ENDOGENOUS GOVERNMENT SPENDING AND RICARDIAN EQUIVALENCE

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Abstract

Many analyses of debt policy assume exogenous government expenditures. Instead, we use an optimizing model in which the government selects values of taxes, spending, and debt to maximize welfare. If demand for publicly provided goods is elastic, a debt-financed tax cut increases consumption, because individuals rationally expect some reduced government spending in future. Even though future taxes rise, they do not offset the expansionary effect of the current tax cut on consumption. Depending on preferences, the marginal propensity to consume out of tax cuts can take any value between zero and the marginal propensity out of ordinary income.

1. Introduction

The main implication of the Ricardian theory of government debt policy (Barro (1974)) is the result that consumers do not spend more when they receive a deficit-financed tax cut. Instead, rational forward-looking individuals save the additional disposable income, anticipating that the increased government debt is financed by future tax increases. Holding government spending fixed, the current tax cut is exactly offset in present value terms by higher taxes in the future. Permanent disposable income and therefore consumption remain unchanged.

We show that assumptions about government spending are crucial in this argument. The presence of endogenous real government activity makes the Ricardian theory of debt policy inapplicable except under extremely restrictive conditions.

The voluminous literature following Barro's (1974) seminal contribution has explored how his basic line of argument can be adapted to more complicated environments with capital market imperfections, different bequest motives, uncertainty about the incidence of future taxes, or distortionary taxation (see, e.g., Abel (1986), Barro (1979), (1981A), Barsky, Mankiw, Zeldes (1986), Carmichael (1982), Chan (1983), Gilles and Lawrence (1984), Judd (1987), and the recent survey by Aschauer (1988)). Although tax cuts may increase or even decrease consumption in some of these richer environments, it seems difficult to disprove the claim that "overall, the Ricardian theorem stands up theoretically as a plausible first-order proposition" (Barro (1981A), p. 228).

Surprisingly, this entire literature has adopted Barro's focus on the government's financial policy (debt and taxes). Government spending or other real activities are treated as exogenous or are simply excluded from the model.[1] If government spending is endogenous, however, significant deviations from Ricardian neutrality are possible: We show that the marginal propensity to consume out of a tax cut can take any value between zero and the marginal propensity to consume out of ordinary income.

To interpret some change in the economy as caused by a tax cut, one necessarily has to hold constant all other factors that could cause the same effect, i.e., invoke the "ceteris paribus" clause. This means that current government spending must be held constant to identify effects of a tax cut on consumption. This is a logical necessity, not a restriction. The strong assumption in Ricardian theory is all future government spending is held constant in analyzing the effect of tax changes.[2]

To evaluate this assumption, the key question is the following. Suppose consumers receive a tax cut today and see the corresponding increase in government debt. What are their expectations about future government activity? Can they rationally believe that the debt is served and/or repaid entirely out of future tax increases (relative to the originally expected levels) or should they expect that government spending is reduced later?

The answer depends critically on the model of government behavior (see Sargent (1984), Sims (1986)). If one postulates that the government follows (and can commit to) arbitrary rules of behavior, Ricardian neutrality can be "saved" in a trivial way: Just let the rule specify the exogenous path of government spending. However, when one listens to current political debates, it seems that tax rates, federal expenditures, and the deficit are jointly determined in a political process that works in a discretionary way.[3]

To formalize this process, we adopt Barro's (1979) assumption that the government maximizes social welfare. Taxes, government spending, and debt are determined endogenously as the outcome of the government's optimization problem. In contrast to Barro (1979), who excludes real government activity, we assume that government spending is also a choice variable.

To obtain a unique optimal policy, we follow Barro (1979, 1986) in assuming distortionary taxation. In reality, taxation is clearly distortionary (at least on the margin). The appeal of Ricardian neutrality as an empirically relevant result stems from the fact that it seems to be robust to small changes in assumptions. Indeed, if government spending is exogenous, e.g., because of a spending rule, small distortions do not significantly affect Ricardian neutrality (as we will confirm). But this is not true, if government spending is endogenous.[4]

Our welfare-maximizing model of government dramatically alters the way in which debt policy affects private behavior. Consumers know that the initial level of debt is a constraint on government activity. A current debt-financed tax cut implies higher payments on interest and/or principal on government debt. Unless future government spending is reduced, taxes will have to be increased. But higher taxes increase the excess burden of taxes on the margin. Since the marginal excess burden indicates relative social cost of private and public spending, tax increases make government spending more expensive relative to private goods. Therefore, any optimal policy that plans an increase in taxes must also include a cut in spending. In particular, the Ricardian argument that lower taxes and higher deficits today are exactly offset by higher future taxes does not hold. Instead, an increase in government debt does not only signal higher taxes, but lower future spending as well. A current tax cut is never a purely financial policy change--Ricardian theory cannot be applied.

How far the results differ from the predictions of a model with an exogenous spending rule depends on the assumptions on preferences over privately and publicly provided goods and services. We find that the effect of debt financed tax reductions on consumption is large if the price elasticity of government expenditures is high. Only in the extreme case that government expenditures are spent on goods that cannot be substituted for private goods, does a neutrality result emerge. At the other extreme, when demand for public goods becomes infinitely elastic, individuals may rationally ignore effects of debt on future taxes. In planning consumption, they may treat disposable income from tax cuts like any other income.

Two remarks on modeling are appropriate. First, notice that the economic intuition for non-neutrality is more general than our welfare-maximizing model. Even if a government has many other objectives, the argument for reduced government spending in response to higher initial debt holds as long as welfare is one component of the government's objective function.[5] We concentrate on strict welfare-maximization to sidestep questions of what "other" objectives the government may have and how they may be motivated rigorously.

Second, it is technically complicated to analyze tax policy in a model of optimizing policy. To observe the experiment of an equal change in taxes and debt, one must assume some unexpected shocks to the economy that motivate such a policy shift. But these shocks must not affect any other variables, like government spending or private income, that could independently affect consumption. We have to introduce at least as many independent sources of noise as we have choice variables. Even in a basic model with only consumption, government spending, and taxes, we need three different random shocks to preferences and technology. Thus, to prove rigorously that high debt implies lower future government spending, we need a relatively complicated model (even if we defer discussion of many technical issues to an appendix).

In Section 2, we develop a general equilibrium model with a representative, infinitely lived consumer in which changes in taxes, government spending, government debt, private consumption, and capital accumulation are driven by shocks to preferences and technology. Optimal government policy is determined in Section 3 and interpreted in Section 4. We show that individuals generally increase consumption in states of nature in which the government reduces taxes. The deviations from the Ricardian result can be substantial: Everything between the Ricardian response and "naive" disregard for future taxes may be optimal consumer behavior. In Section 5, we show in a slightly simplified model that the rule of minimizing excess burden of Barro (1979) arises as a special case when the demand for public goods is completely inelastic. The conclusions are reviewed in Section 6; technical results are given in an appendix.

2. The Model

We will construct a model that allows us to study government activity and its implications for rational private behavior. To make the setup as favorable as possible for Ricardian analysis, we omit all complications that may lead to non-Ricardian results, except that we abandon the assumption of exogenous government spending. That is, if government spending were exogenous, we would return to models that replicate Barro (1974) and Barro (1979).

The economy consists of identical, infinitely lived individuals and a government. Each individual has preferences over two goods c_t and g_t. In period t, individuals want to maximize the utility function

$$E_t\left[\sum_{i=0}^{\infty} \delta^i \cdot u(c_{t+i}, g_{t+i}, \gamma_{t+i})\right], \tag{1}$$

where $u(\cdot)$ is concave in c_t and g_t, γ_t is a random shock to preferences, and $0 < \delta < 1$ is the rate of time preference. We assume that individuals buy the good c_t privately, but that the good g_t is provided by the government in equal quantity to all individuals.[6] Individuals take government decisions about taxation and spending as given. They have initial wealth and earn after tax income y_t (both defined below) to pay for the private good and to invest in capital K_t and government bonds D_t.

Each unit of capital K_t yields $R > 1$ units of goods (public or private) in period $t + 1$. To simplify, we assume that the interest rate R is constant and that the government can commit itself not to tax capital (to eliminate time consistency issues). Then government bonds and capital are perfect substitutes for individuals. The government must pay interest R on debt D_t. Analogous relations hold for all other periods. Initial private wealth is $R \cdot (K_{t-1} + D_{t-1})$ and the individual budget constraint in period t is

$$c_t + K_t + D_t = R \cdot (K_{t-1} + D_{t-1}) + y_t \tag{2}$$

The government uses taxation and debt financing to pay for the publicly provided good g_t. Without loss of generality, we can define units so that c_t and g_t have identical production cost per capita and we can normalize prices to one. Given initial government debt $R \cdot D_{t-1}$, the budget constraint in period t is then[7]

$$g_t + R \cdot D_{t-1} = T_t + D_t , \tag{3}$$

where T_t denotes tax revenues.

It is important that taxes may not be lump-sum. Our assumptions are motivated by the notion that taxes create welfare losses because they distort relative prices. The following structure of production generates allocative distortions of the type assumed in Barro (1979) and (1986) and leads to a relatively simple function for excess burden.[8] Suppose individuals are endowed with Y_t units of raw products (ℓ) that they inelastically supply to two production processes, denoted by $\ell_{1t} + \ell_{2t} = Y_t$. We assume that Y_t is exogenously given and equal to $Y_t = 1 + \psi_t$, where ψ_t is a random shock. Process #1 generates one unit of output per unit input, $y_{1t} = \ell_{1t}$. Process #2 generates $y_{2t} = F(\ell_{2t}, \varepsilon_t)$ units of output, where ε_t is another stochastic shock and $0 \leq \partial F/\partial \ell_{2t} = F_\ell \leq 1$.

The idea is that process #2 is less efficient than process #1, but that the government cannot observe how much a certain individual uses process #2, i.e., that it cannot tax it. For example, process #1 may be regular work while #2 is work at home. With some modifications, ℓ_{1t} and ℓ_{2t} may also be interpreted as work and leisure; this alternative story is described in Appendix 1.

Taxation is generally distortionary, because it can only be levied on process #1 and therefore induces individuals to shift resources to the less efficient process #2. Lump-sum taxes are obtained in the special case when $F_\ell(0, \varepsilon_t)$ converges to zero. Assuming decreasing returns to scale in process #2, higher tax rates cause increasing absolute and marginal distortions. We can define tax revenue as $T_t = \ell_{1t} \cdot \tau_t$ and after tax income as $y_t = y_{1t} + y_{2t} - T_t = (1 - \tau_t) \cdot \ell_{1t} + F(Y_t - \ell_{1t}, \varepsilon_t)$.

