio.

APPENDIX

The following lemma, a well-known result from option pricing theory, (e.g. see Cox and Rubinstein (1975) pp.221 and 229), is used in the proof of Proposition 2.

Lemma 1: If $c(K)$ ($p(K)$ is the Black-Scholes value for a European call (put) with a striking price of K on a security that sells for one dollar and has instantaneous volatility of σ, then

$c'(K) = -(1/R_f)\ N(x - \sigma)$ and

$p'(K) = (1/R_f)\ (1 - N(x - \sigma))$, where

$N(y)$ = probability that a unit normal variable is less than y and

$x = -\ (1/\sigma)\log(K/R_f) + \sigma/2$.

PROOF OF PROPOSITION 1:

(i) In the absence of a cap, the bracketed term in (5), denoted F*, can be rewritten as

$F^* = \max(0,\ b(\tilde{R}_s - R_f) - H)$

This is a convex function of $b(\tilde{R}_s - R_f)$, with the function denoted as $F^*(b(\tilde{R}_s - R_f))$. In a risk neutral world, the argument of this function has a mean and a present value of zero for any b. Moreover, an increase in the absolute value of b induces a mean-preserving spread of $\tilde{R}_s - R_f$, in the sense of Rothschild and Stiglitz (1970). If, as we have assumed, options can be valued in relation to the underlying security as if we are in a risk-neutral world, then the value of the option is

$E[F^*(b(\tilde{R}_s - R_f))/R_f]$.

By Jensen's inequality, this value is an increasing function of the absolute value of b if F*( ) is a convex function.

(ii) When $b > 0$, but near zero, the bonus call option has a negative striking price, implying that $PV(F^*) = -H/R_f - b[c(R_f + Q/b)]$. The first term is unaffected by b, while the second term is the present value of the convex function of $\tilde{R}_s - R_f$ that is represented by

$$\max(0, b(\tilde{R}_s - R_f) - Q).$$

This present value is an increasing function of the absolute value of b by Jensen's inequality. The argument is identical to that for part (i) of the proposition.

When $b < 0$ but near zero, the cap put option has a negative striking price, implying that $PV(F^*) = -b[p(R_f + H/b)]$, which is an increasing function of the absolute value of b. The argument follows the reasoning in the previous paragraph.

QED

PROOF OF PROPOSITION 2:

(i) Using Lemma 1, take the partial derivative of the fee values in eqs. (7a,b) with respect to b. There are three cases to consider when $H > 0$. These cases are delineated by the signs of the striking prices of the bonus and cap options and by the sign of b.

CASE A: $b > 0$ and $b \geq -H/R_f$

$$dF^*/db = b[c'(R_f + H/b)(-H/b^2) - c'(R_f + Q/b)(-Q/b^2)]$$
$$+ c(R_f + H/b) - c(R_f + Q/b).$$

Using Lemma 1 and the Black-Scholes formula, this can be shown to equal

$$[N(x_1 - \sigma)H - N(x_2 - \sigma)Q]/(bR_f)$$
$$+ [N(x_1) - (1 + H/(bR_f))N(x_1 - \sigma)] - [N(x_2) - (1 + Q/(bR_f))N(x_2 - \sigma)]$$

$x_2 = -[\log(1 + Q/(bR_f))]/\sigma + \sigma/2,$

σ = standard deviation of the diffusion process generating $\tilde{R}_s$, and

$N(x)$ = probability that a standard normal distributed random variable will be less than x.

The bracketed terms in (A1) represent two areas under a normal density function--that between x_1 and $x_1 - \sigma$ and that between x_2 and $x_2 - \sigma$ respectively. The larger area is determined by whether x_1 or x_2 is closer to $\sigma/2$, since the standard normal density function is symmetric and monotonically decreasing in the absolute value of its argument.

Clearly, if H is nonnegative, x_1 is closer to $\sigma/2$, implying that the derivative is positive at all nonnegative values of b.

CASE B: $b < -Q/Rf$ and $b \leq -H/R_f$.

