PRICING PHYSICAL ASSETS INTERNATIONALLY

by

Bernard Dumas

(12-88)

RODNEY L. WHITE CENTER FOR FINANCIAL RESEARCH
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6367

First draft: July 1987
Revised: March 1988

PRICING PHYSICAL ASSETS INTERNATIONALLY

Bernard Dumas*

*Wharton School of the University of Pennsylvania, Centre HEC-ISA (France), and National Bureau of Economic Research. The author is grateful to M. Adler, W. Brainard, P. Dybvig, V. Grilli, E. Helpman, D. Henderson, A. Razin, T. N. Srinivasan, A. Stockman and participants of the following workshops: the NBER Summer Institute, the Yale University International economics seminar, the Duke University-University of North Carolina finance seminar and the Penn Macro lunch group. None of them bears any responsibility for my errors. The first draft of this paper was circulated under the title: 'Time to ship and the dynamic behavior of deviations from the Law of One Price'.

5.5.4

Abstract

Transferring physical capital and transferring production and sales activities from one country to the other typically entails large adjustment costs. The model of this paper features two homogeneous stocks of physical capital located in two different countries separated by an 'ocean'. The two physical stocks are optimally invested in a random production process yielding real returns, consumed by local residents, or transferred abroad. Retro-fitting, transferring and re-building capital equipment, and increasing production and sales abroad either takes time (during which capital is idle) or consumes real resources. As a result, the price of capital-consumption goods located in one place is not equal to that of goods located in the other place. The stochastic process for this deviation from the Law of One Price (LOP) is obtained. By construction, this process is compatible with financial market efficiency and with the possibility of (costly) trade in commodities. Whereas empirical studies have found no evidence against the hypothesis that LOP deviations follow a martingale, the theoretical process which I find, exhibits mean reversion (as well as a fair degree of conditional hetero-scedasticity) when investors are risk averse.

5.5.4

1. Introduction

Many, perhaps most, goods have an asset character. Some are storable, some can be invested into a production process, others are consumer durables. When goods have an asset character the dynamic behavior of their price is not simply driven by current flow supply and demand; rather, their price tends to follow a quasi-martingale as do the prices of securities in an efficient market. This rule applies in particular to the relative price of physical capital located in two different countries and, by extension, to any commodity which is a close substitute to physical capital.

This insight lead Roll (1979) to postulate the so-called '*ex-ante*' version of the Purchasing Power Parity doctrine. His reasoning was based on the activities of a risk-neutral speculator who could engage in foreign-exchange transactions and store commodities in two countries, without being allowed to ship them from one place to the other. Despite the absence of direct spatial arbitrage, Roll argued, goods prices in the two countries are not without link: their relative price (the deviation from the Law of One Price) must follow a martingale.[1] And, indeed, deviations from Purchasing Power Parity (arising mainly from deviations from the LOP) have been found empirically to follow a process which cannot be distinguished statistically from a martingale (Rogalski and Vinso (1978), Roll (1979), Adler and Lehman (1983), Huizenga (1986)).

Roll's reasoning, however, is not incontrovertible, for several reasons. It contains one inconsistency. Speculators consume goods to live; they presumably consume the goods available in their respective countries and evaluate returns in real units of these goods. But no capital market equilibrium is possible between risk-neutral investors who evaluate returns in different units, when the relative prices of these units fluctuate randomly.

Roll's reasoning must therefore be extended to incorporate risk-averse consumer-investors. This extension will prove fatal to the strict martingale result (but not to the spirit of ex-ante PPP).

Furthermore, at least two aspects of the Roll reasoning cry out for a generalization. First, it is unrealistic to postulate that storage is the only intertemporal utilization of physical goods. Production and, generally, physical investment opportunities must be made available in both countries. The question is interesting because these investment opportunities' rates of return must have an impact on the conditionally expected rate of change of the deviation from the LOP.

Secondly, the total absence of physical transfers of goods in Roll's model is equally unrealistic. Trade economists should be unwilling to admit the possibility that deviations from the LOP could forever wander away from the zero mark: beyond a point, surely, spatial arbitrage becomes profitable and goods which are nontraded when the deviation is small, get to be traded when the deviation is larger than the cost of trading. Hence there should be some reversion tendency in the LOP deviation. There is an apparent conflict between the LOP reversion produced by trade (no matter how costly) and the Roll argument in favor of a martingale.

The model presented in this paper resolves this conflict by producing a stochastic process for LOP deviations which is compatible with market efficiency and with the possibility of moving goods across the world, while being able also to invest them as productive assets in more than one country. It is found that, under risk aversion, the process of LOP deviations is a not a marginale.[2]

In developing this model, I have borrowed from at least two strands of literature. The first strand comprises International Asset Pricing Models

(IAPMs) of various kinds; it has been reviewed by Adler and Dumas (1983). Generally models of international asset pricing have focused on the fact that investors of different countries consume goods available in their country of residence and, therefore, when PPP does not hold (at the level of consumer prices), evaluate real returns from their investments differently.[3] This feature is mirrored here: the two national categories of investors I will consider will consume goods located in their respective countries. As a result, an IAPM will formally hold. It will not be plagued, however, by some of the inconsistencies which are present in this literature.[4]

Secondly, the macroeconomic literature on capital formation has emphasized an interpretation of Tobin's q theory (cf. Tobin (1961, 1969), Brainard and Tobin (1968)) based on costs or delays incurred when installing capital (cf. Eisner and Strotz (1963), Gould (1968), Lucas (1967a, b) Mussa (1977), Treadway (1969), Kydland and Prescott (1982)). This literature has recently been extended to stochastic settings notably by Pyndick (1982), Abel (1983, 1985) and applied to the Finance field notably by Brennan and Schwartz (1985), Majd and Pyndick (1987) and Myers and Majd (1987). A 'symmetric' literature considers costs incurred when dismantling or retro-fitting capital, the limiting case being the one where investment is irreversible (Nickell (1974), McDonald and Siegel (1986), Pyndick (1986), Bertola (1987)). Such a specification also causes Tobin's q to be different from 1. The deviation from the LOP which I derive here is in the nature of a Tobin's q which would reflect both kinds of costs of adjustment. While Tobin's q is defined as the price of installed capital relative to its market (or consumption) value, my LOP deviation is defined as the price of physical capital located in one country relative to the price of capital located in the other, when it is costly to transfer capital between the two places in either direction.[5,6]

Finally, the model of this paper may be viewed as a real-side alternative to some recent efforts at explaining international capital flows, made by international monetary economists. The work of Svensson (1985) and Stockman-Svensson (1987) relates capital flows to monetary policy via a cash-in-advance constraint. We make no attempt to introduce money in our setting, although that could certainly be done.

For mathematical techniques, I am indebted to the work of Constantinides (1986) and Grossman-Laroque (1987) on portfolio choice under transactions costs, and to Krylov (1981)'s formulation of the optimal-stopping problem.

The outline of the paper is as follows. Section 2 explains and defends the modelling choices (assumptions) which have been made. Section 3 shows the mathematical derivation leading to the kingpin of the model: the indirect welfare function for various types of wealths. Section 4 describes how the goods move across the world in general equilibrium. Section 5 features the main result: the process for deviations from the Law of One Price. Section 6 examines the consequences of LOP deviations for differences in real interest rates between countries.

2. Modelling choices.

I consider a world economy populated with consumers who are identical to each other, except for the fact that they live (in equal numbers) in two different geographic locations (countries), with the constraint that they can only consume goods physically available in their country of residence.[7] There is only one good, except for the fact that one must distinguish two versions of this good, depending on its physical location at any given time. In both locations, the good in question can be consumed, invested in a random, constant-return-to-scale production process, or transferred abroad.

The world is perfectly symmetrical: not only have the consumer-investors of both countries the same risk aversion, which is assumed constant,[8] but their initial endowments are such as to warrant a symmetric treatment (more details below). Furthermore, the production processes of both countries have the same expected rate of return and standard deviation of rate of return. The output shocks in the two countries are uncorrelated.

Shipping capital abroad takes time or consumes resources; this assumption is made as a convenient device to produce LOP deviations and to explore their dynamic behavior in financial market equilibrium. Depending on the interpretation, shipping may entail a real cost or an opportunity cost; viz. while goods are on ship, they do not serve as physical capital in the production process, so that an output flow is foregone.

For reasons of symmetry and portfolio diversification, consumer-investors of both countries would ideally like the two stocks of goods accumulated in the two countries to be equal to each other. Despite this fact, an imbalance can develop, and can even persist, as a result of cumulated random output shocks, and optimal shipments and consumption rates. If and when an imbalance develops between the two stocks of goods, it may not pay to correct it by transferring goods from the country where they are more abundant to the country where they are less abundant. Instead, within a range of tolerance, the consumer-investors who are fortunate enough to be in the country where abundance prevails, consume more, (without having to offer their foreign counterpart any compensation[9]) and, in doing so, contribute to some rebalancing. The first order of business (sections 3 and 4) is to determine this range of tolerance within which no shipping takes place.

Since one is only interested in obtaining prices, it is possible to avoid the painful derivation of portfolio holdings and financial market equilibrium, by assuming that consumer-investors can achieve a (constrained) Pareto optimal allocation of consumption, by means of a sufficiently rich (perhaps complete) capital market. This is a Pareto optimum constrained by the fact that trade between the two countries is costly. The assumption of Pareto optimality is tenable considering that the model includes no hindrance to the exchange of securities or goods between individuals,[10] only to the movements of goods between locations: if a person ships goods from one place to the other, he or she suffers the real or opportunity cost of shipping, whether or not he retains ownership of the goods, whereas, if a person sells a stock of goods to another, without moving them, no cost is born, whether or not the buyer and the seller reside in the same country. Under this assumption, the capital-market and goods-market equilibrium can be replicated advantageously by an appropriate central planning problem.[11] Implicit prices--which would prevail explicitly in decentralized markets--can then be obtained as the derivatives of the appropriate indirect welfare function for (the various forms of) wealths.

3. The central planning problem.

I assume that all consumers initially start their lives with endowments of goods, such that the appropriate central welfare function devotes equal weights to the utility levels of the households of the two countries.[12,13] Accordingly, the central optimization problem is written as follows:

(1) $$V(K, K^*) = \max_{\substack{c, c^* \\ x, x^* \geq 0}} E_t \int_t^\infty e^{-\rho(u-t)}\left(\frac{1}{\gamma} c_u^\gamma + \frac{1}{\gamma}(c_u^*)^\gamma\right) du$$

s.t.:

Whenever $x = x^* = 0$:

(2) $$dK = (\alpha K_t - c_t)dt + \sigma K_t dz_t$$

(3) $$dK_t^* = (\alpha K_t^* - c_t^*)dt + \sigma K_t^* dz_t^* .$$

If, at some time τ, $x > 0$:

(4a) $$K_\tau = K_{\tau-} - x_\tau$$

(4b) $$K_\tau^* = K_{\tau-}^* + sx_\tau \qquad s < 1$$

If, at some time τ, $x^* > 0$:

(5a) $$K_\tau^* = K_{\tau-}^* - x_\tau^*$$

(5b) $$K_\tau = K_{\tau-} + sx_\tau ;$$

where:

$1 - \gamma$ is the degree of risk aversion common to all investors;

c is the rate of consumption of the good located in one location which I arbitrarily label the 'home location';

c^* is the rate of consumption of the good located in the other location arbitrarily labelled the 'foreign location';

x_τ is the (lumpy and positive) amount of goods being transferred, at time τ, from the home country to the foreign country;

x_τ^* is the (lumpy and positive) amount of goods being transferred at time τ, from the foreign country to the home country;

$s < 1$ reflects either shipping cost or production foregone during shipping delay.[14]

K, K^* are the stocks of capital located in the home and foreign countries respectively.

Equation (1) is the equally-weighted welfare function of the central planner. Equations (2), (4a) and (5b) indicate that the stock K of goods located in the home country is depleted by:

-consumption at home (c);

-shipments (x);

and replenished by;

-output $K(\alpha dt + \sigma dz)$, where dz is white noise;

-the arrival of an amount sx^* of goods, transferred from abroad.

This problem has two state variables K and K*. Considering that the utility-of-consumption function is isoelastic, it is reasonable to assume that the 'correct' solution of this problem[15] has a value function V which is homogenous of degree γ in K and K*, and symmetric with respect to these two variables. I now make use of these properties to characterize the solution and to reduce the dimensionality of the problem.

