SEMIPARAMETRIC UPPER BOUNDS FOR OPTION PRICES

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In this paper, we derive upper bounds on call and put options which are priced via the risk-neutral valuation approach of Cox and Ross (1976) and Harrison and Kreps (1979). The upper bounds are shown to obtain across all terminal stock price distributions for which the associated equivalent martingale measures (EMM) have a common variance. Because the proposed upper bound depends only upon the variance of the EMM and not upon the entire distribution, the bound is termed "semiparametric" and must be satisfied for processes with jumps as well as diffusion components. The relation between the variance of the EMM and the empirically observable variance is derived and some illustrative empirical evidence is presented which suggests that these bounds may be of considerable practical value.

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I. Introduction.

In their seminal paper on the valuation of options for alternative stochastic processes Cox and Ross (1976) provided a simpler and more intuitive method for finding the solution to option valuation problems which has since become known as the "risk-neutral valuation" approach to pricing contingent claims. Cox and Ross observed that, since the usual hedging argument for pricing claims did not require any assumptions on the risk preferences of agents in the economy, the subsequent pricing relation must hold for any set of preferences such as risk-neutrality. However, under risk-neutrality any contingent claim's price must equal the expected present value of its payoff, discounted at the riskless rate of interest.[1] Given the payoff structure of the claim and the distribution of the underlying or "fundamental" asset price at maturity, this expected discounted value may often be calculated directly and must coincide with the pricing formula obtained from the standard hedging arguments. This expected discounted value is calculated with respect to a transformed terminal stock price distribution which, in the terminology of Harrison and Kreps (1979), is known as the equivalent martingale measure (EMM) of the stock price process. Although not all contingent claims may be priced in this way, the precise conditions under which such an approach may be implemented were carefully laid out by Harrison and Kreps (1979) and were shown to be quite general indeed.

Of course, the specific pricing formula obtained depends intimately upon the particular stochastic process which the fundamental asset price follows. Not surprisingly, the properties of the option pricing formulas which Cox and Ross derived differ substantially across the various stock price processes they consider. In practice, this may be quite problematic since it is often difficult to identify the exact stochastic process governing stock returns.

Although it has been widely noted that certain moments of the stock price distribution may be estimated quite precisely (such as its volatility), option pricing relations are generally determined by the entire distribution. For example, a lognormal diffusion process and a mixed Poisson-normal jump process may imply the same variance for the terminal stock price distribution, but the two processes have grossly different implications for the value of a deep out-of-the-money option with one week to maturity. This "stochastic process risk" may be particularly serious for the pricing of newly introduced options, such as the options on the Value Line index which began trading as recently as 1985.

In this paper, we attempt to deal with stochastic process risk for option pricing by deriving an upper bound on option prices which are robust to the specification of the stochastic process of stock prices for a given variance of the transformed terminal stock price distribution or EMM. More specifically, for a given variance of the EMM, we propose an upper bound to the option price which must obtain for all stock price processes with that same variance, hence the term "semiparametric" in our title.[2] Of course, the practical value of such an upper bound is determined by how tight the bound is. After all, one trivial upper bound on the price of a call option is the underlying stock price. However, we hope to show that the bounds obtained in this paper are considerably better. To preview some of the results, we note that the Black-Scholes price of a call option with a striking price of $35 and a one week time-to-maturity on a stock with an annual standard deviation of 20 percent, given an annual riskless interest rate of 5 percent, is $5.03. The corresponding semiparametric upper bound is $5.10, a difference of only 7 cents! The tightness of this bound is especially striking when one notes that it must hold for all stock price distributions with the same transformed terminal price variance, including those arising from Merton's (1976a) jump process, Cox and Ross's (1976) Ornstein-Uhlenbeck process with absorbing barrier, etc.

Although several bounds on option prices have been developed in the extant literature, most of these depend functionally upon the distribution of the stock price.[3] That is, although such bounds may be obtained for almost any type of stock price distribution, the actual numerical values of the bounds will differ across different stock price distributions. Merton's (1972) lower bound is perhaps the only one which requires no parametric assumptions whatsoever since it is based purely on a no-dominance argument. Because the upper bounds proposed in this paper are shown to obtain across parametric families of stock price distributions, they may be viewed as a partial complement to Merton's nonparametric lower bounds.

In Section II we present the upper bound and state explicitly the required assumptions, relegating the actual derivation to the Appendix. Since the upper bound depends upon the variance of the transformed distribution which may not be observable, Section III explores the relation between the actual variance of the stock price process and the transformed variance. The usefulness of these bounds is examined in Section IV in comparison to the Black-Scholes formula. Section V presents an empirical example in which the semiparametric upper bounds are computed for actual traded options on five stocks and are compared with corresponding market prices. We conclude in Section VI.

II. An Upper Bound for Option Prices.

Before developing the semiparameter upper bounds, we first review the Cox and Ross (1976) risk-neutral valuation procedure for purposes of exposition and notation. Consider an option at time t on a stock with price S(t) and suppose the option expires at date T. Let $S(t)$ be represented by a specific stochastic process which depends upon a parameter vector θ. For example, if $S(t)$ were a lognormal diffusion with drift μ and variance rate σ^2, then $\theta \equiv [\mu\ \sigma]$. Let $P[S(T)|S(t);\ \theta]$ denote the conditional distribution of $S(T)$ given the current price $S(t)$; clearly P must depend parametrically upon θ. Now suppose that there exists a dynamic portfolio strategy consisting of positions in the stock and an option on that stock which results in a riskless hedge. Assuming frictionless markets and unrestricted borrowing and lending at a fixed riskless rate of interest r, such a riskless hedge must yield the riskless rate of return by arbitrage. This no-arbitrage condition results in the usual partial differential equation, for which the solution is the option price. A key insight of Cox and Ross (1976) is that since this partial differential equation is functionally independent of preferences, the resulting solution must obtain over any set of preferences. In particular it must obtain under risk-neutrality, in which case the option price is simply the conditional expected value of the terminal payoff discounted at the riskless rate. Now this expected value cannot be calculated with respect to P for the simple reason that the expected value of $S(T)$ with respect to P need not equal $S(t)e^{r(T-t)}$ which is required under risk-neutrality. However, Cox and Ross note that if the no-arbitrage condition for the riskless hedge (i.e., the partial differential equation) does not depend upon a subset θ_1 of the parameters $\theta \equiv [\theta_1'\ \theta_2']'$, then it is possible to adjust the parameters θ_1 so as to satisfy the condition that:

$$E_t^*[S(T)|S(t)] \equiv \int_0^\infty S(T)dP[S(T)|S(t);\ \theta_1^*,\ \theta_2] = S(t)e^{r(T-t)} \quad . \qquad (1)$$

Calculating the discounted expected value of the option's terminal payoff with respect to the transformed distribution $P^*(S(T)|S(t)) = P(S(T)|S(t);\ \theta_1^*,\ \theta_2)$ must then yield the option price. This transformed terminal stock price distribution corresponds to Harrison and Kreps' (1979) equivalent martingale measure or EMM. In this section, we derive an upper bound on "risk-neutral valued" options over the set of all EMM's with a given variance V^*.

II.1 Derivation of the Upper Bound

Let $F(S(t),t)$ and $G(S(t),t)$ denote the prices at time t of a European call and a European put option respectively where $S(t)$ denotes the price at time t of the underlying stock and suppose that both options expire at date T, $T > t$. We maintain the following assumptions throughout this paper:

(A1) There exists a continuous-time dynamic portfolio strategy consisting of stock and option which yields a riskless hedge.

(A2) There exists unrestricted riskless borrowing and lending at the instantaneous rate of return $r \geq 0$.

(A3) The no-arbitrage condition resulting from (A1) and (A2) does not depend functionally upon the subset θ_1 of parameters θ of the stock price process $S(t)$.

(A4) $E^*[S^2(T)|S(t)] < \infty$ (finite second moment).

(A5) Markets are perfect, i.e., no taxes, transactions costs, restrictions on short-sales, etc.

Assumptions (A1) and (A3) simply insure that the risk-neutral valuation approach of Cox and Ross (1976) and Harrison and Kreps (1979) obtains. Assumption (A2) is also related to the risk-neutral valuation technique, but has been somewhat relaxed by Harrison and Kreps (1979). Although the fourth assumption (A4) seems rather weak, it does rule out the "fat-tailed" distributions such as the Pareto-Levy which, to some, may be a plausible description of stock returns processes.[4] Finally (A5) is the usual frictionless markets assumption.

Let R(t, T) denote the (random) gross rate of return of S between dates t and T, i.e., $R(t, T) \equiv S(T)/S(t)$. Assumptions (A1) and (A2) insure that we may invoke risk-neutrality without loss of generality, hence:

$$E_t^*[R(t, T)] = e^{r\tau} \tag{2}$$

where $\tau \equiv T - t$ and $E_t^*[\cdot]$ is the transformed conditional expectations operator conditioned upon S(t).[5] Letting V* denote the transformed conditional variance of R, we conclude from (A4) that:

$$V^* \equiv \mathrm{Var}^*[R(t, T)|S(t)] < \infty \, . \tag{3}$$

Note that the conditional variance V* need not be a constant and may in fact depend upon S(t) and other variables in the information set at time t. We now proceed to derive an upper bound for option prices which depends only upon the second moment of the EMM and not its entire distribution.

Given assumptions (A), Cox and Ross (1976) and Harrison and Kreps (1979) have shown that the price of European call and put options with striking price K may be calculated as:

$$F(S(t),t) = E_t^*[e^{-r\tau}\mathrm{Max}[S(T)-K, 0]] \tag{4a}$$

$$G(S(t),t) = E_t^*[e^{-r\tau}\mathrm{Max}[K-S(T), 0]] \, . \tag{4b}$$

More formally, we may rewrite equation (4) as:

$$F(S(t),t) = e^{-r\tau} \int_0^\infty \mathrm{Max}[S(T)-K, 0]dP^*(S(T)|S(t)) \tag{5a}$$

$$G(S(t),t) = e^{-r\tau} \int_0^\infty \mathrm{Max}[K-S(T), 0]dP^*(S(T)|S(t)) \, . \tag{5b}$$

Now consider the collection of all possible EMM's variance $S^2(t)V^*$ and denote this collection **P***, i.e.,

$$\mathbf{P}^*(r, V) = \Big\{ P^*\big(S(T)|S(t)\big): \int_0^\infty S(T)dP^*\big(S(T)|S(t)\big) = S(t)e^{r\tau} , \qquad (6)$$
$$\int_0^\infty \left[S(T) - S(t)e^{r\tau}\right]^2 dP^*\big(S(T)|S(t)\big) = V^* \Big\} .$$

Since, aside from fixing the first two moments, the transformed stock price distribution is otherwise completely arbitrary, **P*** clearly contains an enormous variety of processes for S over which the values F and G might vary dramatically. Now suppose we maximize the values of F and G over all possible distributions P* in **P***, and call these values $\bar{F}$ and $\bar{G}$ respectively. That is,

$$\bar{F}\big(S(t),t\big) = \max_{P^* \in \mathbf{P}^*} \left[e^{-r\tau}\int_0^\infty \mathrm{Max}[S(T)-K,\ 0]dP^*\right] \qquad (7a)$$

$$\bar{G}\big(S(t),t\big) = \max_{P^* \in \mathbf{P}^*} \left[e^{-r\tau}\int_0^\infty \mathrm{Max}[K-S(T),\ 0]dP^*\right] . \qquad (7b)$$

Such upper bounds are of considerable interest since by construction they must obtain for any stochastic process S for which its EMM has the same variance. Of course, it is by no means obvious that such upper bounds exist. However, a simple application of an ingenious result developed by Scarf (1958) in an unrelated context not only insures the existence of these bounds, but allows us to compute them explicitly. We present the bounds in the following two propositions, outline the proofs in the Appendix, and refer interested readers to Scarf's (1958) elegant paper for the details. For completeness, we also restate some of Scarf's original results in the Appendix.

