TWO-PERSON DYNAMIC EQUILIBRIUM: TRADING IN THE CAPITAL MARKET

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Abstract

When several investors with different risk aversions trade competitively in a capital market, the allocation of wealth fluctuates randomly between them and acts as a state variable against which each market participant will want to hedge. This hedging motive complicates the investors' portfolio choice and the equilibrium in the capital market. Although every financial economist is aware of this difficulty, to our knowledge, this issue has never been analyzed in detail. The current paper features two investors, with the same degree of impatience, one of them being logarithmic and the other having an isoelastic utility function. They face one risky constant-return-to-scale stationary production opportunity and they can borrow and lend to and from each other. The behaviors of the allocation of wealth and of the aggregate capital stock are characterized, along with the behavior of the rate of interest and that of the security market line. The two main results are: (1) given the particular menu of assets under consideration, investors in equilibrium do revise their portfolios over time so that some trading takes place, (2) when the two investors 'disagree' about whether the economy should be expanding or contracting, it is possible for the allocation of wealth and the capital stock to admit steady-state distributions. It is also possible for these to randomly oscillate between two extreme attracting points. This is in contrast to the certainty case, where aggregate wealth becomes either very large or very small and one investor in the long run holds all the wealth. The existence of trading opens the way to a theory of capital flows and market trading volume.

1. Introduction

The question of dynamic asset pricing has been addressed so far, mostly under the assumption of identical investors (Lucas (1978), Cox, Ingersoll and Ross (1985)). The asset prices so obtained are then purely virtual prices, since no trading takes place in the capital market. Models such as that of Constantinides (1982) featuring heterogeneous consumers have produced an asset pricing equation based on an aggregation argument; but these models have not so far been disaggregated in order to make portfolio choices explicit. The finance profession, that is, has no theory to offer to account for trading volume and capital flows between capital market participants, under conditions of rational expectations and symmetric information. The present paper aims to fill this gap.

Since our aim is to generate trading in the capital market, we must avoid circumstances which are known to induce constancy of the investors' ownership shares in the various assets, even though investors are not identical to each other. The work of Rubinstein (1974) has outlined these circumstances: if investors all have Hyperbolic-Absolute-Risk-Aversion (HARA) utility functions, with the same impatience parameter and the same cautiousness parameter (but are otherwise different from each other), it is Pareto optimal for them to adopt a linear consumption sharing rule (see also Wilson (1968)). They do so by forever holding a fixed share of the market portfolio, and a fixed amount of a consol bond offering riskless payments. Such portfolio policies obviously require no trading: investors just live off the income generated by their constant portfolio.

There are several ways in which we can choose to deviate from the Rubinstein base case. Investors may differ in their impatience parameter. The question has been examined by Becker (1980) under conditions of certainty;

the result is that the least impatient investor will hold all the wealth in the long run. It seems unlikely that new insights would be gained by the ntroduction of risk into Becker's analysis. We therefore retain the Rubinstein assumption of equal rates of impatience across investors.

Instead, we examine investors whose utility functions are isoelastic, with differing levels of relative risk aversion. Isoelastic utility functions belong to the HARA class of utility functions and, in their case, the cautiousness parameter of Rubinstein is simply equal to the relative risk aversion. In this way, the investors considered here differ in their cautiousness parameter, and we can expect that a linear sharing rule and a constant policy which consists in holding the riskless asset and the market portfolio will not be optimal for them.

Even then, it should be clear that the volume of trading so generated is predicated on the particular menu of asset which we choose. In the present paper investors have access to shares of stock in a risky constant-return-to-scale production technology and they can borrow and lend short-term from each other at the riskless rate. Given this menu, a nonzero volume of trading will occur in response to output shocks. Considering, however, that this menu of securities is sufficient for the capital market to be complete in the sense of Harrison and Kreps (1979), it is always possible to introduce another menu of securities with the same characteristic which will reduce trading down to zero. The non-linear sharing rule which is Pareto optimal when two investors have different but constant relative risk aversions can be easily derived. A single security with a payoff structure replicating that sharing rule would reduce trading down to zero if it were made available.[1] And indeed one can argue that two investors with the given relative risk aversions would naturally get together and design such a contract in order to eliminate the inconvenience of trading.

The particular menu we have chosen to consider (shares of stock and riskless asset) can be defended on the grounds that it involves standardized securities only. By 'standardized securities' we mean securities whose contractual definition requires no knowledge of the distribution of risk aversions in the population of investors.[2] Indeed the present model should be viewed as a simplification of an economy with an arbitrary distribution of risk aversions across the population. In that more general economy trading could be reduced to a volume of zero only by the introduction of as many tailor-made contracts as there would be classes of investors with the same risk aversion (minus one). In general it would be very impractical to achieve this goal because the tailoring of these contracts would require the knowledge of everyone's risk aversion. In contrast, two standardized securities of the type considered here can satisfy every investor provided they are willing to engage in some trading.[3]

Along with some trading volume, our paper will produce a variable distribution of wealth across investors, but one which does not necessarily converge to 100% ownership by one of them. It will also produce a variable rate of interest the stochastic process of which will be fully endogenized.

The model is laid out in section 2. The equilibrium of the capital market is characterized in section 3. Section 4 exhibits the equilibrium relationship between the aggregate capital stock and the distribution of wealth at any time, for a given initial wealth distribution; we dub this relationship 'a wealth sharing rule'. Section 5 provides a derivation of the dynamics of the aggregate economy and, correspondingly, of the distribution of wealth between investors. In section 6, the behavior of the rate of interest is obtained and the amount of trading in the market is quantified.

2. The model

The capital market of our model economy is populated with but two investors, with the same rate of impatience ρ, but different risk aversions. The analysis is greatly simplified and does not lose its illustrative power if we restrict one investor to have a logarithmic utility function, while the other one exhibits any degree of risk aversion $1-\gamma$ where γ is the power of his isoelastic utility function:[4]

$$(1) \qquad \text{Max } E \int_0^\infty e^{-\rho t} \frac{1}{\gamma} c^\gamma dt \; ; \qquad \gamma < 1 \; , \; \rho < 1 \; ,$$

and c is his finite rate of consumption of a single good.

Recall (Merton (1971)) that the logarithmic case can be obtained as the limit of the above case for $\gamma \to 0$; we shall therefore simply write the optimizing equations for the investor with the power utility function.[5]

The two investors consume a single good and have access to two investment opportunities:

- they can buy shares in one[6] constant-return-to-scale production activity, whose random output per unit of capital has a constant gaussian distribution with fixed parameters α and σ;
- they can borrow and lend to and from each other at the equilibrium riskless[7] rate r, **which varies over time in an endogenous fashion.**

Other notations are as follows:

W: wealth of the non-logarithmic investor;

W*: wealth of the logarithmic investor;

S = W+W*: aggregate wealth and capital stock[8];

x, x^*: share of each investor's wealth invested in the risky production opportunity;

c, c^*: consumption rates of the two investors;

(2) $\omega = W/(W + W^*)$: the non-logarithmic investor's share of total wealth.

The dynamics of the aggregate capital stock simply reflect the flows of goods:

(3) $$dS = (\alpha S - c - c^*)dt + \sigma S dz$$

The dynamics of an investor's wealth for a given investment decision x and a given consumption decision c are well known:

(4) $$dW = \{W[r + x(\alpha - r)] - c\}dt + Wx\sigma dz \ ,$$

where dz is the random white noise affecting production. In this equation, the rate of interest r is the market rate. It is not constant over time. In fact, we can reasonably postulate that it is a function of the distribution of wealth: $r = r(\omega)$. The formulation of the two investors' optimization problem must, therefore, incorporate the behavior of the distribution of wealth.[9]

Applying Ito's lemma to the definition (2) of ω and using the equation for the dynamics of wealth (3), as well as the analogous equation for the log investor, we obtain:

(5) $$d\omega = \omega(1 - \omega)\{[(x - x^*)(\alpha - r) - \frac{c}{W} + \frac{c^*}{W^*} - (x - x^*)(\omega x + (1 - \omega)x^*)\sigma^2]dt + (x - x^*)\sigma dz\} \ .$$

Not surprisingly, the allocation of wealth would be constant if the two investors were to hold the same portfolio ($x = x^*$). The allocation of wealth also admits two natural boundaries at $\omega = 0$ and $\omega = 1$: if one investor happened to hold all the wealth, he thereafter would hold all the wealth forever and, conversely, starting from an interior point ($0 < \omega < 1$), the boundaries cannot be reached in finite time. Both of these statements can be proved on the basis of equation (5) by applying standard boundary classification arguments.[10] The only assumption needed for this proof is that x and x* (the fractions of each person's wealth invested in the risky security) are bounded functions. This will be shown to be true for isoelastic investors; indeed such investors never 'go for broke,' taking finite positions in the risky asset even when their wealth is shrinking to zero. We return later to the question of the boundary behavior of this economy.

