Author's current address:
Department of Finance
School of Business Administration
Hebrew University
Jerusalem - Israel

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RISK AVERSION IN THE NOT-SO-SMALL:

BEYOND MEAN AND VARIANCE

by

George G. Szpiro*

* The Wharton School, University of Pennsylvania, and School of Business Administration, Hebrew University. I would like to thank Richard Kihlstrom and Howard Kunreuther for valuable discussions. Financial support through the Wharton Junior Faculty Research Fund is gratefully acknowledged.

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Abstract

The Pratt-Arrow coefficient of risk aversion, r(w), describes decision behavior of individuals under uncertainty, when small amounts of money are involved. In this paper a two-parameter measure is proposed which also takes into account the utility function's third derivative: by thus incorporating a risk's skewness, one receives a better approximation for amounts which are not necessarily very small.

The model provides new theoretical results and also predicts unexpected behavior under certain conditions; this permits the model's empirical verification.

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1. Introduction

In the mid-sixties Pratt and Arrow introduced the coefficient of risk aversion as a function of wealth, $r(w) = -U''(w)/U'(w)$, which has since been widely used as a parameter input for models of decision making under uncertainty. However, this measure is usually employed for finite amounts, even though its derivation relies on a two-term Taylor expansion of the utility function, which ignores third order and higher terms, and is therefore exact only for amounts of infinitesimally small size. For example, the gamble "win 99 dollars with a probability of one percent, or lose one dollar with a probability of 99 percent" would be equivalent to a gamble with the words 'win' and 'lose' interchanged ("lose 99 dollars with a probability of one percent, or win one dollar with a probability of 99 percent"); according to the Pratt-Arrow approach, an individual would pay the same amount to avoid both gambles, as long as his wealth is substantially greater than 99 dollars.

Empirical studies have attempted to measure the shape and value of $r(w)$. (E.g. Friend and Blume [1975], Cohn et al. [1975], Farber [1978], Siegel and Hoban [1982], Morin and Suarez [1983], Szpiro [1986]). Even though most studies show that the coefficient of relative risk aversion is somewhere around two (that is $r(w) = 2/w$), there is still much uncertainty as to its exact value, and estimates range from nearly zero to about 25. Obviously, finite amounts are used as observations to estimate a point measure, which, strictly speaking, should be measured using only amounts of infinitesimal size.[1)]

The main extension of Pratt's infinitesimal approach to large risks is his Theorem 1, which states that if, at <u>every</u> wealth level, a utility function U_1 shows greater local risk aversion than U_2, then the risk premium associated with any risk, irrespective of its size, will be greater for U_1 than for U_2. Hence, the cardinal measure of risk aversion in the small, extends to an ordinal ranking in the large, on condition that the risk aversion functions do not intersect! Hence, inferences for risk aversion in the large are only possible for subsets of utility functions. (See Appendix A.) We conclude, that a better approximation is needed to investigate risk aversion in the not-so-small. Some of the differences between the various empirical estimates may be reconciled, and better inputs for other models of decision making under uncertainty will be found. Moreover, and of at least equal importance, new theoretical insights may be gained.

In this paper I propose to study behavior under risk, using a three-term Taylor approximation. We can then investigate risk aversion for amounts which are not necessarily so small that the third Taylor term can be ignored with impunity. Risk aversion will be defined as a two dimensional measure with the coefficients $r(w) = -U''(w)/U'(w)$, and $s(w) = -U'''(w)/U'(w)$.

In Section 2 the new risk premium will be computed and justified, using a third order approximation; Section 3 presents a graphical interpretation of the results. Section 4 presents a new paradox, which can serve to experimentally verify the model presented in this paper. In Section 5 utility functions will be examined; the recoverability of utility functions, is the subject of Section 6. Section 7 provides a link to related research in finance, and Section 8 summarizes our findings.

2. Third-Order Approximation

Let us consider a decision maker with assets w and a utility function U(w), who is faced with a risk $\tilde{z}$. We impose the two familiar restrictions on the utility function: the decision maker is 'greedy', which means that U'(w)>0, and becomes gradually 'satiated', which means that U"(w)<0. What is the risk premium π, such that he is indifferent between, on the one hand, holding the risk in addition to his wealth, i.e., $w+\tilde{z}$, and, on the other hand, holding $w + E(\tilde{z}) - \pi$? 2)

The expected utility maximizing individual will determine π by solving the following equation:

$$U(w-\pi) = E(U(w+\tilde{z}) \ . \tag{1}$$

Let us expand both sides of equation (1) into a Taylor series; since E(z) = 0, we have,3)

$$-\pi U' + \frac{\pi^2}{2!} U'' - \frac{\pi^3}{3!} U''' + \ldots + (-1)^n \frac{\pi^n}{n!} U^{(n)} = \tag{2}$$

$$\frac{E(\tilde{z}^2)}{2!} U'' + \frac{E(\tilde{z}^3)}{3!} U''' + \ldots\ldots + \frac{E(\tilde{z}^n)}{n!} U^{(n)}$$

The exact solution to the decision problem for a <u>rational man</u> requires solving the preceding equation for π (see also Rubinstein [1973]). Looking at this equation, however, it seems that 'rational man' may be a contradiction in terms. Not only would he need to be rational, but he would also have to possess infinite computational power. Hence, in the spirit of <u>bounded</u> rationality (e.g., Simon [1969]) we make the behavioral assumption that a decision maker truncates the series in equation (3) on both sides, after two terms. That is he, firstly, takes into consideration only the first three moments of the distribution and ignores higher order moments; secondly, he solves the quadratic equation in π, ignoring poynomial terms of higher order.[4)]

It must be emphasized that truncating the series after n terms does not mean that the utility function is replaced by an n-th degree polynomial. Rather, the original utility function is locally approximated at every wealth level by such n-th degree polynomials; hence the utility function is an envelope of such polynomials, and not a polynomial itself. (See Figure 1.) Obviously cubic polynomials have convex and concave regions (Levy [1969]). Hence local approximation by a cubic may result in risk taking behavior even by a global risk averter; this provides an explanation for the coincidence of insurance and gambling.[5)]
As is well known, Pratt [1964] and Arrow [1965] solved equation (3) for small (actually infinitesimal) risks by truncating on both sides after the first term, to receive,

$$\pi = \tfrac{1}{2}\ \mathrm{Var}\ r \ , \qquad (3)$$

where $r(w) = -U''(w)/U'(w)$ is the function of risk aversion and Var stands for the risk's variance, $E(\tilde{z})$. For small risks the "crude approximation" may be quite good (Samuelson [1967,p.12], [1970] and Tsiang [1972]). The quality of the approximation can be improved, however, if truncation occurs only after two terms (Samuelson [1967,p.9] [1970,p.542]). Alternatively, we can say that the methodology can be extended to larger risks when truncation occurs only after two terms (hence, risk aversion in the not-so-small). We receive,

$$-\pi U' + \frac{\pi^2}{2} U'' = \frac{E(\tilde{z}^2)}{2} U'' + \frac{E(\tilde{z}^3)}{6} U''' \qquad (4)$$