Proposition 1: Under assumptions (A) the solution to equation (7a) exists and is given by:

$$\bar{F}(t) = \begin{cases} \dfrac{S(t) - Ke^{-r\tau} + S(t)V^*e^{-2r\tau}}{1 + V^*e^{-2r\tau}} & \text{if } \dfrac{S(t)}{K} \geq \dfrac{2e^{-r\tau}}{1 + V^*e^{-2r\tau}} \\ \tfrac{1}{2}\left[S(t) - Ke^{-r\tau} + \sqrt{(Ke^{-r\tau} - S(t))^2 + S^2(t)V^*e^{-2r\tau}}\right] & \text{if } \dfrac{S(t)}{K} < \dfrac{2e^{-r\tau}}{1 + V^*e^{-2r\tau}} \end{cases} \tag{8}$$

Proof: See Appendix and Scarf (1958).

Proposition 2: Under assumptions (A) the solution to equation (7b) exists and is given by:

$$\bar{G}(S(t),t) = \bar{F}(S(t),t) - S(t) + Ke^{-r\tau} . \tag{9}$$

Proof:

$$\text{Max}[S(T)-K,\ 0] = S(T) - K + \text{Max}[K-S(T),\ 0] \tag{10}$$

$$E_t^*[e^{-r\tau}\text{Max}[S(T)-K,\ 0]] = E_t^*[S(T)e^{-r\tau}] - Ke^{-r\tau} + E_t^*[e^{-r\tau}\text{Max}[K-S(T),\ 0]] \tag{11}$$

$$= S(t) - Ke^{-r\tau} + E_t^*[e^{-r\tau}\text{Max}[K-S(T),\ 0]] \tag{12}$$

$$\Rightarrow \max_{P^* \in P^*}\left[E_t^*[e^{-r\tau}\text{Max}[S(T)-K,\ 0]]\right] = S(t) - Ke^{-r\tau} + \max_{P^* \in P^*}\left[E_t^*[e^{-r\tau}\text{Max}[K-S(T),\ 0]]\right] \tag{13}$$

$$\therefore\ \bar{G}(S(t),\ t) = \bar{F}(S(t),\ t) - S(t) + Ke^{-r\tau} \tag{14}$$

QED

In the next section, we explore the impact of changes in the underlying parameters upon the upper bounds $\bar{F}$ and $\bar{G}$.

II.2 Comparative Static Analysis of $\bar{F}$ and $\bar{G}$.

In order to examine the behavior of the upper bounds $\bar{F}$ and $\bar{G}$ when the underlying parameters are perturbed, we differentiate equation (8) with respect to those parameters, yielding the following results:

$$\frac{\partial \bar{F}}{\partial V^*} \geq 0 \qquad (15a)$$

$$\frac{\partial \bar{F}}{\partial \tau} \geq 0 \qquad (15b)$$

$$\frac{\partial \bar{F}}{\partial r} \geq 0 \qquad (15c)$$

The intuition behind these relations are well-known. Recall that $\bar{F}$ is the upper bound for call option prices over all EMM's with variance V*. Clearly if we increase V*, $\bar{F}$ must also increase since the value of a call is increasing in the risk of the underlying stock, which is confirmed by equation (15a). Equation (15b) states that as the time to maturity increases, the call upper bound also increases. This is not surprising since call option values must also be increasing functions of time-to-maturity regardless of the underlying stock price process. Finally, because implicit in a call option is a "limited liability" loan, the call value is an increasing function of the riskless rate of interest for any stock price process, hence the upper bound must also increase with r as stated in (15c).

Consider now the properties of $\bar{F} = \bar{F}(S,K)$ when the parameters (r, t, V*) are fixed. In particular, suppose that V* is a fixed parameter which does not depend upon S(t) or any other variables in the information set at time t. It is clear from equation (8) that F(S,K) is homogeneous of degree one, i.e., $F(tS,tK) = tF(S,K)$ for any $t > 0$. This result is a direct consequence of our assumption that V* is a constant. To see this, observe that F(S,K) is the call upper bound over all distributions $P^*(S(T)|S(t))$ with mean $S(t)e^{rt}$ and variance V*. Now consider doubling both the stock price S(t) and the striking price K and searching over the distributions $P^*(S(T)|S(t))$ with mean $2S(t)e^{rt}$ and variance V* so as to maximize the expected discounted value of the terminal payoff. Since V* (which is defined as $Var^*[S(T)/S(t)|S(t)]$) is to be kept constant, doubling S(t) necessarily implies that S(T) is doubled in which case we have:

$$F(2S,2K) = \underset{P^*}{\text{Max}}\ E_t^*\left[e^{-r\tau}\text{Max}[2S(T)-2K,\ 0]\right] \tag{16a}$$

$$= 2\underset{P^*}{\text{Max}}\ E_t\left[e^{-r\tau}\text{Max}[S(T)-K,\ 0]\right] = 2F(S,K)\ . \tag{16b}$$

Of course, the assumption that V* is fixed is made merely for convenience and is not required in order for Propositions 1 and 2 to obtain. However, when this assumption is appropriate, it enables us to calculate a table of standardized upper bounds for various $(r,\ \tau,\ V^*)$ combinations. More specifically, let $x = S(t)/K$ and define the function $\bar{f}$ as:

$$\bar{f}(x) \equiv \frac{1}{K}\,\bar{F}(S,\ K) = \bar{F}(\tfrac{S}{K},\ 1)\ . \tag{17}$$

Then values of $\bar{f}$ may be tabulated for empirically relevant combinations of $(x,\ r,\ \tau,\ V^*)$ from which the actual upper bounds $\bar{F}$ may be computed by simple multiplication. An example of such a table is given in Table 1.

INSERT TABLE 1 HERE

As a function of S and K, it may be shown that $\bar{F}$ satisfies the relations:

$$\frac{\partial\bar{F}}{\partial S} \geq 0 \quad , \quad \frac{\partial\bar{F}}{\partial K} \leq 0\ . \tag{18a}$$

Since $\bar{F}(S,K)$ is homogeneous of degree one, the properties of the standardized value $\bar{f}(x)$ may be readily deduced as:

$$\bar{f}'(x) = \frac{\partial\bar{F}}{\partial S} \geq 0 \quad , \quad \bar{f}''(x) \geq 0\ . \tag{18b}$$

Finally, using the put-call parity relation for the upper bounds $\bar{F}$ and $\bar{G}$, the comparative static analysis for $\bar{G}$ may be performed using the previous relations for $\bar{F}$:

$$\frac{\partial \bar{G}}{\partial V^*} = \frac{\partial \bar{F}}{\partial V^*} \geq 0 \tag{19a}$$

$$\frac{\partial \bar{G}}{\partial \tau} = \frac{\partial \bar{F}}{\partial \tau} + Kre^{r\tau} \geq 0 \tag{19b}$$

$$\frac{\partial \bar{G}}{\partial r} = \frac{\partial \bar{F}}{\partial \tau} + K\tau e^{r\tau} \geq 0 \tag{19c}$$

$$\bar{g}(x) \equiv \frac{1}{K}\,\bar{G}(S,K) = \bar{f}(x) - x + e^{r\tau} \tag{19d}$$

$$\bar{g}'(x) = \bar{f}'(x) - 1 \gtrless 0 \qquad \text{if} \qquad \bar{f}'(x) \gtrless 1 \tag{19e}$$

$$\bar{g}''(x) = \bar{f}''(x) \geq 0 \tag{19f}$$

Because the EMM variance V^* is usually unobservable, practical applications of the above results may seem somewhat limited. However, in the next section, we show how V^* may be related to the actual variance V of the stock's holding period returns which may often be estimated.

III. <u>The Relation Between V^* and V.</u>

Because V^* is the variance of the equivalent martingale measure, it may not be empirically observable. However, the semiparametric upper bounds may still be implemented by considering possible values for V^* across distinct parametric families of the original terminal stock price distribution. This is most easily understood by means of a simple example. Suppose there is some uncertainty as to whether the stock price process follows a lognormal diffusion process $S_1(t)$ or a pure jump process $S_2(t)$, described by the following two stochastic differential equations respectively:

$$dS_1 = \eta S_1 dt + \sigma S_1 dW \tag{20a}$$

$$dS_2 = (k - 1)S_2 dN_\lambda \qquad k \geq 1 \tag{20b}$$

where N_λ is a Poisson counter with rate λ. By applying Itô's differentiation rule and the Dynkin operator, the variances of the holding period return for these two processes may be computed as:

$$V_1 = e^{2\eta\tau}[e^{\sigma^2\tau} - 1] \tag{21a}$$

$$V_2 = e^{\lambda(k^2-1)\tau} - e^{2\lambda(k-1)\tau} \tag{21b}$$

It may readily be shown that the variances of the corresponding equivalent martingale measures are given by:

$$V_1^* = e^{2r\tau}[e^{\sigma^2\tau} - 1] \tag{22a}$$

$$V_2^* = e^{\lambda(k+1)r\tau} - e^{2r\tau} \tag{22b}$$

Suppose we compute the semiparametric upper bounds $\overline{F}$ and $\overline{G}$ using V_1^*. This upper bound obtains over all lognormal diffusions S_1 with arbitrary η and variance rate less than or equal to σ^2. Moreover, the upper bound also obtains over all pure jump processes S_2 with parameters (λ, k) such that:

$$\lambda(k + 1) \leq \frac{\sigma^2}{r} + 2 \tag{23}$$

To see why this is so, note that when the inequality (23) is strictly binding, the EMM variances V_1^* and V_2^* are equal whereas V_1^* is greater than V_2^* when (23) is a strict inequality. Since the upper bounds $\overline{F}$ and $\overline{G}$ are monotonically increasing in V^*, they must obtain for all lognormal diffusions and pure jump processes which satisfy the relation (23).

In general, the procedure to determine the bounds $\bar{F}$ and $\bar{G}$ which must obtain over a given set of distinct parametric families of stock price distributions follows the above example analogously. For each parametric family $S_j(t)$, derive the variance V_j of the holding period return and compute the EMM variance V_j^* by appropriately adjusting the free parameter vector $\theta_1^{(j)}$, and by choosing a reasonable value for the parameter vector $\theta_2^{(j)}$ (perhaps by econometric estimation). Then compute the upper bounds using the variance $V^* \equiv \underset{j}{\text{Max}}\ V_j^*$. These upper bounds must then obtain over all processes $S_j(t)$.