The maximization of (1) subject to (4) and (5) with respect to c and x (**but taking the behavior of r and therefore of ω as given** in order to represent pure competition) is a standard dynamic program. Existence and uniqueness of the solution of this program is not guaranteed by the Weierstrass theorem because the budget set is not compact. Cox and Huang (1986) offer an alternative proof of existence (but not uniqueness) within the class of policies which are such that investors never reach zero wealth in finite time.[11] The partial differential equation for its (undiscounted) value function $J(W, \omega)$ is a Hamilton-Jacobi equation which can be written easily:

$$(6) \qquad 0 = \max_{c,x} \Big\{\frac{1}{\gamma} c^{\gamma} - \rho J + \frac{\partial J}{\partial W}\big[W(\gamma + x(\alpha - \gamma)) - c\big] + \frac{1}{2}\frac{\partial^2 J}{\partial W^2} W^2 x^2 \sigma^2 + \frac{\partial^2 J}{\partial W \partial \omega} W x \sigma \hat{\omega}\Big\} + \frac{\partial J}{\partial \omega}\mathring{\omega} + \frac{1}{2}\frac{\partial^2 J}{\partial \omega^2}\hat{\omega}^2 .$$

where $\mathring{\omega}$ and $\hat{\omega}$ are the drift and the diffusion coefficients of ω, as shown in equation (5) above.

One can then proceed in two steps, optimizing consumption first, and then the portfolio. The first-order condition with respect to consumption is:

$$(7) \qquad c^{\gamma-1} = \frac{\partial J}{\partial W} .$$

Substituting the optimal consumption decision into the original Hamilton-Jacobi equation, one can verify that a function of the form:

$$(8) \qquad J(W, \omega) = \frac{1}{\gamma} W^{\gamma} I(\omega)$$

is a solution,[12] I( ) being a function, yet to be determined. We simply assume from now on that the solution is unique within the class of solvent policies envisaged by Cox and Huang (1986).[13] One is then left with the second problem of optimization, with respect to the portfolio x:

$$(9) \qquad 0 = -\rho + (1 - \gamma) I^{\frac{1}{\gamma-1}}$$

$$+ \gamma \operatorname*{Max}_{x} \left\{ r + x(\alpha - r) + \tfrac{1}{2}(\gamma - 1)x^2\sigma^2 + \frac{I'}{I} x\sigma\hat{\omega} \right\}$$

$$+ \frac{I'}{I} \mathring{\omega} + \frac{1}{2} \frac{I''}{I} \hat{\omega}^2$$

The first-order condition with respect to x is evidently:

$$(10) \qquad \alpha - r + (\gamma - 1)x\sigma^2 + \frac{I'}{I} \sigma\hat{\omega} = 0$$

so that the optimal portfolio is:

$$(11) \qquad x = \frac{\alpha - r + \frac{I'}{I} \sigma\hat{\omega}}{(1 - \gamma)\sigma^2} .$$

The optimal portfolio is of the well-known form applying to an isoelastic investor, except for the last term of the numerator. This term represents hedging against shifts in the distribution of wealth, which induce shifts in the rate of interest.

The Hamilton-Jacobi equations (6) and the first-order conditions (7) and (10) are[14] sufficient to guarantee that the market allocation will be Pareto optimal. We quickly verify that fact now. Define:

$$(12) \qquad p = \frac{\partial J}{\partial W} = W^{\gamma-1} I(\omega)$$

$$(13) \qquad p^* = \frac{\partial J^*}{\partial W^*} = \frac{1}{W^*} \frac{1}{\rho}$$

These are of course the two investors' marginal utilities of wealth. We aim to show that they are proportional to each other at all times. This result is obtained by applying Ito's lemma to (12) and (13), and substituting into the resulting equations for dp and dp*, the Hamilton-Jacobi equations (6) (differentiated with respect to the investor's wealth) as well as the first order conditions (7) and (10).[15] In the end one gets:

$$(14) \qquad \frac{dp}{p} = \frac{dp^*}{p^*} = (\rho - r)dt - \frac{\alpha - r}{\sigma} dz \ .$$

Whatever be the value of the ratio $k = p/p^*$ at the initial point in time, this ratio thereafter remains forever at the same value.[16] Hence we have:

$$(15) \qquad W^{\gamma-1} I(\omega) = k \ 1/(\rho W^*) \ .$$

The Pareto-optimality result confirms that this continuous-trading market is complete. Indeed it is a 'Black-Scholes economy' (see Harrison and Kreps (1979)). Equation (15) will be useful below where deriving the behavior of aggregate and individual wealths.

3. Equilibrium

The equilibrium concept used in this paper is the standard rational-expectations equilibrium defined by Radner (1972).[17] It is a quadruple of functions $\{x^*(\omega), x(\omega), r(\omega), I(\omega)\}$ satisfying equations (16), (17), (18) and (24) below.

Instantaneously, the equilibrium is characterized by:

a) the non-logarithmic investor's portfolio optimality condition (8), which reads as follows, when the form of the diffusion coefficient $\hat{\omega}$ is made explicit:

$$\alpha - r = (1 - \gamma)\sigma^2 x - \frac{I'(\omega)}{I(\omega)} \omega(1 - \omega)\sigma^2(x - x^*) \; ; \tag{16}$$

b) the logarithmic investor's portfolio condition: it is well known since Hakansson (1971) (see also footnote #10 above) that such an investor exhibits no hedging motive, i.e., that his function I* is a constant (equal to $1/\rho$); hence:

$$\alpha - r = \sigma^2 x^* \; ; \tag{17}$$

c) a 'supply equals demand' condition:

$$\omega x + (1 - \omega)x^* = 1 \; . \tag{18}$$

Solving these three equations simultaneously, one obtains all the endogenous variables as a function of ω:

$$x^* = \frac{1}{\omega\lambda + 1 - \omega} \tag{19}$$

$$x = \frac{\lambda}{\omega\lambda + 1 - \omega} \tag{20}$$

$$r = \alpha - \frac{\sigma^2}{\omega\lambda + 1 - \omega} \; , \tag{21}$$

where:

$$\lambda = \frac{1 - \frac{I'}{I}\,\omega(1 - \omega)}{1 - \gamma - \frac{I'}{I}\,\omega(1 - \omega)} \tag{22}$$

is best interpreted as the non-logarithmic investor's risk tolerance "adjusted for the hedging motive," since, in effect, **his decision is mean-variance optimizing, in the static sense, but at a level of risk aversion different from** $1 - \gamma$. Similarly, by analogy with the CAPM, $\omega\lambda + 1 - \omega$ can be seen as the market's risk tolerance, also adjusted for the hedging motive.

The reason why the standard mean-variance framework survives, with a minor change,[18] despite the introduction of one more state variable and the nonstationarity of the rate of interest, is that this additional state variable ω is perfectly correlated with wealth (cf. equations (3) and (5)).[19]

Over time, the equilibrium is further characterized by the dynamics (4) of the distribution of wealth (with x, x^*, r, c/W, c^*/W^* substituted in) and by the two functions I and I^*. We mentioned above that I^* is a constant equal to $1/\rho$; therefore $c^*/W^* = \rho$. Once these elements are taken into account, the dynamics of ω, for a given I function, are:

$$d\omega = \omega(1 - \omega)\left\{\left(- \frac{\omega\sigma^2(\lambda - 1)^2}{(\omega\lambda + 1 - \omega)^2} - I^{\frac{1}{\gamma-1}} + \rho\right)dt + \frac{\lambda - 1}{\omega\lambda + 1 - \omega}\,\sigma dz\right\} \tag{23}$$

and, finally, substituting equations (13) to (15), and (17) above, we obtain the differential equation to be satisfied by the unknown function $I(\omega)$:

$$0 = -\rho + (1 - \gamma)I^{\frac{1}{\gamma-1}} \tag{24}$$

$$+ \gamma\left[\alpha - \frac{\sigma^2}{\omega\lambda + 1 - \omega} - \tfrac{1}{2}(\gamma - 1)\sigma^2\left(\frac{\lambda}{\omega\lambda + 1 - \omega}\right)^2\right]$$

$$+ \frac{I'}{I}\,\omega(1 - \omega)\Big[- \frac{\omega\sigma^2(\lambda - 1)^2}{(\omega\lambda + 1 - \omega)^2} - I^{\frac{1}{\gamma-1}} + \rho\Big]$$

$$+ \tfrac{1}{2}\,\frac{I''}{I}\,\Big[\omega(1 - \omega)\,\frac{\lambda - 1}{\omega\lambda + 1 - \omega}\,\sigma\Big]^2 ;$$

where λ is given by (22).