I first rewrite problem (1 - 5) as an Optimal Stopping Problem (Krylov (1981)):

$$(6) \qquad V(K, K^*) = \max_{\substack{c,c^* \\ x,x^*>0}} E_t \int_t^{\tau} e^{-\rho(u-t)}\left(\frac{1}{\gamma} c_u^{\gamma} + \frac{1}{\gamma}(c_u^*)^{\gamma}\right)du$$

$$+ E_t e^{-\rho(\tau-t)} V(K_\tau - x_\tau + sx_\tau^*, K_\tau^* - x_\tau^* + sx_\tau)$$

subject to constraints (2) and (3). $\tau > t$ is the first time at which x or $x^* > 0$.

As has been noted by Constantinides (1986), the first order conditions with respect to the shipping decisions x and x* are:

$$(7a) \qquad V_1(K - x, K^* + sx) = sV_2(K - x, K^* + sx) \ ;$$

$$(7b) \qquad sV_1(K + sx^*, K^* - x^*) = V_2(K + sx^*, K^* - x^*) \ .$$

These equations implicitly assume that x and x^* are not simultaneously positive. Considering the homogeneity and symmetry of the function V, it is clear that these first-order conditions will be satisfied along (at least) one pair of rays:

(8a) $$K - x = \lambda(K^* + sx) \ ;$$

(8b) $$K^* - x^* = \lambda(K + sx^*) \ ;$$

of slopes $\lambda(>1)$ and $1/\lambda$ in the (K, K^*) plane. These two rays delimit a cone ($K \leq \lambda K^*$ and $K^* \leq \lambda K$), within which the optimal decision is $x = x^* = 0$. If, however, $K > \lambda K^*$, or $K^* > \lambda K$, the optimal decision is to choose $x > 0$ or $x^* > 0$ satisfying equations (8a) and (8b) respectively; i.e., **the optimal decision is to ship in such a way as to get back to the frontier of the cone.**

Outside the cone, the indirect indifference 'curves' are straight lines, as shown in figure 1. I.e. there exists a function U() such that:

(9a) $$V(K, K^*) = U(K^* + sK) \text{ when } K > \lambda K^* \ ;$$

and:

(9b) $$V(K, K^*) = U(K + sK^*) \text{ when } K^* > \lambda K \ .$$

Outside the cone, when shipping is optimal, wealth, which otherwise must be measured by two numbers K and K^*, can be measured as one number equal to the amount of the good located in the country where it is scarce, plus the "translated amount" of the good located where it is abundant. The translation factor in both cases is $s < 1$, which is the shipping loss factor. More specifically, based on the postulated homogeneity of degree γ of the function V, the function U which applies outside the cone must be of the following form:

(10a) $$U(K^* + sK) = \mu(K^* + sK)^{\gamma}/\gamma \qquad \text{when } K > \lambda K^* \ ;$$

(10b) $$U(K + sK^*) = \mu(K + sK^*)^{\gamma}/\gamma \quad \text{when } K^* > \lambda K .$$

where μ is a positive number to be determined.[16]

FIGURE 1 GOES HERE

When the solution x, x* of (8), (as well as the solution c, c* of obvious first-order conditions for the consumption rates),[17] have been substituted in, the optimization problem reduces to a choice of two parameters μ and λ and the Hamilton-Jacobi equation characterizing the function V can be written as follows for values of K and K* in the interior of the cone:[18]

(11) $$0 = (\frac{1}{\gamma} - 1)V_1^{\frac{\gamma}{\gamma-1}} + (\frac{1}{\gamma} - 1)V_2^{\frac{\gamma}{\gamma-1}} - \rho V$$

$$+ V_1\alpha K + V_2\alpha K^* + \tfrac{1}{2}V_{11}\sigma^2K^2 + \tfrac{1}{2}V_{22}\sigma^2K^{*2} ;$$

$$K^*/\lambda \leq K \leq \lambda K^*$$

subject to:

(12a) $$V_1(K, K^*) = sV_2(K, K^*) \quad \text{when } K = \lambda K^* ;$$

(12b) $$sV_1(K, K^*) = V_2(K, K^*) \quad \text{when } \lambda K = K^* ;$$

(13a) $$V(K, K^*) = \frac{\mu}{\gamma}(K^* + sK)^{\gamma} \quad \text{when } K = \lambda K^* ;$$

(13b) $$V(K, K^*) = \frac{\mu}{\gamma}(K + sK^*)^{\gamma} \quad \text{when } \lambda K = K^* .$$

where equations (12a, b) follow from (7a, b), while equations (13a, b) follow from (9a, b) and (10a, b).

Since the unknown function V(K, K*) is required to be symmetric, equation (11) determines it up to one constant of integration. For a given value of μ, one of the two boundary conditions (13a, b) determines this constant, while condition (12a) or (12b) determines λ. If μ were given, that is, we would have a standard optimal-stopping problem in the manner of Krylov (1980) and boundary conditions (13) and (12) could be interpreted as the 'value-matching' and 'smooth-pasting' (or 'high-contact') conditions respectively. These conditions were made famous in the finance literature by Samuelson (1965), McKean (1965) and Merton (1973) in their treatment of the American-put pricing problem, with optimal early exercise. The smooth-pasting condition (12) is the first-order condition with respect to the optimal stopping time (in our case, this is the time to **start** shipping). It guarantees that, at the stopping point, the indirect-utility function is not only continuous (value matching) but also that its first derivatives are continuous.[19]

In our context, however, as in Constantinides (1986) and in Grossman-Laroque (1987), the parameter μ is to be chosen optimally, in such a way that the level of the indirect-utility function $V(K, K^*)$ is maximized. This opens the quest for an additional boundary condition which would represent the first-order condition with respect to that parameter.

To this aim, consider again system (11 - 13) but regard it now as being parameterized by λ instead of μ. Call $V(K, K^*, \chi)$ the general (symmetric) solution of (11), where χ is the constant of integration. Assume that the value of V is monotonically related to this constant of integration, so that maximizing V is equivalent to choosing the largest possible value of χ. This function V satisfies the following system of boundary conditions, for $K = \lambda K^*$ (and an analogous one for $K^* = \lambda K$)[20]:

(14a) $$V(\lambda, 1, \chi) = \frac{\mu}{\gamma}(1 + s\lambda)^{\gamma}$$

$$(15a)\quad \begin{aligned} V_1(\lambda,\ 1,\ \chi) &= \mu(1 + s\lambda)^{\gamma-1}s \\ V_2(\lambda,\ 1,\ \chi) &= \mu(1 + s\lambda)^{\gamma-1} \end{aligned} \quad .$$

Now, totally differentiate this system with respect to the choice of λ, imposing that $d\chi/d\lambda = 0$. Equation (14a) gives:

$$(16)\quad V_1(\lambda,\ 1,\ \chi) = \frac{\mu}{\gamma}(1 + s\lambda)^{\gamma} s + \frac{d\mu/d\lambda}{\gamma}(1 + s\lambda)^{\gamma}$$

which, in conjunction with (15a), imposes that $d\mu/d\lambda = 0$ as well. Next, differentiate (15a) under both $d\chi/d\lambda = 0$ and $d\mu/d\lambda = 0$. This yields several dependent equations.

$$(17a)\quad V_{11}(\lambda,\ 1,\ \chi) = \mu(\gamma - 1)(1 + s\lambda)^{\gamma-2}s^2$$

$$(17b)\quad V_{21}(\lambda,\ 1,\ \chi) = \mu(\gamma - 1)(1 + s\lambda)^{\gamma-2}s$$

$$(17c)\quad V_{22}(\lambda,\ 1,\ \chi) = \mu(\gamma - 1)(1 + s\lambda)^{\gamma-2} \ .$$

These equations are dependent under (15a) by virtue of Euler's theorem for homogeneous functions, so that only one of them can be kept or one combination of them. No matter what form is chosen, equations (17) impose now the condition that the **second derivatives of the value function be continuous at the stopping point.**

This is a higher form of contact than is encountered in traditional optimal stopping problems (e.g. the early exercise of options) where the boundary at stopping time is fixed, and only the time at which to stop at this boundary is to be chosen optimally. Here, the boundary itself (the value of μ) is an object of choice. A 'higher-order contact' or 'super-contact' is then imposed to determine this additional dimension of choice. This condition apparently has not been recognized by Constantinides (1986) or Grossman-Laroque (1987). Constantinides ultimately imposes the condition by his choice of integration constants (see below our rendition of Constantinides) but he does not recognize the resulting super contact. Grossman-Laroque optimize the choice of μ (their M) after solving a system analogous to (11 - 13). They apparently do not realize either that higher contact results from this optimization.

Knowing that not only the value of the function, and its first derivatives, but also its second derivatives are continuous at the boundary, we can deduce that, at the boundary point, the boundary function $(\mu/\gamma)(K^* + sK)^\gamma$ (when $K = \lambda K^*$) must satisfy the second-order differential equation (11). This generates a relationship between μ and λ which is very useful in obtaining solutions[21]:

$$(18) \qquad 0 = \left(\frac{1}{\gamma} - 1\right)(\mu)^{\frac{1}{\gamma-1}} \left(s^{\frac{\gamma}{\gamma-1}} + 1\right) - \rho \frac{1}{\gamma} + \alpha + \tfrac{1}{2}(\gamma - 1) \frac{1 + s^2\lambda^2}{(1 + s\lambda)^2} \sigma^2$$

Finally, one can make use again of the homogeneity of the V function to introduce a change of variable and of unknown function:

$$(19a) \qquad \omega = \frac{K}{K + K^*}$$

$$(19b) \qquad V(K, K^*) = \frac{1}{\gamma} (K + K^*)^\gamma I(\omega)$$

$$\frac{\mu}{\gamma} (K^* + sK)^\gamma = \frac{\mu}{\gamma} (K + K^*)^\gamma (1 - \omega + s\omega)^\gamma$$

$$\frac{\mu}{\gamma} (K + sK^*)^\gamma = \frac{\mu}{\gamma} (K + K^*)^\gamma \left(\omega + s(1 - \omega)\right)^\gamma ,$$

in order to reap a major advantage: the transformation of the unknown function into a function of only one variable, and the transformation of the partial differential equation (11) into an ordinary differential equation. Details of this substitution, and the way in which the resulting boundary-value problem is solved numerically,[22] are laid out in Appendix II.

For purposes of comparison, it is useful to have a lower bound for the scaled indirect utility function $I(\omega)$. One lower bound is provided by an approximation wherein shipments are optimally chosen but consumption rates in the two countries are suboptimal. This approximation is in the spirit of the one used by Constantinides (1986). Specifically, if one sets the rates of consumption, out of the two stocks of goods, to be constant and equal in both countries,[23]

(20) $$c/K = c^*/K^* = \beta ,$$

then the Hamilton-Jacobi equation pertaining to the inside of the cone, subject to boundary conditions (12) has an explicit solution in the following form:

(21) $$V(K, K^*, \beta, \lambda) = \nu(\beta)(K^\gamma + K^{*\gamma}) + \delta(\beta, \lambda)(K^{\gamma-\varepsilon}K^{*\varepsilon} + K^\varepsilon K^{*\gamma-\varepsilon}) ,$$

where $\varepsilon = \varepsilon(\beta)$ and $\gamma - \varepsilon$ are the two solutions (assumed to be real and positive) of the following characteristic equation:

(22) $$0 = -\rho + \gamma(\alpha - \beta) + \tfrac{1}{2}(\gamma - \varepsilon)(\gamma - \varepsilon - 1)\sigma^2 + \tfrac{1}{2}\,\varepsilon(\varepsilon - 1)\sigma^2$$

while ν and δ are given by:

(23) $$\nu(\beta) = -\frac{\beta^\gamma/\gamma}{-\rho + \gamma(\alpha - \beta) + \tfrac{1}{2}\gamma(\gamma - 1)\sigma^2}$$

(24) $$\delta(\beta, \lambda) = \nu\gamma\,\frac{-s + \lambda^{\gamma-1}}{s(\gamma - \varepsilon)\lambda^\varepsilon - \varepsilon\lambda^{\varepsilon-1} + s\varepsilon\lambda^{\gamma-\varepsilon} + (\gamma - \varepsilon)\lambda^{\gamma-\varepsilon-1}} .$$

The opening of the cone λ and the constant rate of consumption β are then chosen to maximize V. As far as λ is concerned, this can be done uniformly and the question reduces to that of maximizing $\delta(\beta, \lambda)$ with respect to λ.[24] The result depends on β. Maximizing V (equation (21) above) with respect to the artificial consumption rate β cannot be done uniformly; i.e. one obtains a different answer for different values of K/K^*. A particular value can be obtained for the perfect-diversification situation $K/K^* = 1$. This procedure will be referred to below as 'the Constantinides approximation'.