Dividing both sides by -U', writing $s = -U'''/U'$, and solving for π, we obtain

$$\pi = -\frac{1}{r} + \left(\frac{1}{r^2} + Var + \frac{Skw}{3}\frac{s}{r}\right)^{1/2} \qquad (5)$$

where Skw stands for the risk's third moment $E(\tilde{z}^3)$.[6]

We see that in the not-so-small an individual's attitude towards risk is characterized by two components, r(w) and s(w), the first being always positive by the postulates of 'greed' and 'satiability'. It is often said that U''' must be negative for rational individuals (for instance, decreasing absolute risk aversion for small risks would imply this), which would make s(w) negative throughout, but we do not require this assumption for the present paper; hence s(w) may be of any sign, or even change sign. A surprising phenomenon emerges from equation (5): the term containing Skw could, under certain circumstances, offset, or more than offset the term containing Var. More on that below.

Obviously π grows with increasing variance of the risk, i.e.,

$$\partial\pi/\partial Var > 0 \, . \qquad (6)$$

The dependence of π on the risk's skewness, however, is not so clear: if $s(w)>0$ then $\partial\pi/\partial Skw$ is positive, and vice versa. Thus,

$$sgn(\partial\pi/\partial Skw) = sgn(s) = -sgn(U''') \, . \qquad (7)$$

The often observed preference for positive skewness on a macro level, that is, $\partial\pi/\partial Skw < 0$, would imply that $U''' > 0$ and, hence, that $s(w)$ is negative. (See Section 7, below.)

In closing this section, one may ask why the term containing π^2 in the LHS of equation (4) should be included at all. The reason is, that this term is of the same order of magnitude as the one on the RHS which contains skewness, and possibly of the opposite sign.

3. The rs-Plane and the 'Risk Grid'

Since an individual's risk attitude in the not-so-small is characterized by two components, r and s, we can depict behavior under risk in an 'rs-plane'. Also, we no longer call r(w) the degree of risk aversion, rather we shall call it the degree of variance aversion, and s(w) the degree of skewness aversion (which, for $s < 0$, is tantamount to the degree of love for positive skewness).[7] We define the 'path' of a utility function, as the locus of points which is described by [r(w),s(w)] in the rs-plane. The HAVA utility function $U(w)=w^{1-k}/1-k$, for example, has $r(w)=k/w$, and $s(w)=-k(k+1)/w^2$; thus its path is described by $s=-r^2(k+1)/k$.

Let us see how π varies with a change in the value of r. For small risks, where $\pi = (\frac{1}{2})\mathrm{Var}\ r$, π grows with r. In the not-so-small, things are not quite as simple.

$$\frac{\partial\pi}{\partial r} = \frac{1}{r^2} - \frac{1}{2}\left(\frac{2}{r^3} + \frac{Skw}{3}\frac{s}{r^2}\right)\left(\frac{1}{r^2} + Var + \frac{Skw}{3}\frac{s}{r}\right)^{-1/2} \tag{8}$$

The right hand side is positive whenever

$$|s| < 6\frac{\sigma}{|Skw|}\ , \tag{9}$$

where σ is the standard deviation of the risk.[8] In contrast to Pratt's methodology, we observe that whether $\partial\pi/\partial r$ is positive or not, depends on the specific risk.

We now examine how π varies with a change in s. From equation (5) we get,

$$\frac{\partial \pi}{\partial s} = \left(\frac{Skw}{3r}\right)\frac{1}{2}\left(\frac{1}{r^2} + Var + \frac{Skw}{3}\frac{s}{r}\right)^{-1/2} , \qquad (10)$$

and since r is always positive, $\partial\pi/\partial s$ has the same sign as the risk's skewness.[9)]

To analyze 'isopremium' combinations of r and s for a risk with variance Var and skewness Skw, we look for combinations such that $\pi(w)$=constant for all levels of wealth w. Hence, by equation (5) let us write,

$$-\frac{1}{r} + \left(\frac{1}{r^2} + Var + \frac{Skw}{3}\frac{s}{r}\right)^{\frac{1}{2}} = \pi_i = \text{constant} . \qquad (11)$$

Solving for s as a function of r we obtain,

$$s = \frac{6\pi_i}{Skw} + \frac{3(\pi_i{}^2 - Var)}{Skw} r , \qquad (12)$$

a collection of straight lines, which flare out in the rs-plane (intercepts $6\pi_i/Skw$, and slopes $3(\pi_i{}^2\text{-}Var)/Skw$). We shall call this family of lines, a 'risk grid'.[10)] Since r is always positive, only the right half of the rs-plane is relevant, and the intersections of the isopremium lines, which occur in the left half, may be disregarded.

Let us examine some special cases of equation (12). For $\pi = 0$, i.e., risk neutrality we get

$$s = -3\frac{Var}{Skw} r . \qquad (13)$$

Thus, whenever $s < -3(Var/Skw)r$ the individual exhibits risk loving even if variance aversion r, the conventional measure of risk aversion in the small, is positive. This means that under certain circumstances, the positive effect of the risk's skewness on the risk premium, may offset the variance's negative effect! Beyond that point, as the expression under the square root of equation (11) becomes smaller still, the value of the risk premium, which is now positive, grows, until it reaches $1/r$. Finally, if $(1/r^2)+Var+(Skw/3)(s/r) < 0$, i.e., if

$$s > -\left(\frac{3}{Skw}\right)\frac{1}{r} - \left(\frac{3Var}{Skw}\right)r \,, \tag{14}$$

the risk premium becomes imaginary. The smallest value of s where π is imaginary occurs at $r=1/\sigma$, at which point $s=-6\sigma/Skw$. For a consequence of this, see the discussion on the Allais paradox in the next section.

Another special case arises when $\pi = \sigma$. In this case we get

$$s = \frac{6\sigma}{Skw} \,, \tag{15}$$

or, in other words, π will equal σ, independently of the value of r. Figure 2 depicts the rs-plane with the various special cases, for a risk with negative skewness.11)

4. A New Paradox and Verifiability of the Model

The following section is set out in the HAVA framework, even though the propositions hold for other forms of utility as well.