The first-order condition for individually optimal supply of ℓ_{1t} is $1 - \tau_t = F_\ell(1 + \psi_t - \ell_{1t}, \varepsilon_t)$, which implies optimal values of inputs ℓ_{1t}, income y_t, and tax revenue T_t as functions of $(\tau_t, \varepsilon_t, \psi_t)$. If we invert the function for tax revenue to obtain the tax rate as a function of the desired level of revenue, we obtain income as a function of current shocks and government revenue requirements. Intuitively, higher desired revenue increases the tax rate, which leads to increased distortions and lower income. As a benchmark, income without distortions is equal to $Y_t - T_t$. Therefore, we will write actual income as

$$y_t = y(T_t, \psi_t, \varepsilon_t) = Y_t - T_t - h(T_t, \psi_t, \varepsilon_t) \;, \tag{4}$$

where $h(\cdot)$ can be interpreted as total excess burden, a function that indicates the loss in income because of distortionary taxes. Under some auxiliary assumptions, which are described in Appendix 1, it has partial derivatives $h_T \geq 0$, $h_{TT} \geq 0$, and $h_{T\varepsilon} \geq 0$.

The role of the three types of stochastic shocks, γ_t, ψ_t and ε_t, is to generate interesting movements in macroeconomic variables that can be interpreted as unexpected changes in government policy. We assume that the shocks are independent and white noise. The shock ψ_t increases income and therefore aggregate economic activity. The shock ε_t affects the relative efficiency of the two production processes and therefore tax revenue. Assuming $F_{\ell\varepsilon} > 0$, a positive realization increases private opportunities to escape taxes and increases the government's cost of raising revenue. It will be the main motivation for changes in debt and taxes. The shock γ_t increases preferences for the publicly provided good. We assume that it does not increase marginal utility for the private good directly, i.e., $u_{g\gamma} > 0$ and $u_{c\gamma} = 0$.

Over time, macroeconomic variables are realized as a result of a game between individuals and the government. Each period, individuals determine current consumption, capital, and bond purchases to maximize utility (1) subject to the budget constraint (2). Each individual takes initial wealth and current and past actions of the government and other individuals as given and forms rational expectations about all relevant future variables. The government determines current government spending, debt, and the tax rate to maximize welfare, i.e. the same utility function (1), subject to constraint (3). It takes initial debt and private behavior as given and also forms rational expectations.

3. Optimal Government Policy

We want to characterize the behavior of the government and of consumers in this setting. The analysis of the game is made tractable by the specific structure of production and by the assumption that the government is welfare maximizing. The individual decision to allocate inputs between processes #1 and #2 depends only on endowments Y_t, the shock ε_t, and the tax rate τ_t. It can be determined independently of all other decisions. The resulting welfare loss is completely summarized by the function $h(T_t, \varepsilon_t, \psi_t)$. Since the private incentive to minimize taxes is the only aspect of the game where individual and social objectives differ, the equilibrium allocation can be obtained by solving a social planner's problem in which the government is allowed to choose consumption but takes $h(T_t, \varepsilon_t, \psi_t)$ as given. That is, if we allow the government to choose consumption, it will select exactly the path of consumption that a rational consumer would also choose, given the same constraints.[9]

The social planner's problem of finding equilibrium consumption, government spending, taxes, capital, and debt can be solved by dynamic programming. State variables are initial capital and debt and the random shocks. Let

$$V(K_t, D_t) = \max E_{t+1}\left[\sum_{i=1}^{\infty} \delta^i \cdot u(c_{t+i}, g_{t+i}, \gamma_{t+i})\right] \tag{5}$$

be the value function. It indicates the maximum welfare from period t + 1 on, where the maximum is over choices of consumption, government spending and taxes subject to budget constraints (2) and (3) and the condition that the function for output (4) is satisfied in each period. The problem in period t is to solve

$$V(K_{t-1}, D_{t-1}) = \max u(c_t, g_t, \gamma_t) + \delta \cdot E_t V(K_t, D_t) \quad (6)$$

subject to[10]

$$K_t = R \cdot K_{t-1} + Y_t - c_t - g_t - h(T_t, \psi_t, \varepsilon_t) \quad (7)$$

$$D_t = g_t + R \cdot D_{t-1} - T_t \,. \quad (8)$$

The first order conditions for optimal consumption, government spending and taxation are

$$u_c(c_t, g_t, \gamma_t) - \delta \cdot E_t V_K(D_t, K_t) = 0 \,, \quad (9)$$

$$u_g(c_t, g_t, \gamma_t) + \delta \cdot E_t\left[V_D(D_t, K_t) - V_K(D_t, K_t)\right] = 0 \,, \quad (10)$$

$$\delta \cdot E_t\left[-h_T(T_t, \psi_t, \varepsilon_t) \cdot V_K(D_t, K_t) - V_D(D_t, K_t)\right] = 0 \,. \quad (11)$$

The solution to this problem is a mapping Γ from the state variables (K_{t-1}, D_{t-1}, ψ_t, ε_t, γ_t) to optimal values (c_t, K_t, g_t, T_t, D_t) = Γ (K_{t-1}, D_{t-1}, ψ_t, ε_t, γ_t). We make two simplifying assumptions to characterize the paths of the endogenous variables more explicitly.

First, the focus of the paper is on the link between private and public goods through taxes. Therefore we want to assure that direct substitution or complementarity in utility plays a secondary role. The intuition is brought out most easily under the assumption of separable utility ($u_{cg} = 0$). If $u_{cg} \neq 0$, equilibrium paths of c_t and g_t must be adjusted for the substitution effect. In the verbal interpretation we will ignore this effect; but all results hold and are proven under more general conditions stated in Appendix 2 that allow some substitution.

Second, the system is highly nonlinear so that random shocks may have complicated effects through expectations and risk attitudes. The only function of random shocks is to provide a motivation for some noise in the macroeconomy. We will therefore assume that the distribution of shocks is sufficiently tight, whenever ambiguities would arise otherwise.[11]

We will first study the mapping under "normal" conditions and then turn to degenerate cases. In the non-degenerate case, we assume that taxes cause strictly positive distortions that strictly increase with the tax rate and that marginal utilities for the privately purchased good c_t and the publicly provided good g_t are positive and finite for positive values of c_t and g_t. Formally, $h_T > 0$, $h_{TT} > 0$, $h_{T\varepsilon} > 0$, $u_c > 0$, $u_g > 0$, $u_{cc} < 0$, and $u_{gg} < 0$.

We assume that a solution Γ exists and that the value function is strictly concave at the solution.[12] Then the solution determines unique values of the endogenous variables, which are characterized by the first order conditions. The derivatives of the endogenous variables with respect to the state variables have signs as indicated in Table 1 (see Appendix 2 for the derivation and details).

Intuitively, more initial capital means that the government can increase taxes and spending somewhat and still leave a larger disposable income to individuals, which they use to increase both consumption and new capital. Higher initial debt implies that the government has to issue more new debt and increase taxes. In addition, government spending is more costly and is, therefore, reduced.

The shocks also have economically sensible effects. A positive value of the supply shock ψ_t has a similar effect as high capital K_{t-1}. Unexpectedly high preferences for current public goods, a positive value of γ_t, imply an increase in government spending and a reduction in private spending. High cost of current taxation, a positive value of ε_t, causes a reduction in current taxes and shifts taxation into the future through higher debt.

4. Interpretation

Our main question is how individuals will react to fluctuations in taxes and debt. In particular, will they increase consumption, if an unexpected debt-financed tax cut occurs?

Unexpected changes in taxes occur whenever shocks have nonzero realizations. Notice, however, that shocks typically change all variables, including government spending, so that innovations in financial policy should be observed very rarely. While this may be taken as support of Buchanan's (1976) doubts that pure financial changes will ever be observed, it is still useful to decompose realizations of shocks into a component that leaves taxes T_t unchanged and "other" changes. The ratio of the change in consumption to the change in taxes may then be interpreted as a marginal propensity to consume. Or in an econometric interpretation, one may ask what coefficient an empirical researcher would find in a regression of consumption on taxes and other variables.

Define innovations in endogenous variables by

$$(\Delta c_t, \Delta K_t, \Delta g_t, \Delta T_t, \Delta D_t) = \Gamma(K_{t-1}, D_{t-1}, \psi_t, \varepsilon_t, \gamma_t) - \Gamma(K_{t-1}, D_{t-1}, 0, 0, 0) ,$$

where Γ is the mapping from predetermined to endogenous variables that was characterized in Table 1.

We define a debt-financed tax cut as a realization of $(\varepsilon_t, \gamma_t)$ that just leaves spending unchanged, $\Delta g_t = 0$, and reduces taxes, $\Delta T_t < 0$. Notice that the change increases debt, $\Delta D_t > 0$. Typically, the tax cut will be a positive realization of ε_t which increases the marginal welfare cost of current taxation combined with a change in γ_t that offsets the side-effect of ε_t on government spending. As our first main result, we can show that any such realization of shocks increases consumption for a given level of endowments ($\psi_t = 0$ w.l.o.g.).

Theorem 1:

Consider all realizations of $(K_{t-1}, D_{t-1}, \psi_t, \varepsilon_t, \gamma_t)$ that satisfy $\Delta g_t = 0$ and $\psi_t = 0$. Then $\Delta c_t > 0$ if and only if $\Delta D_t = -\Delta T_t > 0$. In this sense, every debt-financed tax cut increases consumption.

Proof: See Appendix 3.1.

An unexpected tax cut increases debt, which implies that the government cannot maintain future spending and taxes at the level previously planned. The formal analysis (see Table 1) shows that taxes will rise in subsequent periods.[13] This is consistent with Ricardian analysis. But higher debt also reduces government spending in future periods. Each period, the marginal tradeoff between privately and publicly purchased goods is given by

$$\frac{u_g(c_t, g_t, \gamma_t)}{u_c(c_t, g_t, \gamma_t)} = 1 + h_T(T_t, \psi_t, \varepsilon_t)$$

which is implied by the first order conditions (9) - (11). The relative price of publicly provided goods is unity (production cost) plus the marginal cost of distortions h_T. Since higher debt raises future tax rates, it raises h_T. Therefore the marginal cost of the good g increases and there is substitution from good g to the private good c. Future taxes increase by less than the amount that would offset the effect of the current tax cut. Rational individuals will increase consumption immediately.

Suppose we define the marginal propensity to consume out of a tax cut as the ratio

$$MPC_T = -1/(1 + h_T) \cdot \Delta c_t/\Delta T_t \text{ ,}$$

where the changes in c_t and T_t are the results of a pure debt-financed tax cut. Then our result means that the marginal propensity to consume out of a tax cut is unambiguously positive. The reason for defining MPC_T this way is that $-(1 + h_T) \cdot \Delta T_t$ is the change in disposable income caused by the tax cut.

In the econometric interpretation, the theorem says that a regression of consumption on taxes (or debt), endowments, government spending, initial capital, and initial government debt will yield a negative coefficient on taxes (or a positive one on debt).