Of course, $\bar{F}$ and $\bar{G}$ clearly depend upon the values of the parameters $\theta_2^{(j)}$ chosen as well as the set of processes for which V^* is the maximum. That computing the upper bounds requires pre-specifying a particular set of parametric families of processes is not a significant limitation since only a few stock price processes are employed in practice. However, the pre-specification of the parameter vectors $\theta_2^{(j)}$ does render the computations somewhat more cumbersome. Nevertheless under certain circumstances even this limitation may be overcome. Specifically, if it is the case that the EMM variances V_j^* are monotone increasing (decreasing) functions of $\theta_2^{(j)}$ as in the example above, then by setting the $\theta_2^{(j)}$'s equal to their empirically-determined upper (lower) bounds and computing $\bar{F}$ and $\bar{G}$ based on V^*, using these values yields considerably more robust upper bounds.

Finally, a relation between V and V^* may also be derived by examining the implications of economic equilibrium within a specific asset-pricing framework. One such example may be found in Lucas (1978) representative-agent model of equilibrium asset prices. In that model, the time-t price p_t of an asset which has payoff P_T at time T must satisfy the following relation:

$$p_t(y_t) = E_t\left[\beta^\tau \frac{U'(y_T)}{U'(y_t)} P_T(y_T)\right] = \int \beta^\tau \frac{U'(y_T)}{U'(y_t)} P_T(y_T)\psi(y_T|y_t)dy_T \qquad (24)$$

where $\tau \equiv T - t$, β is the reciprocal of one plus the rate of time preference, $U'(y_t)$ and $U'(y_T)$ are the marginal utilities of aggregate consumption at the levels of aggregate output y_t and y_T at time t and T respectively, and $\psi(y_T|y_t)$ is the probability density of y_T conditional upon y_t. Since (24) must hold for all assets in equilibrium, it must hold for the risk-free asset in particular. Defining the risk-free return between t and T as $R_f(\tau)$, (24) implies that:

$$1 = E_t\left[\beta^\tau \frac{U'(y_T)}{U'(y_t)} R_f(\tau)\right] = \int \beta^\tau \frac{U'(y_T)}{U'(y_t)} R_f(\tau)\psi(y_T|y_t)dy_T \qquad (25)$$

Now define the new function $\psi^*(y_T|y_t)$ as the following:

$$\psi^*(y_T|y_t) \equiv \beta^\tau \frac{U'(y_T)}{U'(y_t)} R_f(\tau)\psi(y_T|y_t) \qquad (26)$$

Observe that since U' is nonnegative, ψ^* is nonnegative on the support of y_T. Also, equation (25) implies that $\int \psi^* dy_T = 1$, hence ψ^* is also a density function. It is in fact the equivalent martingale measure under which all assets must earn the risk-free rate of return, i.e.,

$$p_t(y_t) = \int \beta^\tau \frac{U'(y_T)}{U'(y_t)} P_T(y_T)\psi(y_T|y_t)dy_T = R_f^{-1}\int P_T(y_T)\psi^*(y_T|y_t)dy_T \qquad (27a)$$

$$= R_f^{-1}E_t^*\left[P_T(y_T)\right] .$$

It may then be shown that:

$$V^* = V + \int [\frac{P_T}{P_t}]^2[\beta^\tau \frac{U'(y_T)}{U'(y_t)} R_f(\tau) - 1]\psi dy_T - (R_f^2 - [\int \frac{P_T}{P_t} \psi dy_T]^2) \ . \quad (28)$$

Therefore, restricting the form of randomness driving aggregate output y_T and the class of preferences U results in restrictions on the EMM variance V* via relation (28). Of course, similar relations may be derived within other asset-pricing paradigms such as the Cox, Ingersoll, and Ross (1985) continuous time general equilibrium model.

In the next section, we consider the usefulness of these upper bounds by comparing them to corresponding Black-Scholes prices. Because of the degree of generality with which the bounds $\bar{F}$ and $\bar{G}$ were constructed, their practical relevance may seem doubtful. However, Section IV presents some rather surprising experimental evidence to the contrary.

IV. Comparison of $\bar{F}$ and $\bar{G}$ with Theoretical Black-Scholes Prices.

One method of gauging the quality of the semiparametric bounds is to compare them to the Black-Scholes prices when the stock price process is driven by a lognormal diffusion. Specifically, suppose that the stock price dynamics may be described by the stochastic differential equation (20a) with variance V given by (21a). Comparing the Black-Scholes prices to the corresponding upper bounds will then provide some indication as to the seriousness of misspecifying the return-generating process since the upper bound is defined over all possible stock price processes which yield the same variance V* for R(t,T). In Tables 2a-d, such comparisons are performed for hypothetical calls and puts on a stock with current price S = 40 for a variety of interest-rate/striking-price/variance/time-to-maturity combinations.

INSERT TABLES 2a-d HERE

Consider the entries in Table 2a. Given a risk-free interest rate of 2 percent and an annualized compound standard deviation of 20 percent for R(t,T), the largest difference between the upper bound and the Black-Scholes price is 83 cents, which occurs for the call and put options with striking price 35 and time to maturity of 24 weeks. This difference falls to 6 cents for the same options with one week to go!

Now for the same standard deviation consider a call option with striking price 50 which expires in one week. Since it is deep-out-of-the-money it is not surprising that the Black-Scholes price is essentially 0.00. Because the Black-Scholes model assumes that stock prices follow a diffusion, and because the sample paths of diffusions are continuous, the likelihood that the option will be in-the-money in one week given that it is currently deep-out-of-the-money is negligible. If, however, the stock price process contained a jump component, a deep-out-of-the-money call option may be considerably more valuable. Indeed, Merton's (1976b) findings indicate that the percentage error of the Black-Scholes formula may be enormous for deep-out-of-the-money options when jumps are possible. Nevertheless, the upper bounds presented in Table 2a demonstrate that even if only the variance of the total returns is known, a fairly tight upper limit to the call price may be established. In the case of the one-week call option with strike price 50, Table 2a shows that its value cannot exceed 3 cents and this holds for all stock price processes which yield a transformed variance V* corresponding to a σ of 20 percent, including jump processes.

It is clear from Table 2a that the bounds worsen as the time-to-maturity increases. This is not surprising since the impact of "deviations" from lognormality upon the terminal stock price becomes more pronounced as more time is allowed to elapse between the current date and the expiration date.

The results in Table 2a also indicate that the difference between the Black-Scholes prices and the semiparametric upper bounds increases with the standard deviation of the total returns. For a compound standard deviation of 80 percent (which is empirically unrealistic) the smallest difference between the Black-Scholes prices and their upper bounds is 43 cents and the largest difference is $4.45.

Tables 2b-d present the same comparisons as in Table 2a but for riskless interest rates of 4, 6, and 8 percent respectively. As the interest rate increases, so do the upper bounds. However, even for a riskless rate of 8 percent the upper bound of a call option with strike price 50 and one week to go is still 3 cents given a 20 percent compound standard deviation of returns. The results of Tables 2 seem generally to support the empirical relevance of the proposed semiparametric bounds. Of course, in order to implement these bounds, some knowledge of the stock returns process is required, i.e., the variance V^* of total returns $R(t,T)$. Note, however, that restrictions on this quantity may often be estimated quite accurately from sample data. Given the relative dearth of information which is required about the stock price process, the sharpness of the bounds presented in Table 2 are quite striking. Of course, these results were generated from purely hypothetical calls and puts. In the next section, we compare the bounds to actual market prices of traded options.

V. Comparison of $\overline{F}$ and $\overline{G}$ with Traded Options Prices.

To compute the upper bounds $\overline{F}$ and $\overline{G}$ for actual traded options, the conditional variances V of their underlying stocks' holding period returns must first be statistically estimated. This is examined in Section V.1 and in Section V.2 the variances and upper bounds are estimated for five particular stocks for which there are traded options.

V.1 Estimation of Variances.

We consider first the estimation of the conditional variance of R(t,T) for the lognormal diffusion process given by (20a). Of course, if the actual stock price process were known to be a lognormal diffusion then the relevance of semiparametric upper bounds would seem questionable. However, this case is explored purely for expositional purposes so as to clarify the econometric issues at hand. The strong parametric assumption of a lognormal diffusion will be weakened considerably below.

Suppose we observe the continuous stock price process S(t) at regularly spaced discrete intervals h so that a sample of n+1 observations may be represented by the vector $[S(0), S(h), S(2h), \ldots, S(nh)]$. Let $T \equiv nh$ and denote by Z_k the quantity $\ln \frac{S(kh)}{S([k-1]h)}$. Since S(t) follows the lognormal diffusion (19), it is well-known that the Z_k's are independently and identically distributed normal random variables with expectation λh where $\lambda = \eta - \frac{1}{2}\sigma^2$ and variance $\sigma^2 h$. It then follows that under weak regularity conditions the maximum likelihood estimators of $(\lambda, \sigma^2, \eta)$ exist, have the usual properties of consistency and asymptotic normality, and are given by:

$$\hat{\lambda} = \frac{1}{T} \sum_{k=1}^{n} Z_k \tag{29a}$$

$$\hat{\sigma}^2 = \frac{1}{T} \sum_{k=1}^{n} (Z_k - \hat{\lambda} h)^2 \tag{29b}$$

$$\hat{\eta} = \hat{\lambda} + \tfrac{1}{2}\hat{\sigma}^2 \; . \tag{29c}$$

However, the parameters $(\lambda, \sigma^2, \eta)$ are not of direct interest since the computation of $\overline{F}$ and $\overline{G}$ require the conditional variance V of the holding period returns R(t,T). But for the lognormal diffusion process, it may be shown that the conditional mean and variance of R(t,T) are given by:

$$E_t[R(t,T)] = e^{\eta\tau} \tag{30a}$$

$$Var_t[R(t,T)] = e^{2\eta\tau}[e^{\sigma^2\tau} - 1] \; . \tag{30b}$$

Substituting the estimators $(\hat{\lambda}, \hat{\sigma}^2, \hat{\eta})$ from (24) into the right-hand side of (30) then yields consistent estimators $\hat{\bar{R}}(t,T)$ and $\hat{V}(t,T)$ for the conditional mean and variance of $R(t,T)$ respectively.

The above estimation procedure is quite straightforward. This is due to the fact that we have imposed a great deal of structure upon the conditional moments of the process via equation (20a). In fact, (20a) is a parametric distributional restriction of the $S(t)$ process which, loosely speaking, is analogous to assuming normally-distributed disturbances in a linear regression framework. Of course, some structure must be placed upon the conditional mean and variance in order to estimate them statistically. However, since our ultimate use of the variance estimator is in calculating the upper bounds $\bar{F}$ and $\bar{G}$ which do not depend upon the distribution of S, we do not wish to place any parametric restrictions on the stochastic process S.

Fortunately, only a few relatively weak conditions are required in order to be able to estimate V and these conditions are satisfied by a very large class of processes for S. In order to state these conditions precisely, we require some new notation. Denote by $R_t(\tau)$ the holding period return $R(t,T)$, where the new symbol is meant to emphasize the fact that the sequence of returns $\{R_t(\tau)\}$ may be viewed as a separate stochastic process for each holding period $\tau \equiv T - t$. We assume the following:

(B1) For all $\tau > 0$, the process $R_t(\tau)$ is stationary and ergodic.

(B2) For all t, the conditional moments $E_t[R_t(\tau)]$ and $E_t[R_t^2(\tau)]$ do not depend functionally upon any variables in the information set at time t.