As we saw above, HAVA utility functions describe hyperbolic paths in the rs-plane, moving towards the origin with increasing wealth, according to

$$s = -r^2(k+1)/k \quad . \tag{16}$$

When superimposing such a path onto a risk grid (with positive skewness) a surprising phenomenon appears: for relativeley low wealths the risk premium increases with increasing wealth, indicating IARA; above a certain critical wealth level, the risk premium decreases with increasing wealth, which conforms to the customary assumption of DARA.[12)]

Let us formalize the above. From equations (12) and (16) we see that the highest premium π will be paid when the path is tangent to one of the 'constant premium' lines of the risk grid, that is when the two intersections of the path with the line coincide. The intersections of the line with the path are found by solving the following equation for r:

$$-\frac{k+1}{k}r^2 = \frac{6\pi_c}{Skw} + \frac{3(\pi_c{}^2-Var)}{Skw}\,r \tag{17}$$

and the intersections coincide if

$$\pi_c{}^4 - 2\pi_c{}^2Var + Var^2 + 24\frac{k+1}{9k}\pi_c\,Skw = 0 \quad , \tag{18}$$

which must be solved for π_c. At that point,

$$r = -\frac{3(\pi_c{}^2-\text{Var})}{2k\text{Skw}}(k+1) \quad , \tag{19}$$

and for HAVA utility, $r=k/w$, we obtain the critical level of wealth,

$$W_c = -\frac{2k^2}{3(k+1)}\frac{\text{Skw}}{\pi_c{}^2-\text{Var}} \tag{20}$$

Hence, we should observe DARA behavior for $W>W_c$, but IARA behavior whenever $W<W_c$. For moderately variance averse individuals, i.e., for low k, W_c may be so low, relative to the risk, that even the method of 'not-so-small' is not applicable, which could be the reason why this phenomenon has escaped the attention of economists, or why one thinks it sounds absurd.[13)] However, for an individual with, say, $U(w) = -.1(w^{-10})$, we see from Table 1 that the risk premium, computed on the basis of variance and skewness rises and then falls with increasing wealth. (It is noteworthy, that this phenomenon arises, even though s is consistently negative.) The table also sets out the risk premiums which are obtained (a) by applying the infinitesimal approach, that is, taking into account only the risk's variance, and (b) by computing the 'true' risk premium which takes into account all moments. Experiments along these lines should be able to ascertain the existence of both DARA and IARA regions of wealth, and thus to validate the model presented in this paper!

Another version of this seemingly paradoxical phenomenon will now be described, which presents a further opportunity to test the model. As has been pointed out before, a third-degree polynomial, such as we use to approximate utility at current wealth, has convex and concave regions. Because of the assumptions of greed and satiability, the polynomial is convex near current wealth and, for negative s, concave somewhere to the right of current wealth.[14)] Let us compute, for simple gambles, how large the potential gain, G, must be, in order for the individual to exhibit risk taking behavior. From the Taylor development of equation (1) we receive,

$$U(w+G) = U + GU' + \frac{G^2}{2!}U'' + \frac{G^3}{3!}U''' . \qquad (21)$$

By taking the second derivative, and setting equal to zero, we find the inflection point of this approximation, which is where the concave region starts:

$$\frac{d^2U(w+G)}{dG^2} = U'' + GU''' = 0 ; \qquad (22)$$

hence,

$$G_c = -U''/U''' = -r/s . \qquad (23)$$

Thus, risk taking behavior will not occur, unless the potential gain is larger than G_c, and possibly substantially larger, depending on the potential loss and the respective probabilities. To take the analysis one step further, let us examine fair bets: the individual is faced with a possiblitiy of gaining G with probability p, and L with probability (1-p), where L = (p/p-1)G, such that the expected gain is zero. The three-term approximation to expected utility can be written as,

$$EU(w+\tilde{z}) =$$
$$= U + \frac{1}{2}[pG^2 + (1-p)\frac{p^2}{(p-1)^2}G^2]U'' + \frac{1}{6}[pG^3 + (1-p)\frac{p^3}{(p-1)3}G^3]U'''$$

$$= U + p(1-p)[\frac{1}{2} G^2U'' + \frac{1}{6} G^3(1-p)(1-2p)U'''] . \qquad (24)$$

Setting the left hand side equal to $U(w-\pi) = U - \pi U' + \pi^2 U''/2$, we receive,

$$\pi = -\frac{1}{r} + \left(\frac{1}{r^2} + p(1-p)G^2\left[1 + \frac{1}{3}G(1-p)(1-2p)\frac{s}{r}\right]\right)^{1/2} . \quad (25)$$

To find the point where risk taking starts occurring, we set π equal to zero, and receive,

$$G_o = -\frac{r}{s}\frac{3}{1-3p+2p^2} . \quad (26)$$

We have found that a fair bet, as described above, will be accepted if and only if G is greater than G_o. Obviously, no fair bet will ever be accepted if $p = 0.5$ (skewness = 0), or if $s = 0$ (skewness neutrality). The seeming paradox now becomes apparent if we posit HAVA utility functions, and $p < 0.5$. In this case we have $r/s = -w/k+1$, and G_o becomes,

$$G_o = \frac{w}{k+1}\frac{3}{1-3p+2p^2} , \quad (27)$$

which decreases with increasing k! Hence, individuals who, in Pratt's terminology were said to be more risk averse (and in our terminology are said to be more variance averse) will start accepting fair bets 'sooner' than less variance averse individuals. This phenomenon arises from the fact that for HAVA utility, greater variance aversion ($r = k/w$) goes hand in hand with greater aversion to negative skewness ($s = -k(k+1)/w^2$); in our example the bets are positively skewed if $p < 0.5$, which gives rise to the phenomenon. Furthermore, since G_o increases with w we should, again, observe increasing absolute risk aversion, since gambles which are accepted at lower wealth, may not be accepted at higher wealths. However, for low values of k,

G_0 is of the same order of magnitude as total wealth, which makes this situation rare and difficult to observe.

As an aside, let us show how the model of this paper may be able to help explain behavior, which is commonly referred to as the Allais paradox. Allais [1953] and others (see Machina [1983], p.63) have shown that people are not consistent in choices involving gambles which typically are of the following form: a 10% chance of winning \$5 million, an 89% chance of winning \$1 million, and a 1% chance of winning nothing. The first, second, and third moment of this gamble are, respectively, 1390, 15×10^{11}, and 46×10^{17}. Hence, the risk grid for this gamble shows that the risk-neutrality line ($\pi = 0$) has a very flat slope: $-3Var/Skw = -10^{-6}$; consequently, most of the area of the second quadrant of the rs-plane ($r>0$, $s<0$) is made up of values for r and s which result in imaginary premiums. In fact, the contour of the area for imaginary π is given from equation (14) by,

$$s = -(7 \times 10^{-19})\frac{1}{r} - (10^{-6})r \quad . \tag{28}$$

Superimposing r and s of, say, HAVA utility functions onto the Allais risk grid, we observe that, but for very high wealths and/or high values of k, the paths lie inside the domain of imaginary risk premiums. And, imaginary numbers being what they are, we would not expect people to be consistent in their choices. To summarize: for high values of k and relatively low wealth, the risk premium is imaginary, and becomes negative and finally positive with increasing wealth. For lower values of k the risk premium may be imaginary throughout, or may be imaginary for low wealths and become negative for high wealth.[15)]