A critical reader may have noticed that the positive value of ε_t that caused the tax cut might directly increase disposable income y_t--namely, if $h_\varepsilon < 0$. To demonstrate that such an income effect is not responsible for the change in consumption, we will strengthen the result slightly. Let T^* be the tax revenue if all shocks are zero. Then shocks have no direct income effect, if

$$\psi_t = h(T^*, \psi_t, \varepsilon_t) - h(T^*, 0, 0) \tag{12}$$

Define a "pure" debt-financed tax cut as a realization of the shocks $(\psi_t, \varepsilon_t, \gamma_t)$ that, in addition to leaving government spending unchanged and reducing taxes, satisfies (12). Typically, this means that we consider a tax cut that is accompanied by a slight fall in endowments.[14] Still, the same result is obtained:

Theorem 2:

Consider all realizations of $(K_{t-1}, D_{t-1}, \psi_t, \varepsilon_t, \gamma_t)$ that satisfy $\Delta g_t = 0$ and (12). Then $\Delta c_t > 0$ if and only if $\Delta D_t = -\Delta T_t > 0$. In this sense, every pure debt-financed tax cut increases consumption.

Proof: See Appendix 3.2.

Again, the marginal propensity to consume out of a tax cut is positive. In the regression interpretation, endowment must be replaced as regressor by an "adjusted endowment" defined as $Y_t - h_\varepsilon/(1 - h_\psi) \cdot \varepsilon_t$. Compared to Theorem 1, not much is changed, provided h_ε is small.[15]

Next, we would like to relate our results to the literature and show that the deviations from Ricardian neutrality are economically significant. The Ricardian theory implies that individuals should largely ignore the tax cut, because taxes will be increased later to repay the government debt. On the other hand, there is the "naive" hypothesis that individuals disregard the level of debt in determining their consumption, which was widely accepted before Barro's (1974) contribution.

Let us define the marginal propensity to consume out of income as $MPC_y = \Delta c_t/\Delta y_t$, where the innovations in consumption and income are the results of a realization of shocks that leaves fiscal policy unchanged, $\Delta T = \Delta g = 0$.[16] It can be interpreted as coefficient on "adjusted endowments" in the regression of consumption on taxes, "adjusted endowments," government spending, initial capital, and initial government debt. Also define $\theta = MPC_T/MPC_y$. The ratio θ indicates what fraction of the tax cut is considered as increase in funds available for consumption. That is, an increase in disposable income by $1/MPC_y$, which would increase consumption by 1 unit if the change were due to higher endowments, increases consumption by θ units if the income comes from tax cuts. Ricardian neutrality is equivalent to $\theta = 0$; the "naive" hypothesis predicts $\theta = 1$.[17] We obtain

Theorem 3:

a. $0 < \theta < 1$, or equivalently, $0 < MPC_T < MPC_y$.

b. As $u_{gg} \to 0$, $\theta \to 1$.

PROOF: See Appendix 3.3. Recall that we generally assume $0 > u_{gg} > -\infty$ and $h_{TT} > 0$; this is critical for (a).

The theorem confirms that both marginal propensities to consume are positive. A tax cut has a smaller effect on consumption than an increase in endowment income. The reason is that some increase in future taxes is expected. But if the demand for the good g is very sensitive to cost (u_{gg} small in absolute value), the higher debt will not lead to future tax increases but rather to future reductions in spending. If this case characterizes preferences, the result is far from Ricardian neutrality. Individuals will spend as much income from tax cuts as they spend from other disposable income. But this behavior should not be interpreted as a contradiction of the Ricardian theorem. It is simply the rational behavior in an environment in which current tax cuts have no relation to future changes in taxes.

Results close to Ricardian neutrality can be obtained in two other limiting cases. The two cases correspond to two versions of Ricardian theory which are most clearly expressed in Barro (1974) and Barro (1979), respectively.

According to Barro (1974), debt policy is completely irrelevant, if taxes are lump-sum. This is also true in our model. If taxes were non-distortionary ($h(\cdot) = 0$), the paths of government debt and taxes would be irrelevant. Also, D_{t-1} could be omitted as state variable and D_t and T_t would not be uniquely determined by the mapping Γ (as one can verify from the equations in Appendix 2).

According to Barro (1979), distortionary taxes uniquely determine debt policy, but tax changes do not have a first-order effect on consumption. In our setting, this corresponds to a case in which preferences are such that government spending is very inelastic at some exogenous value g_t^*, i.e.,

$$u(c_t, g_t, \gamma_t) = \tilde{u}(c_t) , \qquad \text{if } g_t \geq g_t^*$$
$$= -\infty \qquad , \qquad \text{if } g_t < g_t^*$$

where g_t^* is an exogenous sequence and $\tilde{u}(\cdot)$ is concave. Alternatively, this case can be obtained as limit $u_{gg}(c_t, g_t^*, \gamma_t) \rightarrow -\infty$.

If taxes are distortionary ($h_T > 0$), the level of debt matters even with exogenous spending. Each dollar in interest on the government debt paid to consumers must be financed by one dollar in taxes received from consumers. This has offsetting welfare effects. In addition, taxes induce individuals to shift to inefficient production, which creates a welfare loss of h_T per unit of tax revenue (on the margin). Since marginal losses increase in tax rates, the optimal policy will have a relatively smooth path of taxes, as in Barro (1979).[18]

Unexpected debt-financed tax cuts can still occur, if a shock ε_t temporarily increases the marginal cost of taxation. Then marginal tax rates become more uneven over time but the net present value of tax revenue remains unchanged. One should therefore suspect that $\Delta c_t \cong 0$. Indeed, the proofs for the strict inequality $\Delta c_t > 0$ in Theorems 1 and 2 do not hold for the limiting case of infinite u_{gg}. Unfortunately, the uncertainty of future shocks makes it impossible to obtain precise results. But we can obtain the Ricardian neutrality result in this limiting case, if we simplify the stochastic structure slightly. This is done in the next section.

5. A Comparison of Marginal Propensities to Consume

Uncertainty about shocks in future periods has complicated the analysis considerably. So far, we assumed that shocks follow stationary stochastic processes. This seems a natural assumption. But if shocks are small as we assumed already, optimal decisions should not be affected much by uncertainty. In addition, the stochastic shocks are only needed to provide a motivation for unexpected policy changes in the current period. Therefore, we will make a simplification: Shocks only occur in period t, the period under consideration. Also, we assume that the rate of time discount equals the interest rate, $\delta R = 1$, and define $R - 1 = r$.

Under these assumptions, the future path of the economy for any realization of current state variables can be computed more easily and a direct comparison to Barro (1979) is possible. We can explicitly compute the marginal propensities to consume.

Theorem 4:

Let $A = -u_{gg} + 2(1 + h_T)u_{cg} - (1 + h_T)^2 u_{cc} + h_{TT}u_c$, $|u| = u_{cc}u_{gg} - u_{cg}^2$, and $B = -u_{cc} \cdot A + r \cdot h_{TT}u_c \cdot (-u_{cc}) + r \cdot |u|$. Then

$$MPC_y = r \cdot (|u| - u_{cc} \cdot h_{TT}u_c)/B > 0,$$

$$MPC_T = r \cdot h_{TT}u_c \cdot \left(u_{cg} - (1 + h_T) \cdot u_{cc}\right)/\left(B \cdot (1 + h_T)\right) \geq 0, \quad \text{and}$$

$$\theta = \frac{h_{TT}u_c\left(u_{cg} - (1 + h_T) \cdot u_{cc}\right)}{\left(h_{TT}u_c(-u_{cc}) + |u|\right) \cdot (1 + h_T)} .$$

Proof: See Appendix 3.4.

Under "normal" conditions (i.e., u_{gg} finite and $h_{TT} > 0$) individuals respond less to changes in income caused by tax policy than to other changes, i.e., $0 < \theta < 1$. The difference is primarily due to the fact that debt increases future taxes. In addition, if $u_{cg} < 0$, a tax cut reduces current consumption relative to future consumption, because individuals want to compensate for the expected low future government spending. This reduces the marginal propensity to consume out of tax cuts further.[19]

Another feature of optimal consumption is interesting: In a permanent income model with exogenous government spending, one would have $MPC_y = r/(1 + r)$. Here, we have $MPC_y = r/[1 + r + (u_{cg} - (1 - h_T)u_{cc})^2/(|u| - u_{cc}h_{TT}u_c)] < r/(1 + r)$. The potential for future changes in government spending reduces the impact of any type of aggregate change in income. The reason is that some of the higher income is saved in the form of capital and that higher capital increases optimal government spending and taxes in the future.[20] Even if an increase in disposable income is entirely due to higher aggregate endowments, a fraction $[1 - (1 + r)/r \cdot MPC_y] > 0$ must be set aside for future taxes.

Barro's results are obtained by taking the limit as demand for good g becomes very inelastic, i.e., as $u_{gg} \to -\infty$: We see that $B \to \infty$, hence $MPC_T \to 0$ and $\theta \to 0$. This holds trivially if taxes are lump sum ($h(\cdot) = 0$) as in Barro (1974), but it even holds if taxes are distortionary (h_T, $h_{TT} > 0$) as in Barro (1979).[21] Thus, as $u_{gg} \to -\infty$, the Ricardian proposition holds even with distortionary taxes. This shows that the assumption on preferences is crucial. If individuals have less extreme preferences over publicly provided goods (i.e., finite derivative u_{gg} for positive values of g), tax cuts have an expansionary effect on consumption.

As before, we have $\theta = 1$, if demand for publicly provided goods is infinitely elastic (u_{gg}, $u_{cg} \to 0$). The two marginal propensities to consume are also equal in another special case. Suppose taxes become very distortionary on the margin at the level of taxes currently expected for the future, i.e. h_{TT} is large for given h_T. Under separable utility, we get $\theta \to 1$ as $h_{TT} \to \infty$.[22] Taxes will not be raised later even if debt is increased now.

Overall, a marginal propensity to consume out of tax cuts that has any value between zero and the marginal propensity to consume out of other income is consistent with rational behavior for some utility function over goods c and g.

6. Conclusions

Rational consumer responses to changes in government debt or tax levels depend heavily on the model of government behavior. If the government can commit to rules of behavior and if it chooses the particular rule of keeping expenditures constant, a current change in financial policy has no effect on consumption. That is the message of the Ricardian proposition. But what happens if the government cannot precommit to fixed spending?

In a model with optimizing government policy, individuals will take into account government incentives and the resulting expected optimal future policy actions when they determine current consumption. The government will set spending in future periods at whatever level is optimal at that time. As a result, one cannot assume an arbitrary set of values for future government variables (including spending) when determining consumer reactions to current policy changes. In particular, there is no pure financial policy. Even if current government expenditures are held constant, current shifts in financial policy will affect future real spending. Therefore, we obtain the main result: Debt policy has real effects.

These effects have a magnitude that is important for policy discussions. In our model, the effects of a debt-financed tax-cut may range from the "naive" prediction of increasing consumption regardless of the level of government debt to the Ricardian neutrality result, depending on preferences. The Ricardian result is obtained in the special cases of completely inelastic government expenditures and lump sum taxes. If demand for publicly provided goods is somewhat elastic, individuals will consider the effect of debt on future taxes, but they perceive an increase in net wealth and consume more when taxes are reduced.

Footnotes

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[1]Another limitation of Ricardian analysis involves money creation. Aiyagari and Gertler (1985) show in a model with exogenous spending that an increase in debt has wealth effects, if bonds are partly backed by money creation. However, they take the degree of tax-financing of bonds as given, whereas it is determined endogenously here.