Assumption (B1) is a technical requirement that is satisfied by most processes of empirical relevance. Assumption (B2) implies that the conditional and unconditional moments are equal, which will enable us to estimate the mean and variance by their sample counterparts. This assumption, although rather strong, is a natural non-distributional generalization of the parametric form of (30b) which gives the holding period variance as a specific function of the holding period τ but which is independent of the stock price $S(t)$. It is, of course, possible to consider processes for which the conditional and unconditional variances differ (such as Engle's (1982) ARCH process), but not without imposing much more structure upon the $S(t)$ process than we have so far.

Under assumptions (B1) and (B2), it may be shown that the following estimators are consistent and asymptotically normal.[6]

$$\hat{\bar{R}}(\tau) = \frac{1}{T-\tau+1} \sum_{t=\tau}^{T} \frac{S(t)}{S(t-\tau)} \tag{31a}$$

$$\hat{V}(\tau) = \frac{1}{T-\tau+1} \sum_{t=\tau}^{T} \left[\frac{S(t)}{S(t-\tau)} - \hat{\bar{R}}(\tau)\right]^2 \tag{31b}$$

In the next section, we calculate the semiparametric upper bounds for five stocks which have listed options using the estimators (29).

V.2 An Empirical Example.

In Table 3a-c, the estimated variances $\hat{V}(\tau)$ and $\hat{V}^*(\tau)$ of holding period returns are presented for five stocks: Digital Equipment, National Semiconductor, Tandy, Teledyne, and Telex.[7] In addition to having listed options trading on them, these stocks were chosen because they have paid no dividends in the sample period (312 weekly observations from January 1, 1980 to December 27, 1985). This was done so as to render the simple Black-Scholes option pricing formula applicable. The variance estimates $V(\tau)$ in Table 3a are based only upon assumptions (B1) and (B2); no further assumptions are required. The variance estimates in Table 3b were derived under considerably more structure; they are the maximum-likelihood estimates of $V(\tau)$ for the lognormal diffusion process (20a), i.e., they are estimates of (30b). Finally, Table 3c presents the equivalent martingale measure transformed variances $V^*(\tau)$ which correspond to the variances of Table 3b.

Tables 4a-e present the actual market prices, Black-Scholes prices, and upper bounds for the five stocks on Friday, February 7, 1986, using the parameter estimates from Table 3c.

INSERT TABLES 3 AND 4 HERE

Consider these figures for Digital Equipment reported in Table 4a. For a put option with a strike price of 165 and two weeks to go, the upper bound is only 64 cents above the current market price. This implies that given a 7.1 percent transformed standard deviation for two-week holding period returns, any put pricing formula can exceed the current market price by at most 64 cents, regardless of the specific stock price process. This suggests that for purposes of pricing this particular option, misspecifying the stock returns process may not be serious as long as the holding period variance is estimated accurately. This, however, is not the case for other Digital options. For example, the upper bound exceeds the market price of a call option with strike price 155 and six weeks to go by almost \$5.00! Although there are several cases in which the semiparametric upper bounds differ only slightly from true market prices, in general the differences in Tables 4a-e are quite substantial. This indicates that, given the accuracy with which holding period returns variances may be estimated, misspecifying the underlying stock price process may result in significantly biased pricing formulas.

VI. Conclusion.

In this paper, we have derived upper bounds for options which do not depend upon the specific process of the underlying stock price, but only upon its EMM variance. Moreover, these semiparametric upper bounds must obtain for any option pricing formula which may be derived via a "risk-neutral valuation" approach, which includes almost all pricing formulas of practical relevance. Because these upper bounds do not depend upon the particular distribution of the underlying stock price process, the class of processes for which the bounds apply include discontinuous or "jump" processes as well as the usual diffusion processes. In light of this, the comparison of theoretical Black-Scholes prices with their upper bounds yield the rather surprising conclusion that their difference may be as little as 3 cents.

As an empirical example, the proposed upper bounds were computed for actual options on five non-dividend paying stocks. Although for several options the differences between actual market prices and upper bounds were small, in general they seemed to differ by several dollars. This suggests that the pricing of such options via contingent claims analysis will be quite sensitive to the underlying stochastic process generating stock returns. Of course, one important application of these semiparametric bounds is for newly-issued options which have not previously existed. In such cases, since there are no historical benchmarks to guide the pricing of new options, an upper bound which does not depend upon the underlying distribution of the asset returns may be of particular use.

FOOTNOTES

[1]This is assuming, of course, that borrowing and lending at the riskless rate of interest is unrestricted. The precise conditions under which it is possible to price contingent claims solely by arbitrage is given in Harrison and Kreps (1979).

[2]Our use of the term "semiparametric" may differ somewhat from its use in the statistics literature. Since the proposed upper bound is valid across all families of terminal stock price distributions with the same variance, the upper bound is not strictly parametric. It will become clear, however, that this result is due to the fact that we maximize the option price over all such families so that the bound obtains across parametric families by construction.

[3]For example, Perrakis and Ryan (1984) develop option-pricing bounds in discrete time under Rubinstein's (1976) assumptions of competitive and Pareto-efficient financial markets. Although their bounds obtain for a stock price distributions with a "nonnegative beta" property, the actual values of the bounds depend critically upon the specific stock price distribution assumed and are therefore parametric. Levy (1985) also considers discrete-time option-pricing bounds for "nonnegative beta" distributions but uses a stochastic dominance approach. His bounds are also parametric since its particular values depend upon the distribution function of the stock price. Ritchken (1985) uses a linear-programming approach to obtain bounds which also require parametric assumptions (state probabilities) to calculate numerical values. Finally Perrakis (1986) extends the results of Perrakis and Ryan (1984) by tightening the bounds under further restrictions upon the stock price distribution (finite upper limit, positive lower limit) and by considering bounds on American put options. These are also parametric in nature.

[4]Of course, (A3) does not rule out the possibility of a time-varying volatility which has been shown to generate leptokurtotic distributions for stock returns.

[5]Note that the actual expected return of S need not be equal to the riskless rate. This is merely a fiction which allows us to derive the option price most conveniently. In Harrison and Kreps' (1979) terminology, we seek the unique equivalent martingale measure (which satisfies equation (1)) under which the option price is merely an expected discounted value.

[6]Since the summands in (23) are constructed from overlapping data, they are not independent even if the original $S(t)$ process is. However, using Hansen and Singleton's (1982) generalized method of moments framework, these estimators may be shown to be consistent and asymptotically normal with a given asymptotic covariance matrix (see Hansen (1982, 1985), White and Domowitz (1984), and Newey and West (1985)). Perhaps the first use of this method in the econometrics literature was by Hansen and Hodrick (1980).

[7]The variance estimates in Table 3a were calculated using equations (29). The standard errors of the variance estimates were obtained using the asymptotic covariance estimator of Newey and West (1985) which is guaranteed to be positive definite in finite samples. The estimates in Table 3b

correspond to the usual maximum likelihood estimates of the parameters of (20).

APPENDIX - PROOF OF PROPOSITION 1

The proof of Proposition 1 involves a straightforward application of an ingenious result due to Scarf (1958) and, for completeness, we restate it here in notation slightly different from the original:

Lemma (Scarf): Let c, μ and δ be fixed. Then there exists a quadratic function $Q(X) = \alpha + \beta X + \gamma X^2$ such that $Q(X) \leq \text{Min}[X, c]$ for $X \geq 0$ with equality holding at only two points a and b. Moreover there exists a two-point distribution situated at a and b with mean μ and standard deviation δ.

Proof: See Scarf (1958).

Corollary (Scarf): Let the two-point distribution in the preceding Lemma be denoted by H(X). Then H(X) minimizes:

$$\int_0^\infty \text{Min}[X, c]dP(X) \qquad \text{(A1)}$$

over all distributions P(X) with mean μ and standard deviation δ.

Proof:

$$\int_0^\infty \text{Min}[X, c]dP(X) = \int_0^\infty \left(\text{Min}[X, c] - Q(X)\right)dP(X) + \int_0^\infty Q(X)dP(X) \qquad \text{(A2)}$$

$$\geq \int_0^\infty Q(X)dP(X) = \alpha + \beta\mu + \gamma(\mu^2 + \sigma^2) \qquad \text{(A3)}$$

by the above Lemma. But H(X) attains this lower bound since Q(X) and Min[X, c] are equal on those points a and b where H has all its weight.

QED

From these two results, Scarf demonstrates that:

$$\mathrm{Min}\left(E\left[\mathrm{Min}[X,\ c]\right]\right) = \begin{cases} \dfrac{\mu^2 c}{\mu^2 + \delta^2} & \text{if} \quad c \le \dfrac{\mu^2 + \delta^2}{2\mu} \\[2ex] \dfrac{\mu + c}{2} - \tfrac{1}{2}\sqrt{(c - \mu)^2 + \delta^2} & \text{if} \quad c > \dfrac{\mu^2 + \delta^2}{2\mu} \end{cases} \tag{A4}$$

where $\mu = E[X]$ and $\delta^2 = E[X - \mu]^2$ and the minimization is performed over all distributions P with the same mean μ and variance δ^2. Armed with this result, we now prove Proposition 1:

Proof:

$$\mathrm{Max}[X-C,\ 0] + \mathrm{Min}[X,\ c] = X \tag{A5}$$

$$E\left(\mathrm{Max}[X-c,\ 0]\right) = \mu - E\left(\min[X,\ c]\right) \tag{A6}$$

$$\mathrm{Max}\left[E\left(\mathrm{Max}[X-c,\ 0]\right)\right] = \mu - \mathrm{Min}\left[E\left(\mathrm{Min}[X,\ c]\right)\right] . \tag{A7}$$

Thus we have

$$\bar{F} \equiv \mathrm{Max}\left[e^{-r\tau}E\left(\mathrm{Max}[X-c,0]\right)\right] = \begin{cases} \dfrac{\mu^2(\mu - c) + \mu\delta^2}{\mu^2 + \delta^2}\, e^{-r\tau} & \text{if} \quad c \le \dfrac{\mu^2 + \delta^2}{2\mu} \\[2ex] \tfrac{1}{2}\left[\mu - c + \sqrt{(c - \mu)^2 + \delta^2}\right]e^{-r\tau} & \text{if} \quad c > \dfrac{\mu^2 + \delta^2}{2\mu} \end{cases} \tag{A8}$$

Now since we have assumed:

$$E^*\left[S(T)\,|\,S(t)\right] = S(t)e^{r\tau} \tag{A9}$$

$$\mathrm{Var}^*\left[S(T)\,|\,S(t)\right] = S^2(t)V^* \tag{A10}$$

we set:

$$\mu = S(t)e^{r\tau} \tag{A11a}$$

$$\delta^2 = S^2(t)V^* \tag{A11b}$$

$$c = K \tag{A11c}$$

in equation (A8) and simplify, yielding the desired result in equation (7).

<u>QED</u>

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TABLE 1

Standardized semiparametric upper bounds $\bar{f}$ and $\bar{g}$ for call and put options respectively with various combinations of stock-price/strike-price ratios X, times-to-maturity τ, and annual transformed standard deviations $\sqrt{V^*}/\tau$, given a 6 percent annual riskless rate of interest.