5. Paths in the rs-Plane

Let us say a utility function has a path in the rs-plane which is described by the relationship,

$$s = f(r). \tag{29}$$

In order to determine the utility function itself, this path must be 'integrated', which leads to a differential equation of the form,

$$U''' + U'f(-U''/U') = 0\ . \tag{30}$$

However, not all paths in the rs-plane are admissible. Firstly, only paths which satisfy the requirements of greed and satiability are acceptable, which rules out paths which lie, or partially lie in the left half of the rs-plane (where $r < 0$).[16] Secondly, since $r(w)$ and $s(w)$ are both derived from $U(w)$, there exists an interrelationship between r and s. Pratt showed that , since $r = -d(\ln U')/dw$,

$$U(w) = \int e^{-\int r(w)dw} dw\ . \tag{31}$$

By the same token, $s(w)/r(w) = -d(\ln U'')/dw$, from which follows,

$$U(w) = -\iint e^{\int s/r\ dw} dwdw\ . \tag{32}$$

The RHS's of equations (31) and (32) must be identical, and we obtain,

$$-\int e^{\int s/r\ dw} dw = e^{-\int rdw}\ , \tag{33}$$

or,

$$-e^{\int s/r\ dw} = -re^{-\int rdw} = -e^{-\int rdw + \ln r}\ . \tag{34}$$

Finally,

$$\int s/r\ dw = -\int rdw + \ln r\ , \tag{35}$$

and[17]

$$s = -r^2 + r'\ . \tag{36}$$

Hence, only paths which satisfy equation (36) are possible. If a path lies above the curve $s = -r^2$, wholly or partially, the utility function describes IAVA (i.e., $r'>0$) in that region, and vice versa. It follows immediately that IAVA is a necessary (but not sufficient) condition for aversion to positive skewness ($s > 0$).[18)]

Since we do not place any restrictions on the sign of r' in this paper, any path is admissible, as long as the assumptions of greed and satiability are satisfied. To illustrate: for paths of the form $s = cr^2$, we have, by equation (36), $r' = r^2(1+c)$. Rearranging, and integrating yields

$$\int \frac{r'}{r^2} dw = -\frac{1}{r} = (1+c)w \quad \text{or} \quad r = -\frac{1}{1+c}\frac{1}{w}\,. \tag{37}$$

For this to be an admissible path (i.e., $r>0$), it is required that $1+c < 0$, hence

$$c < -1\,.^{19)} \tag{38}$$

We must point out, that our paths are projections of rsw-space onto the rs-plane. Hence if $U(w)$ satisfies equation (36), then so does $U(w+\text{constant})$. In general, we do not need to know the specific utility function to qualitatively describe behavior under risk, the path suffices: the class of utility functions $U(w+\text{constant})$, have the same qualitative properties, albeit at different levels of wealth.

Let us now discuss utility functions which exhibit CARA. We showed in Section 3, that combinations of r and s which leave π unchanged for a certain risk, form a risk grid made up of straight lines. This means that CARA in the not-so-small is exhibited by utility functions whose paths form straight lines in the rs-plane. Obviously, CARA in the not-so-small is risk specific, depending as it does on the risk's variance and skewness. Hence, in the not-so-small, there exist no general CARA utility functions: CARA for one risk implies decreasing absolute risk aversion (DARA) or increasing absolute risk aversion (IARA) for other risks. By the same token, we recognize that for every utility function whose path is a straight line, there exists a risk, for which this utility function exhibits CARA. If such a path passes through the origin, there exists a risk for which the utility function is risk neutral. Now, for a straight line path, s = a + br, the differential equation is,

$$U''' - bU'' + aU' = 0 \ , \tag{39}$$

and the solutions are given by utility functions of the following form:

$$U = ce^{g_1 w} + de^{g_2 w} + hw \ , \tag{40}$$

where g_1 and g_2 are the roots of the equation,

$$g^2 - bg + a = 0 \ . \tag{41}$$

The assumptions of greed and satiability are satisfied for all levels of wealth if c, d, g_1, and g_2 are negative. (h can be chosen to be zero.) Such a path never enters the North-East quadrant, and U''' is always positive. However, there exist utility functions of the form (40) which satisfy the assumptions of greed and satiability for all positive wealth levels, whose 's' does change sign. (See the example in Appendix B)[20] The above utility functions exhibit CARA for risks where a = 6π/Skw, and b = $3(\pi^2$-Var)/Skw, that is, when,

$$b = \frac{a^2}{12}Skw - 3\frac{Var}{Skw} , \qquad (42)$$

or when

$$Var = \left(\frac{aSkw}{6}\right)^2 - \frac{bSkw}{3} . \qquad (43)$$

In this case,

$$\pi = aSkw/6 . \qquad (44)$$

If the path goes through the origin, 'a' in the expressions above vanishes, and so does g_2; the utility function becomes the customary CAVA utility,

$$U = ce^{bw} + hw , \qquad (45)$$

(or, more generally, $U = (c+dW)e^{bW} + hW$) which is risk neutral for a risk with $Skw = -3Var/b$. The requirements of greed, $U' = cbe^{bw} + h > 0$, and satiability, $U'' = cb^2e^{bw} < 0$, are satisfied if c and b are negative. (h, again, can be chosen zero.) Note that the path of (45) degenerates to a single point in the rs-plane.

Turning to relative risk aversion, we seek paths such that the risk premium as a fraction of wealth is constant when risk is a constant fraction of wealth, $\tilde{z}w$. With $r = c/w$ and $s = d/w^2$, we obtain

$$\pi = -\frac{w}{c} + \left(\frac{w^2}{c^2} + w^2Var + w^3\frac{Skw}{3}\frac{w}{c}\frac{d}{w^2}\right)^{\frac{1}{2}}$$

$$= w\left[-\frac{1}{c} + \left(\frac{1}{c^2} + Var + \frac{Skw}{3}\frac{d}{c}\right)^{\frac{1}{2}}\right] = w\cdot const \qquad (46)$$

Hence, the path $s = (d/c^2)r^2$ describes CRRA. (Equation (38) requires that $d < -c^2$.) Such paths correspond to the utility functions $w^{1-k}/1-k$, with $c=k$ and $d=-k(k+1)$, which are the well-known CRVA utility functions.

6. Recovering Utility Functions

Since risk aversion in the not-so-small is defined by two components, we have to ascertain the magnitudes of both r and s at various wealth levels, in order to define paths in the rs-plane. By eliciting from a person with wealth w_o, the risk premium he is willing to pay in a certain risky situation we determine on which line of the risk grid he lies. To pinpoint the position in the rs-plane however, his response to another risk is needed, which again determines a line, this time in the new risk grid. The intersection of these two lines gives us the values of $r(w_o)$ and $s(w_o)$.