4. The equilibrium process for the allocation of physical capital

Everything one may want to know about the equilibrium behavior of the economy, which I am considering, can be derived from the knowledge of the function $I(\omega)$. In this section, I concentrate on the dynamics of the physical stocks of capital, whereas the main object of our exercise--the dynamics of prices--will be considered in the next two sections. As far as physical quantities are concerned, two topics are of interest: the size of the cone of no shipping and the behavior of quantities inside the cone. Even though no shipping takes place inside the cone, endogenous consumption rates do tend to rebalance the stocks of goods.

The size of the cone, i.e., the value of λ, the tolerated imbalance--has been obtained, by the numerical process described in Appendix II, for varying degrees of risk aversion and varying degrees of risk. The results are displayed in table 1.

Observe that, under certainty ($\sigma = 0$), increasing risk aversion (which is, in fact, decreasing the rate of intertemporal substitution and decreasing the rate of inter-personal substitution) monotonically reduces the opening of the cone.[25] Under any level of risk, the variation as a function of risk aversion is still monotonic and operates in the same way. In particular, observe that as one approaches risk neutrality, the cone of no shipping widens indefinitely so that shipping occurs almost never.[26]

But note the manner in which increasing risk affects the size of the cone: it increases it.[27] This is somewhat surprising because increasing risk apparently has an effect opposite to that of increasing risk aversion. Furthermore, in the context of portfolio choice under transactions costs, Constantinides (1986) found that increasing risk decreased the opening of the cone of no transactions (see his Table 3, page 854). But there are several differences between Constantinides' problem and ours. Our problem is symmetric as far as the choice of assets is concerned: we choose between two physical investments of equal volatility but located in different places, whereas his investor faced a choice between a risky and a riskless asset. Increasing risk in his context meant increasing the risk of one asset only, not both as we do here. Furthermore, our problem is symmetric as far as the origin of consumption is concerned: people of each country consume out of the stock of goods available locally, whereas Constantinides' investors consumed only out of the wealth invested in one asset: the riskless one. Finally, Constantinides utilized an approximate optimization procedure yielding a suboptimal result, while our results are optimal.[28]

In order to verify that these differences between the two settings account for the difference in the results, we applied the 'Constantinides approximation' to the present problem.[29] The results are displayed in Table 2. They confirm that the opening of the cone decreases with increasing risk aversion but increases with higher risk. The values obtained for λ in Table 2 are uniformly lower than those of the exact Table 1. This makes sense: when differential consumption rates in the two countries are allowed to bring about some rebalancing, less shipping is needed. The difference between Table 1 and

Table 2 reflects the effect of constraining consumption rates out of available stocks, to being equal in the two countries (and constant over time).

TABLES 1 AND 2 GO HERE

The behavior of ω (the allocation of the goods between the two places) inside the cone, as a result of production shocks and differential consumption rates, can be obtained easily from the knowledge of the value function. Applying Ito's lemma to the definition (19a) of ω produces:

(25) $$d\omega = \omega(1-\omega)\left[\frac{dK}{K} - \frac{dK^*}{K^*} - \omega\left(\frac{dK}{K}\right)^2 + (1-\omega)\left(\frac{dK^*}{K^*}\right)^2\right] .$$

Based on (2) and (3), this is also:

(26) $$d\omega = \omega(1-\omega)\left\{\left[-\frac{c}{K} + \frac{c^*}{K^*} + (1-2\omega)\sigma^2\right]dt + \sigma\sqrt{2}\, d\bar{z}\right\}$$

where $d\bar{z} = (dz - dz^*)/\sqrt{2}$ is a standardized white noise. Finally, defining the following quantities, in order to save on notations:

(27) $$N(\omega) = I(\omega) + \frac{1}{\gamma} I'(\omega)(1-\omega)$$

(28) $$D(\omega) = I(\omega) + \frac{1}{\gamma} I'(\omega)(-\omega)$$

we have, in sequence (based on (15b)):

(29a) $$V_1 = (K + K^*)^{\gamma-1} N(\omega)$$

(29b) $$V_2 = (K + K^*)^{\gamma-1} D(\omega)$$

(30a) $$\frac{c}{K} = \frac{1}{\omega}\left(N(\omega)\right)^{\frac{1}{\gamma-1}}$$

(30b) $$\frac{c^*}{K^*} = \frac{1}{1-\omega}\left(D(\omega)\right)^{\frac{1}{\gamma-1}}$$

and substituting (39a, b) into (35) fully determines the behavior of ω inside the cone.

I have displayed in figures 2 and 3 the conditional expected change (drift) and the conditional standard deviation (diffusion coefficient, including a sign reflecting the direction of the effects of the shocks dz and dz*) of the ω process as functions of the current level of ω (which puts the ω process in the form of an autoregressive process of order 1), for the numerical values of the parameter indicated in Appendix II. These describe the process inside the cone. Observe the following:

-the drift is zero at the centerpoint ($\omega = 0.5$) when the two stocks of goods are balanced. It is constant in a neighborhood of the edges of the cone. The sign of the drift, which reflects purely the differential consumption rates, is always such as to draw ω towards the centerpoint or equivalently towards the interior of the cone. The process is AR(1) but necessarily a **non linear AR(1);**

-the diffusion coefficient (i.e., the volatility of the allocation of goods between the two locations) is variable (equal to $\omega(1 - \omega)\sqrt{2}$), so that the process must be classified as AR(1) **with conditional heteroscedasticity.** The coefficient is always of the same sign and is largest in absolute value at the centerpoint. On the edges, the diffusion coefficient does not vanish, so that the edges do not act as a natural boundary: there is a positive probability of reaching them in finite time. They are in fact 'regular boundaries' in the sense of Feller.[30]

FIGURES 2 AND 3 GO HERE

Because of the nature of the boundaries--i.e., because they can be reached--it behooves us to specify the behavior of the process on the boundaries. This behavior follows from our earlier description of optimal shipping decisions: when ω is close to a boundary (on the inside of the cone) a random output shock ($dz > \lambda dz^*$ on the right boundary; $\lambda dz < dz^*$ on the left boundary) can create such an imbalance as to trigger immediate shipping. The amount of the shipment will be exactly such as to bring ω back to the frontier immediately (see figure 4). Nonetheless, the process cannot spend a finite time on the frontier, because the probability of $dz - \lambda dz^*$ (or $\lambda dz - dz^*$, as the case may be) being of the same sign steadily during a finite period of time is zero. In other words, the time spent on the boundary is not zero (as it would be if the boundary were a reflecting one); it is infinitely small but strictly positive,[31] i.e. it is not strictly equal to zero, as would be the case if the boundary were regular and reflecting (or were a natural boundary).

FIGURE 4 GOES HERE

5. The equilibrium process of deviations from the Law of One Price

Even though I have formulated the optimization problem as a centralized one, one can infer the prices which would prevail in a decentralized market economy, by looking at the first derivatives of the value function $V(K, K^*)$. This was, of course, the main purpose of the determination of this function.

Indeed, define p as the price of physical capital located at home relative to capital located abroad, (the price of a unit of K relative to a unit of K*). This price is obviously given by:

$$p = \frac{V_1(K, K^*)}{V_2(K, K^*)} . \tag{31}$$

Because of the homogeneity of V, p is a function of ω only. This function is (recall (29a, b), (27), (28)):

$$p(\omega) = \frac{N(\omega)}{D(\omega)} . \tag{32}$$

The Law of One Price prevails when p=1.

Even though p is the price which would indeed prevail in the goods/ capital market, if one were organized in this economy, it would be inconvenient to study the stochastic process of p itself, whose definition is asymmetric in nature: interchanging the two goods changes p into 1/p, which is a non linear transformation. The symmetry will be preserved if, instead, one studies the behavior of the **relative** deviation from the LOP. This relative deviation π will be defined as:

$$\pi = \frac{p - 1}{(p + 1)/2} = 2 \frac{N(\omega) - D(\omega)}{N(\omega) + D(\omega)} ; \tag{33}$$

p being the price of a unit of K, 1 being the price of a unit of K*, π is the difference between these two prices, relative to the average price. Knowing the $I(\omega)$ function, equations (27), (28) and (33) provide π as a function of ω. Since the process of ω is known,[32] it is an easy matter to obtain the process of π.

The functions $p(\omega)$ and $\pi(\omega)$ are displayed in figures 5 and 6. The π function is indeed symmetric whereas the p function, in fact, is not. As expected, and as has been imposed by the boundary conditions (13a, b) p reaches the values s and 1/s at the two extremities: under active shipping, the good located in the country of abundance (from where shipping originates) is s times less valuable than the good located in the country where it is scarce. At the boundaries, π reaches, of course, the corresponding values $\pm 2(s - 1)/(1 + s)$. **The slope $\pi'(\omega)$ at those points is zero by virtue of the super contact condition (17).**[33] When no shipping takes place, the price is somewhere between s and 1/s, depending on the degree of imbalance in the two stocks of goods. Under perfect balance in the quantities, $\pi = 0$ and the LOP prevails.

FIGURES 5 AND 6 GO HERE

I have displayed in figures 7 and 8 the conditional expected change (drift) and the conditional standard deviation (in fact the signed diffusion coefficient) of the π process as functions of the current level of π (which puts the π process in the form of an autoregressive process of order 1), for the now usual numerical values of the parameters. Observe the following:

-the drift of the process reflects **a reversion tendency**, since the drift in figure 7 is negative for a positive deviation from the LOP and **vice-versa**. Mean reversion is particularly strong near the borders. This is true despite capital market efficiency (or rational expectations). The reason for this phenomenon is twofold: first, ω itself exhibits mean reversion (see section 3) as a result of differences in consumption rates between the two countries; secondly, the $\pi(\omega)$ function relating the LOP deviation to the quantity imbalance tapers off as one reaches the boundaries (see figure 6), reflecting the fact that the price, as a result of shipping, will not be able to escape from the interval [s, 1/s]. The mean reversion of the LOP deviation displayed in figure 7 shows conclusively that **the process of deviations from the LOP is not a martingale.**[34]

-figure 8 indicates that the process of deviations **is strongly heteroscedastic:** the conditional standard deviation is much smaller near the boundaries than at the centerpoint. In fact, the standard deviation of the process becomes zero at the boundaries. It has been noted above that the slope of the $\pi(\omega)$ function is zero at the boundaries; this accounts for a diffusion coefficient of π equal to zero, even though that of ω is not zero. Despite the zero volatility at the boundaries, the extreme values s and 1/s have a positive probability of being reached in finite time. The boundaries, here again, are regular boundaries and boundary behavior is qualitatively the same as for the ω process. In fact, considering the shape of the $\pi(\omega)$ function noted above (see again figure 6), and considering the transformation of distances which it induces, the π process is 'near' its boundaries more frequently than did the ω process.[35]

FIGURES 7 AND 8 GO HERE

6. The real interest rate differential

As soon as the relative price of two goods, or two varieties of a good, fluctuates over time, the rate of interest measured in units of one of them is not equal to the rate of interest measured in units of the other.[36] Furthermore, if the fluctuations of the relative price are random, a financial asset which would be riskless when its rate of return is evaluated in units of one of the goods, no longer is when its rate of return is evaluated in terms of the other. One must, therefore, be careful to distinguish four quantities which are conceptually quite different:

-quantity 1: the rate of interest on an asset which is riskless in terms of K (the good located at home); this is the **own rate** measured in units of K itself;

-quantity 2: the expected value of the rate of return on this K-riskless asset measured in terms of K* (the good located abroad);[37] quantities 1 and 2 differ only by the expected value of the change in the relative price; i.e; they differ if and only if the price does not follow a martingale process;

-quantity 3: the own rate of interest on an asset which is riskless in terms of K^*; quantities 2 and 3 differ by a 'risk premium';[38]

-quantity 4: the expected rate of return on the K^*-riskless asset measured in terms of K; quantity 4 differs from quantity 3 by the expected price change;[39] quantity 4 differs from quantity 1 by a risk premium.[40]

Below, I measure the spread between quantity 1 (denoted r) and quantity 3 (denoted r^*), and I call this spread 'the' real interest rate differential. But I break down this total (diagonal) spread into a component related to the expected price change and one which constitutes a risk premium. In this way, the field is completely covered and the other quantities can be reconstructed, if one so desires.