From the put-call parity theorem, it is easily verified that the derivative of PV(F*) with respect to b is the additive inverse of that in eq. (A1). Consequently, if $H > 0$, the derivative is negative and the fee maximizing b is at minus infinity.

CASE C: $-H/R_f < b < 0$

The derivative is zero at all values of b in this region.

Since these are the only cases with $H > 0$, the derivative is nonnegative in the absolute value of b.

(ii) and (iii): Take the derivative of eqs. (7a,b) with respect to b. Cases A and B in part (i) are also cases where H can be negative. Hence, for these cases, the derivative is represented by equation (A1) or its additive inverse. There are also two other regions where H can be negative, denoted as Cases D and E.

CASE A: If H is negative, then, since Q is positive, the relative by the relative sizes of

$$1/[1 + H/(bR_f)] \text{ and } 1 + Q/(bR_f). \tag{A2}$$

One derives a linear equation in b (β) when these two expressions are set equal to each other. This equation has the unique solution

$$b = -HQ/[R_f(H + Q)], \tag{A3}$$

which we henceforth denote as b*.

b* lies in the region for Case A if $H < 0$ and $H + Q > 0$. Assume these two conditions hold and note that for b slightly larger than $-H/R_f$, x_1 is much further than x_2 from $\sigma/2$. From the analysis in part (i) of the proposition, the derivative must be negative around this value. The linearity of the equation thus implies that the derivative of PV(F*) with respect to b is first negative and then positive. On the other hand, if $H < 0$ and $H + Q < 0$, the linear equation derived from (A2) has no solution for Case A and the derivative is always negative in this region.

CASE B: The closeness of x_1 and x_2 to $\sigma/2$ is determined by a comparison of the terms in (A2). The equation derived from comparing the terms in (A2) has no solution in the case B region if $H + Q > 0$. Note that for b slightly less than $-Q/R_f$, x_2 is much further from $\sigma/2$ than x_1, implying that the derivative is negative at this point. This implies that the derivative is negative for the entire region. On the other hand, if H + Q is negative, there is a unique solution to the equation derived from (A2), described by (A3). Thus, the derivative is positive for b's that are more negative than this solution and negative for b's that are less negative.

CASE D: $0 < b \leq -H/R_f$

Values of b in this region lie between the values of b for cases A and B. The derivative with respect to b can be shown to equal

$$-[N(x_2) - N(x_2 - \sigma)],$$

which is always negative if there is a finite cap.

CASE E: $-Q/R_f < b < 0$ and $H \leq 0$

Values of b in this region lie between the values of b for Cases A and B. The derivative is equal to

$$-[N(x_1) - N(x_1 - \sigma)],$$

which is always negative.

It is easily verified that PV(F*) is continuous in b for all of the cases and that there is no discontinuity at $b = 0$. Piecing together these cases implies that there is a solution to the equation derived from (A2) in the region for b defined by either Case A or B. Call the solution b*. If the solution for b* is in the Case A region, $(H + Q > 0)$, the derivative of PV(F*) with respect to b is negative at all values of $b < b^*$ and positive at all values of $b > b^*$. On the other hand, if b* lies in the Case B region, $(H + Q < 0)$, the derivative is positive for all values of $b < b^*$ and negative at all values of $b > b^*$. Thus, the value maximizing b is at b*.

QED

LEMMA 2: Let $c(K, \sigma)$ represents the Brennan (1979) value of a European call option on a self-financing security that is normally-distributed with an annualized standard deviation of sigma and a striking price of K. Then the partial derivative of the call option value with respect to volatility, σ, is

$$c_\sigma(K, \sigma) = n(H/\sigma)/R_f,$$

where n( ) is the standard normal density function.

Proof: Follows immediately from a partial differentiation of eq. (9) with respect to σ and the identity $n'(x) = -x\, N'(x)$.