$\sqrt{V^*}/\tau$	Time to Maturity τ = 1 Week			Time to Maturity τ = 6 Weeks			Time to Maturity τ = 12 Weeks			Time to Maturity τ = 24 Weeks		
	X	$\bar{f}$	$\bar{g}$	X	$\bar{f}$	$\bar{g}$	X	$\bar{f}$	$\bar{g}$	X	$\bar{f}$	$\bar{g}$
0.2	0.80	0.0006	0.1995	0.80	0.0037	0.1970	0.80	0.0075	0.1942	0.80	0.0153	0.1887
0.2	0.90	0.0015	0.1004	0.90	0.0091	0.1024	0.90	0.0177	0.1044	0.90	0.0341	0.1076
0.2	0.95	0.0033	0.0522	0.95	0.0171	0.0604	0.95	0.0307	0.0673	0.95	0.0536	0.0771
0.2	1.00	0.0144	0.0133	1.00	0.0375	0.0308	1.00	0.0553	0.0419	1.00	0.0829	0.0564
0.2	1.05	0.0550	0.0039	1.05	0.0742	0.0175	1.05	0.0919	0.0285	1.05	0.1208	0.0442
0.2	1.10	0.1035	0.0024	1.10	0.1191	0.0124	1.10	0.1355	0.0221	1.10	0.1640	0.0375
0.2	1.20	0.2027	0.0016	1.20	0.2158	0.0091	1.20	0.2307	0.0173	1.20	0.2584	0.0319
0.4	0.80	0.0024	0.2013	0.80	0.0141	0.2074	0.80	0.0273	0.2139	0.80	0.0515	0.2249
0.4	0.90	0.0059	0.1048	0.90	0.0301	0.1234	0.90	0.0531	0.1397	0.90	0.0902	0.1637
0.4	0.95	0.0115	0.0604	0.95	0.0463	0.0896	0.95	0.0746	0.1112	0.95	0.1176	0.1411
0.4	1.00	0.0283	0.0272	1.00	0.0714	0.0647	1.00	0.1031	0.0897	1.00	0.1502	0.1236
0.4	1.05	0.0644	0.0132	1.05	0.1054	0.0487	1.05	0.1380	0.0746	1.05	0.1873	0.1108
0.4	1.10	0.1097	0.0086	1.10	0.1457	0.0390	1.10	0.1777	0.0644	1.10	0.2280	0.1014
0.4	1.20	0.2067	0.0056	1.20	0.2362	0.0295	1.20	0.2662	0.0528	1.20	0.3167	0.0901
0.6	0.80	0.0054	0.2043	0.80	0.0296	0.2229	0.80	0.0545	0.2412	0.80	0.0967	0.2702
0.6	0.90	0.0126	0.1114	0.90	0.0561	0.1494	0.90	0.0930	0.1797	0.90	0.1495	0.2230
0.6	0.95	0.0220	0.0709	0.95	0.0774	0.1207	0.95	0.1196	0.1562	0.95	0.1819	0.2054
0.6	1.00	0.0422	0.0410	1.00	0.1053	0.0986	1.00	0.1510	0.1377	1.00	0.2179	0.1914
0.6	1.05	0.0762	0.0251	1.05	0.1393	0.0826	1.05	0.1868	0.1234	1.05	0.2569	0.1804
0.6	1.10	0.1189	0.0177	1.10	0.1780	0.0713	1.10	0.2261	0.1127	1.10	0.2984	0.1719
0.6	1.20	0.2130	0.0119	1.20	0.2646	0.0579	1.20	0.3121	0.0987	1.20	0.3872	0.1607
0.8	0.80	0.0094	0.2083	0.80	0.0486	0.2419	0.80	0.0859	0.2725	0.80	0.1458	0.3193
0.8	0.90	0.0208	0.1197	0.90	0.0840	0.1773	0.90	0.1345	0.2212	0.90	0.2097	0.2832
0.8	0.95	0.0336	0.0825	0.95	0.1091	0.1524	0.95	0.1649	0.2016	0.95	0.2464	0.2699
0.8	1.00	0.0560	0.0549	1.00	0.1393	0.1326	1.00	0.1990	0.1857	1.00	0.2857	0.2592
0.8	1.05	0.0892	0.0381	1.05	0.1740	0.1173	1.05	0.2364	0.1731	1.05	0.3273	0.2508
0.8	1.10	0.1299	0.0288	1.10	0.2125	0.1058	1.10	0.2765	0.1631	1.10	0.3709	0.2444
0.8	1.20	0.2214	0.0202	1.20	0.2975	0.0908	1.20	0.3628	0.1494	1.20	0.4626	0.2361

TABLE 2a

Comparison of Black-Scholes call and put option prices F_{BS}, G_{BS} with corresponding semiparametric upper bounds $\overline{F}$, $\overline{G}$ for options with striking prices K = 30, 35, 40, 45, 50 and times-to-maturity τ = 1, 12, and 24 weeks on a hypothetical stock with price S = 40 and (annual) compound standard deviations σ = .2, .4, .6, .8 given a 2 percent annual riskless rate of interest.

Time to Maturity τ = 1 Week

Std. Deviation σ	CALL K	CALL F_{BS}	CALL $\overline{F}$	PUT K	PUT G_{BS}	PUT $\overline{G}$	UPPER BD. MINUS PRICE[a]
0.2	30	10.01	10.05	30	0.00	0.03	0.03
0.2	35	5.01	5.08	35	0.00	0.06	0.06
0.2	40	0.45	0.56	40	0.43	0.55	0.11
0.2	45	0.00	0.06	45	4.98	5.04	0.06
0.2	50	0.00	0.03	50	9.98	10.01	0.03
0.4	30	10.01	10.14	30	0.00	0.13	0.13
0.4	35	5.02	5.25	35	0.01	0.24	0.23
0.4	40	0.89	1.12	40	0.88	1.10	0.23
0.4	45	0.01	0.24	45	5.00	5.22	0.22
0.4	50	0.00	0.12	50	9.98	10.10	0.12
0.6	30	10.01	10.29	30	.00	0.27	0.27
0.6	35	5.08	5.52	35	0.07	0.51	0.44
0.6	40	1.33	1.68	40	1.32	1.66	0.34
0.6	45	0.13	0.51	45	5.11	5.49	0.38
0.6	50	.00	0.27	50	9.99	10.25	0.27
0.8	30	10.02	10.49	30	0.01	0.48	0.47
0.8	35	5.24	5.86	35	0.23	0.85	0.62
0.8	40	1.78	2.23	40	1.76	2.22	0.46
0.8	45	0.35	0.85	45	5.33	5.83	0.50
0.8	50	0.04	0.47	50	10.02	10.45	0.43

Time to Maturity τ = 12 Weeks

Std. Deviation σ	CALL K	CALL F_{BS}	CALL $\overline{F}$	PUT K	PUT G_{BS}	PUT $\overline{G}$	UPPER BD. MINUS PRICE
0.2	30	10.14	10.54	30	0.00	0.40	0.40
0.2	35	5.28	5.83	35	0.12	0.67	0.55
0.2	40	1.62	2.03	40	1.44	1.85	0.41
0.2	45	0.24	0.68	45	5.03	5.48	0.44
0.2	50	0.02	0.37	50	9.79	10.14	0.35
0.4	30	10.32	11.50	30	0.18	1.37	1.18
0.4	35	6.15	7.27	35	0.99	2.11	1.12
0.4	40	3.15	3.99	40	2.96	3.81	0.84
0.4	45	1.39	2.17	45	6.19	6.97	0.78
0.4	50	0.54	1.36	50	10.32	11.13	0.81
0.6	30	10.94	12.89	30	0.80	2.76	1.96
0.6	35	7.35	9.05	35	2.19	3.89	1.70
0.6	40	4.66	6.01	40	4.48	5.82	1.34
0.6	45	2.83	3.98	45	7.62	8.77	1.15
0.6	50	1.65	2.78	50	11.42	12.55	1.12
0.8	30	11.83	14.59	30	1.69	4.45	2.76
0.8	35	8.64	11.02	35	3.48	5.86	2.37
0.8	40	6.17	8.11	40	5.99	7.93	1.93
0.8	45	4.34	5.96	45	9.13	10.76	1.62
0.8	50	3.01	4.49	50	12.78	14.26	1.48

Time to Maturity τ = 24 Weeks

Std. Deviation σ	CALL K	CALL F_{BS}	CALL $\overline{F}$	PUT K	PUT G_{BS}	PUT $\overline{G}$	UPPER BD. MINUS PRICE
0.2	30	10.30	11.05	30	0.02	0.78	0.76
0.2	35	5.70	6.53	35	0.38	1.21	0.83
0.2	40	2.34	2.94	40	1.98	2.58	0.60
0.2	45	0.69	1.28	45	5.28	5.87	0.59
0.2	50	0.15	0.73	50	9.69	10.28	0.58
0.4	30	10.92	12.81	30	0.65	2.53	1.88
0.4	35	7.23	8.88	35	1.92	3.56	1.65
0.4	40	4.49	5.77	40	4.12	5.41	1.29
0.4	45	2.63	3.73	45	7.22	8.32	1.10
0.4	50	1.48	2.56	50	11.02	12.11	1.08
0.6	30	12.16	15.21	30	1.89	4.94	3.04
0.6	35	9.05	11.67	35	3.73	6.36	2.63
0.6	40	6.61	8.77	40	6.25	8.40	2.15
0.6	45	4.77	6.57	45	9.36	11.16	1.80
0.6	50	3.41	5.02	50	12.96	14.57	1.61
0.8	30	13.65	18.10	30	3.38	7.83	4.45
0.8	35	10.92	14.82	35	5.61	9.50	3.89
0.8	40	8.71	12.02	40	8.35	11.65	3.31
0.8	45	6.94	9.74	45	11.53	14.33	2.80
0.8	50	5.53	7.96	50	15.07	17.50	2.43

[a]Note that the difference between the Black-Scholes call price and its upper bound is numerically identical to the difference between the put price and its upper bound for a given (S, K, σ, τ) combination (due to the put-call parity relation). Also, since the actual numerical calculations were performed in double precision, the numbers reported in the "Upper Bd. Minus Price" column may not be identical (to two decimal places) to the difference of the numbers reported in the upper bound and price columns respectively because of rounding.

TABLE 2b

Comparison of Black-Scholes call and put option prices F_{BS}, G_{BS} with corresponding semiparametric upper bounds $\overline{F}$, $\overline{G}$ for options with striking prices K = 30, 35, 40, 45, 50 and times-to-maturity τ = 1, 12, and 24 weeks on a hypothetical stock with price S = 40 and (annual) compound standard deviations σ = .2, .4, .6, .8 given a 4 percent annual riskless rate of interest.