It has been shown by Cox, Ingersoll and Ross (1985) (and their proof could be replicated in our context) that, in a decentralized economy, the riskless interest rate is equal to the discount factor of utilities (ρ in our notation; see equation (1)) minus the conditionally expected rate of change in the (undiscounted) marginal indirect utility of wealth. It follows that[41]:

$$(34) \qquad r^* - r = \frac{1}{V_1}\frac{E(dV_1)}{dt} - \frac{1}{V_2}\frac{E(dV_2)}{dt}$$

But, applying Ito's lemma successively to V_1 and V_2 given by (29a, b), then again to π given by (33), and a third time to N and D, one obtains[42]:

$$(35) \qquad r^* - r = \frac{1}{N}\frac{E(dN)}{dt} - \frac{1}{D}\frac{E(dD)}{dt}$$

$$(36) \qquad r^* - r = \frac{4}{4-\pi^2}\frac{E(d\pi)}{dt} - \{- \frac{\pi+2}{4}\frac{1}{dt}(\frac{dN}{N})^2 + \frac{-\pi+2}{4}(\frac{dD}{D})^2$$

$$+ \frac{1}{2}\frac{1}{dt}\frac{dN}{N}\frac{dD}{D}\}$$

(37) $$r^* - r = \frac{4}{4 - \pi^2}\frac{E(d\pi)}{dt}$$

$$- \{- \frac{\pi + 2}{4}\left[\frac{N'(\omega)}{N(\omega)}\right]^2 + \frac{-\pi + 2}{4}\left[\frac{D'(\omega)}{D(\omega)}\right]^2$$

$$+ \frac{1}{2}\frac{N'(\omega)D'(\omega)}{N(\omega)D(\omega)}\} \, 2\sigma^2\omega^2(1 - \omega)^2$$

In equation (37), the first term reflects the expected price change (obtained in section 5 above), while the second term reflects the risk premium.

Every element on the right-hand side of (37) is already known as a function of ω.[43] This equation, therefore, yields the real interest-rate differential as a function of the physical imbalance ω. The differential and its components are plotted as figure 9, for the usual numerical values of the parameters (recall that these parameters included a risk aversion equal to 2). Observe the following:

-the differential, the expected price change and the risk premium are zero at the centerpoint $\omega = 0.5$; this is the result of the symmetric definition of the price variable π;

-the rate differential behaves very much like the expected price change. They both increase monotonically with the physical imbalance; they both reach a maximum on the edges of the cone. A large real-rate differential is an indication of strong expected reversion in the LOP deviation.[44] It can equally well be said that shipping is triggered when the interest rate differential reaches its largest possible value, or when the deviation from the LOP reaches its largest possible value;

-the risk premium, which is the difference in the expected rates of return of home vs foreign assets **measured in the same units,**[45] behaves in an interesting way: (i) its sign is always such as to reduce the absolute value of the interest rate differential; it is equal to zero and changes sign when the physical imbalance ω passes the centerpoint $\omega = 0.5$; (ii) **it is equal to zero again on the edges**[46] (when the price variable π has a zero volatility; see figure 8 and the comments in section 5); (iii) it reaches its largest absolute value somewhere between the centerpoint and the edges, so that the risk premium and the expected price change do not co-vary in the same way in a neighborhood of the centerpoint and in a neighborhood of the extremities; (iv) despite the fairly large amount of risk assumed in our numerical example, the risk premium remains small in comparison to the expected price change and its possible fluctuations over time are also small in comparison with those of the expected price change. These results are of some interest in view of Fama (1984)'s remark that risk premia and expected exchange rate changes must co-vary in a systematic way (and undergo fluctuations of comparable magnitudes), in order to account for the discrepancies which have been observed, in the foreign exchange markets, between forward rates and subsequent spot rates.

FIGURE 9 GOES HERE

7. Conclusion

The main result of this paper pertains to the dynamic behavior of deviations from the Law of One Price under sluggish quantity adjustment. The deviations do not follow a martingale process. Considering the Markov specification of the model, it is not surprising that the process found should be AR(1). It is also not surprising that the deviation should remain between two boundary values. The more interesting aspects concern the non linearity and heteroscedasticity of the AR(1) process. The conditionally expected change of the deviation is a non linear function of the current deviation (see figure 7); its sign is such as to produce mean reversion (which is strongest when the deviation is largest, near and at the boundaries). The conditional standard deviation of the deviation is largest when the deviation is zero; it is zero when the deviation is at its largest possible value, at the boundaries. As a result, the process is most of the time at or near one of the boundaries which means that LOP deviations typically last "a long time." Boundary behavior is also interesting: the boundaries can be reached in finite time with positive probability; once it is at the boundaries, the deviation spends there an infinitely small, but strictly positive, amount of time.

Econometricians will have to determine whether empirical techniques (in the space or frequency domains) which have been applied in the past to the study of LOP deviations, remain valid for this type of process. Even if they do, there is no doubt that the knowledge of the exact form of the process should help greatly increase the power of the tests. At the very least, the theory indicates that observations of changes in LOP deviations should be segregated according to the current value of the absolute deviation. For instance, the empirical analysis would have the strongest power in detecting mean reversion when the absolute deviation is large; for this is the situation where the reversion tendency is largest and the standard deviation smallest.

Throughout this article, one has stopped short of calling the LOP deviation 'the real exchange rate', for the reason that the model features no currencies.[47] But this may be unnecessary timidity. Existing monetary models fall into two categories: the 'flexible-price' models introduce money but often assume that Purchasing-Power Parity holds. Since one major interest of introducing money in an international model would be to explain the very high observed correlation between nominal and real (i.e. purchasing-power deflated) exchange rates, such an assumption, which sets the real rate at 1, guts these models.[48]

Another category of monetary models succeeds in 'explaining' the high correlation between nominal and deflated exchange rates. I am referring to the 'sticky-price' models. Unfortunately, they are silent on the reasons which cause commodities prices to be sticky. Surely, it has something to do with the trading technology. While the technology which has been chosen here may not be the most adequate to generate price stickiness, one essential idea has nonetheless been captured: to the extent that costs of adjustments in moving capital from country to country are higher than in moving capital within a country, we can expect international prices of physical assets to be more volatile than **intranational** prices.

One aspect does necessitate an apology: trade, in our model, often does not take place and, when it does, it is jittery, since it is a direct response to the last output shocks in the two countries. These features surely contradict casual experience. Furthermore, they cause this model to fall short of a simple reconciliation of the traditional flow and stock approaches to international capital movements.[49] The jittery behavior of trade is the direct product of the linearity (actually proportionality) of shipping costs which was assumed for convenience.[50] Under a concave cost structure, shipments would be smoothed out. This problem, however, remains to be solved.

Appendix I:

The 'time-to-ship' interpretation

Instead of constructing the model around an out-of-pocket shipping cost, it is equally possible to regard the international transfer of capital as an activity which takes time. During that transfer time capital is idle so that an opportunity cost comes into the picture.

In this formulation, the objective function is still, of course, as in equation (1). But the equations describing the dynamics of the system are as follows:

whenever $x = x^* = 0$,

(I1) $$dW_{1t} = (W_{1t}\alpha - c_t + \phi W_{3t})dt + W_{1t}\sigma dz$$

(I2) $$dW_{2t} = -\phi W_{2t}dt + W_{2t}\sigma dz^*$$

(I3) $$dW_{3t} = -\phi W_{3t}dt + W_{3t}\sigma dz$$

(I4) $$dW_{4t} = (W_{4t}\alpha - c^*_t + \phi W_{2t})dt + W_{4t}\sigma dz^* \;;$$

if, at some time τ, $x > 0$:

(I5) $$W_{1\tau} = W_{1\tau-} - x_\tau$$

(I6) $$W_{2\tau} = W_{2\tau-} + x_\tau \;;$$

if, at some time τ, $x^* > 0$:

(I7) $$W_{4\tau} = W_{4\tau-} - x^*_\tau$$

(I8) $$W_{3\tau} = W_{3\tau-} + x^*_\tau \;;$$

where:

x_τ is the (lumpy and positive) amount of goods being loaded, at time τ, into ships which will sail from the home country to the foreign country;

x^*_τ is the (lumpy and positive) amount of goods being loaded, at time τ, into ships which will sail from the foreign to the home country;

$W_{1t}, W_{1\tau}$ and $W_{1\tau-}$ are the stocks of goods located in the home country at time τ and $\tau-$ respectively;

$W_{2t}, W_{2\tau}, W_{2\tau-}$ represent the stocks of goods in transit, being transported from the home to the foreign country;

$W_{3t}, W_{3t}, W_{3\tau-}$ represent the stocks of goods in transit, being transported from the foreign to the home country;

$W_{4t}, W_{4\tau}, W_{4\tau-}$ are the stocks of goods located in the foreign location.

Equations (I1) and (I5), for instance, indicate that the stock W_{1t} of goods located in the home country is depleted by:

-consumption at home (c_t);

-shipments (x_t);

and replenished by:

-output $W_{1t}(\alpha dt + \sigma dz)$ (where dz is white noise);

-the arrival of an amount $\phi W_{3t} dt$ of goods, representing a constant fraction ϕ of the goods in transit in the direction of the home country.

Equations (I2) and (I6) indicate that the amount of goods W_2, in transit from the home to the foreign country, is what it is as a result of shipments x from one end, and arrivals $\phi W_2 dt$ at the other. In addition, the stock of goods aboard the ships receives random proportionals shocks $W_2 \sigma dz^*$, which are perfectly correlated with the output shocks of the destination country. The purpose of this specification is that stocks of goods aboard ships will not constitute investments media; shipping as a financial asset will be dominated by production at the destination point; shipping will therefore be undertaken solely for the purpose of conveying capital from one place to the other as needed.[51] In our model, there will be only two sources of risk: the output shocks in the two countries, which will affect the goods already in the country, as well as those about to reach it. The minute capital is loaded onto ship it transfers from one risk area to another, even though it will start earning returns only upon arrival.

This assumption reduces the dimensionality of the problem: it leads to a simple aggregation of the stocks of goods in transit with those located at the destination point. Indeed multiply equations (I2) and (I3) by a factor s equal to:

$$s = \frac{\phi/\alpha}{1 + \phi/\alpha} < 1 , \tag{I9}$$

and add them to equations (I4) and (I1) respectively. Then define the quantities:

$$K = W_1 + sW_3 \tag{I10}$$

$$K^* = W_4 + sW_2 . \tag{I11}$$

K is the stock of goods already at home or which will eventually reach the home country, given the 'shipping loss factor' s. The result of this grouping of equations is identical to equations (2) - (5b) of the text.

Appendix II:

Reducing the dimension of the boundary-value problem

The implementation of the change of variable (19a) and of the change of unknown function (19b) produces the following result

(II1)
$$0 = (1-\gamma)\Big[\Big(1 + \frac{1}{\gamma}\frac{I'}{I}(1-\omega)\Big)^{\frac{\gamma}{\gamma-1}}$$

$$+ \Big(1 + \frac{1}{\gamma}\frac{I'}{I}(-\omega)\Big)^{\frac{\gamma}{\gamma-1}}\Big] I^{\frac{1}{\gamma-1}}$$

$$- \rho + \gamma\big[\alpha - \tfrac{1}{2}\sigma^2(1-\gamma)(\omega^2 + (1-\omega)^2)\big]$$

$$+ \sigma^2 \frac{I''}{I}\omega^2(1-\omega)^2$$

$$- \sigma^2(1-\gamma)\frac{I'}{I}\omega(1-\omega)(2\omega - 1)$$

subject to 'value matching":

(II2a)
$$I(\omega) = \mu(1 - \omega + s\omega)^{\gamma} \text{ when } \omega = \frac{\lambda}{1+\lambda}$$

(II2b)
$$I(\omega) = \mu(\omega + s(1-\omega))^{\gamma} \text{ when } \omega = \frac{1}{1+\lambda}$$

'smooth-pasting':

(II3a)
$$\frac{I'}{I} = \gamma\frac{-1+s}{1-\omega+s\omega} \text{ when } \omega = \frac{\lambda}{1+\lambda}$$

(II3b)
$$\frac{I'}{I} = \gamma\frac{1-s}{\omega+s(1-\omega)} \text{ when } \omega = \frac{1}{1+\lambda}$$

'super contact':

$$\left(\frac{I'}{I}\right)' = -\gamma \frac{(-1 + s)^2}{1 - \omega + s\omega} \quad \text{when } \omega = \frac{\lambda}{1 + \lambda}$$

$$\left(\frac{I'}{I}\right)' = -\gamma \frac{(1 - s)^2}{\omega + s(1 - \omega)} \quad \text{when } \omega = \frac{1}{1 + \lambda}$$

Note, however, that the symmetry of the function can be exploited to solve the equation on one side only (e.g., for $\omega \leq 0.5$)) and to replace the boundary conditions with:

(II4) $$I(\omega) = \mu\big(\omega + s(1 - \omega)\big)^{\gamma} \quad \text{when } \omega = \frac{1}{1 + \lambda}$$

(II5a) $$I'(0.5) = 0$$

(II5b) $$\frac{I'}{I} = \gamma \frac{1 - s}{\omega + s(1 - \omega)} \quad \text{when } \omega = \frac{1}{1 - \lambda}$$

while the 'super contact' condition can be replaced by the relationship (18) between μ and λ derived in the text.

