RATIONAL PONZI GAMES

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Abstract

When can a government borrow a dollar and never pay back any interest or principal? We call such an arrangement under perfect foresight a rational Ponzi game. We use the transversality condition facing individual agents to show that rational Ponzi games require an infinity of lenders. The horizon of individual agents is unimportant; Ponzi games cannot be ruled out by assuming that agents have infinite horizons. We point out both the basic similarity and some key differences between rational Ponzi games and asset price bubbles or fiat money. With reference to the international debt issue, the analysis implies that conditions in the borrower's economy are irrelevant to the feasibility of rational Ponzi games; what matters is the relationship between the paths of interest rates and population and productivity growth rates in the lenders' economy.

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1. Introduction

What are feasible paths of debt for a government that borrows either internally or externally? The question is suggested by recent concerns about the international debt crisis and high federal budget deficits in the U.S. In this paper, we analyze the benchmark case in which all market participants have perfect foresight, so that only risk-free lending is done. We study the conditions under which the borrower's opportunities include strategies with positive net present value.

The strategies we investigate are perfect foresight versions of the "Ponzi schemes" discussed by Minsky (1982) and Kindleberger (1978), where individuals or companies pay out funds to some parties by borrowing funds from others.[1] Since the perfect foresight assumption rules out schemes based on imperfect information (e.g., swindles) or irrationality of lenders (e.g., fallacies of composition), we are asking under what circumstances these Ponzi games can continue indefinitely. When, in other words, is it feasible for a government to incur debt and never pay back any principal or interest? We call such a policy, where all principal repayments and interest are forever "rolled over," i.e., financed by issuing new debt, a "rational Ponzi game."

Examples of rational Ponzi games are not hard to find in the growth theory literature. In the Diamond (1965) overlapping generations model, for example, there are steady states in which the interest rate is below the growth rate of the labor force and government debt per capita is positive. In these steady states, the total stock of government debt is increasing at a rate higher than the interest rate. New debt finances all of the interest

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[1]For an interesting discussion of the life and times of Charles Ponzi, see Russell (1973).

payments on the existing debt plus some additional transfers to the young.[2] Our aim in this paper is to uncover the general conditions that make Ponzi games feasible and to apply these results in a variety of growth contexts. We make the following observations.

First, the feasibility of a rational Ponzi game depends on some key characteristics of the economy whose agents are going to hold the debt. For the case of external debt, this means that the characteristics of the borrower's economy are irrelevant to the feasibility of perpetual debt rollover. Related to the Third World debt situation, this means that the feasibility of perpetual rollover of debt depends on the performance of the economy of the lenders--in particular, on the relationships between real interest rates, population growth rates, and growth in per-capita income--and not that of the borrowing countries.

Second, for a Ponzi game to work, agents in the lending economy at all points of time must be willing to hold the outstanding stock of debt. This means that the debt must have a holding period return equal to the rate of interest, that the outstanding stock of debt must grow at the rate of interest, and that the aggregate desired wealth of the economy must be growing at least as fast, asymptotically, as the rate of interest.

When the lender acts as an infinitely lived representative consumer, a necessary condition for his optimal plan is that his wealth not grow faster

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[2]This means that the government can cut current taxes a small amount without ever raising future taxes to finance the increased interest payments. This possibility is the crux of the controversy between Sargent and Wallace (1981) and Darby (1984) over the inflationary implications of steady state government budget deficits. When the real interest rate (r) exceeds the population growth rate (n) in a closed economy, the steady state real non-interest deficit and the steady state rate of expansion of the monetary base cannot be chosen independently. When $r < n$, steady state deficits need not imply monetary growth.

than the rate of interest. This is sufficient to rule out his participation as a lender in a rational Ponzi game, regardless of the relationship between the interest rate and the rate of growth of income or population.

When we relax the representative consumer assumption, it becomes more difficult to rule out Ponzi games. We consider two types of economies. An essential characteristic of each is that over the course of time there are an infinite number of decision makers. One economy consists of a growing number of infinitely lived agents, and the other consists of an infinite sequence of two period lived agents. In either case, agents care only about themselves, and not about other family members. There are conditions under which rational Ponzi games can exist in either of these economies (the feasibility conditions in the endowment economy versions of each of these models are quite similar). It is therefore not the length of the horizons that is relevant, but rather it is whether there is an infinite number of agents as opposed to a finite number of (effective) agents who care only about lifetime consumption.

The issues raised in this paper have appeared in a variety of contexts in the growth theory and monetary theory literatures. In fact, the questions of whether fiat money can be valued, whether bubbles can exist on assets, whether the government can independently choose the steady state deficit and rate of monetary expansion, and whether a government can run a rational Ponzi game all turn out to be basically the same question. Our analysis draws on the existing literature in these areas and especially on Tirole's (1985) study of rational asset price bubbles in the overlapping generations model.[3] In characterizing the relationship between Ponzi games and asset price bubbles,

[3]See also McCallum's (1984) paper, which analyzes feasible government debt paths in a version of the Sidrauski model. Diamond (1965), Sargent and Wallace (1981), Darby (1984) and Anderson, Ando and Enzler (1984) analyze related issues of steady state government finance.

we show that any monetary equilibrium can be replicated by a Ponzi game equilibrium with finitely lived debt, and derive a strong irrelevancy result for open market operations between money and Ponzi game debt. We also show, however, that there exist some monetary equilibria that cannot be replicated by a Ponzi game equilibrium with positive coupon consols. In addition, the payments on debt contracts reduce the indeterminacy inherent in monetary equilibria. Governments can select among the range of possible equilibria by issuing a certain amount of finitely lived debt rather than "money." Some indeterminacy remains in the price of infinitely lived debt, since the price can be greater than the present discounted value (PDV) of the coupon payments. The distinction between "bubbles" and "fundamental" is not relevant here, however. Identical real equilibria exist where consols are priced at their "fundamental" and where they are priced above their fundamental (i.e., with "bubbles"). The reason the bubble/fundamental distinction is not relevant here is that for the case of rational Ponzi games, the entire debt is acting like a bubble.

Finally, we note that when borrowers are running rational Ponzi schemes, this does not imply that lenders are in any sense losing out. Rational Ponzi games are only feasible, in general, when the economy is in a dynamically Pareto inefficient equilibrium. The existence of perpetually rolled over debt will never make the lending economy worse off and will in general make it better off.

The paper is organized as follows. In the next section, we set out notation and give a rigorous definition of a rational Ponzi game. In Section 3, we study the optimal borrowing problem for an individual agent and present the basic result that Ponzi games can only exist in economies with an infinite number of agents over time. The application of this result to various growth models is immediate, and we do this in section 4. One interesting implication is that rational Ponzi games can exist in economies with infinitely lived agents if there is population growth and agents do not care about the utility of their children. We give an explicit example of this phenomenon, which also implies that fiat money can be valued and asset bubbles can exist even when agents have infinite horizons. In section 4 we also briefly examine the feasibility of rational Ponzi games in the overlapping generations model with intergenerational altruism first studied by Barro (1974). In section 5, we show the similarity between the feasibility conditions for Ponzi games and for asset price bubbles and valued fiat money. We discuss the possible indeterminacy of bond prices and examine the related distinction between "fundamentals" and "bubbles" on debt that constitutes a Ponzi game. In Section 6 we use the results of the previous sections to comment on Neihans' (1985) analysis of LDC borrowing. Section 7 concludes the paper.

2. Notation and a definition

In perfect foresight, all forms of wealth must bear a common real rate of return. We define A_t as total net financial wealth, i.e., financial assets minus outstanding debt. A_t can be positive or negative. An individual's wealth accumulates according to[4]

$$A_t = (1 + r_t)A_{t-1} + (Y_t - C_t), \tag{2.1}$$

where Y_t = income or endowment

C_t = consumption

r_t = real interest rate between periods t-1 and t.

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[4]In continuous time, we simply replace $(A_{t+1} - A_t)$ with dA_t/dt. Throughout this paper, we will use the discrete time formulation for convenience; none of our results hinge on this choice.