As an example, let us take two risks with Var_i and Skw_i (i=1,2). The decision maker, with wealth w_o, choses π_1 and π_2, respectively, as the risk premiums associated with these risks. The two lines in the rs-plane are,

$$s_i = \frac{6\pi_i}{Skw_i} + 3\frac{(\pi_i{}^2 - Var_i)}{Skw_i} r_i \qquad (i=1,2) . \tag{47}$$

Setting the RHS's equal, we obtain,

$$r(w_o) = 2\frac{Skw_1\pi_2 - Skw_2\pi_1}{Skw^2(\pi_1{}^2 - Var_1) - Skw_1(\pi_2{}^2 - Var_2)} , \tag{48}$$

$$s(w_o) = \frac{\pi_2(\pi_1{}^2 - Var_1) - \pi_1(\pi_2{}^2 - Var_2)}{Skw^2(\pi_1{}^2 - Var_1) - Skw_1(\pi_2{}^2 - Var_2)} . \tag{49}$$

If, for testing purposes, we take symmetric gambles, that is $Var_1 = Var_2 = Var$ and $Skw_1 = -Skw_2 = Skw$, equations (48) and (49) simplify to

$$r(w_o) = 2\frac{\pi_1 + \pi_2}{2Var - (\pi_1{}^2 + \pi_2{}^2)} , \tag{50}$$

$$s(w_o) = 6\frac{Var+\pi_1\pi_2}{Skw}\,\frac{\pi_1 + \pi_2}{2Var-(\pi_1{}^2+\pi_2{}^2)}\ . \tag{51}$$

Once r(w) and s(w) have been elicited through experiments or by econometric methods at various wealth levels, a curve can be fitted to the observations in the rs-plane, subject to being an admissible path, which must then be 'integrated' to receive U(w) (see Section 5).[21)] Note again, that all utility functions of the form U(w+constant) correspond to the same path as U(w).

7. Implications for and from Financial Research

The utility function's third derivative has been studied by many researchers, both on a theoretical and on an empirical level. It is well known and easy to prove that DAVA implies positive U"' and, consequently, negative 's' (Arditti [1967]). Recently Scott and Horvath [1980] have shown that U"' must be positive 'on average'. Adding an assumption (nowhere justified) about the consistency of investors' preferences for the direction of the third moment, they conclude that U"'(w) must be positive for all wealth levels. (See, however, Appendix B.)

Whether U"' is positive in the aggregate can be shown empirically by investigating whether investors like or dislike financial assets which are positively skewed. Empirical evidence for positive skewness preference was presented by Arditti, and by Levy and Sarnat [1972]. Experimental support for positive U"' has been offered by Coombs and Pruitt [1960] and by Alderfer and Bierman [1970].

An asset pricing model which takes skewness into account (but disregards variance) was developed first by Jean [1971]. Simonson [1972], uses this model and shows that positive skewness may offset a lower mean. Levy and Arditti [1972] criticize Jean's model for its failure to take the first three moments into account simultaneously. This shortcoming is corrected by Kraus and Litzenberger [1976] who incorporate the effect of skewness on valuation, and develop a three moment capital asset pricing model.22) In the empirical part of their paper they test this model and find that the market price for coskewness is positive.[23] This study was critically examined by Friend and Westerfield [1980], who also found that, in the aggregate, investors may pay a premium for positive skewness in their portfolio. Barone-Adesi [1985] improves on the methodology, and finds evidence as well that investors pay a premium to hold securities with positive coskewness.[24]

If investors prefer positive skewness, as seems to be the case in aggregate, an answer may be found to the question why investors do not diversify to the degree that the mean-variance approach would suggest: Simkowitz and Beedles [1978] and Conine and Tamarkin [1981] show that diversification destroys skewness, in addition to decreasing variance. To the degree that investors like the former but dislike the latter, they will hold only a limited number of securities in their portfolio.

In this paper we used three-term approximations to the utility function. Such approximations have been examined by Levy [1969] and by Hanoch and Levy [1970].[25] As a global description of the individual's behavior under uncertainty these authors (as well as Kraus and Litzenberger) reject such cubic poynomials because of their undesirable economic properties: U'' eventually becomes positive, and IARA may obtain even for wealth levels where $U''<0$. It must again be pointed out, that in the present paper the three term polynomial is used only as a local approximation to the 'inherent' utility function, which satisfies $U'>0$ and $U''<0$ throughout (and which needs to be assessed, if at all, without taking recourse to choices under risk). Furthermore, we do not take the position in this paper that IARA is inherently absurd; as is shown in Appendix B, a perfectly reasonable utility function, that is, a utility function satisfying greed and satiability, may exhibit IARA for low wealths, and DARA for high wealths. The implication for this case is, that the premium one is willing to pay to avoid a risk increases with wealth for a poor person, and decreases with wealth for a rich person. Hence, risky assets would be inferior goods for low wealths, and normal goods for high wealths.

In a paper closely related to the subject matter discussed here, Tsiang [1972] discusses the importance of skewness. He shows that if $U'''>0$, the positive influence of increased skewness partly offsets the negative influence of increased dispersion. (In our paper, skewness may wholly offset variance.) In fact, to disregard skewness may lead to absurd results (e.g., Borch [1969], Feldstein [1969]). With $U'''>0$ positively skewed, assets are 'penalized' if the Taylor series is truncated after two terms: the truncation results in a downward bias of expected utility (p.362). Tsiang concludes that indifference curves in mean-variance space must nowhere have a slope exceeding 45°. The proof relies, however, on a positve third derivative of the utility function. With $U'''<0$, it is the negatively skewed asset which is penalized, and the truncation results in an upward bias of expected utility. Hence, indifference curves in mean-variance space could have slopes less than or greater than one, the latter implying that $U'''<0$. The sign of U''' (and hence of our 's') changes at the point when the indifference curve's slope becomes unity.

Discussing Feldstein's assumptions of a log utility function, and a lognormal distribution of investment outcomes, Tsiang states that "it appears that risk aversion might eventually decrease as risk itself is increased." (p.367) Since the size of risk stands for variance, and realizing that risk aversion can be identified with our risk premium, this situation can easily be explained by referring to our risk grid. For lognormal distributions, skewness is an increasing function of variance; hence, with r and s given, the risk premium will initially increase with increasing variance. Eventually, since $s<0$, the decrease in the term (Skw/3)(s/r) more than offsets the increase in the term Var, and the risk premium starts to decrease. (In fact, the assumption of log utility is not even needed.)