The problem of the determination of equilibrium is thus reduced to that of solving the nonlinear ordinary second-degree differential equation (24) (coupled with (22)), subject to two boundary conditions,[20,21] corresponding to the two natural barriers $\omega = 0$ and $\omega = 1$:

(25) $$I(0) = \left|\frac{1 - \gamma}{\rho - \gamma\left(\alpha - \sigma^2 + \frac{1}{2}\,\frac{\sigma^2}{1 - \gamma}\right)}\right|^{1-\gamma}$$

(26) $$I(1) = \left|\frac{1 - \gamma}{\rho - \gamma\left(\alpha - \frac{1}{2}(1 - \gamma)\sigma^2\right)}\right|^{1-\gamma} .$$

Everything one might want to know about the equilibrium path, will follow from this $I(\omega)$ function: once it is known, equations (15) (19) to (23), and the equation of footnote #9, give aggregate and individual wealths, portfolio choices, the rate of interest, the market price of risk, the dynamics of the allocation of wealth, and consumption choices.

It is unlikely that equation (24) subject to boundary conditions (25) and (26) should have a known analytical solution. Considering, however, that the domain of variation of ω is a closed set, and that the behavior of I on the boundary is well specified, this two-point boundary value problem of the Dirichlet type lends itself to numerical analysis.[22] We choose to present the results not in the form of the I function itself[23] but in terms of the stochastic behaviors of aggregate wealth and of the distribution of wealth (section 5). At equilibrium these two variables are in fact related to each other; our first task (section 4) is to obtain this relationship which we call 'a wealth sharing rule'.

4. The equilibrium wealth sharing rule

Along the equilibrium path the distribution of wealth and the aggregate capital stock fluctuate in tandem. In effect, when and if the capital stock increases, the more highly levered investor (the one with the smaller risk aversion; see section 6 below) reaps a larger share of the increase than does the less levered investor. The manner in which aggregate wealth is dynamically distributed between the two investors is the direct product of the way in which aggregate consumption is distributed between them. The concept of 'consumption sharing rule' has been introduced by Wilson (1968) and Rubinstein (1974); it is designed precisely to describe the way in which consumption is allocated. In a complete (and therefore Pareto optimal) market, marginal rates of substitution are equated across individuals. This implies that, at any time, the levels of the marginal utilities of the various investors are proportional to each other.[24] The proportionality relationship is valid both for the marginal utilities of consumption and for the marginal utilities of wealth, since the two are equated at the optimal level of consumption (see equation (7) above).

In our setting the consumption sharing rule is written as:

$$c^{\gamma-1} = k \frac{1}{c^*} \tag{27}$$

where k is the coefficient of proportionality between marginal utilities. We have already written and proved the proportionality relationship between marginal utilities of wealth in the form of equation (15). By substituting into this equation the definition of ω, we obtain now the relationship between aggregate wealth S and the distribution of wealth:

(28) $$S^{\gamma}\omega^{\gamma-1}I(\omega) = \bar{k}\,\frac{1}{\rho(1-\omega)}$$

This relationship which we label the 'wealth sharing rule'[25] is, of course, valid at all times, including the initial time. It allows one to determine the constant k on the basis of the initial wealths of the two investors.

FIGURE 1 GOES HERE

Figure 1 provides a plot of the wealth sharing rule for the two cases $\gamma > 0$ and $\gamma < 0$, and for given initial wealths. It is apparent that whenever the aggregate capital stock S increases, the wealth of the person with the lower risk aversion increases more than proportionately. In other words:

(29) when $\gamma > 0$, S and ω are positively related;

when $\gamma < 0$, S and ω are negatively related.

In fact, if S goes to infinity, the person with the smaller risk aversion ends up owning almost all the wealth, and if the aggregate capital stock goes to zero, the person with the larger risk aversion captures it almost entirely.

5. Expanding vs. contracting economies and the associated behavior of the distribution of wealth

The dynamics of the aggregate capital stock $S = W + W^*$ were written as equation (3) above. Substituting consumption behavior into (3) gives:

(30) $$\frac{dS}{S} = g(\omega)dt + \sigma dz ,$$

where: $$g(\omega) = \alpha - \frac{\omega}{I(\omega)^{1/(1-\gamma)}} - (1-\omega)\rho$$

or equivalently:

(31) $$d\ln S = \left[g(\omega) - \frac{1}{2}\sigma^2\right]dt + \sigma dz .$$

The drift term $g(\omega)$ in (30) is henceforth called 'the expected rate of growth of the economy'.

Whenever the expected rate of growth g is (strictly) larger than $\sigma^2/2$ in a neighborhood of $\omega = 0$ and in a neighborhood of $\omega = 1$, we shall say that the economy is an **expanding** one; whenever it is (strictly) smaller in two such neighborhoods, we shall say that the economy is a **contracting** one. This terminology is justified by the following observation: in circumstances where the expected rate of growth is uniformly larger than $\sigma^2/2$ for all values of S larger than some fixed value, the aggregate capital stock has a positive probability of becoming infinitely large[26] and when the expected rate of growth is uniformly larger than $\sigma^2/2$ for all values of S smaller than some fixed positive value, S has a zero probability of reaching zero[27]; whereas when the expected rate of growth is smaller than $\sigma^2/2$ uniformly in neighborhoods of 0 and $+\infty$, the aggregate capital stock has a positive probability of reaching zero and a zero probability of becoming infinite.[28]

These assertions can be verified by applying boundary classification techniques (cf. Karlin and Taylor (1981), pp. 226 ff). But they can be understood intuitively on the basis of equation (31) which gives the behavior of the logarithm of the capital stock. Note that the diffusion coefficient in the stochastic differential equation for lnS is constant. In that case boundary and asymptotic behavior are properly understood on the basis of the drift term alone; its sign determines whether the economy is expanding or contracting. Its functional form will also allow us to decide whether or not the economy reaches a steady state.[29]

When $\omega = 0$ (i.e. the logarithmic investor is alone), the expected rate of growth is seen to be equal to:

(32) $$g(0) = \alpha - \rho .$$

When $\omega = 1$ (i.e. the non-logarithmic investor is alone) the expected rate of growth is found from equations (30) and (26) to be equal to:

$$g(1) = = \frac{\alpha - \rho - \frac{1}{2}\gamma(1-\gamma)\sigma^2}{1-\gamma} \tag{32}$$

$$= \frac{g(0)}{1-\gamma} - \tfrac{1}{2}\gamma\sigma^2 \ . \tag{34}$$

Intuition suggests that in all cases the expected rate of growth of the economy $g(\omega)$ is a continuous function which reaches a finite number of maxima and minima between $g(0)$ and $g(1)$. In fact, numerical analysis indicates that the function of $g(\omega)$ is typically monotonic between $g(0)$ and $g(1)$ but that it is possible to construct examples (for instance, by choosing parameter values for which $g(0) = g(1)$) in which $g(\omega)$ has one maximum or minimum. Observe further, by examining equation (34), that the direction of the inequality between $g(0)$ and $g(1)$ does not simply hinge upon the sign of γ.

Finally, recall from figure 1 and our analysis of equation (28) (the wealth sharing rule) that ω and S are positively related when $\gamma > 0$ and negatively related when $\gamma < 0$.

We have now gathered all the information needed to describe the behavior of our economy. Four cases will have to be distinguished[30]:

Case #1: $g(0), g(1) < \sigma^2/2$:

The economy is a contracting one.[31] There is not steady state distribution for the capital stock or for any of the variables in this economy. No matter what the initial conditions may be, the probability density for the capital stock forever recedes towards zero. The probability of eventually reaching a zero capital stock (in infinite expected time!) is equal to one. Correspondingly, the probability is equal to one that the person with the is an attracting barrier; if $\gamma < 0$, $\omega = \bar{1}$ is attracting).