Our solution procedure is more efficient than that outlined by Grossman-Laroque(1987):

Step 1. For some trial values[52] of λ, use (18) to obtain the corresponding value of μ and use (II5b) and (II4) to obtain I and I' at the point $\omega = 1/(1 + \lambda)$.

Step 2. These values of I and I' provide initial conditions for (I1), so that this equation can then be continued[53] until $\omega = 0.5$. If $I'(0.5) = 0$ (i.e., boundary condition (II5a) is satisfied), exit. Otherwise, pick a new value of λ[54] and go back to step 1.

Based on the following numerical values:

(II6) $$\rho = 0.15 \qquad \gamma = -1$$

$$\sigma = 0.5 \qquad \phi = 0.5$$

$$\alpha = 0.11 \ ,$$

so that s = 1.22, the optimum is obtained, in this particular case, for μ = 1047.9422 and λ = 2.66325. The interpretation is that the optimal program allows the stock of goods in one country to be as high as 2.66325 times the stock of goods in the other, before any shipping is decided.[55] For lower values of μ (e.g., μ = 1000), it is not possible to find a λ such that I'(0) = 0.5; and higher values (e.g., μ = 1048, 1125, 1250, 1500) are, by definition, not optimal, as they yield a lower level of V(K, K*).[56]

Footnotes

[1] I.e. a process without mean reversion.

[2] This is not totally surprising, considering that asset prices themselves, under risk aversion, do not follow a martingale; cf. Lucas (1978).

[3] In a recent contribution to this literature, Benninga and Protopapadakis (1986) have introduced explicitly delays in international trading. They derived an IAPM but did not solve for the resulting general equilibrium explicitly.

[4] For instance most international portfolio choice models (including Adler-Dumas (1983)) introduce deviations from the Law of One Price and different loci of consumption but do not recognize the distribution of goods between locations (and the distribution of wealth between people) as state variables. This criticism was already formulated in Adler and Dumas' concluding section.

[5] In order to simplify, I do not distinguish within each country between investment goods (installed or not installed) and consumption goods. Furthermore, I assume constant returns to scale, implying no rents. Hence, by construction, in each country, the traditional Tobin's q is equal to 1. But this assumption could be relaxed.

[6] In the international context, Dixit (1987a, b)'s work on market exit and entry is related to this literature, at least as far as the mathematical apparatus is concerned. It differs from the present work in that it incorporates set-up (or sunk) costs, while I consider proportional costs.

[7] But they can own stocks of goods physically located abroad.

[8] The assumption of constant relative risk aversion (isoelastic utility) is made only in order to reduce the dimensionality of the optimization program; cf. Appendix II.

[9]In the literature dealing with 'the demand for storage' (cf. Brennan (1958)), this advantage is called a 'convenience yield'.

[10]This is in contrast to the models of Black (1974) or Stulz (1981).

[11]This shortcut has been used previously by Lucas and Prescot(1971), and Constantinides (1982).

[12]In a Pareto-optimal market, these weights are constant over time and across states of nature.

[13]No attempt is made here to relate the welfare weights to the initial endowments. Note, however, that these weights reflect not only the relative wealths of the investors, but also the locational composition of the aggregate physical stocks. For instance, under equal welfare weights, if, at any time, a person lives in the country where goods happen to be more abundant, this locational advantage must be compensated--initially or because of risk-sharing contracts--by lower personal wealth.

[14]Appendix I spells out the 'Time-to-Ship' interpretation.

[15]As in all financial investment problems, solvency constraints are implied. Investors with an isoelastic utility always satisfy them. The 'correct' solution for V is the one which satisfies the solvency constraints.

[16]Equations (10), in turn, imply that (7) must have an infinity of solutions such as (8) indexed by their corresponding value of λ; for, if one considers a solution (8) corresponding to a given value of λ, then all the other values $\lambda' > \lambda$ generate a solution as well, by virtue of (10). In other words, if a given cone is optimal, a wider one also is. These observations were already made by Constantinides (1986). In what follows, I shall conventionally utilize the smallest cone; i.e., the smallest value of λ.

[17]Which are:

$$c^{\gamma-1} = V_1 \qquad c^{*\gamma-1} = V_2.$$

[18]Existence and uniqueness of the solution have not been proven. This can presumably be done along the lines of Krylov (1981) and Grossman-Laroque (1987). In particular, the parameter ρ, the discount factor of utilities of equation (1), must be larger than some number, in order to guarantee existence of the solution of (12) below, as well as the convergence of the integral in (1). See footnote 21.

[19](12a) can equivalently be written as a pair of equations:

(12a')
$$\left.\begin{aligned} V_1(K, K^*) &= \mu(K^* + sK)^{\gamma-1}s \\ V_2(K, K^*) &= \mu(K^* + sK)^{\gamma-1} \end{aligned}\right\} \quad \text{when } K = \lambda K^*$$

The equivalence, under (13a), results from Euler's theorem for homogeneous function. (12'a) says that the first derivative has the same value, to the right and to the left of the stopping point.

[20]See previous footnote. Under (14a), the two equations of (15a) are dependent.

[21]The existence of a solution to this equation must be imposed as a condition for the existence of a solution to the optimization problem.

[22]There is no known explicit solution using standard functions. This is true even in the logarithmic case $\gamma = 0$.

[23]Rather than being optimally chosen at each point in time, and different in the two countries, in such a way as to bring about some rebalancing.

[24]When δ is maximum, 'super contact' obtains. The reader can verify this fact, if he is willing to refer to the original text of Constantinides. On page 849, equations (13) and (14) express the first-order high contact. Then, as explained in the paragraph which follows equation (15), these equations are differentiated in order to determine the proper integration constants A_1 and A_2. This is tantamount to imposing higher-order second-degree contact.

[25]Under certainty ($\sigma = 0$), it is valid to assume that shipment takes place once only and thereafter rebalancing is done simply by means of differential consumption rates; consequently the value function inside the cone is:

$$I(\omega) = \mu'\left(\omega^{\gamma} + (1 - \omega)^{\gamma}\right)$$

for some μ'. The value of λ which satisfies equations (12a, b) is:

$$\lambda = (1/s)^{\frac{1}{1-\gamma}}.$$

[26]The limiting situation would be the one envisaged by Roll (1979). Under risk neutrality, strictly speaking, no equilibrium of the capital market exists.

[27]A wider cone, when risk increases, does not necessarily translate into less frequent shipping. The larger volatility makes reaching the boundaries of a given cone more likely, while a wider cone has the opposite effect. Which effect dominates, as far as the frequency of shipping is concerned, has not been determined.

[28]Except for the approximation entailed in the use of a numerical technique.

[29]Please refer to the end of the previous section for details of the specification.

[30]See Cox and Miller (1965) pages 219ff or Karlin and Taylor (1981) pages 226ff.

[31]The boundaries are obviously not absorbing or reflecting, which are standard boundary types. They seem to be of the 'sticky' type; cf. Karlin and Taylor (1981) page 233 and page 257.

[32]See the previous section.

[33]Outside the cone, of course, $\pi(\omega) = s$ or $1/s$ and $\pi'(\omega) = 0$. The continuity in the first derivative of $\pi(\omega)$ arises from the continuity, already noted, in the second derivatives of $V(K, K^*)$ at the boundary.

[34]Observe also that the behavior of the drift is such that sampling at discrete points in time would produce serial correlation of the price changes; observations taken discretely would not follow an AR(1) process.

[35]This is not in contradiction with the fact, already noted a propos the ω process, that occurrences of actual boundary occupation last an infinitely small amount of time.

[36]This is obvious; but, for an explanation of this relationship and a recent empirical investigation of real rate differences across countries, see Cumby and Obstfeld (1984).

[37]This is the expected 'real' rate of return on the K-riskless asset, from the point of view of someone (presumably residing in the foreign country) who consumes the good located abroad.

[38]In what follows, the term 'risk premium' will not necessarily mean that the designated quantity would be equal to zero under risk neutrality, only that it would be zero in the absence of risk.

[39]I shall continue to measure the price by means of the symmetrical variable π, so that the expected price change mentioned here is simply the opposite of the expected price change mentioned previously a propos quantity 2 vs quantity 1. In this way, I spare the reader the ludicrous arguments arising from Jensen's inequality.

[40]And one which is exactly the opposite of the premium mentioned earlier a propos quantities 2 and 3. Again, this desirable feature is the result of the symmetric definition of the price variable π.

[41]In the following equations, E is the expected value operator conditional on the current values of K and K*.

[42]Recall that the diffusion coefficient of ω is available in equation (26).

[43]The derivatives $N'(\omega)$ and $D'(\omega)$ are easily obtained from (27) and (28).

[44]Frankel (1979) reached a similar conclusion but in the context of a monetary model.

[45]Equation (37), that is, is a special case of an International Capital Asset Pricing Model. The general form of the IAPM will not be developed here; but it is clear that it involves three covariance terms: one is the covariance of a security's rate of return with the price variable p (or π) and the other two are the covariances with the K and K* outputs. To the extent that, in general equilibrium, the price p is itself functionally related to K and K*, these three terms collapse into two. In the special case where the security in question is the 'foreign' real rate, the risk premium involves the covariances of the price p with the two outputs. These are the elements contained in the second term of (37); cf. Benninga and Protopapadakis (1986).

[46]This could be shown analytically.

[47]One person offered the comment that my "model has nothing to do with the real appreciation of the dollar during the 80s." That person, no doubt, had a point.

[48]As a substitute, international monetary economists traditionally focus on the behavior of the relative price of traded vs. non-traded commodities within a given country, and call this price the real exchange rate.

[49]For instance, we cannot define an elasticity of capital flows with respect to the exchange rate or with respect to the interest rates.

[50]It generated tractable boundary conditions.

[51]An alternative but more complex specification would have achieved the same result by introducing a riskless output activity in each country, along with riskless shipping.

[52]The trial values we used were the suboptimal values of the Constantinides approximation (see end of Section 3).

[53]This was implemented numerically by means of the Runge-Kutta method of order four. See Abramovitz and Stegun (1972), p. 897.

[54]Based on the observed discrepancy between I'(0.5) and 0.

[55]Note that the opportunity cost of shipping in this numerical example is fairly high: $\phi = 0.5$ (I am referring here to the 'time-to-ship' interpretation of Appendix I); in each period of time, only half of the goods in transit reach their destination. Transportation is quite slow!

[56]Note that, in this numerical example, γ is negative.

References

A. B. Abel, "Optimal Investment under Uncertainty," American Economic Review, March 1983, 228-233.

A. B. Abel, "A Stochastic Model of Investment, Marginal q and the Market Value of the Firm," International Economic Review, June 1985, 305-322.

M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover, 1972.

M. Adler and B. Dumas, "International Portfolio Choice and Corporation Finance: a Synthesis," The Journal of Finance, XXXVIII, no. 3 (June 1983), 925-984.

M. Adler and B. Lehman, "Deviations from Purchasing Power Parity in the Long-Run," The Journal of Finance, XXXVIII (December 1983), 1471.

S. Benninga and A. Protopapadakis, "The Equilibrium Pricing of Exchange Rates and Assets when Trade Takes Time," working paper, 1986.

F. Black, "International Capital Market Equilibrium with Investment Barriers," Journal of Financial Economics, 1 (December 1974), 337-352.

W. C. Brainard and J. Tobin, "Pitfalls in Financial Model Building," American Economic Review, 58 (May 1968), 99-122.

M. J. Brennan, "The Demand for Storage," American Economic Review, 48 (March 1958), 50-72.

M. J. Brennan and E. S. Schwartz, "Evaluating Natural Resource Investments," Journal of Business, 58, 2 (1985), 135-157.

G. M. Constantinides, "Intertemporal Asset Pricing with Heterogeneous Consumers and Without Demand Aggregation," Journal of Business, 55, no. 2 (1982), 253-267.