Time to Maturity τ = 12 Week

Std. Deviation σ	CALL K	CALL F_{BS}	CALL $\overline{F}$	PUT K	PUT G_{BS}	PUT $\overline{G}$	UPPER BD. MINUS PRICE[a]
0.2	30	10.02	10.06	30	0.00	0.04	0.04
0.2	35	5.03	5.09	35	0.00	0.06	0.06
0.2	40	0.46	0.57	40	0.43	0.54	0.11
0.2	45	0.00	0.06	45	4.97	5.03	0.06
0.2	50	0.00	0.03	50	9.96	9.99	0.03
0.4	30	10.02	10.15	30	0.00	0.13	0.13
0.4	35	5.03	5.26	35	0.01	0.24	0.23
0.4	40	0.90	1.13	40	0.87	1.10	0.23
0.4	45	0.01	0.24	45	4.98	5.20	0.22
0.4	50	0.00	0.12	50	9.96	10.08	0.12
0.6	30	10.02	10.30	30	.00	0.28	0.28
0.6	35	5.10	5.53	35	0.07	0.51	0.44
0.6	40	1.34	1.68	40	1.31	1.65	0.34
0.6	45	0.13	0.51	45	5.09	5.47	0.38
0.6	50	.00	0.27	50	9.97	10.23	0.27
0.8	30	10.03	10.50	30	0.01	0.48	0.47
0.8	35	5.25	5.87	35	0.23	0.85	0.62
0.8	40	1.78	2.24	40	1.75	2.21	0.46
0.8	45	0.35	0.85	45	5.32	5.82	0.50
0.8	50	0.04	0.47	50	10.00	10.44	0.43

Time to Maturity τ = 12 Weeks

Std. Deviation σ	CALL K	CALL F_{BS}	CALL $\overline{F}$	PUT K	PUT G_{BS}	PUT $\overline{G}$	UPPER BD. MINUS PRICE
0.2	30	10.27	10.72	30	.00	0.45	0.44
0.2	35	5.42	5.99	35	0.11	0.68	0.57
0.2	40	1.71	2.13	40	1.35	1.77	0.42
0.2	45	0.26	0.71	45	4.86	5.30	0.45
0.2	50	0.02	0.38	50	9.57	9.93	0.36
0.4	30	10.45	11.68	30	0.17	1.41	1.23
0.4	35	6.26	7.43	35	0.95	2.11	1.16
0.4	40	3.23	4.10	40	2.87	3.74	0.87
0.4	45	1.44	2.23	45	6.04	6.83	0.79
0.4	50	0.57	1.39	50	10.12	10.94	0.82
0.6	30	11.05	13.06	30	0.78	2.79	2.02
0.6	35	7.45	9.20	35	2.13	3.88	1.75
0.6	40	4.75	6.12	40	4.39	5.76	1.38
0.6	45	2.89	4.06	45	7.48	8.65	1.17
0.6	50	1.69	2.83	50	11.24	12.38	1.14
0.8	30	11.93	14.76	30	1.65	4.49	2.83
0.8	35	8.73	11.17	35	3.41	5.85	2.44
0.8	40	6.25	8.24	40	5.89	7.87	1.99
0.8	45	4.40	6.06	45	9.00	10.65	1.66
0.8	50	3.06	4.56	50	12.61	14.11	1.50

Time to Maturity τ = 24 Weeks

Std. Deviation σ	CALL K	CALL F_{BS}	CALL $\overline{F}$	PUT K	PUT G_{BS}	PUT $\overline{G}$	UPPER BD. MINUS PRICE
0.2	30	10.56	11.41	30	0.02	0.87	0.85
0.2	35	5.96	6.86	35	0.33	1.23	0.90
0.2	40	2.52	3.17	40	1.81	2.45	0.64
0.2	45	0.77	1.37	45	4.96	5.56	0.60
0.2	50	0.17	0.77	50	9.28	9.87	0.60
0.4	30	11.14	13.15	30	0.61	2.61	2.00
0.4	35	7.44	9.19	35	1.81	3.56	1.75
0.4	40	4.65	6.02	40	3.94	5.30	1.36
0.4	45	2.75	3.89	45	6.95	8.09	1.14
0.4	50	1.56	2.66	50	10.66	11.77	1.10
0.6	30	12.35	15.55	30	1.81	5.01	3.20
0.6	35	9.22	11.99	35	3.60	6.36	2.76
0.6	40	6.77	9.03	40	6.05	8.31	2.26
0.6	45	4.90	6.78	45	9.09	10.98	1.88
0.6	50	3.52	5.19	50	12.62	14.29	1.67
0.8	30	13.82	18.45	30	3.28	7.92	4.64
0.8	35	11.08	15.14	35	5.45	9.51	4.06
0.8	40	8.85	12.31	40	8.14	11.59	3.46
0.8	45	7.07	9.99	45	11.26	14.19	2.93
0.8	50	5.64	8.17	50	14.75	17.27	2.53

[a] Note that the difference between the Black-Scholes call price and its upper bound is numerically identical to the difference between the put price and its upper bound for a given (S, K, σ, τ) combination (due to the put-call parity relation). Also, since the actual numerical calculations were performed in double precision, the numbers reported in the "Upper Bd. Minus Price" column may not be identical (to two decimal places) to the difference of the numbers reported in the upper bound and price columns respectively because of rounding.

TABLE 2c

Comparison of Black-Scholes call and put option prices F_{BS}, G_{BS} with corresponding semiparametric upper bounds $\overline{F}$, $\overline{G}$ for options with striking prices K = 30, 35, 40, 45, 50 and times-to-maturity τ = 1, 12, and 24 weeks on a hypothetical stock with price S = 40 and (annual) compound standard deviations σ = .2, .4, .6, .8 given a 6 percent annual riskless rate of interest.

Time to Maturity τ = 1 Week

Std. Deviation σ	CALL K	CALL F_{BS}	CALL $\overline{F}$	PUT K	PUT G_{BS}	PUT $\overline{G}$	UPPER BD. MINUS PRICE[a]
0.2	30	10.03	10.08	30	0.00	0.04	0.04
0.2	35	5.04	5.11	35	0.00	0.07	0.07
0.2	40	0.47	0.58	40	0.42	0.53	0.11
0.2	45	0.00	0.06	45	4.95	5.01	0.06
0.2	50	0.00	0.03	50	9.94	9.97	0.03
0.4	30	10.03	10.17	30	0.00	0.13	0.13
0.4	35	5.04	5.28	35	0.01	0.24	0.23
0.4	40	0.91	1.14	40	0.86	1.09	0.23
0.4	45	0.02	0.24	45	4.96	5.19	0.22
0.4	50	0.00	0.12	50	9.94	10.07	0.12
0.6	30	10.03	10.32	30	.00	0.28	0.28
0.6	35	5.11	5.55	35	0.07	0.51	0.44
0.6	40	1.35	1.69	40	1.30	1.65	0.34
0.6	45	0.13	0.51	45	5.08	5.46	0.38
0.6	50	.00	0.27	50	9.95	10.22	0.27
0.8	30	10.04	10.52	30	0.01	0.48	0.48
0.8	35	5.27	5.89	35	0.23	0.85	0.62
0.8	40	1.79	2.25	40	1.75	2.21	0.46
0.8	45	0.35	0.85	45	5.30	5.80	0.50
0.8	50	0.04	0.48	50	9.99	10.42	0.43

Time to Maturity τ = 12 Weeks

Std. Deviation σ	CALL K	CALL F_{BS}	CALL $\overline{F}$	PUT K	PUT G_{BS}	PUT $\overline{G}$	UPPER BD. MINUS PRICE
0.2	30	10.40	10.89	30	.00	0.49	0.49
0.2	35	5.57	6.16	35	0.10	0.69	0.59
0.2	40	1.80	2.24	40	1.27	1.71	0.44
0.2	45	0.29	0.73	45	4.68	5.13	0.45
0.2	50	0.02	0.39	50	9.35	9.72	0.37
0.4	30	10.57	11.85	30	0.17	1.45	1.28
0.4	35	6.38	7.58	35	0.91	2.11	1.20
0.4	40	3.32	4.21	40	2.78	3.68	0.90
0.4	45	1.49	2.29	45	5.89	6.69	0.80
0.4	50	0.59	1.42	50	9.93	10.75	0.83
0.6	30	11.15	13.23	30	0.75	2.83	2.08
0.6	35	7.54	9.35	35	2.08	3.88	1.81
0.6	40	4.83	6.24	40	4.29	5.71	1.42
0.6	45	2.95	4.14	45	7.35	8.54	1.19
0.6	50	1.74	2.88	50	11.07	12.22	1.15
0.8	30	12.02	14.92	30	1.62	4.52	2.90
0.8	35	8.82	11.32	35	3.35	5.85	2.50
0.8	40	6.33	8.36	40	5.79	7.83	2.04
0.8	45	4.46	6.16	45	8.86	10.56	1.70
0.8	50	3.11	4.64	50	12.44	13.97	1.53

Time to Maturity τ = 24 Weeks

Std. Deviation σ	CALL K	CALL F_{BS}	CALL $\overline{F}$	PUT K	PUT G_{BS}	PUT $\overline{G}$	UPPER BD. MINUS PRICE
0.2	30	10.81	11.76	30	0.02	0.96	0.95
0.2	35	6.22	7.18	35	0.29	1.26	0.96
0.2	40	2.71	3.40	40	1.65	2.34	0.69
0.2	45	0.86	1.46	45	4.66	5.27	0.61
0.2	50	0.20	0.81	50	8.87	9.48	0.61
0.4	30	11.36	13.49	30	0.56	2.69	2.13
0.4	35	7.64	9.50	35	1.71	3.57	1.86
0.4	40	4.82	6.26	40	3.76	5.20	1.44
0.4	45	2.88	4.06	45	6.68	7.87	1.18
0.4	50	1.64	2.77	50	10.31	11.44	1.13
0.6	30	12.53	15.89	30	1.74	5.10	3.36
0.6	35	9.40	12.30	35	3.47	6.37	2.90
0.6	40	6.92	9.30	40	5.86	8.24	2.38
0.6	45	5.03	7.00	45	8.84	10.80	1.97
0.6	50	3.62	5.35	50	12.30	14.02	1.73
0.8	30	13.98	18.80	30	3.18	8.01	4.83
0.8	35	11.23	15.47	35	5.30	9.54	4.24
0.8	40	8.99	12.60	40	7.93	11.54	3.61
0.8	45	7.19	10.25	45	11.00	14.05	3.05
0.8	50	5.75	8.39	50	14.43	17.06	2.63

[a] Note that the difference between the Black-Scholes call price and its upper bound is numerically identical to the difference between the put price and its upper bound for a given (S, K, σ, τ) combination (due to the put-call parity relation). Also, since the actual numerical calculations were performed in double precision, the numbers reported in the "Upper Bd. Minus Price" column may not be identical (to two decimal places) to the difference of the numbers reported in the upper bound and price columns respectively because of rounding.

TABLE 2d

mparison of Black-Scholes call and put option prices F_{BS}, G_{BS} with corresponding semiparametric upper bounds $\bar{F}$, $\bar{G}$ for options with striking prices K = 30, 35, , 45, 50 and times-to-maturity τ = 1, 12, and 24 weeks on a hypothetical stock with price S = 40 and (annual) compound standard deviations σ = .2, .4, .6, .8 ven a 8 percent annual riskless rate of interest.