Equation (2.1) can be iterated forward from some initial wealth level A_0 to give

$$(2.2) \qquad A_0 = \sum_{s=1}^{T} \Gamma(0,s)(C_s - Y_s) + \Gamma(0,T)A_T .$$

where $\Gamma(k,s)$ is the discount factor in period k applicable to income received in period s:

$$(2.3) \qquad \Gamma(k,s) = \prod_{j=k+1}^{s} (1 + r_j)^{-1}$$

We will use the simpler notation $\Gamma(s)$ when discounting from period zero, i.e., $\Gamma(s) \equiv \Gamma(0,s)$, and we set $\Gamma(k,k) \equiv 1$. The term $\Gamma(T)A_T$ will play a crucial role in this paper.

In this paper, we focus on borrowing and lending by individuals. Consider an agent who is a price taker in the loan market (i.e. who takes interest rates as parametric) and who is formulating a plan for borrowing and lending over the infinite future. Any plan will imply a sequence of net cash flows Z_t from lenders. We will assume for most of the paper that all contracts are perfectly enforceable, so that promised payments and actual payments coincide.[5] We define the agent's net indebtedness at the end of period t, D_t, as the present value of the net cash flow that the borrower has already received as the result of past credit market transactions. Net

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[5]We are implicitly specifying some structure of default costs that induces individuals to abide by their contracts. This rules out important aspects of actual credit markets, an issue to which we return in Section 6.

indebtedness D_t by this definition, assuming initial indebtedness is zero, is equal to the accumulated value of cash flows up until time T and is given by:[6]

(2.4)
$$D_T = \left[\sum_{s=1}^{T} \Gamma(s)Z_s\right] \Gamma(T)^{-1} .$$

Viewed from period zero, equation (2.4) states that the present (time 0) value of time T debt equals the present value of the stream of prospective cash inflows for the borrower between 0 and T:

(2.5)
$$\Gamma(T)D_T = \sum_{s=1}^{T} \Gamma(s)Z_s .$$

In an infinite horizon setting, a common approach in the literature (e.g., Blanchard (1985)) is to impose the constraint that $\Gamma(T)D_T$ be non-positive as t approaches infinity:[7]

(2.6)
$$\lim_{T\to\infty} \Gamma(T)D_T \leq 0 .$$

When the interest rate is constant over time, this takes the familiar form

(2.6a)
$$\lim_{T\to\infty} (1 + r)^{-T}D_T \leq 0 .$$

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[6]It is easy to verify that D_t, like any other form of financial wealth, satisfies a version of (2.1): $D_t = (1 + r_t)D_{t-1} + Z_t$. One could alternatively define net indebtedness as the present value of current and future net cash flows on existing loan contracts held by the individual. Net indebtedness by the alternative definition satisfies the same recursion. The two definitions are equivalent whenever all loan contracts held by the individual yield present value zero over the life of the contract. This need not be the case; in certain cases there may be a "bubble" on the individual's debt. We discuss this possibility with respect to government debt in Section 5.

[7]The limit may not exist; we treat this problem more carefully in definition 2.1. In addition, condition (2.7) is usually imposed with equality; we argue below that the correct condition is the weak inequality.

For a sovereign borrower, the implication of (2.6) is that debts must be paid off by future savings, either in the form of non-interest government budget surpluses or, in the absence of a government/private sector distinction, non-interest current account surpluses. Condition (2.6) is usually viewed as being imposed by lenders to rule out Ponzi games. We will adopt the implied definition of rational Ponzi games as borrowing strategies in which the limit in (2.6), i.e., the present value of all cash flows associated with a borrowing strategy, is strictly positive. More formally,

Definition 2.1 A rational Ponzi game is a sequence of loan market transactions with positive net present value to the borrower in the following sense: there exists an $\varepsilon > 0$ and a t' such that $\Gamma(t)D_t \geq \varepsilon$ for all $t \geq t'$. If the limit of $\Gamma(t)D_t$ exists, then a rational Ponzi game is a strategy such that $\lim_{t\to\infty} \Gamma(t)D_t > 0$.

By engaging in a Ponzi game, the borrower is able to extract positive resources (in present value terms) from the lender(s). The classic form of this scheme consists of borrowing money (issuing short or long term debt) and financing all promised payments of principal or interest by issuing new debt. In other words, in each period promised cash outflows (interest, coupon payments, etc.) are offset by other cash inflows (new borrowing).[8]

Our definition of rational Ponzi games intentionally leaves aside the issue of how individual loan contracts are priced. For most of the paper we will be concerned with Ponzi game equilibria in which contracts are priced at the present value of the stream of promised payments (the "market fundamental"

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[8]But note that the definition also includes the case in which some payments are made, as long as the present value of net payments is less than the amount initially received by issuing debt.

(Tirole [1982])). In many cases, however, Ponzi game equilibria also exist in which the contracts are priced above their fundamental. In fact, even if all promised payments are made through net transfers from the borrower to lenders (rather than through new borrowing), a Ponzi game exists whenever a borrower can issue debt priced above its fundamental. We discuss the issues of price indeterminacies and multiple equilibria in Section 5.

3. The Transversality Condition and the Number of Traders

Under what circumstances will lenders agree to be part of a rational Ponzi game? The basic result we establish in this section is that Ponzi games require the participation of an infinity of agents. This follows from work by Cass (1972) and Tirole (1982, 1985). The result stems from the fact that no single agent will wish to form the lending side of a rational Ponzi game; to do so would involve a sacrifice of consumption with no offsetting benefit. The same argument can be shown to apply to the joint behavior of any finite set of agents. This means that in an economy with a finite number of agents, each agent faces a constraint of the form (2.6) as the result of optimal behavior by all other agents. With an infinite number of agents over time, by contrast, there always exists a new set of agents with whom to trade, and this may make it possible for debt to be rolled over perpetually.

This argument can be formalized in two steps by making two assumptions to dispose of uninteresting cases.

Assumption 3.1 (Finiteness) There are a finite number of individual decision makers in the credit market in each period.

Assumption 3.1 forces us to focus on Ponzi games that occur over time, i.e., that exploit the potential infinity of traders due to the birth of new agents.

Assumption 3.2 (Nonsatiation) The utility of individuals depends only on consumption, and individuals prefer more consumption to less in each period.

Assumption 3.2 leads to a familiar definition of consumption efficiency (see Cass (1972)), which we apply here specifically to borrowing/lending strategies:

Definition 3.1 A borrowing/lending strategy is consumption efficient if it is impossible for the agent, leaving the remainder of his portfolio strategy unchanged, to rearrange loan market transactions so as to raise consumption in any period without lowering it in some other period(s). Otherwise the strategy is consumption inefficient.

By nonsatiation, a rational agent will never choose a consumption inefficient consumption plan. More formally, we have the following proposition:

Proposition 3.1 (The Transversality Condition) Under Assumption 3.2, the following condition is necessary for optimality of an individual's borrowing/lending strategy:

(3.1) $$\liminf_{T\to\infty} \Gamma(T)D_T \geq 0 \ .$$

The proposition, which holds regardless of the agent's horizon[9] is a restatement of Cass's (1972) condition for consumption inefficiency in a one-

[9] For a finitely lived agent, the condition is simply $D_T \geq 0$, or equivalently $\Gamma(T)D_T \geq 0$, where T is the agent's terminal date.

good economy with a linear technology.[10] Cass's condition applies directly to our case, since an individual taking interest rates as parametric effectively faces a linear technology for transferring consumption intertemporally.

The intuition behind Proposition 3.1 is straightforward. Violation of (3.1) implies that the present value of debt is bounded above by some strictly negative number $-\varepsilon$ from some period t' onwards (i.e., the present value of <u>assets</u> is bounded <u>below</u> by some $\varepsilon > 0$ from t' onwards). If this were true, the agent could consume an extra amount ε in period t', leave all consumption beyond t' unchanged, and still have assets nonnegative for all t. Since such a plan is feasible and by nonsatiation is preferred to the original plan, the original plan cannot be optimal.

