THE OPTIMAL NON-LINEAR BANK

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I. INTRODUCTION

An earlier paper with Robert G. King, "Banking and Insurance" derives and discusses the optimal linear bank, i.e., a bank that pays depositors one interest rate independent of deposit size or depositor type. This note derives and discusses the optimal bank when interest rates may depend on deposit size, which is itself a function of depositor type, θ. Once again, the key is to exploit the analogy between maximizing expected utility and maximizing social welfare, applying the techniques of the optimal income tax literature, notably Mirrlees (1971) as exposited by Atkinson and Stiglitz (1980).

The mapping from the public finance problem of the optimal income tax to the contractual question of optimal deposit rates makes the analysis tractable. The question becomes, in a sense, one on the optimal tax on savings, with the consumer choosing consumption in two periods instead of consumption and leisure. The social welfare function, which in the income tax problem represents the government's distaste for an unequal distribution of income naturally becomes an individual's aversion to uncertain income. These two uses of a concave function indicating preferences over a distribution are quite well known. In fact, the original uses of a concave social welfare to express preferences about the inequality of income distribution stressed the analogy to the risk aversion of individuals. This mapping, which we invert, was first introduced by Kolm (1969) and Atkinson (1970). Having made this analogy, the techniques developed by Mirrlees (1971) and exposited by Atkinson and Stiglitz (1980) lead to a solution of the control problem for the non-linear case.

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II. MODEL

The basic economy follows that of "Banking and Insurance" <u>exactly</u>, so the description here will be brief.

The economy has 3 periods: a planning period (T = 0) and two periods with production and consumption (T = 1, 2). The infinite number of agents have the following identical valuation of consumption goods in T = 1, 2: U = G(u) where $u = u(c_1, c_2) = [c^{1-\frac{1}{\sigma}} + \beta c^{1-\frac{1}{\sigma}}]^{\sigma/\sigma-1}$. $G(\cdot)$ is chosen to add risk aversion to u so that U is a concavification of u. Each individual has an endowment of the single good in each period. At periods 0 and 2, agents have identical endowments ϕ and y_2. At T = 1 each agent receives a <u>privately</u> observable income level $y_1(\theta)$ with $y_1^1(\theta) > 0$. y_1 has compact support on $(\underline{\theta}, \overline{\theta})$ with strictly positive density function $f(\theta)$. The large number of traders means that per capita income is the certain $E(y_1(\theta))$: there are no aggregate shocks. Agents have two storage technologies; one liquid, one illiquid. The first (process A) transforms one unit of good in T to one unit in T+1. The other (process B) transforms 1 unit in T to R > 1 unit in T+2. No output can be retrieved from process B in T+1.

Given the uncertainty in the economy, agents will produce an institution--or write a contract--to share risk. Each agent's shock is private information, however, so a conventional insurance market will not work. We next derive the optimal incentive compatible contract, which we further claim is a bank.

III. SOLUTION

This section presents a heuristic solution to problem, adapting Atkinson's and Stiglitz's (1980, Sec. 13-4) summary of Mirrlees' (1971) work. This solution, though formal, is heuristic because it neglects the conditions under which we may use the "First Order Approach" to incentive compatibility. (See Mirrlees (1979).) Because the bank chooses an interest rate function, we must solve an optimal control problem (see Intrilligator, 1971, or Takayama, 1985).

First, the individual's problem:

$$\max_{c_1, c_2} u(c_1 c_2) \quad \text{s.t.} \quad c_1 + p_2(\theta)c_2 = T + y_1(\theta) + p_2(\theta)y_2$$

$$\text{or} \quad c_1(\theta) = T + y_1(\theta) + p_2(\theta)\big(y_2 - c_2(\theta)\big) \ .$$

Substituting the constraint into the utility function and differentiating yields the first order condition:

(1) $$\frac{du}{dc_2} = u_1(\cdot)\big(-p_2(\theta)\big) + u_2 = 0 \ .$$

Maximizing expected utility (in T = 0, the planning period, when individual differences have yet to arrive) subject to the production or resource constraint is done by solving

$$\int_\theta G(u)dF$$

s.t.

$$\int_\theta \left[c_1(\theta) + \frac{1}{R} c_2(\theta) - y_1(\theta) - (\frac{1}{R} y_2 + \phi)\right]dF = 0 \ .$$

Finally, we need an "f-constraint" or a differential equation on u, the state variable, which we use as the incentive compatibility constraint. Differentiating the utility function and the individual budget constraint, and applying the first order condition for utility maximization (1) yields

$$(2)\qquad \frac{du}{d\theta} = u_1 \frac{dc_1}{d\theta} + u_2 \frac{dc_2}{d\theta} = u_1 \cdot \left(y_1^1(\theta) + p_2(\theta)\left(-\zeta^1(\theta)\right)\right) -$$

$$+ \left(y_2 - c_2(\theta)p_2^1\right) + u_2c_2(\theta)$$

$$= u_1y_1^1(\theta) + u_1\left(y_2 - c_2(\theta)\right)p_2^1(\theta)$$

$$= u_1\left(y_1^1(\theta) + p_2^1(\theta)\left((y_2 - c_2(\theta)\right)\right)$$

$$= \frac{u_2}{p_2}\left(y_1^1(\theta) + p_2^1(\theta)(y_2 - c_2(\theta)\right) .$$

Now form the Hamiltonian

$$H = G(u)f + \lambda\left[c_1(\theta) + \frac{1}{R}c_2(\theta) - y_1(\theta)\right]f + \zeta\frac{du}{d\theta}$$

with $c_2(\theta)$ the <u>control variable</u> and $u(\theta)$ the <u>state variable</u>.