[2]Public choice theory also questions the omission of real government activity, e.g., see Buchanan (1976). Buchanan seems to doubt the empirical relevance of policy changes that are purely financial. While our model supports this argument from a theoretical perspective (see below), our focus is on expectations, given that a tax cut has occurred.

[3]There is also a welfare argument against an exogenous spending rule. We show below that the relative cost of public and private goods depends on the level of debt, if taxes are even slightly distortionary. Higher debt makes public spending more expensive, which provides a powerful economic incentive to abandon the rule and to reduce government spending. Given the apparent absence of an enforcement mechanism for spending rules in the political process, such rules are not likely to be maintained.

[4]As in Barro's model, a unique optimal debt policy exists if taxation imposes a (possibly very small) excess burden on the economy that increases with tax rates. On the other hand, optimal debt policy is indeterminate, if all taxes are lump-sum. In our model, the level of future government spending is independent of debt policy if all taxes are lump-sum. But this result is not robust to small perturbations: Debt policy may influence government spending significantly, if taxes are even slightly distortionary.

[5]For example, a systematic bias of politicians towards higher spending could easily be accommodated by adding appropriate expressions to the government's objective function. Regardless of the level, it is only important for our results that government spending responds to changes in the cost of public goods in the direction suggested by economic incentives.

[6]This good may be a public good in the sense that no person can be excluded from consuming it. Then taxation arises naturally as solution of the externality problem. But it may be any other good or service that is financed by taxes.

[7]We assume throughout the paper that the relevant transversality conditions are satisfied.

[8]In general, excess burden may be a complicated function of many endogenous variables. Our assumptions make it a specific function of taxes only. This will allow us to sign effects of shocks on macroeconomic variables easily and greatly simplify the analysis of the game. It also excludes the possibility of multiple solutions of the mapping of shocks to endogenous variables in a natural way and maintains easy comparability to Barro's results.

[9]The assumption of welfare maximization is sufficient but not necessary. Any modification that does not distort the intertemporal pattern of private consumption and leaves $u(\cdot)$ as part of the objective function does not change our qualitative conclusions. For example, if politicians had a bias in favor of government spending and maximized $u(c_t, g_t, \gamma_t) + \zeta \cdot g_t$, where $\zeta > 0$, nothing would change except that u_g would have to be replaced by $u_g + \zeta$ in equation (10) below.

[10]Equation (7) is obtained from (2) by substituting debt from (3).

[11]For details, see Appendix 3.1. These technical complications would arise identically in a model with exogenous government spending; hence they seem to be unrelated to our main issue.

[12]An existence problem might arise if financing the debt is not feasible; i.e., we assume that initial debt is sufficiently small. The value function is always weakly concave and it is strictly concave if the game ends after a finite number of periods. We exclude the degenerate case that $V_{KK}V_{DD} - V_{DK}^2$ converges to zero in the transition from a finite to an infinite-period economy.

[13]Table 1 implies that T_{t+1} and D_{t+1} rise in response to higher D_t. Then higher D_{t+1} implies an increase in T_{t+2}, and so on.

[14]Technically, this is equivalent to just assuming that $h_\varepsilon = 0$. But because of $h_{T\varepsilon} > 0$ the condition $h_\varepsilon = 0$ cannot always be satisfied.

[15]The treatment of distortions may create empirical difficulties, if endowments are not observed. Regressions may be misleading, if researchers observe only the sum $y_{1t} + y_{2t}$ or only y_{1t} (the income that the government considers taxable) and if they use one of these income variables as regressor. The reason is that a tax cut would increase these measures of income so that researchers would interpret the resulting rise in consumption as reaction to higher income, even if true endowments remain unchanged. If $y_{1t} + y_{2t}$ is used as regressor instead of Y_t (or $Y_t - h_\psi/(1 - h_\psi) \cdot \varepsilon_t$), the regression coefficient on income is increased, while the coefficient on taxes is reduced by h_T. The measured coefficient of taxes may then be positive or negative. With the simplifications of Section 5, it is reduced by $h_T \cdot MPC_y$ and is positive, if $h_{TT}/(h_T \cdot (1 + h_T)) > u_{gg}/u_g$. This is satisfied, if distortions are small (h_T small) but increasing (h_{TT} large) and if government spending is price elastic (u_{gg}/u_g small). If y_{1t} is used, the results are unpredictable without additional assumptions.

[16]A careful definition is necessary, because disposable income differs from endowments not just by tax revenues, but also by excess burden (see Appendix 3.3).

[17]This is true even in a permanent income approach because all random shocks are i.i.d., hence all changes in disposable income are temporary (provided future taxes are ignored).

[18]If spending is exogenous, equations (7) and (8) imply that the optimal policy is to minimize $E_t \sum_{i=0}^{\infty} \delta^{i-t} h(T_{t+i}, \varepsilon_{t+i}, \psi_{t+i})$ given $E_t \sum_{i=0}^{\infty} \delta^{i-t}(T_{t+i} - g^*_{t+i}) = R \cdot D_{t-1}$. This is just Barro's (1979) objective of minimizing the present value of excess burden.

[19]Under separable utility, this behavior has an interpretation in terms of private net wealth. If we define net wealth as the present value of after tax income plus $R \cdot (D_{t-1}+K_{t-1})$, it must equal the present value of consumption (from equation (2)). Every period, individuals consume $r/(1 + r)$ of net wealth. Disposable income from higher aggregate endowments increases net wealth by $(1 + r)/r \cdot MPC_y < 1$. Income from tax cuts increases net wealth by $(1 + r)/r \cdot MPC_T = \theta \cdot (1 + r)/r \cdot MPC_y$. Thus, if a tax cut increases aggregate disposable income by $1/[(1 + r)/r \cdot MPC_y]$, which would raise net wealth by a unit if it came from higher aggregate endowments, only a fraction θ is considered net wealth, while a fraction $1 - \theta$ is set aside for expected future taxes.

[20]This argument does not involve taxes on capital that would distort individual saving decisions. The point is that a social planner allocates higher aggregate initial capital to both public and private consumption. This could be shown clearly, if we introduced idiosyncratic shocks to endowments that aggregate to zero. An individual receiving an idiosyncratic increase in disposable income would indeed consume a fraction $r/(1 + r)$.

[21]This result is exact for our definition of a "pure" debt-financed tax cut. For a tax cut of the type considered in Theorem 1, it would only be an approximation.

[22]If goods c and g are substitutes, the situation is more complicated, because consumers must save to compensate for the reduced future government spending. Then $\theta \to 1 - u_{cg}/(u_{cc}(1 + h_T)) < 1$ as $h_{TT} \to \infty$.

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Table 1:

Effect of on	K_{t-1}	D_{t-1}	ε_t	γ_t	ψ_t
c_t	+	?	+	-	+
g_t	+	-	?	+	+
T_t	+	+	-	?	+
D_t	?	+	+	+	?
K_t	+	?	?	?	+

Legend: + = the marginal effect is positive;

- = the marginal effect is negative;

? = the marginal effect may be positive or negative.

Appendix

A.1. Labor Supply and Taxes

A labor-leisure interpretation of production can be obtained, if we replace y_t in budget constraint (2) by $y_{1t} - T_t$ and replace preferences $u(c_t, g_t, \gamma_t)$ by $\tilde{u}(c_t, \ell_{2t}, g_t, \gamma_t) = u\big(c_t + F(\ell_{2t}, \varepsilon_t), g_t, \gamma_t\big)$. We have chosen the two-processes interpretation in the text, because the notation with $\tilde{u}(\cdot)$ would be inconvenient.

We assume that the production process $y_{2t} = F(\ell_{2t}, \varepsilon_t)$ is continuous, has continuous partial derivatives, and satisfies the following properties:

Assumption: Given the initial value of debt D_{t-1} and given any values of ε_t, ψ_t on the support of their distributions, there is a value ℓ^*, $0 < \ell^* \le Y_t = 1 + \psi_t$, such that

(1a) $F_{\ell\ell} < 0$, (1b) $F_{\ell\varepsilon} > 0$, (2a) $F_{\ell\ell\ell} \ge 0$, (2b) $F_{\ell\ell\varepsilon} \ge 0$,

(3a) $\alpha = (-F_{\ell\ell})\cdot(Y_t - \ell_{2t}) - (1 - F_\ell) > 0$

for all $\ell^* \ge \ell_{2t} > 0$; moreover

(3b) $\big(1 - F_\ell(\ell^*, \varepsilon_t)\cdot(Y_t - \ell^*)\big) > \max(R\cdot D_{t-1}, 0)$

(3c) $\lim_{\ell_{2t} \to \ell^*} \alpha = 0$.

The assumption looks complicated because of a Laffer curve effect. Tax revenue is supposed to rise initially when the tax rate and ℓ_{2t} are increased. This is condition (3a). But revenue is zero at $\ell_{2t} = Y_t$. Condition (3c) defines ℓ^* as the level of nontaxed activity that maximizes tax revenue. Hence we are only concerned with the properties of F( ) for $\ell_{2t} \le \ell^*$. Conditions (1a, b) guarantee that a shock ε_t raises ℓ_{2t}, (3b) guarantees that it is feasible to finance the debt, and (2a, b) assure that the derivatives of the loss function h( ) have the signs that intuition suggests. Notice that (3b) just requires that initial debt is low enough.

There are functions that satisfy the assumptions, e.g., $F(\ell, \varepsilon) = \ell - a\cdot(\ell - \varepsilon)^2$, $\ell^* = (1 + \psi_t - \varepsilon_t)/2$, provided $|\psi_t| \le 1/8$ and $|\varepsilon_t| \le 1/8$ are bounded and $R\cdot D_{t-1}$ is small enough (e.g., zero).

Suppose tax rates are in the interval $0 \le \tau_t \le 1 - F_\ell(\ell^*, \varepsilon_t)$. Each individual maximizes (1) subject to (2) where

$$y_t - T_t = \ell_{1t}\cdot(1 - \tau_t) + F(\ell_{2t}, \varepsilon_t) , \quad \ell_{1t} + \ell_{2t} \le Y_t , \quad \ell_{1t}, \ell_{2t} \ge 0 .$$

The Kuhn-Tucker conditions for optimal labor allocation are (using the facts that $\ell_{2t} \le \ell^* < Y_t$ and $F_\ell > 0$ for $\ell_{2t} \le \ell^*$)

$$1 - \tau_t - F_\ell(\ell_{2t}, \varepsilon_t) + \lambda_t = 0 , \quad \lambda_t \ge 0 , \quad \lambda_t \cdot \ell_{2t} = 0 .$$

The solution is $\ell_{2t} = 0$, if $\tau_t < 1 - F_\ell(0, \varepsilon_t)$ and $\ell_{2t} = F_\ell^{-1}(1 - \tau_t, \varepsilon_t)$, if $1 - F_\ell(0, \varepsilon_t) \le \tau_t \le 1 - F_\ell(\ell^*, \varepsilon_t)$. Notice that the solution involves only τ_t and ε_t, i.e., it is independent of the choice of consumption or any other variable in the individual's problem.