G. M. Constantinides, "Capital Market Equilibrium with Transactions Costs," Journal of Political Economy, (November 1986), 842.

J. C. Cox, J. E. Ingersoll and S. A. Ross, "An Intertemporal General Equilibrium Model of Asset Prices," Econometrica, 53, no. 2 (March 1985), 385-408.

D. R. Cox and H. D. Miller, The Theory of Stochastic Processes, London: Chapman and Hall, 1965.

R. E. Cumby and M. Obstfeld, "International Interest Rate and Price Level Linkages under Flexible Exchange Rates: a Review of Recent Evidence," in J. F. O. Bilson and R. C. Marston, eds., Exchange Rate Theory and Practice, Chicago: the University of Chicago Press, 1984, 121-151.

R. Eisner and R. H. Strotz, "Determinants of Business Investment," in Commission on Money and Credit, Impacts of Monetary Policy (Englewood Cliffs, N.J.: Prentice-Hall, 1963), 59-337.

E. Fama, "Forward and Spot Exchange Rates," Journal of Monetary Economics, 14 (November 1984), 319-338.

J. Frankel, "On the Mark: A Theory of Floating Exchange Rates Based on Real Interest Differentials," American Economic Review, 69, 4 (September 1979), 610-622.

J. P. Gould, "Adjustment Costs in the Theory of the Firm," Review of Economic Studies, 35 (January 1968), 47-55.

S. J. Grossman and G. Laroque, "Asset Pricing and Optimal Portfolio Choice in the Presence of Illiquid Durable Consumption Goods," working paper, Princeton University and INSEE (Paris, France), 1987.

J. Huizenga, "An Empirical Investigation of the Long Run Behavior of Real Exchange Rates," working paper, University of Chicago, 1986.

S. Karlin and H. M. Taylor, A Second Course in Stochastic Processes, New York: Academic Press, 1981.

N. V. Krylov, Controlled Diffusion Processes, New York: Springer Verlag, 1980.

F. E. Kydland and E. C. Prescott, "Time to Build and Aggregate Fluctuations," Econometrica, 50, 6 (November 1982), 1345-1370.

R. E. Lucas, Jr., "Optimal Investment Policy and the Flexible Accelerator," International Economic Review, 8 (February 1967), 78-85.

R. E. Lucas, JR., "Adjustment Costs and the Theory of Supply," Journal of Political Economy, 75 (August 1967), 321-334.

R. E. Lucas, "Asset Prices in an Exchange Economy," Econometrica, 46 (November 1978), 1429-1446.

R. E. Lucas and E. C. Prescott, "Investment under Uncertainty," Econometrica 39, no 5 (1971), 659-681.

S. Majd and R. S. Pyndick, "Time to Build, Option Value and Investment Decisions," Journal of Financial Economics, 18 (1987), 7-27.

R. McDonald and D. Siegel, "The Value of Waiting to Invest," The Quarterly Journal of Economics, November 1986, 707-727.

H. P. McKean Jr., "Appendix: A Free Boundary Problem for the Heat Equation Arising from a Problem in Mathematical Economies," Industrial Management Review, 6, 2 (Spring 1965), 32-39. Reprinted in the Collected Scientific Papers of P. A. Samuelson, Vol. 3.

R. C. Merton, "Theory of Rational Option Pricing," Bell Journal of Economics, 4, 1 (Spring 1973), 141-183.

M. Mussa, "External and Internal Adjustment Costs and the Theory of Aggregate and Firm Investment," Economica, 44 (May 1977), 163-178.

S. Nickell, "On the Role of Expectations and the Pure Theory of Investment," Review of Economic Studies, 1974, 1-19.

R. S. Pyndick, "Adjustment Costs, Uncertainty and the Behavior of the Firm," American Economic Review, 72 (June 1982), 415-427.

R. S. Pyndick, "Irreversible Investment, Capacity Choice, and the Value of the Firm," NBER Working Paper #1980, June 1986.

R. J. Rogalski and J. P. Vinso, "Empirical Properties of Foreign Exchange Rates," Journal of International Business Studies, 9 (1978), 2, 69-79.

R. Roll, "Violations of Purchasing Power Parity and their Implications for Efficient International Commodity Markets," in M. Sarnat and G.P. Szego, eds., International Finance and Trade, Cambridge, Ma: Ballinger Publishing Co, 1979.

P. A. Samuelson, "Rational Theory of Warrant Pricing," Industrial Management Review, 6, 2 (Spring 1965), 13-31. Reprinted in Collected Scientific Papers of P. A. Samuelson, Vol. 3, R. C. Merton, ed., 1972.

B. H. Solnik, "An Equilibrium Model of the International Capital Market," Journal of Economic Theory, 8 (August 1974), 500-524.

A. C. Stockman and L. E. O. Svensson, "Capital Flows, Investment and Exchange Rates," Journal of Monetary Economics, 19 (1987), 171-201.

R. M. Stulz, "On the Effects of Barriers to International Investment," Journal of Finance, 36 (September 1981), 923-934.

L. E. O. Svensson, "Currency Prices, Terms of Trade, and Interest Rates," Journal of International Economics, 18 (1985), 17-41.

J. Tobin, "Money, Capital and Other Stores of Value," American Economic Review, 51 (May 1961), 26-37.

J. Tobin, "A General Equilibrium Approach to Monetary Theory," Journal of Money, Credit and Banking, 1 (February 1969), 15-29.

A. B. Treadway, "On Rational Entrepreneurial Behavior and the Demand for Investment," Review of Economic Studies, 36 (1969), 227-239.

Table 1

The opening of the cone of no shipping as a function of risk and risk aversion. (N.B. 1.22 = 1/s) . The table gives the value of λ. For numerical values of the parameter, other than γ and σ, see (27) in appendix II. Some values in this table are obtained by an explicit formula; e.g., when $\sigma = 0$, $\gamma = (1/s)^{1/1-\gamma}$. Other values are obtained by a numerical procedure (see appendix II).

Risk Aversion $1 - \gamma$	Risk (σ)				
	0	0.02	0.1	0.45	0.5
2	1.1045	1.3447	1.9813	2.647	2.66438
1 (log)	1.22				3.2609
1/2	1.4884		3.3598		4.1831
0 (neutral)	∞	∞	∞	∞	∞

Table 2

The opening of the cone formed by the Constantinides approximation.

Risk Aversion $1 - \gamma$	σ					
	0	0.02	0.04	0.1	0.4	0.5
2		1.18	1.26	1.38	2.26	NA
1		1.28	1.55			2.525
1/2		1.56				3.22