8. Conclusion

In this paper we computed the premium that an individual is willing to pay in order to avoid a risky situation. For small risks, it has become customary to approximate expected utility by a two-term Taylor series. When the amounts at risk are not infinitesimally small, however, this approach may lead to incorrect results. Thus, I propose to use three-term Taylor series to approximate expected utility for not-so-small risks. This is tantamount to evaluating a risky situation by its mean, its variance and its skewness, while ignoring higher moments. The utility of the certainty equivalent, $U(w-\pi)$, is approximated by a two-term expansion. Solving the resulting equation for the risk premium, we receive an expression which depends on the risk's variance, its skewness, and on two expressions which are derived from the individual's utility function: the well-known $r(w) = -U''/U'$, and the newly introduced $s(w) = -U'''/U'$. As a consequence, risk aversion is now characterized by the two components, variance aversion $r(w)$, which is always positive, and skewness aversion $s(w)$, which could, in principle, be either positive or negative. Hence, since both skewness and $s(w)$ could be of either sign, the positive influence of skewness may offset the negative influence of variance on the risk premium; if $s(w)>0$, this would, for example, lead to higher prices for positively skewed assets.

Combinations of r and s which leave the risk premium for a specific risk constant, lie on straight lines in the rs-plane, which 'flare out'. Some lines correspond to negative risk premiums, which means that an individual with such a combination of r and s is a risk taker for this specific risk: skewness has, in this case, more than offset variance. We call the family of these 'constant risk premium' lines the 'risk grid'. (A part of the rs-plane corresponds to imaginary premiums.) On the other hand, each utility function's r(w) and s(w) correspond to a 'path' in the rs-plane. When superimposing anything but a straight line path onto a risk grid, one sees that some regions of wealth correspond to IARA, others to DARA. (This holds even if U''' is positive everywhere.) This fact permits the empirical verification of the model.

Appendix A

The Infinitesimal Approach as Applied to the Not-So-Small

A limitation of the 'two-term Taylor' approach to the study of risk aversion, can be demonstrated by the following example. Let $U_1(w) = \ln(w)$ and $U_2(w) = -\exp(-w/K)$. For log-utility we get $r_1(w) = 1/w$, and for U_2 we have $r_2(w) = 1/K$. Since the two risk aversion functions intersect at $w = K$, Pratt's theorem is only applicable for wealth levels which are either below K (in which case U_1 shows greater risk aversion than U_2), or for wealth levels above K (in which case U_2 is more risk averse than U_1). More importantly, for not-so-small risks it is not clear, a priori, which utility function results in higher risk premiums. As an example, let us consider an individual who has a chance of winning (or losing) a fraction j of his current wealth with probability p. The true risk premiums can be computed for both utility functions:

$$\pi_1 = w(1-[1+j]^p) - pjw \qquad \pi_2 = K.\ln[1-p+pe^{-jw/K}] - pjw \, .$$

Figure 3 shows which utility function results in greater risk premiums at different wealth levels and at different fractions j, with K=100,000 and p=10%. The result, which could neither have been obtained by the infinitesimal methodology, nor by applying Pratt's theorem, may seem surprising: for any fraction j, even if arbitrarily small, there exists an amount w_o, such that at this wealth level, π_1 is greater than π_2, even though $r_1<r_2$ (and vice versa). The surprise diminishes when one realizes that this happens only if, in spite of j very small, $w_o < K$ and $w_o + jw_o > K$ (and vice versa), which violates the assumption of

Pratt's theorem. We conclude, that under certain circumstances, Pratt's methodology is applicable only to infinitesimal risks, in the most proper sense of the word; even arbitrarily small risks may be too large. This is in opposition to Tsiang's claim, that what is required for a good approximation is that the risk be small in relative terms, and that "it is not necessary that the risk taken should be infinitesimally small in its absolute magnitude." [1972,p.355]

Appendix B

A Utility Function Exhibiting both Positive and Negative Preference for Skewness

Consider a utility function of the form,

$$U = c\, e^{mw} + d\, e^{nw} .$$

Setting

$$w_o = \ln\left[-\frac{c}{d}\left(\frac{m}{n}\right)^i\right]/n-m ,$$

we have $U^{(i)}(w) > 0$ if

and

$$w > w_o \quad \text{if} \quad dn^i > 0$$

$$w < w_o \quad \text{if} \quad dn^i < 0 .$$

As a numerical example, take $m = -.0001$, $n = -.00001$, $c = .01$, $d = -1.0$. Since $w_o = 0$ if $(m/n)^i = -d/c$, we receive, for this utility function,

$$U' > 0 \quad \text{and} \quad U'' < 0 ,$$

for all positive wealths. Thus, the customary requirements of utility are satisfied. For the function's third derivative, however, we obtain,

and

$$U''' < 0 \text{ if wealth} < 25{,}584$$

$$U''' > 0 \text{ if wealth} > 25{,}584.$$

Hence, this utility functions exhibits aversion to positive skewness for low wealths, and preference for positive skewness for higher wealths.

As an aside, it may be noted, that the above utility function is a solution to the differential equation

$$U'' + \alpha U' + \beta U = 0 ,$$

where $\alpha = -(m+n)$, and $\beta = mn$. Writing $r(w)$ for the Pratt-Arrow degree of risk aversion, the latter is equivalent to,

$$r(w) = \alpha + \beta/Q(w) ,$$

where $Q(w)$ has been defined by Aumann and Kurz [1977] as a measure of boldness.

Some readers may be disturbed by the fact that this utility function displays IARA. Obviously, DARA holds if $U''^2 - U'''U' < 0$, which is equivalent to

$$(m - n)^2 > 0 \quad \text{if} \quad cdmn > 0$$

and

$$(m - n)^2 < 0 \quad \text{if} \quad cdmn < 0 \; .$$

For the above utility function 'cdmn' is negative, and we have IARA throughout. However, even though IARA may sound counterintuitive in the Mean-Variance framework, when skewness is taken into account, it may become more acceptable.