The case $\tau_t < 1 - F_\ell(0, \varepsilon_t)$ is the situation with nondistortionary taxes. We clearly have $T_t = \tau_t\cdot Y_t$, $y_t = Y_t$, and $h(\cdot) = 0$. Then marginal changes in ε_t have no effect on the allocation. The more interesting case is $F_\ell(0, \varepsilon_t) \le \tau_t \le 1 - F_\ell(\ell^*, \varepsilon_t)$. Then $\ell_{1t} = Y_t - F_\ell^{-1}(1 - \tau_t, \varepsilon_t)$, $T_t = \tau_t\big(Y_t - F_\ell^{-1}(1 - \tau_t, \varepsilon_t)\big) = T(Y_t, \varepsilon_t, \tau_t)$, and $y_t = Y_t - H(\tau_t, \varepsilon_t)$, where $H(\tau_t, \varepsilon_t) = F_\ell^{-1}(1 - \tau_t, \varepsilon_t) + F\big(F_\ell^{-1}(1 - \tau_t, \varepsilon_t), \varepsilon_t\big)$. Notice that $T_\tau = Y_t - \ell_{2t} + \tau_t/F_{\ell\ell} = 1/(-F_{\ell\ell})\cdot\big[(-F_{\ell\ell})\cdot(Y_t - \ell_{2t}) - (1 - F_\ell)\big] > 0$, hence we can write $\tau_t = \tau(T_t, \psi_t, \varepsilon_t)$. Then $h(T_t, \psi_t, \varepsilon_t) = H\big(\tau(T_t, \psi_t, \varepsilon_t), \varepsilon_t\big)$ is the loss in output as result of distortionary taxation. The partial derivatives of $h(\cdot)$ are

$$h_T = \tau_t/\alpha > 0 , \quad h_\varepsilon = h_T \cdot \ell_{1t} \cdot F_{\ell\varepsilon} - F_\varepsilon , \quad h_\psi = -\tau_t^2/\alpha < 0$$

$$h_{TT} = - F_{\ell\ell}/\alpha + \tau_t/\alpha^3(-2 \cdot F_{\ell\ell} + \ell_{1t} \cdot F_{\ell\ell\ell}) > 0$$

$$h_{T\varepsilon} = T_t \cdot F_{\ell\varepsilon}/\alpha^3(-2 \cdot F_{\ell\ell} + \ell_{1t} \cdot F_{\ell\ell\ell}) + T_t/\alpha^2 \cdot F_{\ell\ell\varepsilon} > 0$$

$$h_{T\psi} = -T_t/\alpha^3(2 \cdot F_{\ell\ell} + \tau_t \cdot F_{\ell\ell\ell}) < 0 .$$

For Theorems 1 and 2 we need that $h_\varepsilon \leq 0$, which can be assured by assuming that F_ε is large enough. The intuition that ε_t largely causes incentive effects and that ψ_t is largely an income change corresponds to a situation in which h_ε, $h_{T\psi}$, and h_ψ are small in absolute value. This is assumed to determine the signs in Table 1.

A.2. Characterization of the Mapping Γ

We prove the results in Table 1 first for a truncated economy by induction and then take the limit. The steps are straightforward but extremely tedious. Essentially, we have to solve 3 first order conditions and two constraints to determine the 5 endogenous variables as functions of the 5 state variables. To find the signs of the derivatives, we have to determine simultaneously some technical properties of the value function.

First, we have to formalize the assumptions on the utility function. We do not want that direct substitution of goods c and g in $u(\cdot)$ dominates the analysis. To limit substitution effects it is sufficient to assume that goods c_t and g_t are not complements in utility and that none of the goods is inferior. Formally, this means $u_{cg} \leq 0$, $u_{cc} - (u_c/u_g)u_{cg} < 0$ and $u_{gg} - (u_g/u_c)u_{cg} < 0$.

As example, consider utility functions of the form $v(c + \alpha g)$ with $0 \leq \alpha < 1$, which are popular in the empirical literature (e.g., Aschauer (1985), Barro (1981B)). They imply that government spending is a partial substitute for private consumption. But functions of this type do not provide a motivation for government spending (the optimal level of g would be zero). If we supplement such a function by some additional preference for the good g that motivates its existence, e.g., $v_1(c + \alpha g) + v_2(g, \gamma)$, the complete utility function satisfies our assumptions.

It is necessary to define some notation. For the infinite horizon economy, let $U_{ij} = \delta E_t V_{ij}$; i, j = D, K. Let $f_T = 1 + h_T$, and

$$U = \begin{pmatrix} U_{KK} & U_{KD} \\ U_{DK} & U_{DD} \end{pmatrix}, \qquad u = \begin{pmatrix} u_{cc} & u_{cg} \\ u_{gc} & u_{gg} \end{pmatrix},$$

$$\tilde{h} = (h_T, 1), \quad \tilde{f} = (f_T, 1),$$

$$\Omega = \begin{pmatrix} u_{cc} + U_{KK} & u_{cg} + U_{KK} - U_{KD} & +h_T U_{KK} + U_{KD} \\ -u_{cc} + u_{cg} - U_{KD} & u_{gg} - u_{cg} + U_{DD} - U_{KD} & -h_T U_{KD} - U_{DD} \\ -f_T u_{cc} + u_{cg} & u_{gg} - f_T u_{cg} & -h_{TT} u_c \end{pmatrix}$$

$$B = \begin{pmatrix} R \cdot U_{DK} & R \cdot U_{KK} & -h_\varepsilon \cdot U_{KK} & 0 & U_{KK} \cdot (1 - h_\psi) \\ -R \cdot U_{DD} & -R \cdot U_{DK} & h_\varepsilon \cdot U_{DK} & -u_{g\gamma} & -U_{DK} \cdot (1 - h_\psi) \\ 0 & 0 & h_{T\varepsilon} \cdot u_c & -u_{g\gamma} & h_{T\psi} \cdot u_c \end{pmatrix}$$

where the j^{th} columns of Ω and B are denoted by Ω_j, B_j, respectively.

Suppose our game is ended after a finite number of periods, T. Terminal values $D_T = K_T = 0$ are given. Define a value function $V^{t-1}(D_{t-1}, K_{t-1})$ by $V^{T-1} = \max u(c_T, g_T, \gamma_T)$, $V^t = \max u(c_t, g_t, \gamma_t) + \delta E_t V^{t+1}$, $t < T$, where the maximum is taken subject to the budget constraints.

For the truncated economy, define Ω^t and B^t analogous to Ω and B with U_{ij} replaced by $U^t_{ij} = \delta E_t V^t_{ij}$. Let c^t_K, c^t_D, g^t_K, g^t_D, T^t_D, and T^t_K be the derivatives of c_t, g_t, and T_t with respect to changes in state variables D_{t-1} and K_{t-1}. For the induction, it is useful to define the expressions

$X^t_1 = (u^t_{gg} - u^t_{cg})U^t_{KK} + u^t_{cg}U^t_{DK}$, $X^t_2 = u^t_{cc}U^t_{DD} - (u^t_{cc} - u^t_{cg})U^t_{DK}$,

$X_3^t = (u_{cc}^t - u_{cg}^t)U_{KK}^t - u_{cc}^t U_{DK}^t$, $X_4^t = U_{DK}^t + h_T^t U_{KK}^t$, and $X_5^t = U_{DD}^t + h_T^t U_{DK}^t$ (where superscripts indicate the period). Note that

$$|\Omega^t| = -u_c \cdot h_{TT} \cdot \left(|u^t| + |U^t| + X_1^t + X_2^t + X_3^t\right) + |u^t| \cdot \tilde{h}U\tilde{h}' + |U| \cdot \tilde{f}u\tilde{f}' .$$

Then we can obtain the following results.

Lemma 1:

For the final period t = T, the derivatives of the endogenous variables with respect to state variables satisfy

$$c_K^t > 0,\ g_K^t > 0,\ g_D^t < 0, \qquad \text{(A1)}$$

and

$$T_D^t > 0,\ T_K^t > 0 . \qquad \text{(A2)}$$

Proof: We have to maximize

$$u\left[R \cdot (K_{T-1} + D_{T-1}) + Y_T - T_T - h(T_T, \varepsilon_T, \psi_T),\ T_T - R \cdot D_{T-1},\ \gamma_t\right] ,$$

which leads to the first order condition $u_g - (1 + h_T) \cdot u_c = 0$. Taking the total differential implies

$$T_K = R/A \cdot \left(u_{cg} - (1 + h_T)u_{cc}\right) > 0$$

$$T_D = R/A \cdot \left(u_{cg} - u_{gg} + (1 + h_T)(u_{cg} - u_{cc})\right) > 0$$

$$g_D = T_D - R = -R/A \cdot \left[h_{TT}u_c - h_T \cdot \left(u_{cc}(1 + h_T) - u_{cg}\right)\right] < 0$$

$$g_K = T_K > 0$$

$$c_K = R - (1 + h_T)T_K = R/A \cdot \left[h_{TT}u_c - u_{gg} + (1 + h_T)u_{cg}\right] > 0$$

$$c_D = R - (1 + h_T)T_D = R/A \cdot \left[h_{TT}u_c + h_T\left(u_{gg} - (1 + h_T)u_{cg}\right]\right. \lesseqgtr 0$$

where $A = h_{TT}u_c - u_{gg} + 2(1 + h_T)u_{cg} - (1 + h_T)^2 u_{cc} > 0$. QED.

Lemma 2:

In period $t = T - 1$ we have

$$U^t_{KK} < 0,\ |U^t| > 0,\ |\Omega^t| < 0, \tag{A3}$$

$$X^t_1 > 0,\ X^t_2 > 0,\ X^t_3 > 0\ . \tag{A4}$$

$$X^t_4 < 0\ ,\quad X^t_5 < 0 \tag{A5}$$

Proof: The envelope conditions are $V^{t-1}_K = R\ E_t u^t_c$ and $V^{t-1}_D = R\ E_t(u^t_c - u^t_g)$. Therefore,

$$U^{t-1}_{KK} = R\delta\ E_{t-1}(u^t_{cc}c^t_K + u^t_{cg}g^t_K)$$

$$U^{t-1}_{KD} = U^{t-1}_{DK} = R\delta\ E_{t-1}(u^t_{cc}c^t_D + u^t_{cg}g^t_D)$$

$$= R\delta\ E_{t-1}[(u^t_{cc} - u^t_{cg})\ c^t_K + (u^t_{cg} - u^t_{gg})g^t_K)]$$

$$U^{t-1}_{DD} = R\delta\ E_{t-1}[(u^t_{cc} - u^t_{cg})\ c^t_D + (u^t_{cg} - u^t_{gg})\ g^t_D)]$$

Using the results of Lemma 1, we obtain for $t = T$

$$U^{t-1}_{KK} = \delta^1 R^2 E_{t-1}[|u^t|/A \cdot (h_{TT}u^t_c u^t_{cc} - |u^t|)]\ .$$