Figure 1. The cone of no shipment

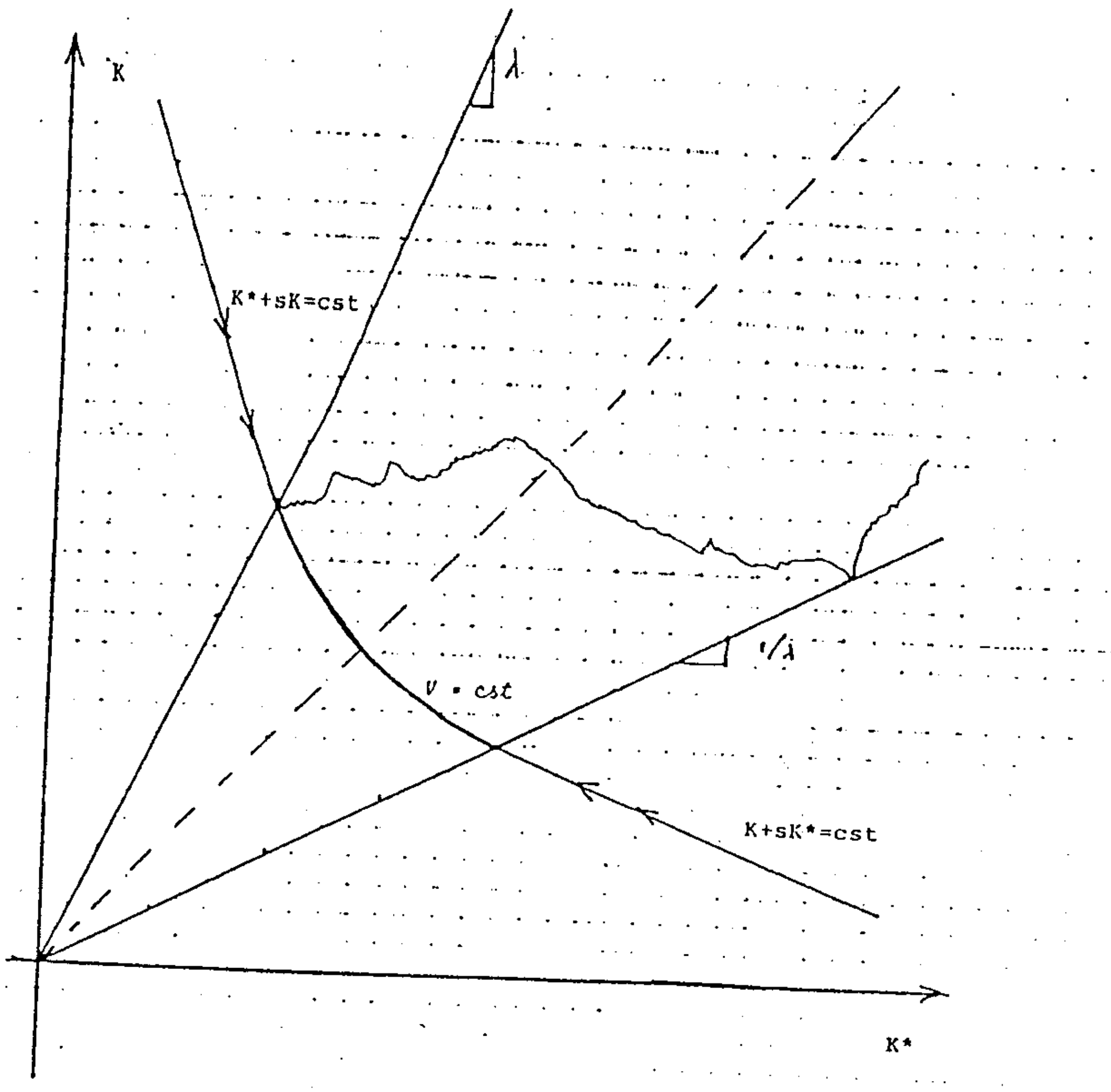


Figure 2: DRIFT OF OMEGA PROCESS

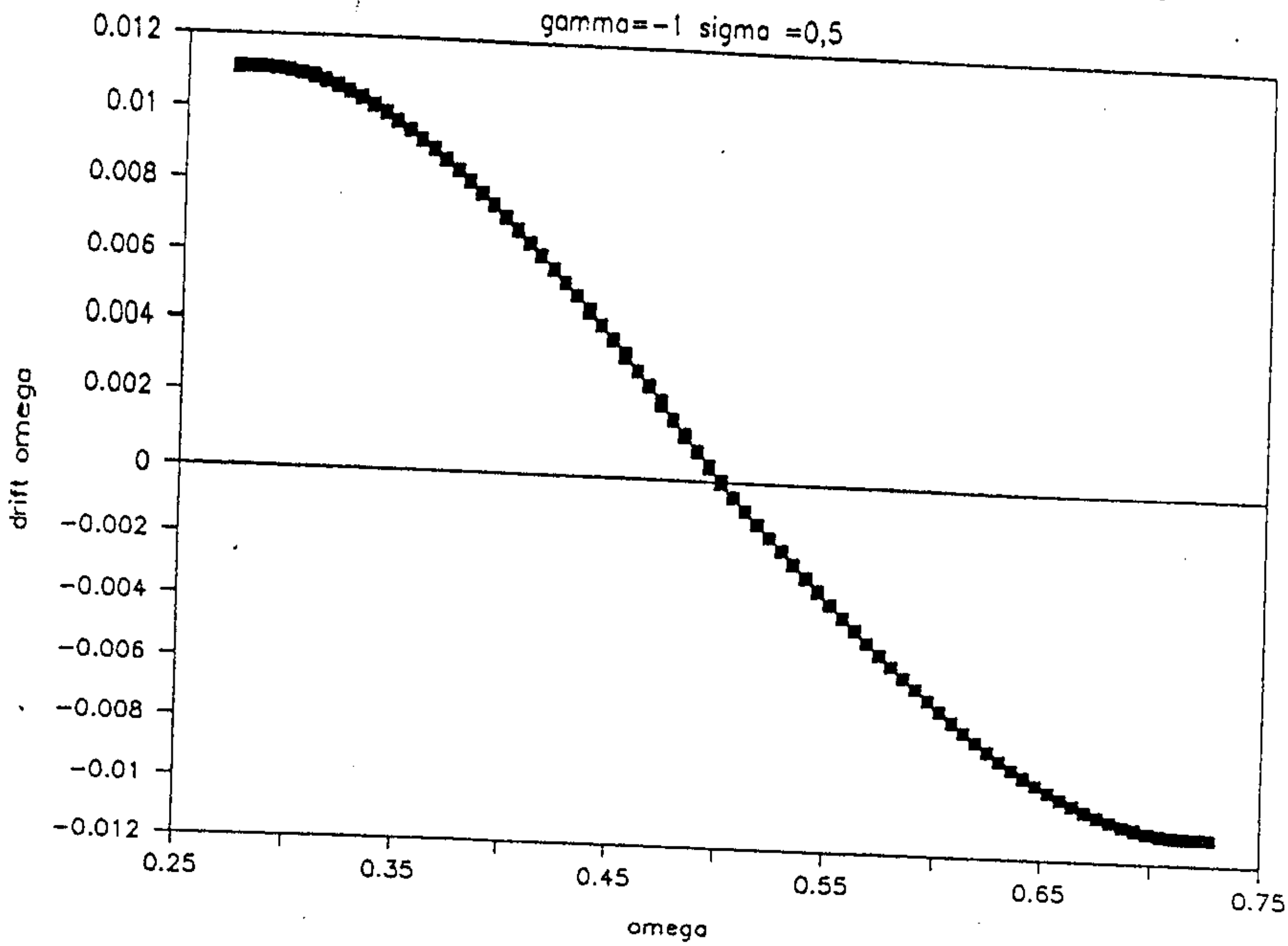
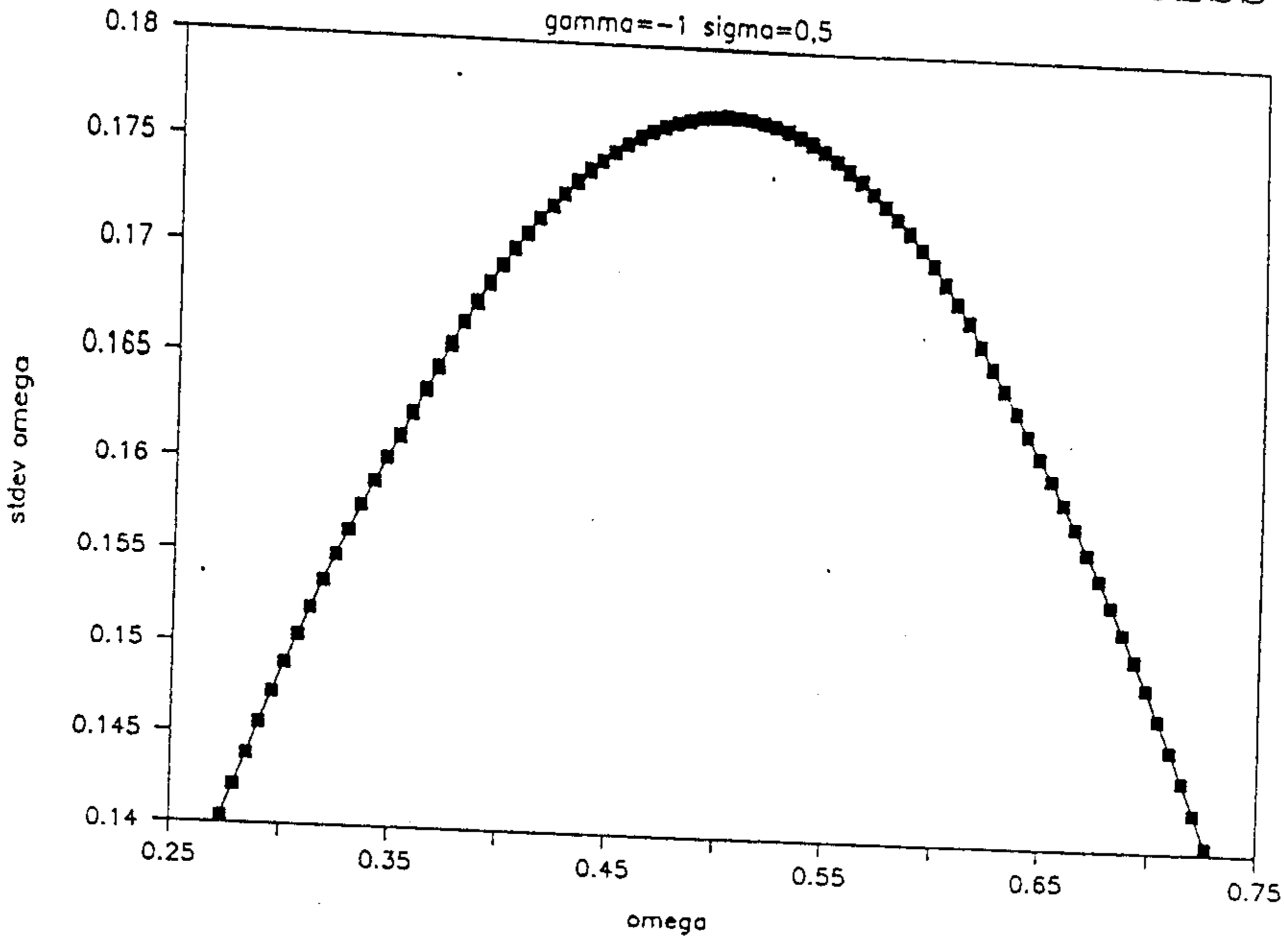


Figure 3: COND ST DEV OF OMEGA PROCESS



BEHAVIOR ON THE EDGES OF THE CONE

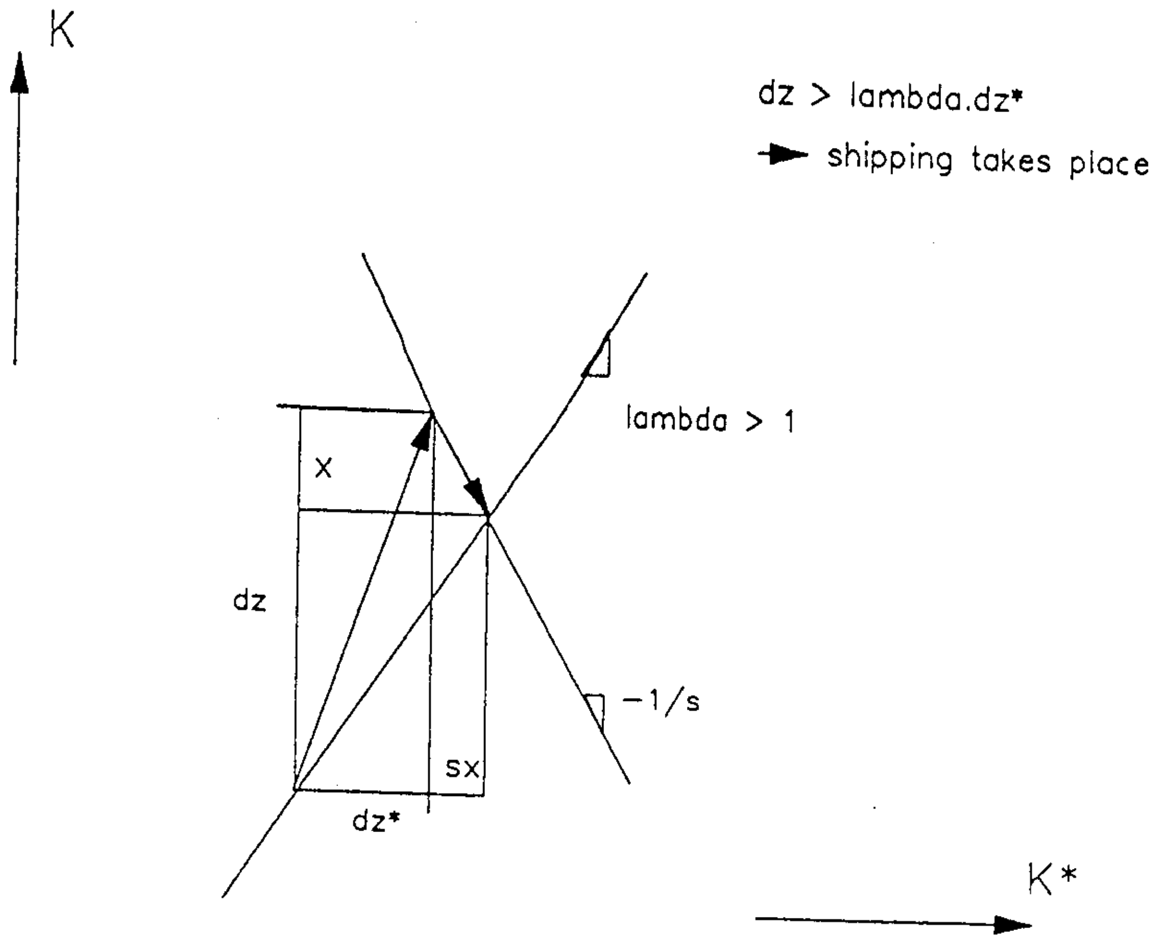


Figure 4

Figure 5: THE FUNCTION P(OMEGA)

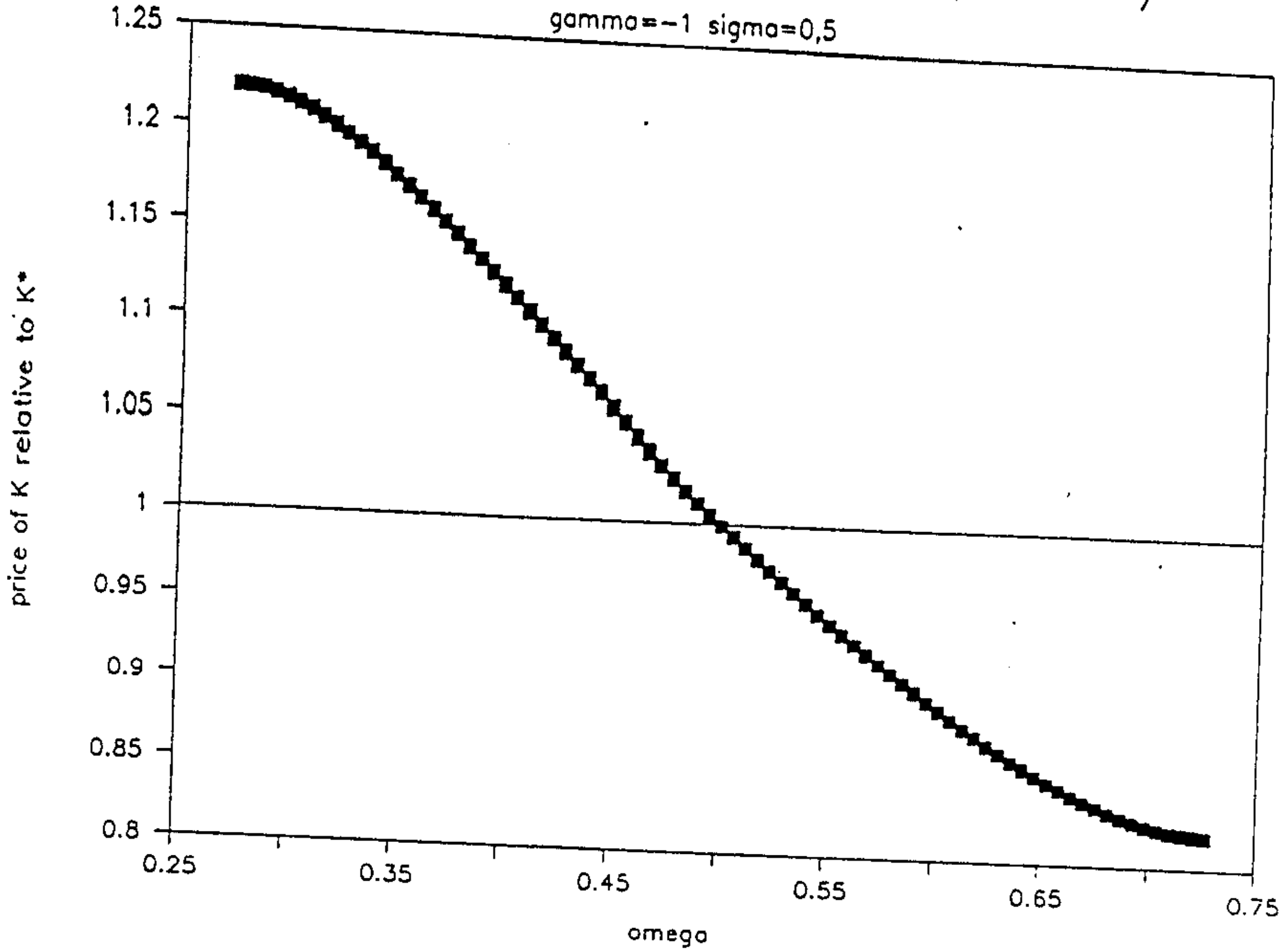


Figure 6: THE FUNCTION PI(OMEGA)