What prevents an individual from running a rational Ponzi game, i.e., from choosing a strategy in which the limit in (3.1) is strictly <u>positive</u>? Since equation (3.1) must hold for all participants in the loan market (currently alive or not), we have the result:

Proposition 3.2 Rational Ponzi games do not exist in a credit market with a finite number of participants over time.

The proof follows from Tirole (1982). The intuition is straightforward. Consider any possible equilibrium sequence of interest rates. As can be seen by definition 2.1, Ponzi games involve borrowing an amount that, after a point at least, grows at least as fast as the inverse of the discount factor. The existence of a rational Ponzi game means that some (finite or infinite) group

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[10] Cass's theorem implies that a strategy is inefficient if the sequence $\{\Gamma(t)D_T\}$ of present values of debt is bounded above by any strictly negative number. This means that efficiency requires that the sequence have no subsequences with negative limit points. Equivalently, the greatest lower bound of the set of limit points of subsequences--the lim inf--must be nonnegative for consumption efficiency.

of lenders is allowing its total lending to grow at least as fast as the inverse discount factor in equilibrium. But by Proposition 3.1, no single agent will allow his own wealth to grow as fast as the inverse discount factor, because this would be a consumption inefficient strategy. By simple addition, this implies that the total net wealth of any finite number of agents must grow more slowly than the inverse discount factor. It follows that lending side of the Ponzi game must be composed of an infinite number of agents entering the economy over time.[11,12] Each individual continues to satisfy his transversality condition, but the economy of lenders as a whole does not satisfy any aggregate transversality condition.

The analysis in this section suggests two further remarks. First, the logic of Proposition 3.2 shifts attention to conditions in the lenders' economy in determining limits on feasible borrowing strategies. The path of the borrower's economy will generally be irrelevant to the analysis. This contrasts sharply with the situation when outstanding debt must be repaid (i.e., when Ponzi games are infeasible). We return to these issues in Section 6.

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[11]This possibility was suggested by Shell (1971), who pointed out that the form of the lifetime budget constraint is not obvious for an infinitely lived agent in a world with an infinity of traders.

[12]Although such a scheme could conceivably work in finite time in an economy with an infinite number of agents (and also an infinite endowment), we have ruled out this uninteresting case by assumption. The following example (similar to Gamow's infinite hotel problem described in Shell(1971)), although Assumption 3.1 rules it out, may provide some helpful intuition on how rational Ponzi games work (the intergenerational schemes we discuss later are isomorphic):

Consider a countably infinite current population, each with an identical positive endowment (the aggregate endowment is therefore infinite). Now array the population side by side to the right of a particularly clever first individual. Let everyone pass one unit of endowment to the person on his left, except for the first individual, who gives up nothing. Consumption of the first individual goes up by one unit, and everyone else's consumption is unchanged. The first individual has successfully run a rational Ponzi game.

Second, the analysis makes it clear that the standard "transversality condition" one often encounters in the literature,

$$\lim_{T\to\infty} \Gamma(T)D_T = 0 \ , \tag{3.2}$$

is really a combination of two things: (1) a genuine transversality condition stating that nonnegativity of the limit in (3.2) is necessary for optimality, (i.e. the individual <u>chooses</u> not to be on the <u>lending</u> side of a Ponzi game), and (2) a restriction on the individual's budget set stating that the limit must be nonpositive (i.e. <u>equilibrium rules out</u> individuals being on the <u>borrowing</u> side of a Ponzi game).[13] The second condition holds when the number of potential loan market participants is finite. Together, the two conditions yield (3.2).[14]

4. Infinite horizons

Proposition 3.2 motivates what lies behind the examples of Ponzi schemes that we present in this section and the next. The key characteristic we focus on is whether the behavior of a possibly infinite group of potential lenders

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[13]The TVC is necessary for optimality for an individual taking interest rates as parametric, and also, therefore, for a central planner facing a linear technology. The issue is more complicated, however, when we consider the planner's problem in an economy with a neoclassical technology. There, the failure of the TVC does not necessarily imply the existence of Pareto dominating consumption plans, because any attempt to run down the capital stock will change the equilibrium interest rate sequence. In certain knife-edge cases, such as the Golden Rule equilibrium ($r = n$), the TVC fails but consumption efficiency nonetheless holds.

[14]When the borrower is the government (Blanchard (1985), McCallum (1984)), the implicit counterpart to Assumption 3.2 is that the government exhibits nonsatiation with respect to government expenditure. Alternatively, one could rule out the government being on the lending side of a rational Ponzi game by assuming that the government has no endowment other than the ability to tax. This would make it impossible for the government to transfer resources with positive present value to the private sector.

can be consolidated into the behavior of a finite set of decision makers over time whose preferences satisfy Assumption 3.2. If so, Proposition 3.2 applies to the consolidated group of agents, and rational Ponzi games can be ruled out. The simplest example of this is the infinitely lived representative consumer model of optimal growth theory. If consolidation does not occur--as for example in Samuelson's (1958) overlapping generations model without intergenerational altruism, or in the infinite horizon model without intergenerational altruism that we present below--the feasibility of rational Ponzi games depends on whether the economy is in a Pareto inefficient equilibrium in the absence of Ponzi games or asset price bubbles (Tirole (1985)).[15] We begin with the infinite horizon, stationary population case.

4.1 Stationary population

Consider an economy with a finite number of infinitely lived agents. Population growth is zero; the economy is composed of the same group of agents each period. Since Proposition 3.2 applies directly to this economy, we can conclude immediately that rational Ponzi games are ruled out.[16]

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[15]Tirole (1985) shows that there exist equilibria with asset bubbles if and only if the bubbleless economy is Pareto inefficient. We analyze the relationship between bubbles and rational Ponzi games in section 5.

[16]McCallum (1984) studies a model such as this in which Ponzi games are ruled out and Ricardian Equivalence holds with respect to debt that does not constitute a Ponzi game. In this setting, he shows that there is no limit on how high debt can go in any period in the finite future. This is because individuals willingly hold all of the debt in order to pay the future taxes. However, this does not mean that the interest and principal can be perpetually rolled over, because in such a circumstance no one would hold the debt.

4.2 Growing population

Consider now an economy consisting of a growing number of infinitely lived agents.[17] Let $L_t = (1+n)L_{t-1}$ be the number of agents alive at t, so that nL_{t-1} children are born in period t. Now suppose that there is a central planner who maximizes a welfare functional defined over the path of aggregate consumption. If the welfare functional shows nonsatiation in aggregate consumption in all periods, Proposition 3.1 holds for the economy as a whole, and it follows that no outsider could run a rational Ponzi game trading solely with individuals in this economy. The same result emerges if we view the economy as consisting of a finite number of initial "dynasties" i.e., families growing at rate n. If the utility function of the initial parents shows nonsatiation and is defined solely over the (total or per-capita) consumption of the family, and if we endow the initial parents with dictatorial power over the consumption of their children, we have reduced an economy with an infinite number of potential lenders over time to one with a finite set of participants whose preferences satisfy Assumption 3.2. Rational Ponzi games are then ruled out by Proposition 3.2.

Alternatively, however, suppose that the economy is decentralized and that there is no intergenerational altruism. New agents are born into the economy and fend for themselves. The key point here is that while each individual will satisfy his own transversality condition, this will not suffice to rule out rational Ponzi games. Even though each individual's wealth will not be growing faster than the inverse discount factor, population growth may make it possible for aggregate desired wealth to grow at the rate

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[17]Since completing an earlier draft of this paper, independent work by Weil (1986) has been brought to our attention. Weil shows the possible dynamic inefficiency of infinite horizon models with population growth and neoclassical technology.

of interest or faster. The following example illustrates the possibility of rational Ponzi games in an economy with infinite-horizon agents.

4.2.1 Ponzi games with infinitely lived lenders: an example

Consider an economy with a constant returns to scale storage technology yielding return $r > 0$, with each infinitely lived agent receiving an initial endowment of goods e_0 and no further endowments.[18] We make the following assumptions:

(i) individuals have time separable utility functions with constant rate of time preference $\delta > 0$, and the population grows at rate $n > 0$;

(ii) $r = \delta$, i.e., the rate of return on storage equals the time preference rate; and interest on beginning-of-period storage is received during the period;

(iii) individuals assume that they cannot run rational Ponzi games, i.e., they optimize subject to the constraint that the limit of the PDV of wealth is nonnegative.