Figure 1

Three-term Taylor Approximations to Utility

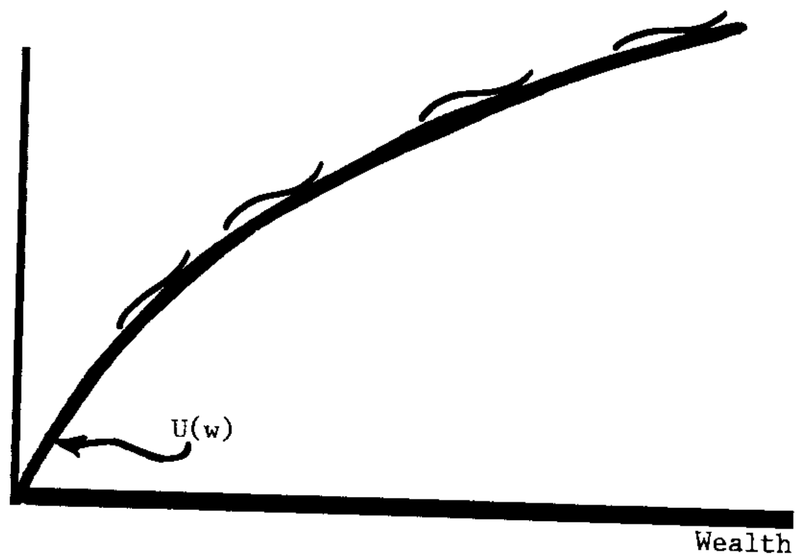

Figure 2

Risk Grid and Paths in the rs-Plane

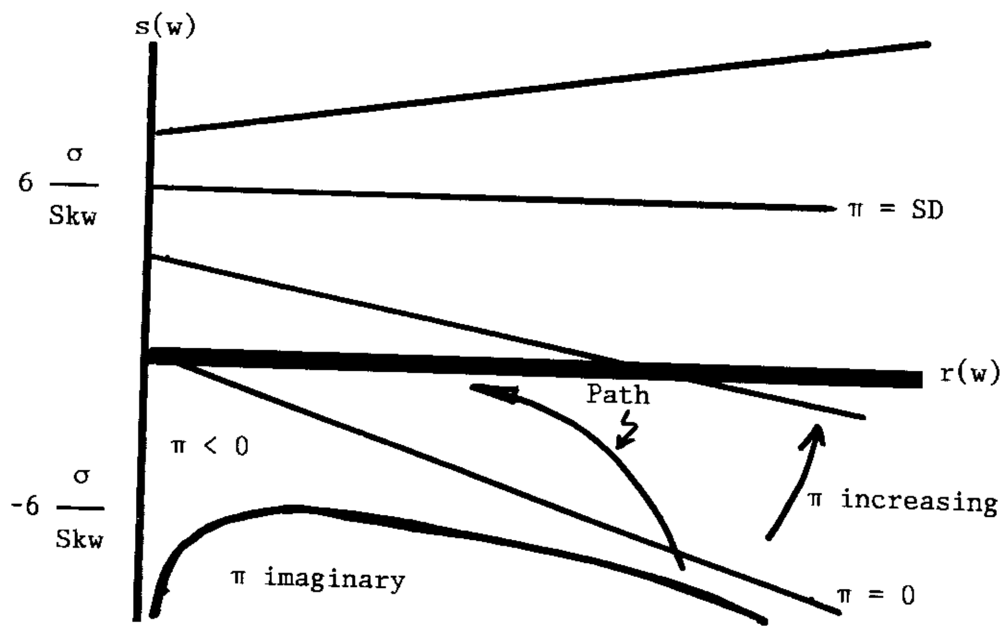

Figure 3

Risk Aversion for Not-So-Small Amounts

$U_1(w) = \ln(w)$, $U_2(w) = -\exp(-w/100{,}000)$

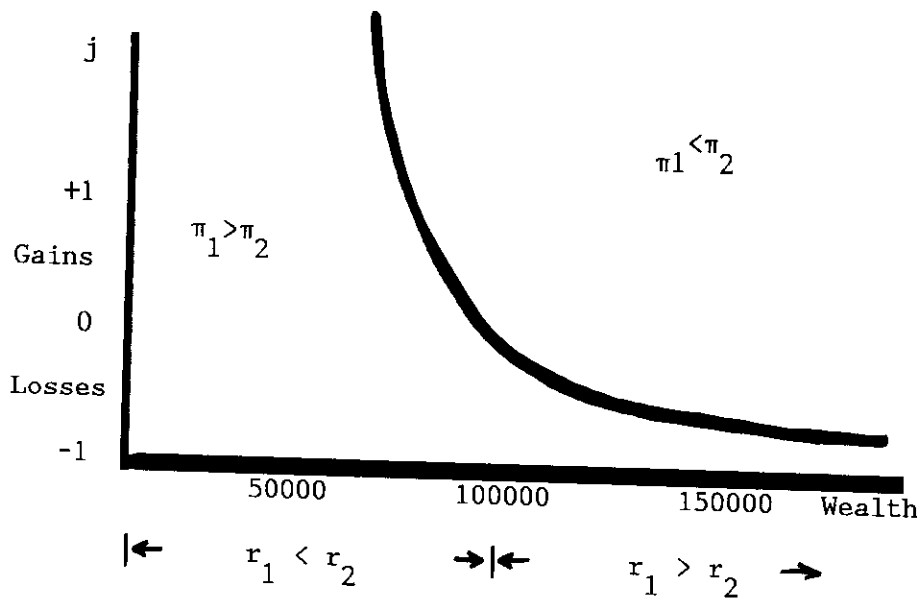

Figure 4

$$U = c\, e^{g_1 w} + d\, e^{g_2 w}$$

where $g_1 = -.0001$, $g_2 = -.00001$, $c = .01$, $d = -1.0$

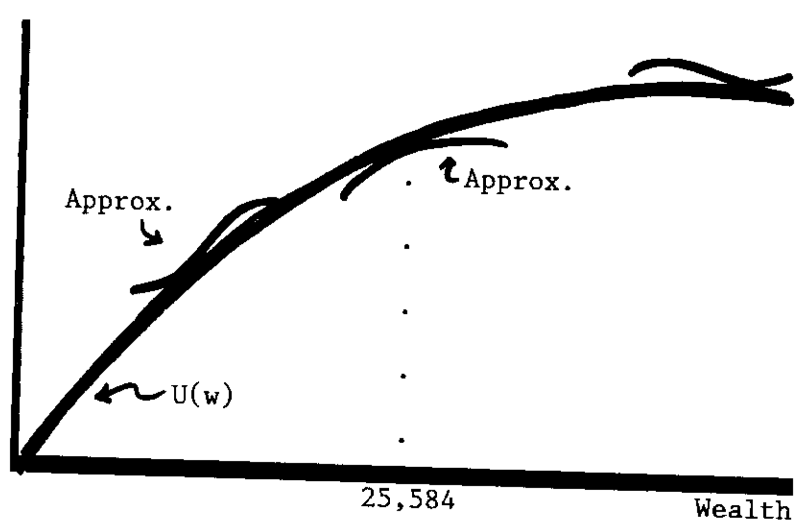

Table 1

Gamble*: Win 15000 with probability 1%
Lose 152 with probability 99%

Utility**: $U(w) = -.1(w^{-10})$

Wealth	Risk Premium, using		
	Var and Skw	Var only	All moments
80,000	40.13 ↑	156.25	85.42
90,000	47.10	138.89	80.27
100,000	50.62 IARA	125.00	75.66
110,000	52.14	113.64	71.52
120,000	52.43 ↓	104.17	67.79
130,000	52.11 ↑	96.15	64.41
140,000	51.30	89.29	61.34
150,000	50.24 DARA	83.33	58.54
160,000	49.04	78.13	55.98
170,000	47.77	73.53	53.63
180,000	46.46 ↓	69.44	51.46

* Mean=0
Var=22,700
Skw=33,700,000

** $W_c = 119,000$
$\pi_c = 52.50$

Footnotes:

1) The notion of 'size-of-risk aversion' tries to circumvent the problem somewhat. (Menezes and Hanson [1970])

2) Without limiting generality we shall analyze actuarially neutral risks in this paper: $E(\tilde{z})=0$.