We know that $|u^t| > 0$ and $u^t_{cc} < 0$ by concavity of u( ); also $A > 0$, $u_c > 0$, and $h_{TT} > 0$. But we do not know the conditional covariances between these expressions. Here we need the assumption that shocks are sufficiently small. The conditional covariances are of the order of magnitude of the variances in the shocks; hence for small shocks $U^{t-1}_{KK} < 0$ follows from $E_{t-1}|u^t| > 0$, $E_{t-1}u^t_{cc} < 0$, $E_{t-1}A > 0$, $E_{t-1}u_c > 0$, and $E_{t-1}h_{TT} > 0$. From here on we will use the assumption of small shocks to sign expressions without mentioning it explicitly. Note that we do compute the exact solution to the dynamic programming problem; the assumption of small shocks is only considered as a sufficient condition for the characterization of the exact solution. Next,

$$|U^{t-1}| = U_{KK}^{t-1}U_{DD}^{t-1} - (U_{KD}^{t-1})^2 = \delta^2R^2E_{t-1}|u^t|(c_D^t \cdot g_K^t - c_K^t \cdot g_D^t)$$

$$\simeq \delta^2R^4E_{t-1}|u^t|/E_{t-1}A \cdot E_{t-1}(h_{TT}u_c^t) > 0 ,$$

$$X_1^{t-1} = \delta RE_{t-1}[((u_{gg}^{t-1} - u_{cg}^{t-1})u_{cc}^t + (u_{cc}^{t-1} - u_{cg}^{t-1})u_{cg}^t)c_K^t$$

$$+ ((u_{gg}^{t-1} - u_{cg}^{t-1})u_{cg}^t + (u_{cc}^{t-1} - u_{cg}^{t-1})u_{gg}^t)g_K^t]$$

$$\simeq \delta R(u_{gg}^{t-1}u_{cc}^{t-1} - u_{cg}^{t-1}u_{cg}^{t-1})E_{t-1}c_K^t > 0$$

where we approximated (using the first order condition $u_g^{t-1} = \delta RE_{t-1}(u_g^t)$ and the assumption of small shocks)

$$E_{t-1}(u_{gg}^{t-1}u_{cc}^t - u_{cg}^{t-1}u_{cg}^t) \simeq \delta R(u_{gg}^{t-1}u_{cc}^{t-1} - u_{cg}^{t-1}u_{cg}^{t-1}) > 0 ,$$

$$E_{t-1}(u_{gg}^{t-1}u_{cg}^t - u_{cg}^{t-1}u_{gg}^t) \simeq 0 .$$

Using similar approximations, we get

$$X_2^{t-1} = \delta RE_{t-1}[((u_{cc}^t - u_{cg}^t)u_{cc}^{t-1} - (u_{cc}^{t-1} - u_{cg}^{t-1})u_{cc}^t)c_D^t$$

$$+ ((u_{cg}^t - u_{gg}^t)u_{cc}^{t-1} + (u_{cc}^{t-1} - u_{cg}^{t-1})u_{cg}^t)g_D^t]$$

$$\simeq -\delta R|u^{t-1}|E_{t-1}g_D^t > 0$$

$$X_3^{t-1} = \delta RE_{t-1}[((u_{cc}^{t-1} - u_{cg}^{t-1})u_{cc}^t - (u_{cc}^t - u_{cg}^t)u_{cc}^{t-1})c_K^t$$

$$+ ((u_{gg}^{t-1} - u_{cg}^{t-1})u_{cg}^t - (u_{cg}^t - u_{gg}^t)u_{cc}^{t-1})g_K^t]$$

$$\simeq \delta R|u^{t-1}|E_{t-1}g_K^t > 0 .$$

Using these results, $|\Omega^{t-1}| < 0$ is immediate. Moreover,

$V_D^{t-1} = RE_t(u_c^t - u_g^t) = RE_t h_T^t u_c^t$ implies

$$U_{KD}^{t-1} = R\delta E_{t-1}\left[-h_T^t(u_{cc}^t \cdot c_K^t + u_{cg}^t \cdot g_K^t) - h_{TT}^t u_c^t T_K^t\right]$$

$$U_{DD}^{t-1} = R\delta E_{t-1}\left[-h_T^t(u_{cc}^t \cdot c_D^t + u_{cg}^t \cdot g_D^t) - h_{TT}^t u_c^t T_D^t\right] .$$

Using these alternative expressions,

$$X_4^{t-1} = \delta RE_{t-1}\left[\left((h_T^{t-1} - h_T^t)u_c^t\right)\left((u_{cc}^t \cdot c_K^t + u_{cg}^t \cdot g_K^t)/u_c^t\right) - h_{TT}^t u_c^t T_K^t\right]$$

Note that $E_{t-1}(h_T^{t-1} - h_T^t)u_c^t = u_g^{t-1} - u_c^{t-1} + \delta E_{t-1}V_D^t = 0$, hence

$$X_4^{t-1} \simeq \delta RE_{t-1}[-h_{TT}^t u_c^t T_K^t] < 0 ,$$

provided shocks are small. Similarly,

$$X_5^{t-1} = \delta RE_{t-1}\left[\left((h_T^{t-1} - h_T^t)u_c^t\right)\left((u_{cc}^t \cdot c_D^t + u_{cg}^t \cdot g_D^t)/u_c^t\right) - h_{TT}^t u_c^t T_D^t\right]$$

$$\simeq \delta RE_{t-1}[-h_{TT}^t u_c^t T_D^t] < 0 .$$

QED.

Lemma 3:

Suppose (A3) and (A4) hold for some period $t = i + 1$, $i + 1 \leq T$. Then (A3) and (A4) also hold for period $t = i$.

Proof: Let $t = i + 1$. The endogenous variables satisfy the first order conditions (9)-(11).

The first order conditions (10) and (11) can be simplified to

$$u_g(u_t, g_t, \gamma_t) - u_c(c_t, g_t, \gamma_t) + \delta \cdot E_t[V_D^t(D_t, K_t)] = 0 ,$$

$$u_g(u_t, g_t, \gamma_t) - (1 + h_T(T_t, \gamma_t, \varepsilon_t)) \cdot u_c(c_t, g_t, \gamma_t) = 0 .$$

Taking the total differential of (9) and these two equations, we obtain

$$\Omega^t \cdot (dc_t \; dg_t \; dT_t)' = B^t \cdot (dD_{t-1} \; dK_{t-1} \; d\varepsilon_t \; d\gamma_t \; d\psi_t)'.$$

Then Cramer's rule implies

$$c_K^t = \frac{\partial c_t}{\partial K_{t-1}} = \frac{-R}{|\Omega^t|} \cdot \left[h_{TT}u_c(X_1^t + |U^t|) - (u_{gg} - f_T^t u_{cg}) \cdot |U^t|\right] > 0$$

$$c_D^t = \frac{\partial c_t}{\partial D_{t-1}} = \frac{-R}{|\Omega^t|} \cdot \left[h_{TT}u_c\left(u_{gg}U_{DK}^t - u_{cg}(U_{DK}^t - U_{DD}^t) + |U^t|\right)\right.$$
$$\left. + h_T(u_{gg} - f_T u_{cg}) \cdot |U^t|\right]$$

$$g_K^t = \frac{\partial g_t}{\partial K_{t-1}} = \frac{-R}{|\Omega^t|} \cdot \left[h_{TT}u_c \cdot X_3^t + |U^t| \cdot (u_{cg} - f_T \cdot u_{cc})\right] > 0$$

$$g_D^t = \frac{\partial g_t}{\partial D_{t-1}} = \frac{-R}{|\Omega^t|} \cdot \left[-h_{TT}u_c \cdot \left(|U^t| + X_2^t\right) - |U^t| \cdot h_T \cdot (u_{cg} - f_T \cdot u_{cc})\right] < 0$$

$$T_K^t = \frac{\partial T_t}{\partial K_{t-1}} = \frac{-R}{|\Omega^t|} \cdot \left[-|u| \cdot X_4^t + |U^t| \cdot (u_{cg} - f_T \cdot u_{cc})\right] > 0$$

$$T_D^t = \frac{\partial T_t}{\partial D_{t-1}} = \frac{-R}{|\Omega^t|} \cdot \left[-|u|X_5^t + |U^t| \cdot (u_{cg} - f_T u_{cc})\right] > 0 .$$

In determining the signs, we also use the fact that $f_T u_{cc} - u_{cg} = f_T(u_{cc} - (u_c/u_g)u_{cg}) < 0$ and $u_{gg} - f_T u_{cg} = u_{gg} - (u_g/u_c)u_{cg} < 0$. Analogous to the Lemma 2 we get

$$U_{KK}^{t-1} = R\delta E_{t-1}(u_{cc}^t c_K^t + u_{cg}^t g_K^t)$$
$$= - R^2 \delta E_{t-1}\left[|U^t|(h_{TT}u_c u_{cc}^t - |u^t|) + U_{KK}^t |u^t|\right]/|\Omega^t| < 0$$

given that (A3) and (A4) hold for period t = i + 1.

$$|U^{t-1}| = \delta^2 R^2 E_{t-1}|u^t|(c^t_D g^t_K - c^t_K g^t_D)$$

$$= \delta^2 R^4 E_{t-1}\Big[(h_{TT}u^t_c)^2\big(|U^t|^2 + |U^t|(X^t_1 + X^t_2 + X^t_3) + X^t_1 X^t_2$$

$$+ X^t_3(u^t_{gg}U^t_{DK} - u^t_{cg}(U^t_{DK} - U^t_{DD}))\big)$$

$$- h_{TT}u^t_c|U^t|(fu^t f|U^t| + hU^t h|u^t|)\Big]$$

$$\simeq - \delta^2 R^4 E_{t-1}|u^t|/E_{t-1}|\Omega^t| \cdot E_{t-1}(h_{TT}u^t_c|U^t|) > 0 \ .$$

The arguments for X^{t-1}_1, X^{t-1}_2, $X^{t-1}_3 > 0$ and $|\Omega^{t-1}| < 0$ are identical to the arguments in Lemma 2. QED.

Notice that Barro's (1974) case, $h(\cdot) = 0$, implies that $\Omega_3 = 0$. Hence $|\Omega| = 0$ and the proof fails. This reflects the fact that then the mapping Γ has no unique solution.

Lemma 4:

Suppose (A3) and (A5) hold for some period $t = i + 1$, $i + 1 \le T$. Then (A5) also holds for period $t = i$.

Proof: Use the same arguments as in Lemma 2.

Corollary: (A1)-(A5) hold for all periods $t \le T$.

Proof: Statements (A3)-(A5) follow by induction from Lemmas 2 and 3. Inspection of the derivatives in the proof of Lemma 3 shows that they imply (A1) and (A2).

Lemma 5:

Take the limit $T \to \infty$. Then (A1)-(A5) still hold as weak inequalities. Under the assumptions $|U| > 0$, all other inequalities in (A1) - (A5) are strict.

Proof: As $T \to \infty$, Lemmas 3 and 4 immediately imply $U_{KK}^{t-1} \leq 0$, $|U^{t-1}| \geq 0$, $|\Omega^{t-1}| \leq 0$, $X_1^{t-1} \geq 0$, $X_2^{t-1} \geq 0$, $X_3^{t-1} \geq 0$, $X_4^{t-1} \leq 0$, $X_5^{t-1} \leq 0$. But the recursion formulas show that the inequalities are strict in period t-1, provided $|U^t| > 0$ holds in period t. QED.