The first two assumptions are chosen for convenience. We relax assumption (ii) below, along with the assumption of constant per-capita initial endowments. Assumption (iii) is required for the optimization problems of individuals to be well defined. These agents take the interest rate as

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[18] We restrict attention here to economies with a constant returns to scale storage technology. This is done to avoid distinguishing between the interest rate before and after the introduction of debt. The two rates will turn out to be the same in the storage economy as long as the amount of initial debt issued does not exceed initial storage.

parametric and would like, given nonsatiation in consumption, to use Ponzi schemes to run up "unbounded" terminal indebtedness (cf. equation (3.2)).[19]

Assumptions (i)-(iii) imply that the optimal consumption strategy is to consume the annuity value of the initial endowment: $c_t = re_0$ for all t (recall that interest on the initial endowment is received in the first period of life). This yields a steady state equilibrium with a per capita capital stock (goods in storage) equal to e_0. Aggregate wealth will be growing at rate n, however. If $n \geq r$ in the economy without debt, then the competitive equilibrium is consumption inefficient, and a rational Ponzi game can exist in equilibrium. A small amount of foreign (or domestic) bonds could be issued in the current period, crowding out an equal amount of storage. Agents holding the original debt would be willing to roll over the principal, and the interest payments could be raised by selling new debt to the newcomers.

Consumption inefficient equilibria and rational Ponzi games can also exist in more general infinite horizon economies with population growth. Consider, for example, a storage economy in which the interest rate is positive but not necessarily equal to the rate of time preference, and initial endowments are growing at rate g. As before, agents receive endowments only at birth. The population is growing at rate n. Assume that agents have identical time separable utility functions, with constant relative risk aversion (CRRA) period utility functions $u(c) = c^{(1-A)}/(1-A)$.

In Appendix 4.1, we derive the optimal capital accumulation for each individual and the implied path of aggregate capital in this economy. Optimal consumption is equal to $c_t = (r-x)k_{t-1}$, where k_{t-1} is any individual's time

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[19] In some equilibria, we will show that Ponzi games are feasible, but we assume that they can be issued only by governments and not private agents. See section 4.2.2 below.

t-1 wealth, and $(1+x) \equiv ((1+r)/(1+\delta))^{1/A}$. This solution holds as long as r exceeds x (if $A \geq 1$, $r > x$ for any positive δ). The growth of an individual's consumption and wealth are each equal to x. We show that if $n > 0$, the growth rate of the aggregate capital stock converges in the limit to the maximum of x and n+g. Since we assumed above that $r > x$, a rational Ponzi game can exist in this economy if and only if $x < r \leq n+g$ and $n > 0$. If these conditions hold, then there exists an initial amount of debt small enough such that the total debt will never grow larger than desired wealth, i.e. desired storage (desired wealth minus outstanding debt) will never be negative.[20] This debt can be rolled over forever at the interest rate on storage. The initial debt crowds out some storage of currently alive generations, and then all subsequent interest payments (and conceivably sales of bonds by existing bondholders) are financed by the endowments (i.e., initial storage) of later generations.

The above results mean that it is perfectly possible for rational Ponzi games to exist in economies with infinitely lived agents, as long as there are an infinite number of independent decision makers over time. This also means that rational bubbles and intrinsically useless money can exist in these economies. It is the infinity of agents--and the related possibility of dynamically inefficient equilibria--that opens up this possibility, not the length of individuals' horizons.

4.2.2 Infinite horizons: strategic issues

A further issue regarding the feasibility of rational Ponzi games in infinite horizon economies with growing populations relates to who is allowed

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[20] $r \leq n + g$ (although not $n > 0$) is also the condition for the existence of rational Ponzi games in a two period overlapping generations model with CRRA utility.

to be on the borrowing side of a Ponzi game. We have skirted this issue partially by allowing only governments (domestic or foreign) to run rational Ponzi games. Theoretically, however, in a world in which rational Ponzi games can exist, any infinite horizon agent can issue debt and perpetually roll it over.[21] All that is required is that other agents believe that new generations will be willing to purchase this agent's debt. The implied strategic issues, which arise both in models with finite horizon agents (with the U.S., Argentina, and Brazil fighting for the Ponzi game proceeds), and in models in which agents have inifinite horizons (U.S. vs. John Doe vs. Charles Ponzi), are beyond the scope of the present paper.[22]

4.3 Rational Ponzi games and intergenerational altruism

Rational Ponzi games are clearly feasible in the standard overlapping generations (OG) model without intergenerational altruism. As pointed out earlier, these can occur in Diamond's (1965) neoclassical OG model, for example, if $r < n$. In a celebrated paper, Barro (1974) argued that if altruistically motivated bequests from parents to children were positive, these resource transfers would serve to reduce the behavior of the infinite sequence of households to that of a single infinitely lived dynasty. By our earlier arguments, this "consolidation" would appear to rule out rational

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[21] Running a rational Ponzi game generally requires the existence of an infinitely lived agent, such as the government, to perpetually roll over the debt. Theoretically, however, a finite lived agent could issue consols with zero coupon (fiat money) and there would be no payments to roll over. The debt could merely circulate after the demise of this agent. Similary, there could clearly be bubbles in a world with no infinite horizon agents.

[22] The question of who gets to run Ponzi games is related to the question of what determines the split of an aggregate bubble across different assets, and whether the ability to reproduce assets which contain bubbles is sufficient to rule out bubbles.

Ponzi games even though such phenomena would be possible in the absence of altruism.

The bequest economy shows the importance of Assumption 3.2, that lenders care only about their consumption. As Gale (1983) has shown, the preferences of the "consolidated" agent are in fact indeterminate in the Barro framework. In a companion to this paper (O'Connell and Zeldes (1986)), we examine a particular example of Gale's in which the dynasty values not only its own consumption but also the limiting value of the discounted aggregate bequest. With these preferences, rational Ponzi games cannot be ruled out.

The explanation for this result is that consumption efficiency is no longer a necessary condition for optimality if preferences do not satisfy Assumption (3.2). The dynasty's taste for the terminal bequest may lead it to reduce consumption in all periods in order to accumulate assets to be passed on from generation to generation.

When altruism runs from children to parents (the "gift economy"), equilibria with positive gifts have exactly the characteristics of Ponzi game equilibria. Gifts are transfers from later generations to earlier ones, justified under perfect foresight by the prospect of future gifts to be received. Similarly, Ponzi games involve a sequence of resource transfers justified by a transfer (via a sale of assets rather than an outright gift) from an infinitely receding "last" generation. Rational Ponzi games can therefore also exist in this setup (see O'Connell and Zeldes (1986)).

5. Bubbles, Money, Ponzi Games, and Multiple Equilibria

In this section, we investigate the feasibility of rational Ponzi games in finite horizon, overlapping generation models. Tirole (1985) has demonstrated that asset price bubbles and intrinsically useless fiat money serve identical roles, and that the conditions for fiat money to be valued are the same as the conditions for the existence of bubbles.[23] The main point of this section is that Ponzi games play basically the same role in an economy as asset price bubbles or intrinsically useless fiat money. We show that what is generally relevant for characterizing the equilibrium is the sum of the initial values of bubbles, money, and debt, as opposed to each separate value. This implies a strong irrelevancy result: open market operations between Ponzi game debt and money will generally leave the set of real and nominal equilibria unchanged.

We then go on, however, to explain an important difference between Ponzi games and bubbles. It is well known that multiple equilibria can exist in models in which bubbles are feasible. Given an initial quantity of debt, however, the payment stream associated with debt implies a floor and sometimes a ceiling on the value of the outstanding debt. These added conditions restrict the set of equilibria with Ponzi games to a subset (sometimes empty) of those implied by Tirole's (1985) analysis of bubbles and money. In some cases, the set of equilibria is reduced to a single equilibrium that can be chosen by the government. These extra restrictions explain differences between models with money that serves strictly as a store of value and models with debt (e.g. Dornbusch (1985)).