Case #2: $g(0), g(1) > \sigma^2/2$:

The economy is an expanding one. There is no steady state distribution for the capital stock or for any of the variables in this economy. No matter what the initial conditions may be, the probability density for the capital stock forever recedes towards infinity. The probability of eventually reaching an infinite capital stock (in infinite expected time!) is equal to one (provided of course that the initial capital stock is not zero). Correspondingly, the probability is equal to one that the person with the lower risk aversion will ultimately own all the wealth (i.e. if $\gamma > 0$, $\omega = 1$ is an attracting barrier; if $\gamma < 0$, $\omega = 0$ is attracting).

FIGURE 2 GOES HERE

Case #3: $g(0) < \sigma^2/2 < g(1)$ and $\gamma > 0$

OR: $g(0) > \sigma^2/2 > g(1)$ and $\gamma < 0$

The economy is neither a contracting one nor an expanding one. The case definition implies that the economy tends to expand when the capital stock is already very large and to contract when the capital stock is very small. This situation is illustrated in Figure 2. While 'instability' is a word which comes to mind to describe this situation, the following is a more rigorous rendition. There is no steady state distribution for the capital stock or for any of the variables in this economy. No matter what the initial conditions may be (other than $\omega = 0$ or $\omega = 1$), the probability density for the capital stock sometimes recedes towards infinity and sometimes recedes towards zero. Even after it has receded towards one boundary (zero or infinity) for some time, there is a positive probability that an appropriate succession of shocks will initiate a transition towards the other boundary. The probability of eventually reaching a zero capital stock (in infinite expected time!) is always positive and so is the probability of reaching an infinite capital stock. Since the allocation of wealth ω is monotonically related to aggregate wealth, it exhibits a similar kind of behavior.

FIGURES 3, 4 GO HERE

Case #4: $g(0) < \sigma^2/2 < g(1)$ and $\gamma < 0$

OR: $g(0) > \sigma^2/2 > g(1)$ and $\gamma > 0$

Again the economy is neither a contracting one nor an expanding one. But it tends to expand when the capital stock is low and to contract when the capital is large. This 'stable' situation is illustrated in figure 3. There exists a steady state density for the capital stock and for all the variables of this economy. **No matter what the initial conditions may be, the probability density for the capital stock converges to the stationary measure.** One such stationary measure (for the logarithm of the capital stock) is displayed in figure 4. The probabilities of eventually reaching a zero or an infinite capital stock are both always zero. The allocation of wealth, like the aggregate capital stock to which it is monotonically related, also admits a stationary measure.

Under certainty ($\sigma = 0$), one of the two endpoints would necessarily be the long-run outcome: when the two investors have the same rate of impatience ρ, the one with the lower risk aversion[32] would end up owning all the wealth, when the rate of impatience ρ is less than the earning rate $\alpha = r$ (expanding economy), and the opposite would be true in the opposite case.[33] It is easy to see that cases #3 and 4 could not have arisen under certainty.[34] In particular, if the volatility of the output were zero, there would be no possibility for a stable economy as in case #4. The case of uncertainty is therefore **qualitatively different** from the case of certainty.

6. Fluctuations in the rate of interest and volume of trading

As the allocation of wealth fluctuates between borrower and lender, the security market line of the traditional CAPM should be viewed as pivoting around one fixed point representing the risky production opportunity, while the variable slope of the line determines the current value of the riskless rate of interest.

As we shall verify in the next section, the person with the smaller risk aversion is under all circumstances the one who borrows. A positive output shock shifts the wealth distribution towards him. At the next point in time, he will still be a borrower and since he is now richer he will borrow more than before, thereby driving up the rate of interest. The behavior of the allocation of wealth is thus mirrored in the stochastic behaviors of the market price of risk $1/(\omega\lambda + 1 - \omega)$ and of the equilibrium riskless rate of interest r, which are monotonic functions of ω via equations (21) and (22) above.[35] They both admit two natural barriers, at 1 and $1 - \gamma$ for the market price of risk, and at $\alpha - \sigma^2$ and $\alpha - (1 - \gamma)\sigma^2$ for the rate of interest. These values correspond to the endpoints $\omega = 0$ and $\omega = 1$, where one of the two investors would impose his risk aversion and his corresponding value of the rate of interest.

It follows from the analysis of the last section that there are four possible long-run behaviors of the rate of interest, depending on the case situation at hand: in cases 1 to 3 one or both of the two boundaries are attracting (but not attainable) while in case 4 the interior region is stable in the stochastic sense. Whenever there exists a stable interior distribution of the allocation of wealth, so is there one for the rate of interest, which wanders between the two extreme values, while tending to return to the stable interior region.

FIGURE 5 GOES HERE

For empirical purposes, it is of interest to formulate the process for the rate of interest in an autoregressive from (the resulting formulation is necessarily AR(1) since every process in this economy is Markovian). This is accomplished in continuous time by writing the stochastic differential equation for r, which contains a drift term and a diffusion term. Examples of the resulting drift function are displayed as Figure 5. The drift function is of necessity nonlinear since it is zero at the boundary points; in addition it can change sign and admit one maximum and minimum (despite the fact that the economy is, in this figure, unambiguously a contracting one).[36] **This is a highly nonlinear AR(1) process.** The diffusion term of the rate of interest is also by no means constant[37]: it is zero at the two barriers and exhibits a maximum somewhere in between. **Hence this is a heteroscedastic AR(1) process.** Although this model is one of the most simple one can conceive, while still exhibiting a variable rate of interest, the process so obtained is much more complex than any of those which have been previously utilized to model interest rate behavior (cf. e.g. the Ornstein-Ohlenbeck process used by Vasicek (1977)).

The current model includes only one risky asset, so that it is not exactly appropriate to discuss the relative pricing of assets. Assets which are in zero net supply may nonetheless be priced. Since the present formulation has been able to generate an interesting behavior for the short-term rate of interest, one might think of applying it to the **pricing of bonds**. Rather than expressing the price of the bond as a function of a driving variable such as the allocation of wealth, which is not observable, it is empirically more useful to express it as function of the short-term rate of interest r, [38] as has been done for instance, by means of a 'pseudo-arbitrage' reasoning, by Vasicek (1977) and Brennan and Schwartz (1982). These authors commonly introduce an assumption designed to disconnect their bond market equilibrium from the general equilibrium of the economy, which they do not wish to model. They assume that the market reward for bearing interest-rate risk is a constant.

It is important to realize that **such an assumption would not be tenable** in a general-equilibrium setting such as the current one where the variation of the interest rate arises from the heterogeneity of individuals. Indeed, the market reward for bearing interest-rate risk would be equal to the market price of risk, as defined above,[39] times the volatility of the output. It follows from what we said and from equation (21) that the market price of risk and the rate of interest are (negatively) linearly related and that their volatilities are proportional to each other. It would therefore not be permissible to assume that the market price of interest-rate risk is constant, in a setting which allows for the kind of interest-rate uncertainty which we have modelled here. In this model, that is, the interest rate cannot be fluctuating if the market price of risk is assumed constant. Unwarranted assumptions regarding the behavior of the market price of risk are one danger of the 'pseudo-arbitrage' approach against which Cox, Ingersoll and Ross (1985b) have already warned.

Asset holdings by the two investors are given by the values of x and x* (equations (20) and (19)), for the non-logarithmic and the logarithmic investors respectively. It is always found that $x > 1$ (while $x^* < 1$) when $\gamma > 0$ and that $x < 1$ (while $x^* > 1$) when $\gamma < 0$. In other words, as has been indicated several times, the less risk averse person levers himself in order to invest more than his wealth into equity. He is a perennial borrower. x and x^* represent the shares of each investor's wealth invested in the risky asset. However, the two investors' shares of ownership in the risky production opportunity are equal to $x\omega$ and $x^*(1 - \omega)$ respectively.