The first order condition becomes

$$\frac{\partial H}{\partial c_2} = 0 \Rightarrow \lambda\left[\left.\frac{dc_1}{dc_2}\right|_{\bar{u}} + \frac{1}{R}\right]f + \zeta\frac{\partial}{\partial c_2}\left(\frac{du}{d\theta}\right) = 0$$

or

$$\left.\frac{dc_1}{dc_2}\right|_{\bar{u}} + \frac{1}{R} = \left[\frac{-\zeta(\theta)}{\lambda f}\right]\left[\frac{\partial}{\partial c_2}\left(\frac{du}{d\theta}\right)\right] .$$

From the implicit function theorem and (1), $\left.\frac{dc_1}{dc_2}\right|_{\bar{u}} = p_2(\theta)$, so the first order condition becomes

$$(3)\qquad p_2(\theta) + \frac{1}{R} = \left[\frac{-\zeta}{\lambda f}\right]\left[\frac{\partial}{\partial c_2}\left(\frac{du}{d\theta}\right)\right] .$$

This corresponds to Atkinson and Stiglitz's (13-52). To simplify matters, let $u_{12} = u_{21} = 0$. Then the <u>canonical equation</u> becomes

$$(4) \qquad -\frac{d\zeta}{d\theta} = \frac{\partial H}{du} = G^1 \cdot f + \lambda \frac{dc_1(\theta)}{du} f + \zeta \frac{\partial}{\partial u}\left(\frac{du}{d\theta}\right)$$

$$= \left(G^1 + \lambda \frac{dc_1}{du}\right) f \; .$$

Now integrate (4) with respect to θ, imposing the <u>transversality condition</u>, $\zeta(\bar{\theta}) = 0$, to get

$$\int_{\theta}^{\bar{\theta}} \frac{d\zeta}{d\theta}\, d\theta = \zeta(\bar{\theta}) - \zeta(\theta) = \int_{\theta}^{\bar{\theta}} \left(G^1 - \lambda \frac{dc_1}{du}\right) dF \; .$$

Rearranging, and noticing that $\frac{dc_1}{du} = \frac{1}{\frac{du}{dc_1}}$, changes the integral to

$$(5) \qquad -\frac{\zeta(\theta)}{\lambda} = \int_{\theta}^{\bar{\theta}} \left(\frac{G^1}{\lambda} - \frac{1}{u_1}\right) dF \; .$$

Substituting into (3),

$$p_2(\theta) + \frac{1}{R} = \frac{1}{f}\left[\int_{\theta}^{\bar{\theta}} \left(\frac{G^1}{\lambda} - \frac{1}{u_1}\right) dF\right]\left[\frac{\partial}{\partial c_1}\left(\frac{du}{d\theta}\right)\right] \; .$$

Then again using $u_{12} = 0$ we can evaluate $\frac{\partial}{\partial c_1}\left(\frac{du}{d\theta}\right)$ as $-p_2 u_1$ or by (1), $-u_2$. This leads to

$$(6) \qquad \frac{1}{u_1} = p_2(\theta) = \left[\int_{\theta}^{\bar{\theta}} \left(\frac{G^1}{\lambda} - \frac{1}{u_1}\right) dF\right]\left[-\frac{u_2}{f}\right] - \frac{1}{R} \; .$$

IV. INTERPRETATION

To begin the interpretation, notice the solution is a deviation from $\frac{1}{R}$, the "natural" or technological rate of interest, the rate that would prevail in a simple market for claims on future income. This lowers interest rates which provides insurance in the following manner. People with a high income at T = 1 save income to smooth consumption, and the lower interest rate extracts some of their surplus. Low income people would tend to borrow, and the low interest rates benefit them. This redistribution insures a risk averse person who *ex ante* does not know his future θ. Next, consider the integral. Suppose we reduce the utility (u) of everyone (or every state) ϕ above θ by a marginal unit. Then we have $\frac{1}{u_1}$ goods to redistribute to everyone. This represents the gain from purchasing insurance. The cost depends on how numerous those states are and how we value them, which is $\frac{G^1}{\lambda}$, which is how risk aversion enters. Thus the bank redistributes income from high to low states. As u_1 and G^1 vary, eventually the small gain in revenue won't be worth the cost, so an interior solution exists. The second term $[-\frac{u_2}{f}]$ describes how the distortion in interest rates (from $\frac{1}{R}$) affects second period consumption. Again, equation (6) weights this by the probability density f. Lowering interest rates, although it increases utility by providing insurance, also decreases utility by distorting the savings choice.

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