At the end of this section, we briefly describe the correspondence between money and debt when the quantity of money is falling. A falling money stock is equivalent to a debt whose coupons are paid not by issuing new debt, but instead by raising future taxes. We show that there can be situations where we can get monetary equilibria (with a falling money stock) but not Ponzi game equilibria.

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[23]Weil (1984a,1984b) has also done extensive related work.

Finally, we describe why the "bubble"/"fundamental" distinction is not economically relevant for the case of infinitely lived debt that constitutes a rational Ponzi game. This is closely related to Tirole's (1985) discussion of bubble accounting.

5.1. Why Ponzi games and bubbles are basically the same

Consider a government that issues an initial amount (X_0^D) of consols with constant real coupon R, and then sells additional (identical) consols in all subsequent periods to finance all required coupon payments. This government is successfully running a rational Ponzi game. Now consider rational Ponzi games with smaller and smaller R. The limit of this process, consols with coupon equal to zero, will also constitute a rational Ponzi game, as long as the consols can be sold for a positive amount. (In this case no future action is required in order to "finance" the coupon payments, and there will be a constant stock of consols outstanding). These zero coupon consols, however, are simply the intrinsically useless fiat money of the overlapping generations literature.

The example just given suggests a basic similarity between rational Ponzi games and fiat money. In fact, perpetually rolled over debt plays the same role in an economy as bubbles and fiat money. To make this point more carefully, we show that there exist real equilibria with Ponzi games that mimic most real equilibria with bubbles and/or fiat money.

We examine two kinds of assets, the key distinction being that one lasts for a finite period of time, and the other lasts forever. The reason that this distinction is important is that all finitely lived assets must be priced at their "fundamental", i.e., at the present value of all associated cash flows. The prices of infinitely lived assets, however, can sometimes exceed this amount (the price may include a "bubble"), and the equilibrium asset price may be indeterminate.[24] Infinitely lived assets raise the possibility of multiple equilibria, each with the asset priced differently, i.e., each with a different sized "bubble".

For simplicity, the finitely lived asset will be a one period pure discount bond that is issued at time t, pays one good at time t+1 and self-destructs at the end of period t+1 if not redeemed.[25] The infinite lived asset will be a consol that pays R goods per period in each period forever. In each case, the bonds are real, i.e. their payoffs are in terms of goods.

Throughout this section, we will use the standard overlapping generations model with capital presented in Diamond (1965). To begin, assume no government debt. Using Tirole's definitions, the "fundamental" on an asset is the present discounted value (at market interest rates) of the payment stream associated with the asset. The "bubble" on an asset is the difference between the market price and the fundamental. Consider an intrinsically useless piece of paper. If this piece of paper (with fundamental equal to zero) is valued in equilibrium, then its value is exactly equal to its bubble. In other words, fiat money and bubbles are identical in this case. Tirole (1985) shows that if the asymptotic interest rate in the economy with the paper valued at zero and no other bubbles (the "bubbleless" economy) is less than the population growth rate, then there also exists a continuum of equilibria with money and/or bubbles, indexed by B_o ($0<B_o \leq B_{omax}$), in which the total initial

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[24] Tirole (1985) emphasizes that durability of the asset is a basic condition for a price bubble.

[25] We assume that the bond self destructs in order to ensure that it is a finitely lived asset, i.e. that it does not continue to circulate forever even though there are no additional promised payments associated with it.

value of money and bubbles is equal to B_o.[26] If B_o is equal to B_{omax}, then the equilibrium is dynamically efficient, and the per capita bubble is constant over time. If B_o is less than B_{omax}, the equilibrium is inefficient and the per capita bubble shrinks over time toward zero, (although it never actually hits zero).

To extend this to include government debt, define X_o^M as the initial nominal quantity of money, P_o^M as the price of a unit of money in terms of goods, and $B_o^M = X_o^M \cdot P_o^M$ as the initial value of the fiat money or bubble. Similarly, define X_o^D as the initial quantity (number of units) of debt, P_o^D as the initial price of a unit of debt, and $B_o^D = X_o^D \cdot P_o^D$ as the initial value of the government debt. Define B_o now as the sum of the initial value of debt and bubble/money ($B_o \equiv B_o^D + B_o^M$). This brings us to the following proposition:

Proposition 5.1 Let X_o^D be the initial quantity of government debt that is to be perpetually rolled over. The set of equilibria is indexed by B_o and consists of any choices of B_o^M and $B_o^D = X_o^D \cdot P_o^D$ satisfying:

(5.1) $\quad 0 \le B_o \le B_{omax}$

(5.2) $\quad P_o^D \begin{cases} \ge \text{PDV (cash flows) for infinitely lived assets} \\ = \text{PDV (cash flows) for finitely lived assets,} \end{cases}$

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26 Bubbles can exist if $r < n$ (asymptotically) in the bubbleless economy. The intuition is the following. Introducing a bubble (or Ponzi game) must raise the asymptotic interest rate (this requires an assumption about preferences and technology (see Diamond (1965), Tirole (1985), or Weil (1984b)), and the change in the interest rate will be continuous in the initial size of the bubble. Therefore if $r < n$ in the bubbleless economy, there exist initial bubbles which would imply $r < n$ in the bubbly economy. The bubble would grow at rate r and the desired aggregate wealth by n (assuming a steady state), and therefore a bubble and/or rational Ponzi game would be feasible.

where the upper limit B_{omax} in (5.1) is identical to the limit in Tirole (1985).

The proof of the proposition is in Appendix 5.1. Free disposal and arbitrage imply that the price of any bond must always be greater than or equal to zero, and that the one period holding return on any bond with positive price must equal the market interest rate (on capital). As we show in Appendix 5.1, these conditions imply (5.2). Note that given an initial quantity of debt, condition (5.2) ties down an exact value of B_o^D for finite lived debt, and a floor on B_o^D for infinitely lived debt.

The key step in the proof is then to show that when all payments associated with debt are made by issuing new debt at the current market price, (and no <u>extra</u> new debt is issued), the value of the total debt outstanding $(X_t^D \cdot P_t^D)$ must rise at the rate of interest. This is true no matter what type of debt is used and whether or not the debt is priced at its "fundamental". The rest of the proposition follows directly from Tirole (1985). The aggregate capital stock is equal to aggregate wealth minus the sum of the aggregate value of the bubble and the aggregate value of debt. The value of the bubble and the value of the outstanding debt each must grow at the prevailing interest rate. Therefore, in any equilibrium with rational Ponzi game debt and fiat money, only the <u>sum</u> of the value of the initial money stock (or bubble), and the initial value of the Ponzi game is relevant for the equilibrium. Equation (5.1) is simply Tirole's characterization of the set of equilibria, expanded to include rational Ponzi games.

Proposition 5.1 leads directly to a strong irrelevancy result on government open market operations between money and (Ponzi game) bonds.

Proposition 5.2 Take any equilibrium in a model with valued fiat money. If the government issues new money and uses it to purchase outstanding Ponzi game debt, or issues new (finitely lived) debt and uses it to purchase outstanding money, there exists a new equilibrium with exactly the same real allocation and price path as the initial equilibrium.[27]

Note that while the irrelevancy holds for Ponzi game debt, it will not in general hold for debt that is not perpetually rolled over.

Proposition 5.2 highlights the incompleteness of these models as models of money. The asset labeled "money" here performs exactly the same role as real government bonds. In models in which money performs a wider range of functions (economizing on transactions costs, saving on information costs, etc.), policies like open market operations that change the relative supplies of money and other assets are more likely to have real effects.

Finally, we have Proposition 5.3:

Proposition 5.3 If there exists a Ponzi game equilibrium, there exists an identical monetary equilibrium.

Proofs of Propositions 5.2 and 5.3 are by construction and appear in Appendix 5.2.