Time to Maturity τ = 1 Week

Std. Deviation σ	CALL K	CALL F_{BS}	CALL $\bar{F}$	PUT E	PUT G_{BS}	PUT $\bar{G}$	UPPER BD. MINUS PRICE[a]
0.2	30	10.04	10.09	30	0.00	0.05	0.05
0.2	35	5.05	5.12	35	0.00	0.07	0.07
0.2	40	0.47	0.59	40	0.41	0.53	0.11
0.2	45	0.00	0.06	45	4.93	5.00	0.06
0.2	50	0.00	0.03	50	9.93	9.96	0.03
0.4	30	10.04	10.18	30	0.00	0.14	0.14
0.4	35	5.06	5.29	35	0.01	0.24	0.24
0.4	40	0.91	1.14	40	0.86	1.08	0.23
0.4	45	0.02	0.24	45	4.95	5.17	0.22
0.4	50	0.00	0.12	50	9.93	10.05	0.12
0.6	30	10.04	10.33	30	.00	0.28	0.28
0.6	35	5.12	5.56	35	0.07	0.51	0.44
0.6	40	1.36	1.70	40	1.30	1.64	0.34
0.6	45	0.13	0.51	45	5.06	5.44	0.38
0.6	50	.00	0.27	50	9.93	10.20	0.27
0.8	30	10.05	10.53	30	0.01	0.49	0.48
0.8	35	5.28	5.90	35	0.22	0.85	0.62
0.8	40	1.80	2.26	40	1.74	2.20	0.46
0.8	45	0.36	0.86	45	5.29	5.79	0.50
0.8	50	0.04	0.48	50	9.97	10.40	0.43

Time to Maturity τ = 12 Weeks

Std. Deviation σ	CALL K	CALL F_{BS}	CALL $\bar{F}$	PUT K	PUT G_{BS}	PUT $\bar{G}$	UPPER BD. MINUS PRICE
0.2	30	10.53	11.06	30	.00	0.54	0.54
0.2	35	5.70	6.32	35	0.09	0.71	0.62
0.2	40	1.90	2.35	40	1.19	1.65	0.45
0.2	45	0.31	0.76	45	4.52	4.97	0.45
0.2	50	0.02	0.40	50	9.14	9.52	0.37
0.4	30	10.68	12.01	30	0.16	1.49	1.33
0.4	35	6.49	7.73	35	0.87	2.12	1.24
0.4	40	3.40	4.32	40	2.70	3.62	0.92
0.4	45	1.54	2.35	45	5.75	6.56	0.81
0.4	50	0.62	1.45	50	9.74	10.57	0.83
0.6	30	11.26	13.40	30	0.73	2.87	2.14
0.6	35	7.64	9.50	35	2.02	3.88	1.86
0.6	40	4.90	6.36	40	4.20	5.66	1.46
0.6	45	3.01	4.22	45	7.21	8.43	1.22
0.6	50	1.78	2.94	50	10.90	12.06	1.16
0.8	30	12.11	15.09	30	1.58	4.56	2.98
0.8	35	8.90	11.47	35	3.29	5.85	2.57
0.8	40	6.40	8.49	40	5.70	7.78	2.09
0.8	45	4.53	6.26	45	8.73	10.47	1.73
0.8	50	3.16	4.71	50	12.28	13.83	1.55

Time to Maturity τ = 24 Weeks

Std. Deviation σ	CALL K	CALL F_{BS}	CALL $\bar{F}$	PUT K	PUT G_{BS}	PUT $\bar{G}$	UPPER BD. MINUS PRICE
0.2	30	11.06	12.11	30	0.01	1.06	1.05
0.2	35	6.48	7.51	35	0.26	1.29	1.03
0.2	40	2.90	3.64	40	1.50	2.25	0.74
0.2	45	0.95	1.56	45	4.38	4.99	0.62
0.2	50	0.23	0.85	50	8.48	9.11	0.62
0.4	30	11.53	13.83	30	0.53	2.78	2.25
0.4	35	7.84	9.80	35	1.62	3.58	1.97
0.4	40	4.98	6.51	40	3.59	5.11	1.52
0.4	45	3.00	4.23	45	6.43	7.66	1.23
0.4	50	1.72	2.88	50	9.98	11.13	1.15
0.6	30	12.71	16.23	30	1.67	5.18	3.51
0.6	35	9.56	12.60	35	3.34	6.38	3.04
0.6	40	7.07	9.56	40	5.67	8.17	2.49
0.6	45	5.16	7.21	45	8.59	10.64	2.05
0.6	50	3.73	5.52	50	11.98	13.77	1.79
0.8	30	14.13	19.15	30	3.09	8.11	5.02
0.8	35	11.38	15.79	35	5.16	9.57	4.41
0.8	40	9.13	12.89	40	7.74	11.50	3.76
0.8	45	7.32	10.50	45	10.75	13.93	3.18
0.8	50	5.87	8.60	50	14.12	16.86	2.74

ote that the difference between the Black-Scholes call price and its upper bound is numerically identical to the difference between the put price and its upper ound for a given (S, K, σ, τ) combination (due to the put-call parity relation). Also, since the actual numerical calculations were performed in double ecision, the numbers reported in the "Upper Bd. Minus Price" column may not be identical (to two decimal places) to the difference of the numbers reported in the per bound and price columns respectively because of rounding.

TABLE 3a

Estimates of holding period return $V(\tau)$ variances for various hold periods τ for Digital, National Semiconductor, Tandy, Teledyne, and Telex stocks using generalized method of moments. The estimates were computed using a sample of 312 weekly returns from January 10, 1980 to December 27, 1985 and have not been annualized. Standard errors are given in parentheses.

STOCK	$V(\tau = 2)$ (Std. Er.)	$V(\tau = 6)$ (Std. Er.)	$V(\tau = 10)$ (Std. Er.)	$V(\tau = 14)$ (Std. Er.)	$V(\tau = 19)$ (Std. Er.)	$V(\tau = 23)$ (Std. Er.)	$V(\tau = 27)$ (Std. Er.)	$V(\tau = 32)$ (Std. Er.)	$V(\tau = 36)$ (Std. Er.)
Digital Equipment	0.00491 (0.00086)	0.01331 (0.00272)	0.02209 (0.00447)	0.03186 (0.00690)	0.04294 (0.00919)	0.05061 (0.01181)	0.06039 (0.01629)	0.06918 (0.02097)	0.07422 (0.02216)
National Semiconductor	0.00948 (0.00143)	0.02916 (0.00452)	0.05359 (0.01176)	0.07714 (0.02192)	0.11181 (0.03363)	0.14685 (0.04241)	0.18533 (0.05154)	0.21803 (0.06083)	0.24372 (0.07326)
Tandy	0.00684 (0.00116)	0.02626 (0.00560)	0.05422 (0.01438)	0.08979 (0.02724)	0.14225 (0.04704)	0.18963 (0.06460)	0.23889 (0.08136)	0.31114 (0.10530)	0.36442 (0.11816)
Teledyne	0.00464 (0.00112)	0.01428 (0.00277)	0.02651 (0.00609)	0.04083 (0.00993)	0.05877 (0.01474)	0.07731 (0.01970)	0.08907 (0.02237)	0.10493 (0.02668)	0.11435 (0.02801)
Telex	0.00973 (0.00136)	0.03552 (0.00749)	0.07150 (0.02064)	0.11399 (0.03895)	0.17418 (0.06611)	0.21414 (0.08941)	0.23532 (0.10372)	0.26620 (0.11043)	0.32118 (0.11818)

TABLE 3b

Estimates of holding period return $V(\tau)$ variances for various hold periods τ for Digital, National Semiconductor, Tandy, Teledyne, and Telex stocks using maximum likelihood. The estimates were computed using a sample of 312 weekly returns from January 10, 1980 to December 27, 1985 and have not been annualized. Standard errors are given in parentheses.

STOCK	$V(\tau = 2)$ (Std. Er.)	$V(\tau = 6)$ (Std. Er.)	$V(\tau = 10)$ (Std. Er.)	$V(\tau = 14)$ (Std. Er.)	$V(\tau = 19)$ (Std. Er.)	$V(\tau = 23)$ (Std. Er.)	$V(\tau = 27)$ (Std. Er.)	$V(\tau = 32)$ (Std. Er.)	$V(\tau = 36)$ (Std. Er.)
Digital Equipment	0.00510 (0.00042)	0.01582 (0.00140)	0.02724 (0.00274)	0.03942 (0.00457)	0.05576 (0.00768)	0.06976 (0.01094)	0.08464 (0.01496)	0.10454 (0.02117)	0.12156 (0.02717)
National Semiconductor	0.00949 (0.00078)	0.02968 (0.00283)	0.05158 (0.00595)	0.07528 (0.01049)	0.10765 (0.01856)	0.13588 (0.02723)	0.16632 (0.03811)	0.20771 (0.05520)	0.24368 (0.07199)
Tandy	0.00616 (0.00050)	0.01965 (0.00177)	0.03482 (0.00362)	0.05184 (0.00630)	0.07598 (0.01111)	0.09781 (0.01638)	0.12210 (0.02316)	0.15628 (0.03410)	0.18698 (0.04514)
Teledyne	0.00428 (0.00035)	0.01354 (0.00118)	0.02376 (0.00232)	0.03503 (0.00387)	0.05072 (0.00659)	0.06466 (0.00949)	0.07995 (0.01315)	0.10109 (0.01896)	0.11977 (0.02473)
Telex	0.00935 (0.00077)	0.03089 (0.00293)	0.05671 (0.00647)	0.08744 (0.01201)	0.13391 (0.02271)	0.17856 (0.03515)	0.23091 (0.05193)	0.30885 (0.08050)	0.38276 (0.11087)

TABLE 3c

Estimates of holding period return $V^*(\tau)$ variances for various hold periods τ for Digital, National Semiconductor, Tandy, Teledyne, and Telex stocks using maximum likelihood. The estimates were computed using a sample of 312 weekly returns from January 10, 1980 to December 27, 1985 and have not been annualized. Standard errors are given in parentheses.

STOCK	$V^*(\tau = 2)$ (Std. Er.)	$V^*(\tau = 6)$ (Std. Er.)	$V^*(\tau = 10)$ (Std. Er.)	$V^*(\tau = 14)$ (Std. Er.)	$V^*(\tau = 19)$ (Std. Er.)	$V^*(\tau = 23)$ (Std. Er.)	$V^*(\tau = 27)$ (Std. Er.)	$V^*(\tau = 32)$ (Std. Er.)	$V^*(\tau = 36)$ (Std. Er.)
Digital Equipment	0.00506 (0.00041)	0.01542 (0.00124)	0.02611 (0.00212)	0.03715 (0.00303)	0.05143 (0.00422)	0.06335 (0.00522)	0.07558 (0.00626)	0.09142 (0.00762)	0.10453 (0.00875)
National Semiconductor	0.00941 (0.00076)	0.02879 (0.00234)	0.04898 (0.00401)	0.06998 (0.00579)	0.09742 (0.00815)	0.12052 (0.01017)	0.14443 (0.01231)	0.17566 (0.01514)	0.20176 (0.01755)
Tandy	0.00605 (0.00049)	0.01847 (0.00149)	0.03131 (0.00254)	0.04459 (0.00365)	0.06182 (0.00509)	0.07621 (0.00631)	0.09102 (0.00759)	0.11024 (0.00926)	0.12618 (0.01066)
Teledyne	0.00423 (0.00034)	0.01287 (0.00104)	0.02178 (0.00176)	0.03096 (0.00252)	0.04283 (0.00350)	0.05270 (0.00432)	0.06283 (0.00517)	0.07591 (0.00628)	0.08672 (0.00721)
Telex	0.00901 (0.00072)	0.02758 (0.00224)	0.04689 (0.00384)	0.06697 (0.00553)	0.09318 (0.00778)	0.11522 (0.00971)	0.13803 (0.01173)	0.16780 (0.01442)	0.19265 (0.01670)

TABLE 4a

Comparison of semiparametric upper bounds with Black-Scholes and actual market prices of call and put options on Digital Equipment Corp. stock. Market prices are Friday February 7, 1986 closing prices for Digital options trading on the Chicago Board Options Exchange (obtained from the February 10, 1986 issue of the Wall Street Journal). Variance estimates used in calculating the upper bounds and the Black-Scholes prices were obtained from Table 3c. For options with time-to-maturity less than 20 weeks, the interest rate used in calculations was the 13-week Treasury bill rate (result of the Monday February 3, 1986 auction reported in the February 10, 1986 WSJ) of 7.39 percent (annualized). For options with 20 weeks or more to go, the 26-week T-bill rate (as reported in the same issue of the WSJ) of 7.54 percent was used.