QED

PROOF OF PROPOSITION 3:

(i) A regression of $\log(\tilde{R}_p)$ on $\log(\tilde{R}_s)$ yields the equation

$\log(\tilde{R}_p) = \alpha + \beta\log(\tilde{R}_s) + \tilde{\epsilon}$, which implies

$\log(\tilde{R}_p) - \log(\tilde{R}_s) = \alpha + (\beta - 1)\log(\tilde{R}_s) + \tilde{\epsilon}$ and

$$\begin{aligned} \mathrm{var}(\log(\tilde{R}_p) - \log(\tilde{R}_s)) &= (\beta - 1)^2\mathrm{var}(\log(\tilde{R}_s)) + \mathrm{var}(\tilde{\epsilon}) \\ &= (\beta - 1)^2\mathrm{var}(\tilde{R}_s) + \mathrm{var}(\tilde{\epsilon}) \end{aligned}$$

by Ito's lemma.

(ii), (iii) Using Lemma 2, immediately above, partially differentiate the present value of the payoff in the bracketed portion of eq. (8) with respect to σ. This yields

$$[n(H/\sigma) - n(Q/\sigma)]/R_f,$$

which is positive if and only if the absolute value of Q exceeds the absolute value of H. Since H is smaller than Q, this can only occur if Q exceeds -H.

QED

1.A recent article in Institutional Investor (Hawthorne (1986)) quotes Roger Bransford, managing director of a pension fund consulting firm as saying, "I wish people would put these (ideas about performance-based fees) out to be tested by academia...". The contents of a recent issue of the Financial Analysts Journal, (Jan./Feb. 1987), which was almost entirely devoted to this topic, also provides evidence of its importance and timeliness.

2.Agency papers by Heckerman (1975) and Bhattacharya and Pfleiderer (1985) study contract design using expected utility theory.

3.Exchange options were first valued by Margrabe (1978). His formula cannot be directly used to study the more general fees of the next section because the values of options that comprise these fees depend on the values of three assets: the managed portfolio, the benchmark portfolio, and a riskless cash payout.

4.For simplicity, we ignore other performance-based contracts that exist, particularly those with multiple hurdles at which different managerial award fractions m become relevant. Until the cap is reached, most of these contracts have the convex reward structure characterized by the contract in equation (4). That is, at the margin, the manager keeps the same or a greater fraction of his portfolio return as the return increases. The essential economic features of these contracts are identical to those for the contract described by equation (4). Extending our results to these contracts is a simple technical exercise.

5.If the portfolio beta is not between zero and one, this assumption may be inconsistent with equilibrium. This is because the possibility of bankruptcy precludes borrowing at a risk-free rate. However, most of the buy-and-hold strategies that will be optimal when contracts are properly designed have betas between zero and one. Hence, our results are affected by this inconsistency only for contracts with very extreme and unlikely parameter values.

6.It is interesting to note that β^* is independent of the volatility of the benchmark return. This also indicates that β^* is independent of the length of time in an evaluation period, except for the effect of time on the discount factor $R_f = \exp(-r_f t)$, where r_f is the instantaneous rate of discount and t is the length of the evaluation period.

With shorter time intervals for evaluation, one would expect the cap and the penalties to be smaller and this would have an effect on β^*. For instance, if H and Q are proportional to t, β^* approaches one as the evaluation period becomes arbitrarily small.

7.For instance, unsystematic risk makes contracts without caps more valuable, conditional on a given β. The argument that proves this is identical to that used to prove part (i) of Proposition 1.

8.For values of $\tilde{R}_p$ and $\tilde{R}_s$ that are close to one or close to each other, $\log(\tilde{R}_p) - \log(\tilde{R}_s)$ is approximately equal to $\tilde{R}_p - \tilde{R}_s$.

9.By Ito's Lemma, this standard deviation is the standard deviation of

10.The pension fund officer expects the returns of an actively managed portfolio to differ from the return of the benchmark. For such an actively managed portfolio, volatility is measured imprecisely, except in very long time series. For this and other theoretical reasons, it would be impossible to determine on the basis of a few observed ex-post returns whether a difference between the two returns is due to a deliberate attempt to game the contract or to active management based on superior investment talent.

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