5.2 Differences between Ponzi game equilibria and bubble equilibria.

Up to this point, we have stressed the basic similarity between bubbles and Ponzi games. We now investigate some subtle differences between the

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[27] We qualify part of the proposition by referring to finitely lived Ponzi game debt. The reason (as seen in Proposition 5.5) is that there may be some cases where Ponzi games cannot be run with positive coupon consols, and introducing these consols to a model that did not previously include them drives out money and changes the real equilibrium.

two. We do this by comparing economies with only money to economies in which rational Ponzi games exist, but money (and asset bubbles) do not.

In a world with only money, there exist a continuum of equilibria (indexed by B_o), and there is no way to determine which equilibrium the economy will choose. In a world with Ponzi games and no money, condition 5.2 restricts the set of equilibria to a subset of the feasible equilibria with money. How small that subset is for a given initial quantity of bonds depends primarily on whether it is finitely or infinitely lived debt, and for infinitely lived debt on the size of the coupon payments, the initial quantity of debt issued, and the interest rate sequence in the corresponding monetary equilibrium. We take up the finitely and infinitely lived cases in turn.

Suppose the Ponzi game is played with finitely lived debt. Since the bonds are priced at their fundamental, the government can set the initial value of the Ponzi game uniquely by choosing the initial quantity of bonds. If there is no money in the economy, this selects a single equilibrium from the set of feasible monetary equilibria.

Proposition 5.4 Any monetary equilibrium can be replicated by a Ponzi game equilibrium with finitely lived debt contracts. There is a unique initial quantity (X_o^D) of debt that will replicate a given monetary equilibrium.

The proof of Proposition 5.4 is in Appendix 5.2. If the Ponzi game is played with infinitely lived debt (consols with coupon R), we cannot tie down the equilibrium exactly, and in general, there still remains a multiplicity of equilibria with consols. But a difference with money remains: given an initial quantity of consols, we can now put a floor on the value of the Ponzi game, because the bonds must be priced at or above their fundamental. This rules out equilibria with low values of B_o. In addition, by choosing a large enough quantity of consols, the government can raise the floor up to B_{omax} and thereby choose the efficient equilibrium, ruling out other equilibria that could not be ruled out if the government just issued money.

A similar result holds in a model in which both Ponzi games and money are allowed to coexist. First, the "fundamental" on the debt puts a floor on the value of B_o and second, by issuing enough debt, this floor could be set to B_{omax}, thus eliminating bubbles and money. To see this, let $\{r_{t*}\}$ be the equilibrium interest rate sequence in the monetary equilibrium, B_o^{M*} be the initial value of the equilibrium money stock, and P^{DF*} be the PDV of payments associated with the debt at interest rate sequence $\{r_{t*}\}$. We then have the Proposition:

Proposition 5.5 For any monetary equilibrium,

(i) if $\sum_{t=o}^{\infty} \Gamma^*(t) < \infty$, (which implies that the fundamental of a positive coupon consol at the interest rate sequence in the monetary economy is finite), then there exists an identical Ponzi game equilibrium with positive coupon consols. For any given coupon R, there exists a range of initial quantities of debt $(0 < X_o^D < B_o^{M*}/P_o^{DF*})$ that that can support this equilibrium.

(ii) if $\sum_{t=o}^{\infty} \Gamma^*(t) = \infty$, then there does not exist a Ponzi game equilibrium with positive coupon consols.

We prove the proposition (Appendix 5.2) by verifying that there exists an initial stock of bonds X_0^D, and an initial price of bonds P_o^D such that $X_o^D \cdot P_o^D = B_o^{M*}$, and $P_o^D < P_o^{DF*}$ (or $P_o^D = P_o^{DF*}$ for finitely lived debt).[28]

An example of the exception in the second part of Proposition 5.5 is the simplest "Samuelson case" of the pure exchange overlapping generations model with no population growth (Gale (1973)). This economy has a stationary monetary equilibrium (the Golden Rule equilibrium) with r=0. This equilibrium can not be replicated with positive coupon consols that are rolled over forever, because the consols in such an equilibrium would have to have infinite value. However, if the population growth rate were positive, then the Golden rule equilibrium could be achieved with a Ponzi game equilibrium with positive coupon consols.[29]

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[28]Recall that in general there can be a multiplicity of equilibria in these models. Given two economies, there can exist an equilibrium in one that is identical to a given equilibrium in the other, but we have no way of knowing whether the economies will choose those identical equilibria.

[29]Dornbusch (1985) uses just such an economy. He assumes that the population growth rate of the economy is zero. The golden rule monetary economy therefore has interest rate equal to zero, which means that the "fundamental" associated with a positive coupon consol is infinite. There is no way to price the consol at or above its "fundamental" in the golden rule equilibrium. Therefore, the equilibrium interest rate in an economy with consols must be positive (>n), the government cannot engage in a Ponzi game (it must raise taxes to pay the coupons), and consols must be priced at their fundamental.

If Dornbusch were instead to allow for positive population growth, then the economy could achieve the golden rule equilibrium with positive coupon consols, and in this equilibrium the government would be engaging in a rational Ponzi game. Since consols need not in general be priced at their "fundamental" when Ponzi games are feasible, there exist equilibria in this economy with different stocks of bonds but identical total values of bonds outstanding. In addition, changes in endowments (and therefore asset demands) need not produce changes in interest rate sequences when the value of bonds is indeterminate. These possibilities are absent in Dornbusch's paper.

5.3 Extensions and Related Issues

5.3.1 Crowding out of bubbles by Ponzi games

Consider an economy with valued fiat money, with equilibrium initial value somewhere between 0 and B_{omax}. What happens to the equilibrium if the government begins a Ponzi game? (Assume that this move was unanticipated.) If the initial value of the money stock were exactly equal to B_{omax} (which would imply the economy was in an efficient equilibrium), then there is no room for an additional Ponzi game. Since positive coupon debt must have positive value, the new debt must reduce the value of the bubble/money by at least an equal amount, leaving the new total value of debt plus bubble no greater than B_{omax}. As Hamilton and Michener (1986) point out, in this case the government, although never explicitly raising taxes, is implicitly taxing away the pre-existing store of value (money/bubble).

Although the above story holds when the initial value of money was B_{omax}, it does not hold when the initial value of money is positive but less than B_{omax}. In this case, there is room for the government to issue new debt of value not exceeding (B_{omax} - B_o^M) and never either implicitly or explicitly raise taxes. While it is true that the per capita B_t shrinks over time for any B_o less than B_{omax}, there seems to be no *a priori* reason to choose the B_{omax} equilibrium from the continuum of feasible equilibria. In general, therefore, it is possible for the government to give a tax cut and never raise taxes, even in a world where a store of value exists.[30] This can be done up to an amount of B_{omax} - B_o^M without implicitly taxing the existing money/bubble.

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[30]Hamilton and Michener (1986) rule out this possibility by assuming that the economy is always at B_{omax}.

5.3.2 The relationship between equilibria with a shrinking nominal money stock and rational Ponzi game equilibria.

In the discussion thus far, we have assumed that the nominal money stock was kept constant in the monetary equilibrium, and that no new Ponzi games were started in the Ponzi game equilibrium. Under these assumptions, we showed the close similarity between fiat money and rational Ponzi games. We now briefly compare money and Ponzi games when the money stock is either growing or shrinking. We assume that the additional money is given in a lump sum fashion (i.e. a helicopter drop) or taken away by lump sum taxes, independent of one's holding of money.[31]

If the nominal money stock (X_o^M) is growing by μ percent per year, the value of each unit of money must continue to rise by r percent per year (although changing μ may change r), so the value of the stock ($B_o^M = X_o^M \cdot P_o^M$) must grow by the factor $(1+r)(1+\mu)$. The analogy to this for Ponzi games would be if, in addition to the new debt required to roll over existing debt, μ percent more debt (new Ponzi games) were issued each year. We could replicate the monetary equilibrium by giving debt-financed tax cuts to the old in amounts equal to the change in the real money stock in the monetary equilibrium.