The other derivatives in Table 1 can be computed directly from

$$\Omega \cdot (dc_t \quad dg_t \quad dT_t)' = (B_3 \quad B_4 \quad B_5) \cdot (d\varepsilon_{t-1} \quad d\gamma_{t-1} \quad d\psi_{t-1})'$$

and the budget constraints (for dD_t and dK_t). Using the results in Lemma 3, we obtain

$$\frac{\partial D_t}{\partial K_{t-1}} = \frac{-R}{|\Omega|} \cdot [h_{TT}u_c X_3^t + |u| \cdot X_4^t]$$

$$\frac{\partial D_t}{\partial D_{t-1}} = \frac{-R}{|\Omega|} \cdot [h_{TT}u_c(|u| + X_1^t + X_3^t) - |U|(u_{gg} - f_T u_{cg}) - |u| h_T X_4^t] > 0$$

$$\frac{\partial K_t}{\partial K_{t-1}} = \frac{-R}{|\Omega|} \cdot [h_{TT}u_c(|u| + X_2^t) - |u| X_5^t] > 0$$

$$\frac{\partial K_t}{\partial D_{t-1}} = \frac{-R}{|\Omega|} \cdot [h_{TT}u_c(u_{cc} - u_{cg}) \cdot (U_{DD} - U_{DK}) + h_T X_5^t |u| - h_T |U| (u_{gg} - f_T u_{cg})]$$

To compute the effects of y_t and ε_t, define

$$A_1 = u_c(f_T|U| + (u_{gg} - u_{cg})X_4^t + u_{cg}X_5^t)/(-|\Omega|) > 0$$

$$A_2 = u_c(h_T X_3^t - X_2^t - |U|)/(-|\Omega|)$$

$$A_3 = u_c(|U| + |u| + X_1^t + X_2^t + X_3^t)/|\Omega| < 0 .$$

Then

$$\frac{\partial c_t}{\partial \psi_t} = h_{T\psi} \cdot A_1 + \frac{1 - h_\psi}{R} \cdot \frac{\partial c_t}{\partial K_{t-1}}, \qquad \frac{\partial c_t}{\partial \varepsilon_t} = h_{T\varepsilon} \cdot A_1 + \frac{h_\varepsilon}{R} \frac{\partial c_t}{\partial K_{t-1}}$$

$$\frac{\partial g_t}{\partial \psi_t} = h_{T\psi} \cdot A_2 + \frac{1 - h_\psi}{R} \cdot \frac{\partial g_t}{\partial K_{t-1}}, \qquad \frac{\partial g_t}{\partial \varepsilon_t} = h_{T\varepsilon} \cdot A_2 + \frac{h_\varepsilon}{R} \frac{\partial g_t}{\partial K_{t-1}}$$

$$\frac{\partial T_t}{\partial \psi_t} = h_{T\psi} \cdot A_3 + \frac{1 - h_\psi}{R} \cdot \frac{\partial T_t}{\partial K_{t-1}}, \qquad \frac{\partial T_t}{\partial \varepsilon_t} = h_{T\varepsilon} \cdot A_3 + \frac{h_\varepsilon}{R} \frac{\partial T_t}{\partial K_{t-1}}$$

$$\frac{\partial D_t}{\partial \psi_t} = h_{T\psi} \cdot (A_2 - A_3) + \frac{1 - h_\psi}{R} \cdot \frac{\partial D_t}{\partial K_{t-1}}, \qquad \frac{\partial D_t}{\partial \varepsilon_t} = h_{T\varepsilon} \cdot (A_2 - A_3) + \frac{h_\varepsilon}{R} \frac{\partial D_t}{\partial K_{t-1}}$$

$$\frac{\partial K_t}{\partial \psi_t} = -h_{T\psi} \cdot (A_1 + A_2 + h_T A_3) + \frac{1 - h_\psi}{R} \cdot \frac{\partial K_t}{\partial K_{t-1}},$$

$$\frac{\partial K_t}{\partial \varepsilon_t} = -h_{T\varepsilon}(A_1 + A_2 + h_T A_3) + \frac{h_\varepsilon}{R} \cdot \frac{\partial K_t}{\partial K_{t-1}}.$$

Note that $A_2 - A_3 = u_c(X_1^t + f_T X_3^t + |u|)/(-|\Omega|) > 0$ and $A_1 + A_2 + h_T A_3 = u_c/|\Omega| \cdot [h_t|u| + f_T X_2^t - u_{gg}U_{DK} + u_{cg}(U_{DK} - U_{DD})]$, which may be positive or negative.

Then the signs in Table 1 are obtained under the assumption that $h_{T\psi}$ is small enough so that the expressions with $(1 - h_\psi)$ determine the sign of the effects of ψ_t and that h_ε is small enough that expressions with $h_{T\varepsilon}$ determine the direction of the effects of ε_t. Finally,

$$\frac{\partial c_t}{\partial \gamma_t} = \frac{-u_{g\gamma}}{|\Omega|} \cdot [h_{TT}u_c(u_{cg} + U_{KK} - U_{KD}) - f_T|U| - u_{cg}\tilde{h}U\tilde{h}]$$

$$\frac{\partial g_t}{\partial \gamma_t} = \frac{-u_{g\gamma}}{|\Omega|} \cdot [-h_{TT}u_c(u_{cc} + U_{KK}) + u_{cc} \cdot \tilde{h}Uh' + |U|] > 0$$

$$\frac{\partial T_t}{\partial \gamma_t} = \frac{-u_{g\gamma}}{|\Omega|} \cdot [|U| + X_2^t - h_T X_3^t]$$

$$\frac{\partial d_t}{\partial \gamma_t} = \frac{-u_{g\gamma}}{|\Omega|} \cdot \left[-h_{TT}u_c(u_{cc}+ U_{KK}) + (f_T u_{cc} - u_{cg}) \cdot X_4^t\right] > 0$$

$$\frac{\partial k_t}{\partial \gamma_t} = \frac{-u_{g\gamma}}{\Omega} \cdot \left[h_{TT}u_c(u_{cc} - u_{cg} + U_{KD}) - u_{cc}f_T X_5^t\right] .$$

A.3. Proofs of the Theorems

The key observation for the proofs is that the assumptions select a one-dimensional subspace, a curve, out of the three-dimensional space of realizations of $(\psi_t, \varepsilon_t, \gamma_t)$. An innovation like Δc_t is a function of the actual realization and can be determined by integrating the partial derivatives of the variable along this curve from (0, 0, 0) to the realization.

The endogenous variables c_t, g_t, and T_t are differentiable functions of $(D_{t-1}, K_{t-1}, \psi_t, \varepsilon_t, \gamma_t)$. Denote them by $c(\psi_t, \varepsilon_t, \gamma_t)$, $g(\psi_t, \varepsilon_t, \gamma_t)$, and $T(\psi_t, \varepsilon_t, \gamma_t)$, respectively, at the given values of D_{t-1}, K_{t-1}. Let innovations be denoted by Δ's, e.g., $\Delta c(\psi_t, \varepsilon_t, \gamma_t) = c(\psi_t, \varepsilon_t, \gamma_t) - c(0, 0, 0)$, and derivatives by subscripts, e.g. $c_\psi = \Delta c_\psi = \partial c(\psi_t, \varepsilon_t, \gamma_t)/\partial\psi_t$.

A.3.1. Proof of Theorem 1

A debt-financed tax cut is a realization of $(\psi_t, \varepsilon_t, \gamma_t)$ in R^3 in the subspace $\{(0, \varepsilon_t, \gamma_t) | \Delta g(0, \varepsilon_t, \gamma_t) = 0\}$. Since $g_\gamma > 0$ (see Appendix 2), $\Delta g = 0$ defines an implicit function $\gamma_t = \gamma^*(\varepsilon_t)$ with derivative $\gamma^*_\varepsilon(\varepsilon_t) = -g_\varepsilon/g_\gamma$; the subspace is the curve $(0, x, \gamma^*(x))$ in R^3, where the index x is any number in the support of ε_t.

Consider a specific realization $(\psi_t, \varepsilon_t, \gamma_t) = (0, \varepsilon_t, \gamma^*(\varepsilon_t)$ on this curve. We have

$$\Delta c_t = \Delta c(0,\ \varepsilon_t,\ \gamma^*(\varepsilon_t)) = \int_0^{\varepsilon_t}[c_\varepsilon(0,\ x,\ \gamma^*(x)) + \gamma^*_\varepsilon(x)\ \cdot c_\gamma(0,\ x,\ \gamma^*(x))dx$$

$$= \int_0^{\varepsilon_t} \frac{dc}{dx}\,(0,\ x,\ \gamma^*(x))dx = \int_0^{\varepsilon_t} c_x(x)dx$$

$$\Delta T_t = \int_0^{\varepsilon_t} [T_\varepsilon(0,\ x,\ \gamma^*(x)) + \gamma^*_\varepsilon(x)\ \cdot\ T_\gamma(0,\ x,\ \gamma^*(x))]dx$$

$$= \int_0^{\varepsilon_t} \frac{dT}{dx}\,(0,\ x,\ \gamma^*(k))dx = \int_0^{\varepsilon_t} T_x(x)dx\ .$$

Note that $\Delta D_t = -\Delta T_t$ by the budget constraint. Thus, it is sufficient to prove that $c_x > 0$ and $T_x < 0$. These derivatives must satisfy the total differential

$$\Omega\ \cdot\ (dc_t\ dg_t\ dT_t)' = (B_3\ B_4\ B_5)\ \cdot\ (d\varepsilon_t\ d\gamma_t\ d\psi_t)'$$

defined in Appendix 2. Here $d\psi_t = dg_t = 0$ are given, and $d\gamma_t/dx = \gamma^*_\varepsilon$ is endogenous. Hence,

$$(\Omega_1\ -B_4\ \Omega_3)(c_x\ \gamma^*_\varepsilon\ T_x)' = B_3$$

is a system of 3 equations for $(c_x\ \gamma^*_\varepsilon\ T_x)$. By Cramer's rule, we get

$$c_x = |(B_3\ -B_4\ \Omega_3)|/A = u_{g\gamma}/A\ \cdot\ [-h_{T\varepsilon}u_c X_4^t - h_\varepsilon(|u|-h_{TT}u_c U_{KK})]$$

$$T_x = |\Omega_1\ \ -B_4\ \ B_3|/A = u_{g\gamma}/A\ \cdot\ [h_{T\varepsilon}u_c(u_{cc} + U_{KK}) - h_\varepsilon u_c X_4^t]$$

where $A = |(\Omega_1\ -B_4\ \Omega_3)| = -|\Omega|\ \cdot\ g_\gamma > 0$.

Then $c_x > 0$ and $T_x < 0$ follow from properties (A3) and (A5) in Appendix 2 and the assumptions that $h_\varepsilon \le 0$ and $h_{T\varepsilon} > 0$.