FIGURES 6a, 6b GO HERE

Trading takes place in the capital market since $x^*(1 - \omega)$ and $x\omega$ are non-constant functions of a fluctuating ω; for, this implies that one investor buys shares from and sells shares to the other, as time passes. In contrast to previous theories of dynamic capital market equilibrium, the present model accounts for (some) trading volume. Indeed figures 6a and 6b display the shares of ownership as functions of the allocation of wealth and it is clear that they are no constant: when one investor owns almost all the wealth, almost all of **his** wealth is allocated to the risky asset and, by necessity, he owns almost all the shares of this asset. The other investor may or may not be a borrower, depending on his risk aversion, but his leverage always remains finite $\left(x < 1/(1 - \gamma),\ x^* < 1 - \gamma\right)$ so that he can only own a small fraction of the shares of the risky asset. As an investor's share of wealth fluctuates, so does his share of ownership of the risky asset; and, of course, his share of wealth does fluctuate because, as a result of different risk aversions, the two investors make up their portfolios differently. Our model provides scope for capital flows between investors; the current-account balance between them is not equal to zero.

7. Conclusion

The current model, to our knowledge, is the first to present a self-contained account of dynamic capital market equilibrium, involving investors with different taste parameters. The theory is self contained in the sense that all state variables are identified and have a well-specified, endogenously determined, stochastic process. The model exhibits a stochastically variable distribution of wealth, which sometimes admits a stable interior distribution, and a variable short rate of interest with the same property. It also produces trading in the capital market.

The agenda for future research includes an extension to the international setting, with several productive assets, endogenous default and deviations from purchasing-power parity, and possibly also several currencies.

FOOTNOTES

[1]That is the **only** security which would achieve that result. In particular, a menu of assets including the stock plus a **long-term** consol would not eliminate trading any more than does the menu considered here.

[2]The existence of riskless borrowing and lending is nonetheless predicated on the fact (assumed known by all parties) that all investors have isoelastic utilities. Only these utilities will guarantee that investors will never be in a situation where they are unable to repay. Otherwise a credit rationing scheme would have to be superimposed on the debt contracting activity in order to allow riskless borrowing and lending. Alternatively the possibility of default could be introduced in the definition of debt instruments; but the menu of securities would then differ from the one we consider here.

[3]The history of capital markets, especially in recent years, is characterized by a gradual evolution from standardized towards specialized securities. The fact that standardized securities anteceded specialized ones empirically demonstrates their practicality. For as long as the move towards specialized securities is not complete--and we should expect that it never will, the bulk of the market capitalization remaining in standardized form--the kind of trading we describe here will effectively take place.

[4]Note that the horizon is infinite.

[5]It is necessary to ensure in any given economy that the integral of equation (1) converges both for the logarithmic investor and for the non-logarithmic one. Since isoelastic utilities are not bounded, the positivity of the discount rate is not a sufficient condition for convergence. Please refer to footnote 16 below.

[6]The present interpretation of the model features **one** centralized production unit which issues shares of stock. The two investors will trade these shares because the aggregate dividend which is distributed is the one which is socially desirable. At any given time, this dividend is excessive for the consumption needs of one investor and insufficient for the needs of the other. The former will then buy some shares from the latter. **In another interpretation**, there would be two identical (and perfectly correlated) production units operated in the backyards of the two investors. Each one could then help himself to the amount of dividend he individually desires and no trading of shares would be needed. The two interpretations are equivalent because of constant returns to scale, but the former one presented here seems more "natural." In both interpreations, there would be the same amount of trading in the short-term riskless asset.

[7]Endogenous default resulting from unwillingness to pay is left for future research. Inability to pay is ruled out by one well-known property of isoelastic utility function: when consumption tends to zero, marginal utility tends to infinity.

[8]Wealth and physical capital stock have the same value because of the assumption of constant returns to scale.

[9]The assumption that the equilibrium interest rate is a function of the distribution of wealth is innocuous. Ultimately in equilibrium all variables are one-to-one functionally and monotonically related (see below). This is because aggregate output follows a Markov diffusion process and is the single source of shocks in this economy. Hence we could have equivalently assumed that the rate of interest is a function of aggregate wealth or of the wealth of one of the two investors. Expressing it as a function of the distribution of wealth is convenient because that variable, in contrast to aggregate or

personal wealth, takes values in the compact set [0, 1] and because it appears naturally in the 'supply equals demand' condition ((18) below).

[10]See Karlin and Taylor (1981), pp. 226 ff.

[11]Their proof does not require that the budget set be compact. It can be applied to our case provided that the riskless rate of interest is bounded from above. This will turn out to be true in our setting; the interest rate is in fact bracketed by the values $\alpha - \sigma^2$ and $\alpha - (1 - \gamma)\sigma^2$ (see below).

[12]N.B.: as a result: $c/W = I^{1/(\gamma-1)}$.

[13]The J function for the logarithmic investor takes a somewhat different from:

$$J^*(W^*,\ 1 - \omega) = \frac{1}{\rho} \ln W^* + K(1 - \omega) \ ,$$

where K( ) is a function which is well defined and finite as long as $\rho > 0$ and r is bounded. In other words, in the case of the logarithmic investor $I^* = 1/\rho$. As a result $c^* = \rho W^*$.

[14]We use the plural to refer to the fact that one such equation obtains for each investor.

[15]The steps are identical to those followed by Cox, Ingersoll and Ross (1985a).

16 $\frac{d(p/p^*)}{p/p^*} = 0$.

[17]See also Duffie and Huang (1985).

[18]As a matter of fact, if we had introduced a multiplicity of assets, we could have proved that a Tobin separation theorem applies to the present situation.

[19]In fact, in equilibrium, ω is functionally related to aggregate wealth and to each person's wealth; see below.

[20]An alternative to (26) is $I'(1) = 0$, which could be obtained by differentiating (24) and inserting the value $\omega = 1$.

[21]Boundedness is the true underlying 'boundary condition' which produces (25) and (26). Once it is assumed that the unknown function I is bounded, the reader will note that (25) and (26) follow from the ODE itself when inserting $\omega = 0$ or $\omega = 1$ into it. This is a consequence of the nature of the boundaries of the stochastic process for ω. Conversely the boundedness condition is compatible with the ODE (24) if and only if it is possible to obtain the values (25) and (26) for I from the ODE. This is possible if and only if:

$$\rho > \gamma\left(\alpha - \sigma^2 + \frac{1}{2}\,\frac{\sigma^2}{1 - \gamma}\right)$$

and

$$\rho > \gamma\left(\alpha - \frac{1}{2}\,(1 - \gamma)\sigma^2\right) .$$

[22]cf. Smith (1978).

[23]Diagrams and tabulations for the I( ) function are available from the author upon request. The function is increasing when $\gamma < 0$ (risk aversion larger than 1) and decreasing when $\gamma > 0$.

[24]See equation (14) above.

[25]Equation (28) gives us the wealth sharing rule from the $I(\omega)$ function. A reverse procedure would have been conceivable: the relationship W(S) or W*(S) between a person's wealth and aggregate wealth can be shown to satisfy a Black-Scholes differential equation. Because the rate of interest is endogenous, however, the equation in question is a nonlinear one. One can then use equation (28) to perform a change of unknown function from W(S) or W*(S) to $I(\omega)$; equation (24) would result from this procedure. The interpretation of the Black-Scholes equation for the wealth sharing rule is that the more risk averse person in effect holds a call on aggregate output. If such a call were available no trading would be needed, as was indicated in the introduction. For more details see Dumas (in preparation).

[26]In other words, $+\infty$ is an 'attracting boundary.'

[27]I.e. zero is a non-attracting boundary.

[28]In the current model, all of these events occur, if at all, after an infinitely large expected time: the boundaries are not 'attainable.'

[29]In a previous version of this paper, the economy had been characterized on the basis of the drift term of the distribution of wealth. Some of the conclusions were incorrect precisely because the diffusion coefficient of ω is not a constant (see equation (23)). I am grateful to Andy Abel for pointing out this fact. No such difficulty arises when we use lnS as a reference variable.

[30]I do not consider the borderline cases $g(0) = \sigma^2/2$ and $g(1) = \sigma^2/2$ because they are special cases arising for specific parameter combinations only and, above all, because their analysis would require the knowledge of the local behavior of $g(\omega)$ close to $\omega = 0$ and $\omega = 1$, knowledge which I do not have.

[31]This, of course, does not mean that capital stock decreases with probability one over any given interval of time. The capital stock does fluctuate up and down but, in accordance with our definition above, the probability of reaching a capital stock equal to zero is positive. As always that event will occur only after an infinite expected time. These warnings apply in the other cases as well.

[32]Risk aversion would act then only as a measure of elasticity of substitution between consumption at different points in time.