If the nominal money stock is shrinking ($\mu < 0$), the government must raise lump sum taxes in order to buy back the money from the public. In this case, there can exist monetary equilibria even if $r > n$, as long as $r + \mu \leq n$ (Wallace (1980)). The reason is that the aggregate stock of money is growing more slowly than r. The analogy with debt would be if the government paid off μ percent of the outstanding debt each period with taxes, and rolled over the rest. In this case, however, the present value of the taxes collected is

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[31] If the distributions are made proportional to existing holdings, the set of equilibria are not affected by the growth rate of money.

independent of μ (as long as $\mu > 0$) and exactly equals the value of the initial debt issued. This implies that $\lim_{t\to\infty} \Gamma(t)\, B_t^D = 0$, and this would not constitute a Ponzi game. So the analogy to a contracting nominal money stock is national debt that does not constitute a Ponzi game, i.e. on which interest payments are made by raising taxes.

5.3.3 The basic irrelevance of the "bubble"/"fundamental" distinction with respect to rational Ponzi games.

In our discussion above, we put quotes around the terms "bubble" and "fundamental" because our main point here is that when the borrower is playing a Ponzi game, the entire value of the debt ("bubble" plus "fundamental") is acting like a bubble. Different debt contracts impose different price floors on a unit of debt, but with this exception national debt that is perpetually rolled over plays exactly the same role as a bubble. Proposition 5.6 (proved in Appendix 5.2) says that there exist equilibria with large "bubbles" and small "fundamentals" that are identical to equilibria with small "bubbles" and large "fundamentals".

Proposition 5.6 Take any Ponzi game equilibrium with consols priced above their "fundamental". There exists an identical equilibrium with a larger initial number of consols each priced exactly at its "fundamental".

The "fundamental"/"bubble" distinction on debt thus does not have economic significance when the debt is perpetually rolled over. This point is related to Tirole's (1985) discussion of bubble accounting. When the debt is perpetually rolled over, the coupon payments appear as fundamentals to the individual holders of the debt, even though they are nothing more than new bubbles from a macro perspective.

5.3.4 Replacing the neoclassical production function with a linear technology.

Because of the one-to-one mapping between the capital stock and the interest rate with a neoclassical production function, Tirole (1985) was able to show that the conditions for bubbles depended only on the asymptotic interest rate in the bubbleless economy. The analysis of rational Ponzi games can be readily extended to the case of a linear storage economy with stationary endowments. If the rate of return on storage (r_s) is less than n, rational Ponzi games can exist in such economies, and they play two different roles. First, Ponzi games can push out storage, eliminating consumption inefficient equilibria. Second, once storage is completely pushed out, increasing the size of the Ponzi game further will raise the interest rate, eliminating dynamically inefficient equilibria.

Consider a storage economy with $r_s < n$ in which the equilibrium involves positive storage. Since individuals in each period face the identical problem, per capita storage will be identical in each period (=s), implying that aggregate storage will be growing at rate $n > r_s$. This is a consumption inefficient equilibrium, and the government could run a Ponzi game up to initial size s, simply pushing out storage. The intuition here is straightforward, and relates directly to our derivation of the transversality condition. If the present value of storage never falls below ε, then ε is wasted storage. Neither the capital stock ε nor any of its proceeds is ever consumed. Capital is growing at least as fast as the discount factor. Since in effect part of the storage is being rolled over forever and never consumed, that storage could just as well take the form of bonds that are perpetually rolled over, enabling the issuer of the debt to consume the proceeds.[32]

If the government runs a Ponzi game of initial size ε, the resulting equilibrium will be consumption efficient, but dynamically inefficient (e.g. Samuelson (1958)). The government could increase the initial size of the Ponzi game, which would increase the interest rate, cause a Pareto improvement, and drive the economy closer to the dynamically efficient golden rule of $r=n$.[33]

6. International lending

The debt servicing difficulties of developing countries (LDCs) have recently attracted a good deal of attention from economists and policymakers. In the academic literature, it has been customary to model the sovereign LDC borrower as a single agent with an infinite horizon (e.g., Eaton and Gersovitz (1981)), a view that fits directly into our setup. In this section we use our results to discuss the feasibility of perpetual rollover of Third World debt. We comment particularly on Niehans' (1985) article on international debt.

Up to this point in the paper, we have assumed that all contracts are honored. Niehans (1985) points out that this assumption presupposes a legal

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[32]If endowments (or wages) are increasing over time (e.g., due to technological progress), then aggregate wealth will be growing faster than rate n, and therefore Ponzi games will be feasible for a wider range of initial r_s. As an example, take a storage/endowment economy with endowments growing at rate g. If saving of the young is proportional to lifetime wealth (as it would be if agents have CRRA utility functions), then the condition for the feasibility of rational Ponzi games would be $r_s \leq n + g$. If saving is log linear in wealth, but not proportional, the condition would be $r_s \leq (n + \alpha g)$, where α is the elasticity of saving with respect to the endowment. In other words, the rate of per capita income growth need not enter with a coefficient of one in the feasibility condition.

[33]This is also true if ε equals zero, as long as the equilibrium interest rate before the introduction of the Ponzi game was less than n.

enforcement structure or other source of default costs that may be largely absent in international credit markets. When there are no default costs and debt claims are unenforceable, Niehans argues, a country will repudiate all existing debts as soon as the present value of net required payments to lenders is positive for all future horizons.[34] This shrinks the set of borrowing strategies to rational Ponzi games. In other words, the only way that international borrowing can exist in this world is if neither interest nor principal will ever be repaid, i.e., if the path of borrowing constitutes a rational Ponzi game.

In discussing the policy implications of his work, Niehans offers, "strong advice for the lending banks: do not acquire unenforceable claims unless the borrower's rate of economic growth exceeds the rate of interest" (p. 76).

The implied condition, that the ratio of a country's external debt to its GNP not grow without bound, is often encountered in studies of government finance that emphasize steady states (e.g., Diamond (1965), Anderson, Ando and Enzler (1984)). When rational Ponzi games are ruled out, the condition has some plausibility whether or not the economy is in a steady state. Ruling out Ponzi games requires that current debt be matched by the present discounted value of future non-interest current account surpluses. If there is a fixed

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[34] In other words, borrowers calculate present values of future cash flows for all possible horizons and plan to default in the period for which the present value, viewed from today, is at a maximum. This rule implies repudiation today if and only if there is no remaining horizon for which the PDV of cash flows is positive.

limit on feasible current account surpluses as a percentage of GNP, this limit will imply a fixed upper bound on the ratio of debt to GNP over time.[35]

Conditions in the borrower's economy, however, are irrelevant to the feasibility of Ponzi game equilibria. Issues of feasibility of repayment never arise because repayment never occurs--outstanding debt is simply rolled over.[36] In order to analyze whether Ponzi games are feasible, e.g., whether equilibria exist in which Argentina's foreign debt can be perpetually rolled over, we need to examine conditions not in Argentina's economy, but in the economy of the lenders, the United States. For example, (in some models examined) the growth rate of the U.S. economy must be greater than the interest rate in order to ensure that the U.S. lenders will forever voluntarily hold a stock of foreign bonds that is growing at the rate of interest. Niehans' advice to banks is therefore beside the point when discussing the feasibility of rational Ponzi games. One would also be hard pressed to find a role in such a world for creditworthiness models of the type used in the empirical debt performance literature, where the supply of funds depends on characteristics of borrowers.

While conditions in the borrower's economy are irrelevant to the <u>feasibility</u> of Ponzi game equilibria, they may help choose among the large set of feasible equilibria. Up to this point we have not emphasized the extreme

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[35]This upper bound on the current account as a percentage of GNP can be thought of as a lower bound on domestic expenditure as a percentage of GNP, or a subsistence requirement of sorts. Lender-imposed limits on debt/GNP ratios are also implied by empirical debt performance models that interpret this ratio as a creditworthiness indicator.

[36]Cohen (1985) proposes as a measure of solvency the constant fraction of a country's exports that must be devoted to debt service forever in order to repay current debt. This "index of solvency" is a decreasing function of the difference between the growth rate of exports in the borrower's economy and the interest rate. While this index is appropriate when debt must be repaid it is irrelevant to the question of whether lenders will in fact <u>require</u> repayment.

coordination of expectations that is required to support a Ponzi game equilibrium when the technical conditions make such an equilibrium feasible. As Niehans himself states in a different section of his paper, "the decisive factor is the willingness of creditors to lend" (p. 73).