A.3.2. Proof of Theorem 2

A pure debt-financed tax cut is a realization of $(\psi_t, \varepsilon_t, \gamma_t)$ in the subspaced $\{(\psi_t, \varepsilon_t, \gamma_t) | \Delta g = 0, \text{(14) holds}\}$. Equation (14) implies

$$d\psi_t - h_\psi d\psi_t - h_\varepsilon d\varepsilon_t = 0. \tag{A6}$$

Since $g_\gamma > 0$ and $h_\psi \leq 0$, the system of equations $\Delta g = 0$, $\Delta y = 0$ defines implicit functions $\gamma_t = \gamma^*(\varepsilon_t)$ and $\psi_t = \psi^*(\varepsilon_t)$ with derivatives $\gamma^*_\varepsilon(\varepsilon_t) = -g_\varepsilon/g_\gamma - g_\psi/g_\gamma \cdot h_\varepsilon/(1 - h_\psi)$ and $\psi^*_\varepsilon(\varepsilon_t) = h_\varepsilon/(1 - h_\psi)$; the subspace is the curve $\left(\psi^*(x), x, \gamma^*(x)\right)$ in R^3, where the index x is any number in the support of ε_t.

Consider a specific realization $(\psi_t, \varepsilon_t, \gamma_t) = \left(\psi^*(\varepsilon_t), \varepsilon_t, \gamma^*(\varepsilon_t)\right)$ on this curve. We have $\Delta c_t = \Delta c\left(\psi^*(\varepsilon_t), \varepsilon_t, \gamma^*(\varepsilon_t)\right) = \int_0^{\varepsilon_t} c_x(x)dx$ and $\Delta T_t = \int_0^{\varepsilon_t} T_x(x)$, where now $c_x(x) = c_\varepsilon + c_\psi \psi^*_\varepsilon(x) + c_\gamma \gamma^*_\varepsilon(x)$, $T_x(x) = T_\varepsilon + T_\psi \psi^*_\varepsilon(x) + T_\gamma \gamma^*_\varepsilon(x)$.

We have to prove that $c_x > 0$ and $T_x < 0$. Using $dg_t = 0$ and $d\psi_t = h_\varepsilon/(1 - h_\psi)d\varepsilon_t$ in the total differential

$$\Omega(dc_t\ dg_t\ dT_t)' = (B_3\ B_4\ B_5)\ (d\varepsilon_t\ d\gamma_t\ d\psi_t)'$$

we get

$$(\Omega_1\ -B_4\ \Omega_3)\ (c_x\ \gamma^*_\varepsilon\ T_x)' = B_3 + h_\varepsilon/(1 - h_\psi)B_5$$

which is a system of three equations for $(c_x\ \gamma^*_\varepsilon\ T_x)$. Note that $B_3 + h_\varepsilon/(1 - h_\psi)B_5 = [0\ 0\ z]'$ where $z = u_c\left(h_{T\varepsilon}(1 - h_\psi) + h_\varepsilon hT_\psi\right)/(1 - h_\psi) > 0$. By Cramer's rule, we get

$$c_x = \begin{vmatrix} 0 & 0 & X_4^t \\ 0 & 1 & X_5^t \\ 1 & 1 & h_{TT}u_c \end{vmatrix} \cdot \frac{u_{g\gamma}z}{A} = u_{g\gamma}(-X_4^t)z/A > 0$$

and

$$T_x = u_{g\gamma}(u_{cc} + U_{KK})z/A < 0 \qquad \text{(QED)}.$$

A.3.3. Proof of Theorem 3

To derive MPC_y, we have to consider realizations $(\psi_t, \varepsilon_t, \gamma_t)$ with the property $\Delta g_t = \Delta T_t = 0$. Noting that $dy_t = d\psi_t - h_\varepsilon d\varepsilon_t - h_\psi d\psi_t$, the total differential in the general model (see A.3.2)

$$\Omega\ (dc_t\ dg_t\ dT_t)' = (B_3\ B_4\ B_5)\ (d\varepsilon_t\ d\gamma_t\ d\psi_t)'$$

reduces to

$$\Omega_1\ dc_t = \big(B_3 + h_\varepsilon/(1 - h_\psi)B_5\big)d\varepsilon_t + B_4 d\gamma_t + B_5/(1 - h_\psi)dy_t\ ,$$

$$\big(\Omega_1,\ -B_4,\ -B_3 - h_\varepsilon/(1 - h_\psi)B_5\big)(dc_t\ d\varepsilon_t\ d\gamma_t)' = B_5/(1 - h_\psi)dy_t\ ,$$

which implies

$$MPC_y = \big|\big(B_5/(1 - h_\psi),\ -B_4,\ -B_3 - h_\varepsilon/(1 - h_\psi)B_5\big)\big| / \big|\big(\Omega_1,\ -B_4,\ -B_3 - h_\varepsilon/(1 - h_\psi)B_5\big)\big|$$

$$= U_{KK}/(u_{cc} + U_{KK}) > 0.$$

In Appendix 3.2 we showed that $MPC_T = -c_x/T_x/(1 + h_T) = X_4^t/(u_{cc}^t + U_{KK}^t)$ $/(1 + h_T) > 0$, hence $\theta > 0$. To show that $\theta < 1$, note that $(1 - \theta)\ MPC_y =$ $MPC_y - MPC_T = 1/\big(f_T(u_{cc} + U_{KK})\big) \cdot (U_{KK} - U_{DK})$, where $u_{cc} + U_{KK} < 0$. Hence, we have to show that $U_{KK} - U_{DK} < 0$ for $u_{gg} < 0$ and $U_{KK} - U_{DK} = 0$ for $u_{gg} = 0$. As in Appendix 2, we start with a truncated economy and use induction. Using the same notation and assumptions as in Appendix 2,
$U_{KK}^{t-1} - U_{DK}^{t-1} = \delta R\ E_{t-1}u_{cc}^t(c_K^t - c_D^t) + u_{cg}^t(g_K^t - g_D^t)$. If the game ends in period T, $U_{KK}^{T-1} - U_{DK}^{T-1} = \delta R^2\ E_{T-1}[h_{TT}u_c^T u_{cg}^T - f_T|u^T|]/A < 0$. If $U_{KK}^t - U_{DK}^t < 0$ for some period $t \le T$, then $U_{KK}^{t-1} - U_{DK}^{t-1} = \delta R^2 E_{t-1}[\,|U^t|\ (h_{TT}u_c^t u_{cg}^t - f_T|u^t|)$

$+ h_{TT}u_c|u^t|(U^t_{KK} - U^t_{DK})]/(-|\Omega|) < 0$. By induction, $U^t_{KK} - U^t_{DK} < 0$ for all periods, hence $U_{KK} - U_{DK} \leq 0$ in the infinite horizon game. But if $U_{KK} - U_{DK} \leq 0$ in period t, the recursion formula implies that $U_{KK} - U_{DK} < 0$ in period t - 1. Hence $U_{KK} - U_{DK} < 0$ holds for all t.

If $u_{gg} \to 0$ the assumption $u_{gg} < (u_g/u_c)u_{cg} \leq 0$ implies $u_{cg} \to 0$. Hence $|u| = u_{cc}u_{gg} - u^2_{cg} \to 0$ and $U^{t-1}_{KK} - U^{t-1}_{DK} \to 0$ in the recursion formula above.

QED.

A.3.4. Proof of Theorem 4

The key to optimal consumption in period t is to predict future taxes. Future taxes and spending clearly depend on the situation the government faces at the start of period t + 1. The state variables at that time are D_t and K_t. Since there is no uncertainty form t + 1 on, we can solve the intertemporal optimization problem directly. Maximization of utility (1) in t + 1 subject to the budget constraints

$$D_t = \sum_{i=1}^{\infty} \delta^i (T_{t+i} - g_{t+i})$$

$$K_t = \sum_{i=1}^{\infty} \delta^i (c_{t+i} + g_{t+i} + h(T_{t+i}) - Y_{t+i})$$

implies the first order conditions

$$u_c(c_{t+1}, g_{t+1}) = u_c(c_{t+i}, g_{t+i})$$

$$u_g(c_{t+1}, g_{t+1}) = u_g(c_{t+i}, g_{t+i})$$

$$u_g(c_{t+i}, g_{t+i}) - (1 + h_T(T_{t+i})) \cdot u_c(c_{t+i}, g_{t+i}) = 0$$

for all periods t + i, i ≥ 1. Using the assumption $h_{TT} > 0$ and the budget constraints, we obtain $c_{t+i} = c_{t+1} = c$, $g_{t+i} = g_{t+1} = g$, and $T_{t+i} = T_{t+1} = T$

for all $i \geq 1$, where $g = T - rD_t$ and $c = r(K_t + D_t) + 1 - T - h(T)$ are functions of T. Finally, T is determined implicitly as function of K_t and D_t by the first order condition

$$u_g\big(r(K_t+D_t) + 1-T-h(T),T-rD_t\big) = \big(1+h_T(T)\big) \cdot u_c\big(r(K_t+D_t) + 1-T-h(T),T-rD_t\big) .$$

Now we can replicate the steps in Lemmas 1 and 2 of A.3.1 (replacing R by r), noting that all uncertainty is resolved in period t. To simplify notation, we omit superscripts for all expressions evaluated in t + 1.

In particular, we get the derivatives

$$T_K = r\big(u_{cg} - (1 + h_T)u_{cc}\big)/A ,$$

$$T_D = - r\big(u_{gg} - (1 + h_T)u_{cg} - u_{cg} + (1 + h_T)u_{cc}\big)/A ,$$

and the following properties of the value function:

$$U^t_{KK} = u_{cc}\big(r - (1 + h_T)T_K\big) + u_{cg}T_K = -r(|u| - u_{cc}h_{TT}u_c)/A < 0$$

$$U^t_{DK} = -h_T U^t_{KK} - h_{TT}u_c T_K = r(-(1 + h_T)|u| + u_{cg}h_{TT}u_c)/A < 0$$

$$X^t_4 = U^t_{DK} + h^t_T U^t_{KK} = -h_{TT}u_c T_K + (h^t_T - h_T)U^t_{KK} = rh_{TT}u_c\big(u_{cg} - (1 + h_T)u_{cc}\big)/A < 0$$

using the first order condition $h_T = h^t_T$.

In A.3.3 we showed that $MPC_T = X^t_4/(u^t_{cc} + U^t_{KK})/(1 + h_T)$ and $MPC_y = U_{KK}/(u_{cc} + U_{KK})$. Hence

$$MPC_T = \frac{-X^t_4 \cdot A}{(-u^t_{cc} \cdot A - U^t_{uu} \cdot A)(1 + h_T)} = \frac{r \cdot h_{TT}u_c \cdot (u_{cg} - (1 + h_T) \cdot u_{cc})}{B \cdot (1 + h_T)}$$

and $MPC_y = r(|u| - u_{cc}h_{TT}u_c)/B$, where $B = -u^t_{cc}\big(-u_{gg} + 2(1 + h_T)u_{cg} - (1 + h_T)^2u_{cc} + h_{TT}u_c\big) + r(|u| - u_{cc}h_{TT}u_c)$.

QED.