STOCK PRICE $S = 159.625$

Strike Price	Time to Maturity $\tau = 2$ Weeks					
	CALL			PUT		
K	Mkt. Price	BS Price	Upper Bd.	Mkt. Price	BS Price	Upper Bd.
150	9.500	11.102	12.617	1.125	1.066	2.581
155	6.500	7.142	8.747	2.250	2.363	3.697
160	4.000	4.541	5.708	4.750	4.478	5.645
165	1.500	2.532	3.721	8.000	7.456	8.644

Strike Price	Time to Maturity $\tau = 6$ Weeks					
	CALL			PUT		
K	Mkt. Price	BS Price	Upper Bd.	Mkt. Price	BS Price	Upper Bd.
150	12.250	14.177	16.793	3.000	3.323	5.939
155	7.750	10.969	13.318	4.750	5.074	7.423
160	7.250	8.262	10.394	6.500	7.326	9.458
165	4.750	6.057	8.088	8.750	10.080	12.111

Strike Price	Time to Maturity $\tau = 10$ Weeks					
	CALL			PUT		
K	Mkt. Price	BS Price	Upper Bd.	Mkt. Price	BS Price	Upper Bd.
150	15.500	16.633	20.103	4.250	4.965	8.435
155	11.250	13.594	16.756	5.500	6.858	10.020
160	8.250	10.947	13.844	8.250	9.142	12.040
165	6.625	8.687	11.409	9.750	11.815	14.537

TABLE 4b

Comparison of semiparametric upper bounds with Black-Scholes and actual market prices of call and put options on National Semiconductor stock. Market prices are February 7, 1986 closing prices for National Semiconductor options trading on the Chicago Board Options Exchange (obtained from the February 10, 1986 issue of the Wall Street Journal). Variance estimates used in calculating the upper bounds and the Black-Scholes prices were obtained from Table 3c. For options with time-to-maturity less than 20 weeks, the interest rate used in calculations was the 13-week Treasury bill rate (result of the Monday February 3, 1986 auction reported in the February 10, 1986 WSJ) of 7.39 percent (annualized). For options with 20 weeks or more to go, the 26-week T-bill rate (as reported in the same issue of the WSJ) of 7.54 percent was used. An entry of "r" indicates that the particular option was not traded, and an entry of "s" indicates that no option was offered with that strike-price/time-to-maturity combination.

STOCK PRICE S = 14.625

Strike Price K	Time to Maturity τ = 2 Weeks					
	CALL			PUT		
	Mkt. Price	BS Price	Upper Bd.	Mkt. Price	BS Price	Upper Bd.
10.000	4.875	4.652	4.771	0.063	0.000	0.118
12.500	2.125	2.186	2.377	0.063	0.027	0.218
15.000	0.438	0.418	0.561	0.750	0.752	0.895
17.500	0.063	0.020	0.168	r	2.847	2.995

Strike Price K	Time to Maturity τ = 14 Weeks					
	CALL			PUT		
	Mkt. Price	BS Price	Upper Bd.	Mkt. Price	BS Price	Upper Bd.
10.000	5.000	4.892	5.579	0.125	0.077	0.764
12.500	2.875	2.857	3.483	0.375	0.494	1.120
15.000	1.313	1.446	1.889	1.125	1.535	1.979
17.500	s	0.649	1.033	s	3.191	3.575

Strike Price K	Time to Maturity τ = 27 Weeks					
	CALL			PUT		
	Mkt. Price	BS Price	Upper Bd.	Mkt. Price	BS Price	Upper Bd.
10.000	4.250	5.239	6.395	0.250	0.244	1.400
12.500	3.125	3.442	4.431	0.625	0.854	1.843
15.000	1.750	2.137	2.874	1.250	1.956	2.694
17.500	s	1.273	1.854	s	3.500	4.081

TABLE 4c

Comparison of semiparametric upper bounds with Black-Scholes and actual market prices of call and put options on Tandy Corp. stock. Market prices are February 7, 1986 closing prices for Tandy options trading on the Chicago Board Options Exchange (obtained from the February 10, 1986 issue of the Wall Street Journal). Variance estimates used in calculating the upper bounds and the Black-Scholes prices were obtained from Table 3c. For options with time-to-maturity less than 20 weeks, the interest rate used in calculations was the 13-week Treasury bill rate (result of the Monday February 3, 1986 auction reported in the February 10, 1986 WSJ) of 7.39 percent (annualized). For options with 20 weeks or more to go, the 26-week T-bill rate (as reported in the same issue of the WSJ) of 7.54 percent was used. An entry of "r" indicates that the particular option was not traded, and an entry of "s" indicates that no option was offered with that strike-price/time-to-maturity combination.

STOCK PRICE $S = 39.875$

Strike Price K	Time to Maturity $\tau = 10$ Weeks					
	CALL			PUT		
	Mkt. Price	BS Price	Upper Bd.	Mkt. Price	BS Price	Upper Bd.
30	r	10.387	11.507	r	0.103	1.224
35	5.750	6.080	7.163	0.375	0.729	1.812
40	2.000	2.953	3.747	2.000	2.533	3.328
45	0.625	1.190	1.917	r	5.702	6.430

Strike Price K	Time to Maturity $\tau = 23$ Weeks					
	CALL			PUT		
	Mkt. Price	BS Price	Upper Bd.	Mkt. Price	BS Price	Upper Bd.
30	11.125	11.317	13.433	r	0.493	2.609
35	6.750	7.555	9.401	0.875	1.573	3.418
40	3.500	4.703	6.124	r	3.562	4.984
45	1.500	2.755	3.915	r	6.456	7.617

Strike Price K	Time to Maturity $\tau = 36$ Weeks					
	CALL			PUT		
	Mkt. Price	BS Price	Upper Bd.	Mkt. Price	BS Price	Upper Bd.
30	s	12.231	15.227	s	0.883	3.880
35	r	8.767	11.351	1.250	2.174	4.759
40	4.500	6.056	8.114	r	4.218	6.276
45	r	4.060	5.713	r	6.977	8.629

TABLE 4d

Comparison of semiparametric upper bounds with Black-Scholes and actual market prices of call and put options on Teledyne Corp. stock. Market prices are February 7, 1986 closing prices for Teledyne options trading on the Chicago Board Options Exchange (obtained from the February 10, 1986 issue of the Wall Street Journal). Variance estimates used in calculating the upper bounds and the Black-Scholes prices were obtained from Table 3c. For options with time-to-maturity less than 20 weeks, the interest rate used in calculations was the 13-week Treasury bill rate (result of the Monday February 3, 1986 auction reported in the February 10, 1986 WSJ) of 7.39 percent (annualized). For options with 20 weeks or more to go, the 26-week T-bill rate (as reported in the same issue of the WSJ) of 7.54 percent was used. An entry of "r" indicates that the particular option was not traded, and an entry of "s" indicates that no option was offered with that strike-price/time-to-maturity combination.

STOCK PRICE S = 329.000

Strike Price	Time to Maturity τ = 2 Weeks					
	CALL			PUT		
K	Mkt. Price	BS Price	Upper Bd.	Mkt. Price	BS Price	Upper Bd.
310	19.250	21.701	24.560	1.125	1.852	4.711
320	12.000	14.219	16.737	2.500	4.343	6.860
330	5.750	8.452	10.646	6.250	8.548	10.742
340	2.500	4.512	6.778	r	14.581	16.846

Strike Price	Time to Maturity τ = 6 Weeks					
	CALL			PUT		
K	Mkt. Price	BS Price	Upper Bd.	Mkt. Price	BS Price	Upper Bd.
310	23.000	27.517	32.452	4.500	5.977	10.912
320	16.000	21.002	25.421	7.250	9.380	13.799
330	11.500	15.545	19.543	r	13.841	17.839
340	8.750	11.150	14.977	12.000	19.364	23.191

Strike Price	Time to Maturity τ = 10 Weeks					
	CALL			PUT		
K	Mkt. Price	BS Price	Upper Bd.	Mkt. Price	BS Price	Upper Bd.
310	30.000	32.218	38.753	r	8.996	15.531
320	22.000	26.040	31.976	r	12.682	18.618
330	16.125	20.688	26.114	r	17.194	22.620
340	13.000	16.158	21.263	22.000	22.527	27.633

TABLE 4e

Comparison of semiparametric upper bounds with Black-Scholes and actual market prices of call and put options on Telex Corp. stock. Market prices are February 7, 1986 closing prices for Telex options trading on the Chicago Board Options Exchange (obtained from the February 10, 1986 issue of the Wall Street Journal). Variance estimates used in calculating the upper bounds and the Black-Scholes prices were obtained from Table 3c. For options with time-to-maturity less than 20 weeks, the interest rate used in calculations was the 13-week Treasury bill rate (result of the Monday February 3, 1986 auction reported in the February 10, 1986 WSJ) of 7.39 percent (annualized). For options with 20 weeks or more to go, the 26-week T-bill rate (as reported in the same issue of the WSJ) of 7.54 percent was used. An entry of "r" indicates that the particular option was not traded, and an entry of "s" indicates that no option was offered with that strike-price/time-to-maturity combination.

STOCK PRICE S = 63.250

Strike Price K	Time to Maturity τ = 6 Weeks					
	CALL			PUT		
	Mkt. Price	BS Price	Upper Bd.	Mkt. Price	BS Price	Upper Bd.
55	9.500	9.653	11.224	0.375	0.952	2.524
60	5.125	6.142	7.461	1.250	2.401	3.719
65	2.125	3.578	4.668	4.000	4.795	5.885
70	1.000	1.913	2.987	8.875	8.090	9.163

Strike Price K	Time to Maturity τ = 19 Weeks					
	CALL			PUT		
	Mkt. Price	BS Price	Upper Bd.	Mkt. Price	BS Price	Upper Bd.
55	10.000	12.644	15.803	1.250	2.980	6.139
60	7.750	9.686	12.411	r	4.893	7.618
65	4.125	7.278	9.602	r	7.356	9.681
70	2.875	5.375	7.431	r	10.325	12.381

Strike Price K	Time to Maturity τ = 32 Weeks					
	CALL			PUT		
	Mkt. Price	BS Price	Upper Bd.	Mkt. Price	BS Price	Upper Bd.
55	r	14.997	19.659	2.250	4.341	9.003
60	9.500	12.260	16.370	4.250	6.385	10.496
65	7.250	9.941	13.523	r	8.848	12.430
70	r	8.007	11.154	r	11.695	14.841