[33]When $\alpha = \rho$, the long-run allocation of wealth would be determined by the initial situation.

[34]Refer to equation (34) above: when $\sigma = 0$, $g(1)$ and $g(0)$ are either both positive or both negative.

[35]r is an increasing function of ω when $1 - \gamma < 1$, a decreasing one otherwise; the opposite is true for the market price of risk. As a function of the accumulated capital stock, however, the equilibrium rate of interest is always increasing (and the market price of risk is a decreasing function). In other words, the correlation coefficient between the risky output and the rate of interest is equal to +1.

[36]Figure 5 represents a situation in which $\gamma = 0.5$ and the non-logarithmic investor is therefore the less risk averse one. In terms of the classification of section 5, the parameter values correspond to case #1 (contracting economy).

[37]But it is always positive (except at the boundaries), reflecting the fact that a positive output shock induces a rise in the rate of interest; see above.

[38]And, of course, of the time to maturity.

[39]I.e. the slope of the security market line $1/(\omega\lambda + 1 - \omega)$; see equation (21) and the explanation which follows.

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FIGURE 1: WEALTH SHARING RULE

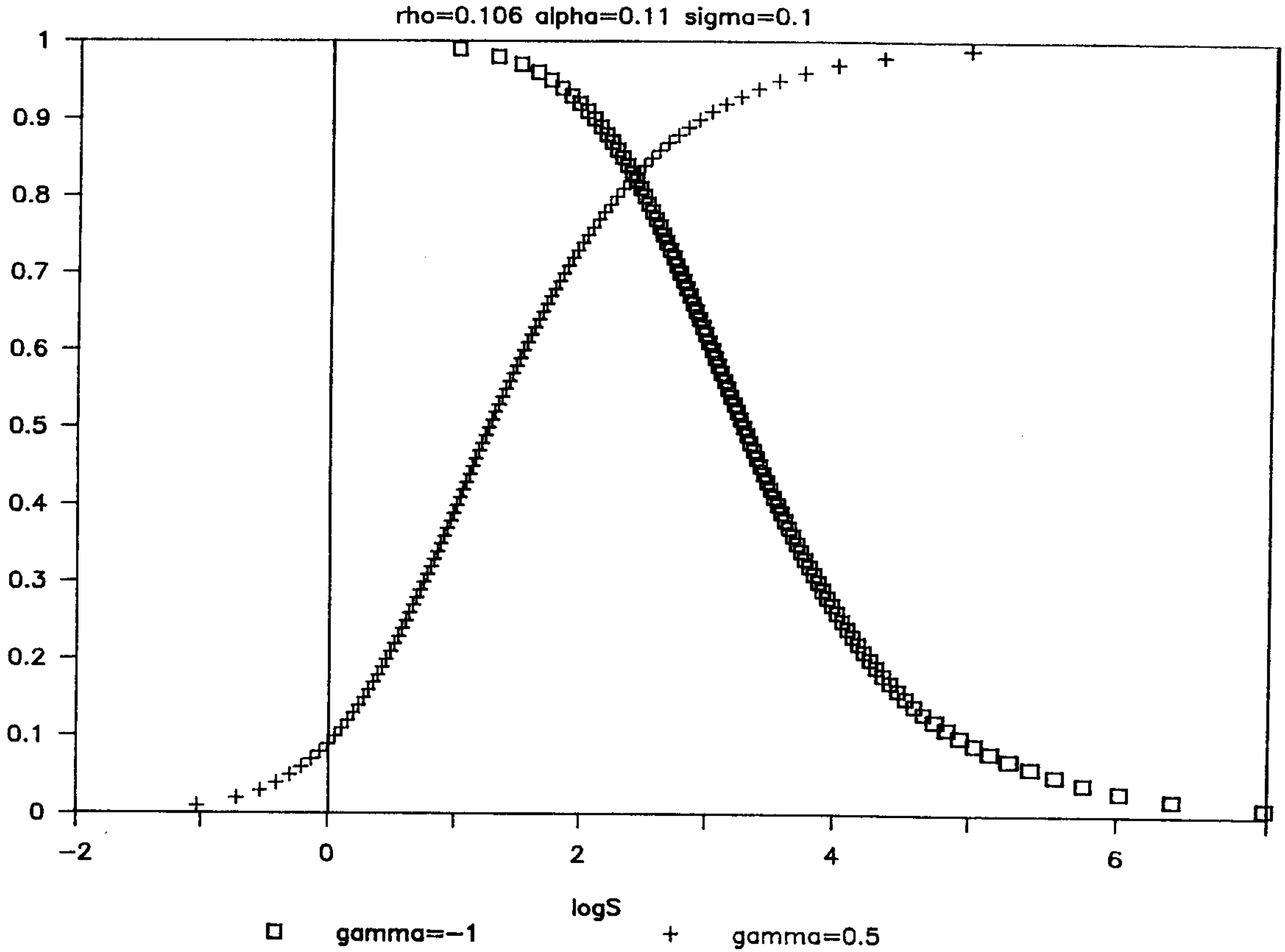

FIGURE 2: GROWTH DIAGRAM

rho=0.106 gamma=0.5 alpha=.11 sigma=0.1

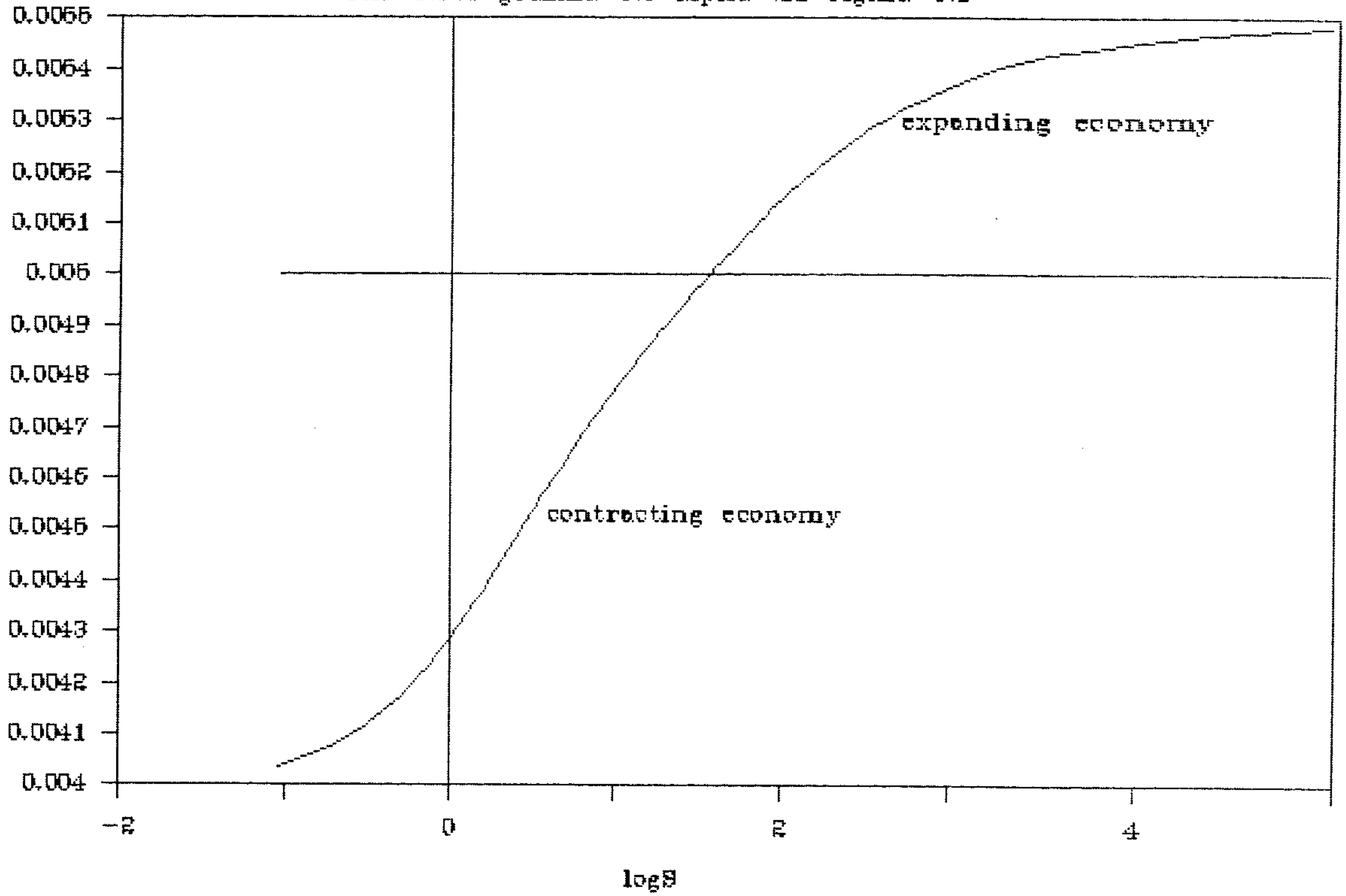

FIGURE 3: GROWTH DIAGRAM

rho=0.106 gamma=-1 alpha=.11 sigma=0.1

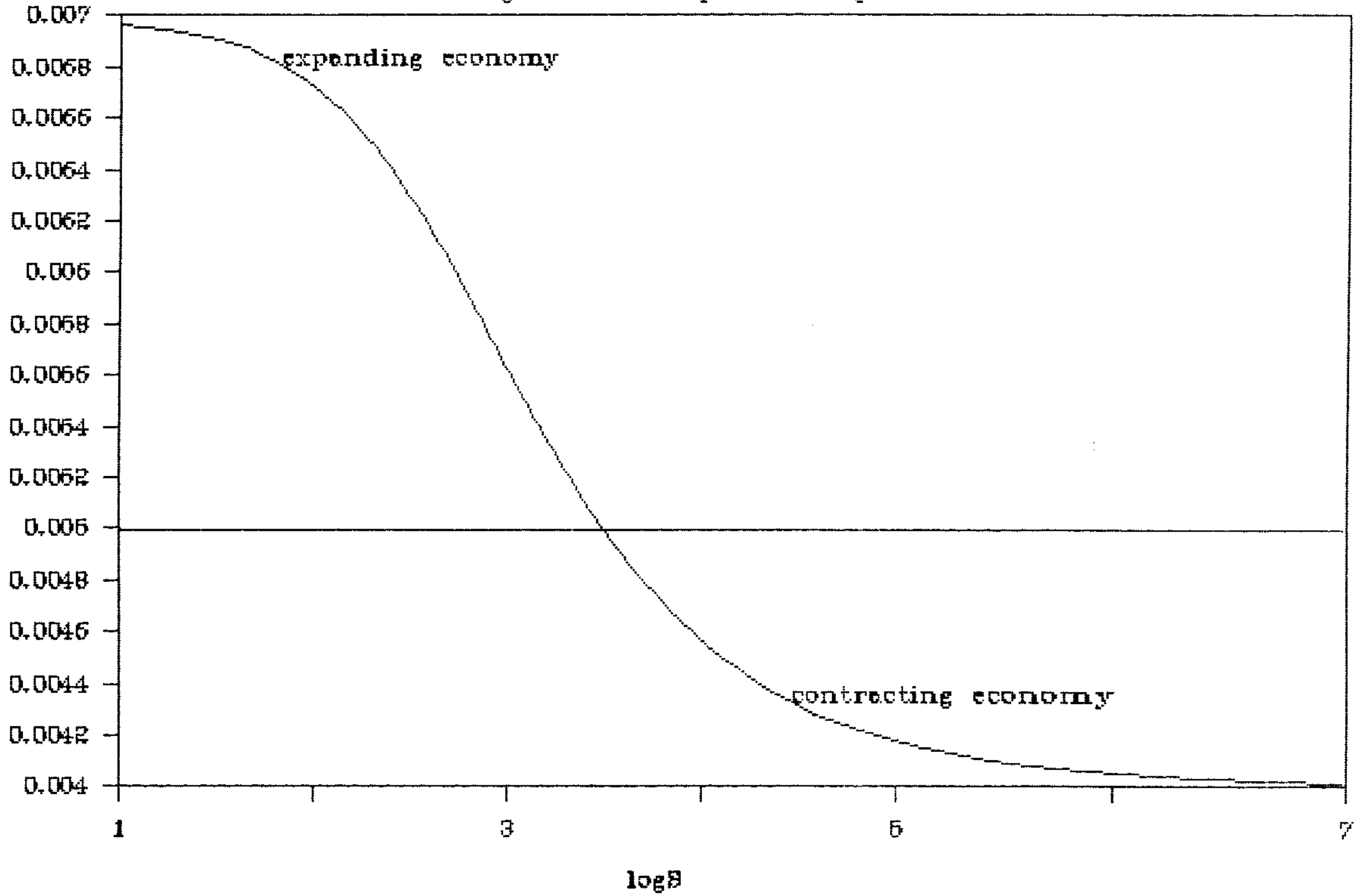

FIGURE 4: STATIONARY DENSITY

rho=0.106 gamma=−1 alpha=0.11 sigma=0.1

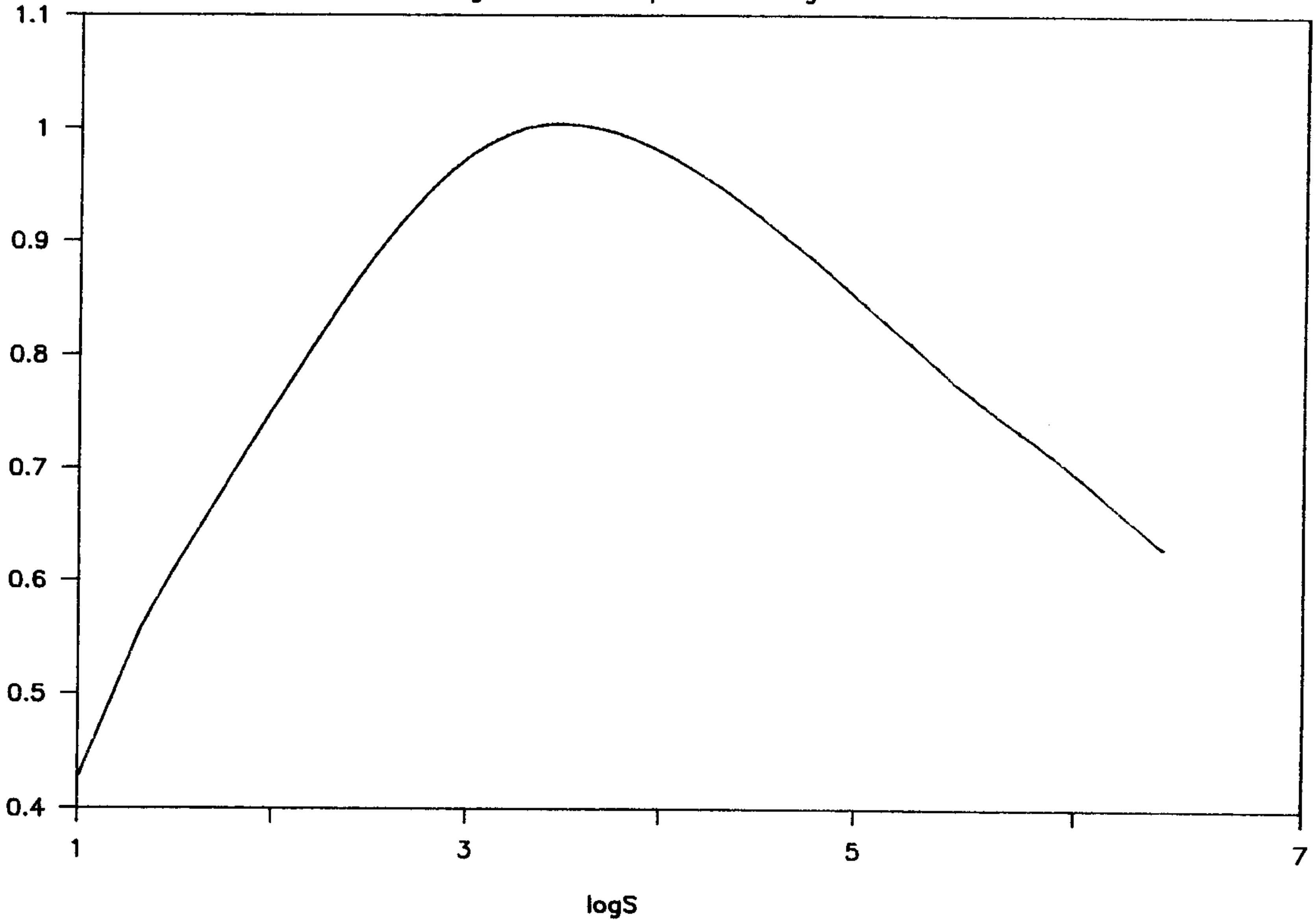

FIGURE 5 : BEHAVIOR OF RATE OF INTEREST

rho=.1075 gamma=0.5 alpha=.11 sigma=0.1

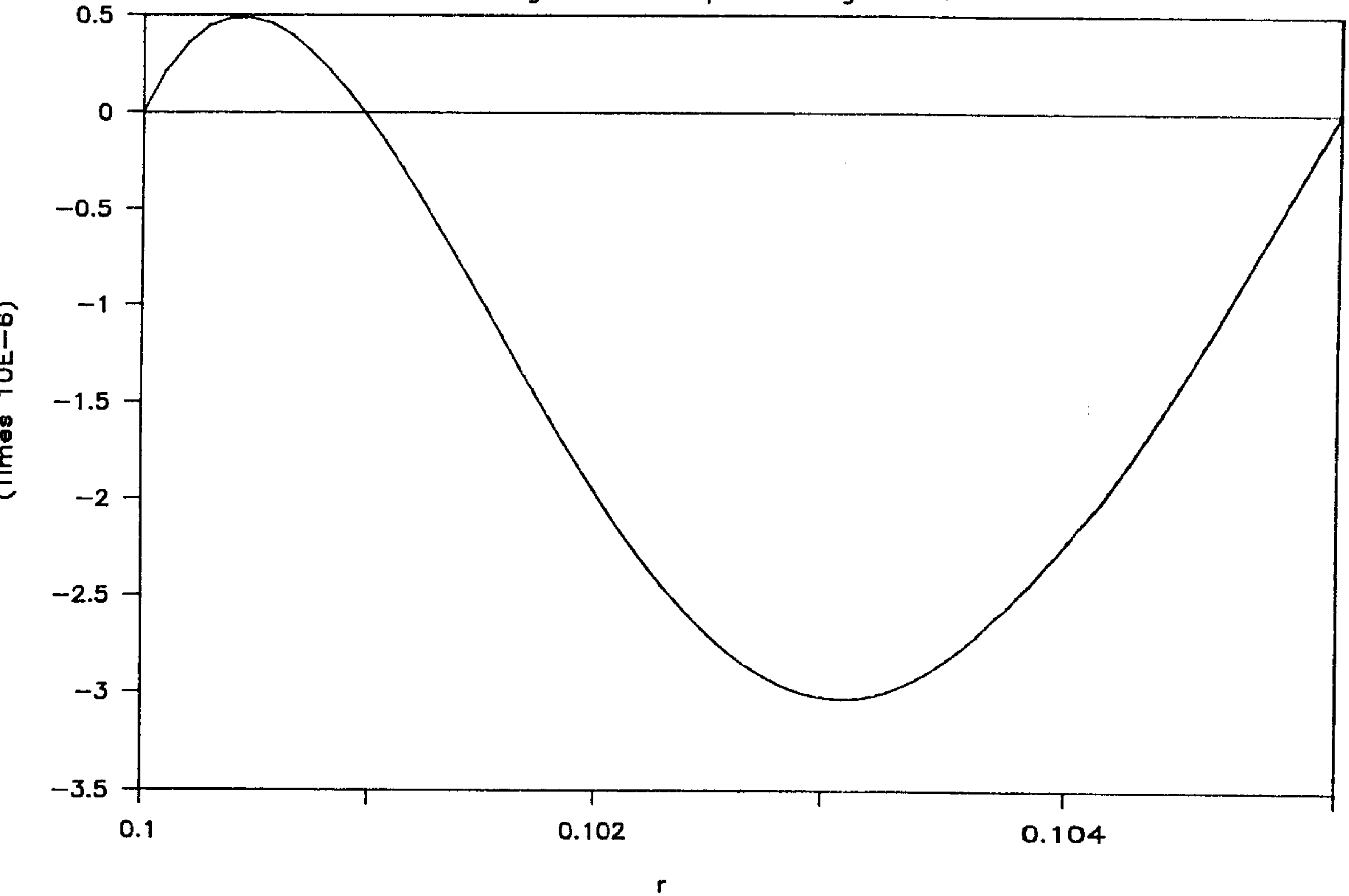

FIGURE 6a: SHARES OF OWNERSHIP

rho=0.106 gamma=-1 alpha=.11 sigma=0.1

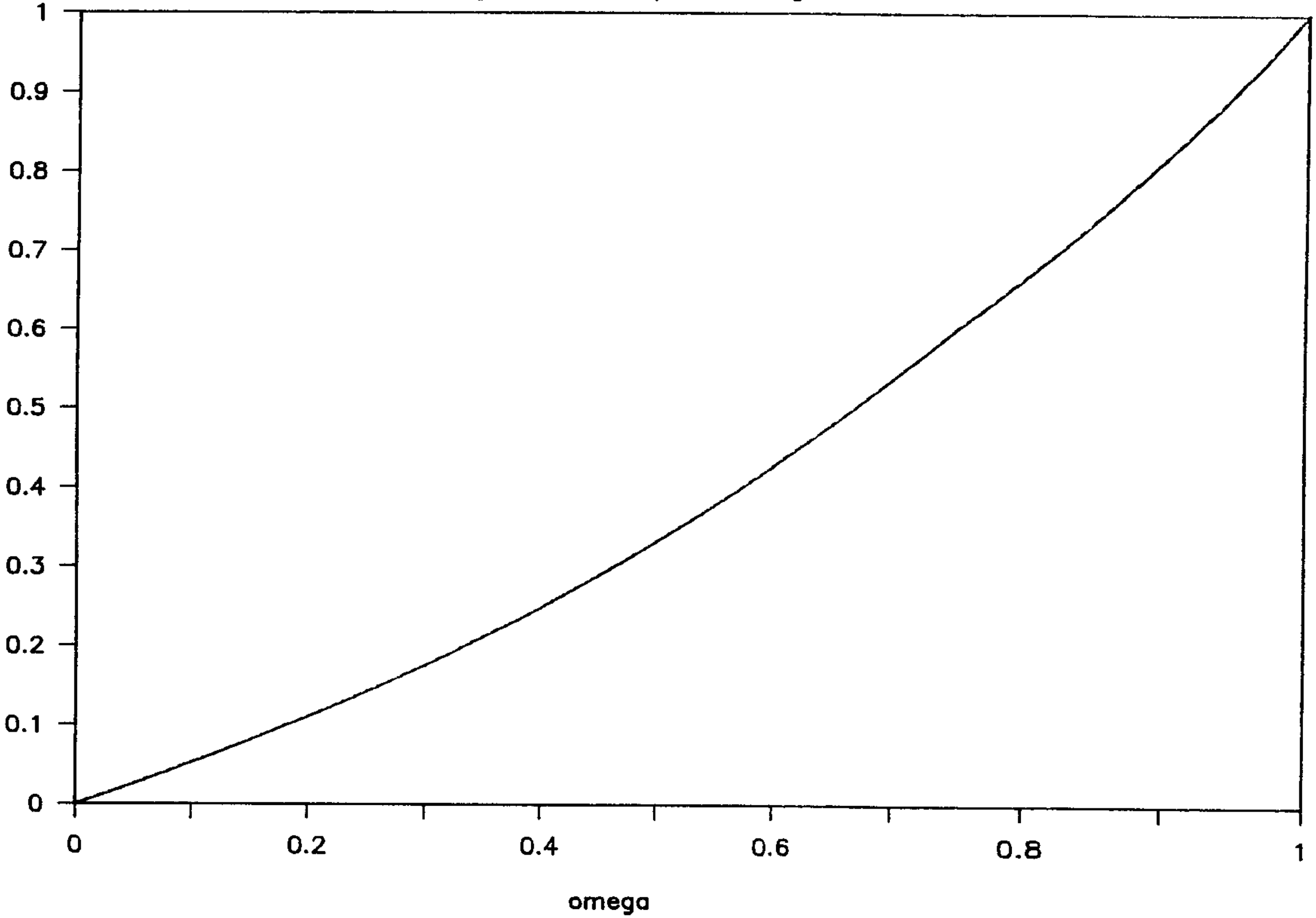

FIGURE 6b: SHARES OF OWNERSHIP

rho=0.106 gamma=0.5 alpha=.11 sigma=0.1