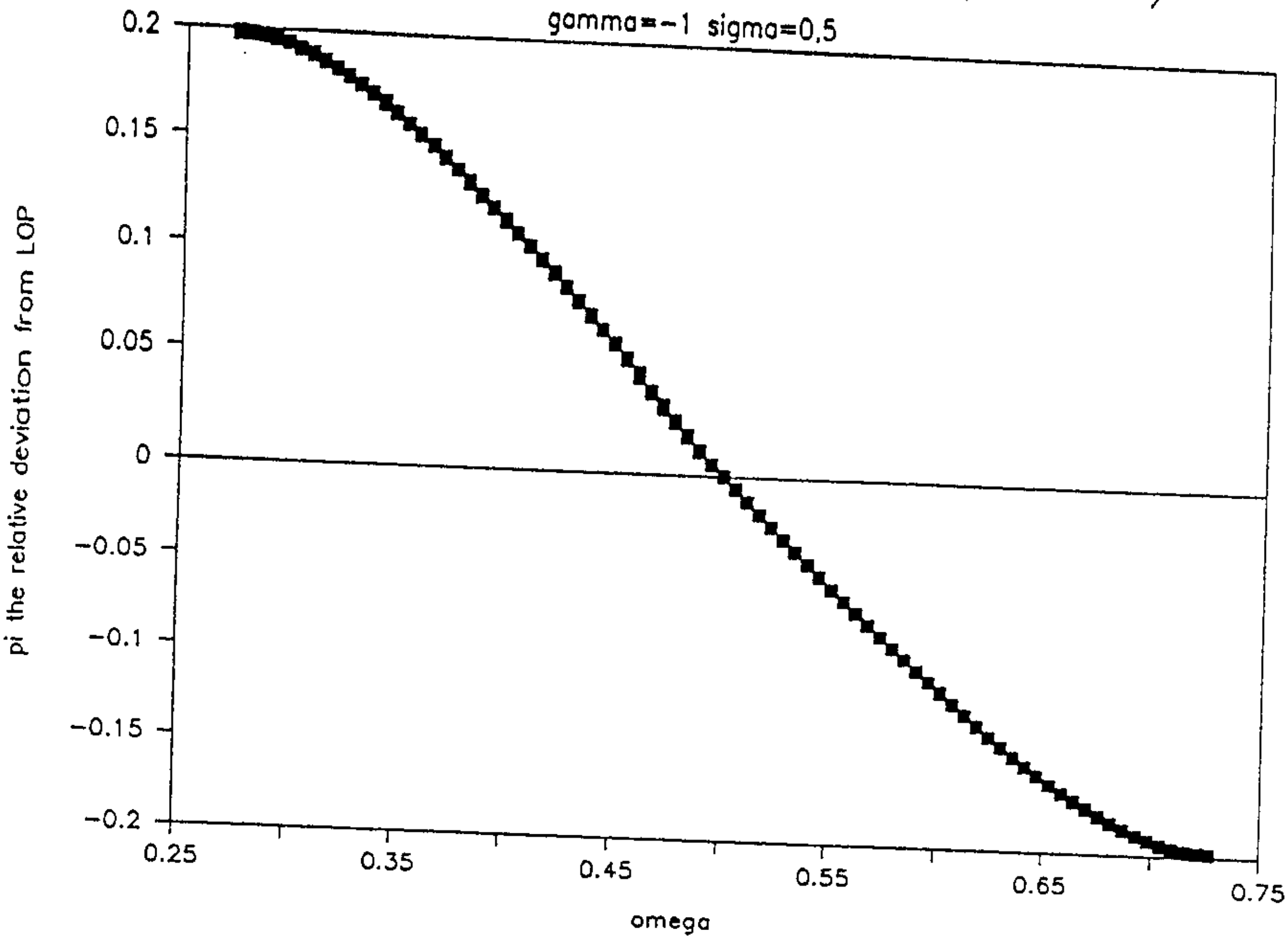


Fig. 7: DRIFT OF LOP DEVIATION PROCESS

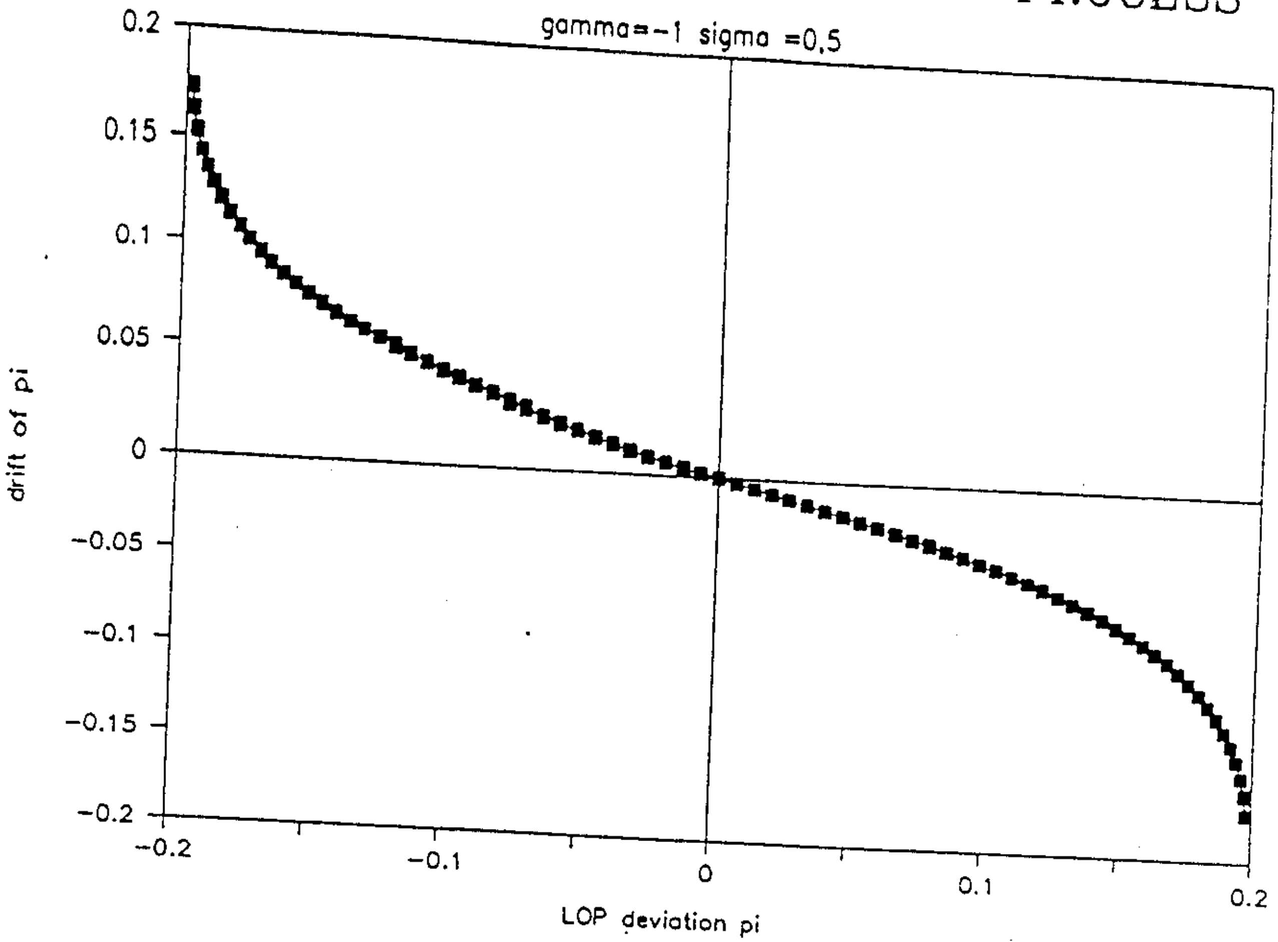


Fig 8:COND ST DEV OF LOP DEVIATION PROC

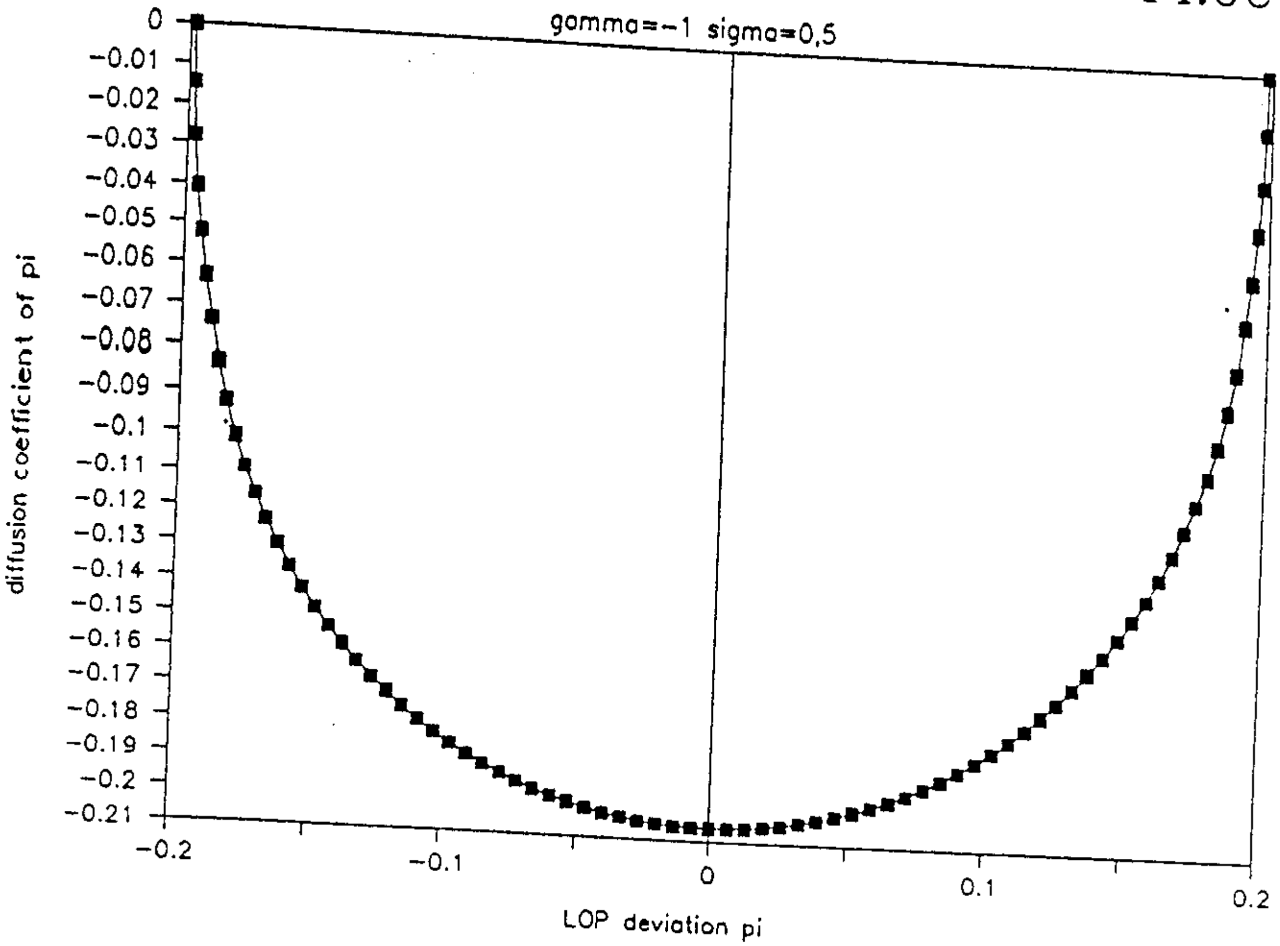


Figure 9: THE REAL RATE DIFFERENTIAL