3) We will sometimes leave away the argument when no misunderstanding can arise, writing U for U(w) etc.

4) In fact, one could argue that this behavior holds also in the large. Hence, we do not necessarily require fourth and higher moments to be small for the 'approximation' to hold. (Whether the results of this paper extend to the large, depends on whether one is willing to accept this behavioral assumption in the large.)

The truncation of the right hand side is quite reasonable, considering that "the additional information provided by the fourth and higher moments is not very clear to the professional statistician, let alone to the uninitiated [decision maker]." (Levy and Sarnat [1983]; see Kaplansky [1945]). The truncation of the left hand side is also justified: by Abel's Impossibility Theorem and by Galois Theory, a closed form solution for equation (3) is only possible, even if extremely complicated computationally, if the left hand side is truncated at most after four terms.

5) Markowitz [1952] proposed a model of behavior under risk which is similar to the one suggested here: at <u>customary</u> wealth, local utility is characterized by a function of a certain shape; this general shape does not vary, no matter what the level of wealth.

6) Note that $s(w)$ is invariant to linear transformations of utility. For small Var and Skw, the first order approximation for π in equation (5) is $(\frac{1}{2})\mathrm{Var}\ r + (1/6)\mathrm{Skw}\ s$

7) By the same token, we shall speak of HAVA instead of HARA utility functions. On the other hand if we refer to CARA, we mean constant absolute risk aversion, using both variance and skewness, etc.

8) From equation (8) we get

$$1 > \left(\frac{1}{r} + \frac{\mathrm{Skw}}{6}s\right)\left(\frac{1}{r^2} + \mathrm{Var} + \frac{\mathrm{Skw}}{3}\frac{s}{r}\right)^{-\frac{1}{2}} .$$

Hence,

$$\frac{1}{r^2} + \mathrm{Var} + \frac{\mathrm{Skw}}{3}\frac{s}{r} > \frac{1}{r^2} + \frac{\mathrm{Skw}}{3}\frac{s}{r} + \left(\frac{\mathrm{Skw}}{6}s\right)^2 ,$$

from which equation (7) follows.

9) How π changes with wealth is more complicated, and involves the utility function's fourth derivative:

$$d\pi/dw = (\partial\pi/\partial r)(\partial r/\partial w) + (\partial\pi/\partial s)(\partial s/\partial w)$$

$$= \left[\pi\left(\frac{s}{r} + r\right) - \frac{\mathrm{Skw}}{6}\left(\frac{s^2}{r} + \frac{U^{(4)}}{U'}\right)\right] / (r\pi + 1)$$

10) We could proceed by writing

$$d\pi = \frac{\partial\pi}{\partial r}dr + \frac{\partial\pi}{\partial s}ds = 0,$$

from which we obtain, after some tedious computations,

$$\frac{ds}{dr} = \frac{3}{\mathrm{Skw}}(\pi^2 - \mathrm{Var}) .$$

11) If Skw>0, Figure 1 would be rotated around the r-axis.

12) This holds for utility functions other than HAVA, and for negative skewness as well, possibly with the IARA and DARA regions interchanged. In fact, all utility functions except those with a straight line path (see Section 5) exhibit such behavior).

I hesitate to use the words 'surprising' and 'paradox' for IARA behavior. Over the years one has come to accept DARA as the only reasonable behavior, especially in the aggregate. However, on an individual basis, one may think, for example, of a relatively poor person, who, when faced with a decrease in wealth, starts saving on insurance premiums by decreasing coverage; this implies IARA. (See also Bowman [1982].)

13) However, see also footnote 4.

14) Note, that Markowitz posited the opposite: a concave region near current wealth, and a convex region somewhere to the right of current wealth; hence he assumes $U''' < 0$.

15) This, again, presents an opportunity to test the model: does the Allais paradox disappear for very high wealths and/or high values of k? Or, when reducing the amounts involved in the gamble, does the paradox disappear sooner for rich individuals than for poor individuals?

16) However, not all paths in the right half of the plane are acceptable: a 'utility' function with $U' < 0$ and $U'' > 0$, would also lie in the right half. The following lemma may help to identify acceptable candidates. *If there exists a wealth level w_o for a path, at which $U' \geq 0$ and $U'' \leq 0$, then the path is acceptable for all wealth levels, if it lies wholly in the right*

half of the rs-plane. Proof: if at w_o, $U' > 0$ and $U'' < 0$, and at w_1, $U' < 0$ and $U'' > 0$, then there exists a wealth level w^* at which both U' and U'' are equal to zero. The zeroes must coincide, since otherwise the path would enter the left half of the plane. However, if they do coincide, U' can not become negative: U'' has become positive, which implies that U' increases again.

17) Another way to see this relationship is to differentiate $r = -U''/U'$:

$$r' = -\frac{U'''U' - (U'')^2}{(U')^2} = s + r^2 .$$

18) Neutrality towards skewness (i.e., $s = 0$) obtains, if $r' = r^2$, which implies, as expected, quadratic utility. Hence the path $s = 0$ is not admissible.

19) More generally, since $U = \int \exp(-\iint r' dwdw)dw$, and writing $D(w)$ for the deviation of s from $-r^2$, the requirements are satisfied if

$$\int D(w)dw > 0 .$$

20) This does not contradict Scott and Horvath's conclusion [1980] that U''' must be consistently positive, it contradicts their assumption: we permit an individual to have different directions of preference at different wealths. (The value of 'a' must be smaller than $b^2/4$ for g_i to be real valued, since, if $g_i = \alpha \pm i\beta$, then $U = ce^{\alpha w}[\sin \beta w + \cos \beta w]$, which is not acceptable.)

21) Solely for illustrative purposes, let me offer the following 'evidence': Friend and Blume, and others, have found $r(w) \simeq 2/w$ using financial risks which are presumably positively skewed (Tsiang, p.355, quoting Cootner). Szpiro found $r(w) \simeq 1.4/w$, in the context of negatively skewed property/liability insurance. If

we took these results at face value, we would conclude that s(w) is positive at the wealth levels examined.

22) Levy and Sarnat also found that moments beyond the third do not seem to influence the investor, i.e., the direction of preference did not turn out to be significant. This vindicates our behavioral assumption to truncate the Taylor series after the third term. Kraus and Litzenberger ignore fourth and higher moments since "behavioral arguments [concerning] ... the fourth and higher moments have not been made." [1976, p.1087]

23) In a later paper (Kraus and Litzenberger [1983]) the authors show that if the characteristic line of a security's returns is quadratic, then mean, variance and skewness may suffice to exactly determine the security's price (i.e., not by approximation).

24) Further econometric problems are discussed by Sears and Wei [1985].

25) Utility functions of such shape had already been conjectured by Friedman and Savage [1948], and by Markowitz [1952].)

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