For a rational Ponzi game to get started, lenders must correctly foresee the entire infinite chain of contracts the borrower intends to write. If some future generation were to fail to keep the chain going, the LDC would immediately repudiate all existing debt, leading to capital losses for existing creditors. If such an event were foreseen, the Ponzi game would never get started. In other words, in situations where a Ponzi game equilibrium exists, there also exists a perfect foresight equilibrium in which there is no Ponzi game because each generation believes that future generations will not continue lending. What, then, sustains the expectations of lenders regarding the behavior of future lenders? One interpretation of creditworthiness indicators or limits on debt to GNP ratios is that these conventions serve to choose among alternative rational Ponzi game equilibria and by doing so to sustain the expectations--and the lending--of lenders. However, in a world in which loan contracts are completely unenforceable, these "creditworthiness" indicators are no different than sunspots: they can affect equilibria merely because agents believe they do.

If we move away from Niehans' assumption that international loan contracts are unenforceable, then the creditworthiness indicators may have merit. First, if Ponzi games are not feasible and all debt must be repaid, the ability of the borrowers to repay is clearly relevant. Second, even if there exist equilibria in which foreigners successfully run a Ponzi game, if lenders know that borrowers are unable to repay their debts there also exists an equilibrium in which foreigners <u>cannot</u> run a Ponzi game. In such an equilibrium, any bonds sold by the foreign government would be worthless, because agents expect (correctly so in equilibrium) that future agents would be unwilling to lend. If the foreign country has both the ability and willingness to repay debt if necessary, then their debt must be valued at least at the present value of these payments, and therefore the non-Ponzi game equilibrium is ruled out. In other words, the ability and willingness to repay debt ensures in this case that the country will never have to carry through and repay!

While introducing a Ponzi game clearly makes the borrowers better off, we have emphasized that Ponzi games are only feasible in equilibria for the lenders' economy that are inefficient before their introduction. (The only time this may not be true is in the "gifts" equilibrium discussed in Section 4.4, where the gifts are acting like a Ponzi game already. See O'Connell and Zeldes (1986).) Therefore, the lenders will not be made worse off and often will be made better off--<u>relative to a world with no Ponzi game</u>. In other words, while Brazil may be able to extract resources from the U.S. economy by running a rational Ponzi game, doing so will not in general hurt the U.S. because these resources were in a sense going to waste anyway. While the U.S. may be better off relative to an equilibrium with no Ponzi game, however, it would clearly be better off running its own Ponzi game on itself, extracting the resources itself rather than letting Brazil do so.

7. Conclusion

Since we summarized the results in the introduction, we will not do so again here. Instead, we briefly discuss some questions that have been left unanswered in the paper.

A government that can run a Ponzi game can cut taxes today without ever having to raise them in the future. Such a policy represents an increase in net wealth for the private sector, and one might therefore expect Ricardian equivalence to fail with respect to tax cuts financed by Ponzi games. We address this issue in a companion paper (O'Connell and Zeldes (1986)) and find, surprisingly, that Ricardian equivalence is unaffected by whether or not future taxes are increased.

A second issue is what happens if more than one individual or government tries to run a rational Ponzi game. The issues involved here also appear in models of fiat money, where they are resolved by assuming that the government has a legal monopoly on the creation of currency. This assumption is unsatisfactory, however, in a world where government obligations (e.g., sovereign debt of the LDCs) cross national boundaries easily. If the U.S. government can run a rational Ponzi scheme in the U.S., what prevents Argentina or Nigeria from doing the same?

A final set of issues relates to the empirical relevance of rational Ponzi games. We often observe situations where the risk-free real interest rate is below the sum of the population growth rate and the growth of per capital income. For Ponzi games to exist, however, this condition must continue to hold infinitely far into the future. How informative is the analysis in this paper in a world in which the future is intrinsically uncertain? To answer this requires that we drop the perfect foresight assumption and model uncertainty explicitly. Weil (1984b) has examined the conditions under which stochastic bubbles can exist in an otherwise non-stochastic real economy. This analysis applies, with minor modifications, to Ponzi games with infinitely lived debt. With finitely lived debt, however, the value of the debt is tied down and therefore these issues do not arise. A natural extension would therefore be to introduce some source of uncertainty into the real economy of the lenders (e.g., uncertain horizon, or shocks to the storage technology, income, or population growth rates). One result we anticipate is an inability to roll over debt completely in some states of the world, so that conditions in the borrower's economy would become relevant to the feasibility of rational Ponzi games.

Appendix 4.1

Aggregate wealth in an infinite horizon storage economy with growing population

In this appendix, we derive the behavior of aggregate wealth in a storage economy with a growing number of infinite horizon agents.

Individuals

Individual agents receive an initial endowment of capital k_0 (and no further endowments) and have access to a storage technology with real rate of return $r>0$. Each agent solves the following problem:

$$(1) \qquad \text{Max} \sum_{i=0}^{\infty} \beta^{i} \frac{c_i^{1-A}}{1-A}$$

s.t.

$$(1.1) \qquad k_t = (1+r)k_{t-1} - c_t$$

$$(1.2) \qquad \lim_{t\to\infty} (1+r)^{t} k_t \geq 0$$

$$(1.3) \qquad k_o \text{ given,}$$

Let k_t be defined as end of period wealth, including both storage and any holdings of others' debt. The discount factor $\beta \equiv 1/(1+\delta)$ is between 0 and 1. Condition (1.2) is the assumption by individuals that they cannot run rational Ponzi games. This condition is needed in order for the optimization problem to be well defined.

The solution to this problem is defined by the Euler equations

$$(2) \qquad u'(c_t) = \beta(1+r)u'(c_{t+1}), \qquad t=1,2,\ldots ,$$

the transversality condition (TVC) $\lim_{t\to\infty} (1+r)^t k_t \leq 0$ (this condition follows from Proposition (3.1)), and the restrictions (1.1) - (1.3) defining the individual's opportunity set.

With CRRA period utility functions ($u(c) = c^{1-A}/(1-A)$), the Euler equations can be solved to get $c_t = (1+x)^{t-1}c_1$, where x is defined implicitly by $(1+x) \equiv [(1+r)/(1+\delta)]^{1/A}$. When $r=\delta$, consumption is flat; consumption rises or falls over time as r is greater or less than δ.

The TVC plus constraint (1.2) together imply that the discounted value of lifetime consumption equals initial wealth k_0 (solve (1.1) forward and set the limiting discounted capital stock to zero). Plugging in the difference equation for consumption yields $c_1 = \mu k_0$, where μ is given by

$$(3) \qquad \mu = [\sum_{i=1}^{\infty} (1+x)^{i-1}/(1+r)^i]^{-1}.$$

We assume $r > x$ to ensure that the infinite sum in (3) converges. (This is equivalent to assuming $(1 + \delta) > (1 + r)^{1-A}$). In this case μ is simply equal to $r - x$.

Initial consumption is therefore proportional to initial wealth. Since the individual solves the same problem each period, and since μ does not depend on time, it follows that consumption is the same proportion of current wealth each period. The individual's wealth thus satisfies the same difference equation as consumption:

$$(4) \qquad k_t = (1+x)^t k_0.$$

Aggregate wealth

Now suppose that there is population growth at rate n and that the initial capital endowments grow at rate g. Since agents already alive increase their capital at rate x, aggregate capital (K_t) satisfies the following difference equation:

$$(5) \qquad K_t = (1+x)K_{t-1} + nL_tE_t.$$

Here $L_t=(1+n)^t$ and $E_t=(1+g)^t$ are the population and the per capita endowment at birth, respectively, so that an amount nL_tE_t of new capital is introduced at time t due to the birth of new individuals. We have chosen the normalizations $L_0=E_0=1$ for convenience.

The limiting behavior of K_t depends on the relationship between (1+x) and (1+n)(1+g). To see this, write the solution to (5) in two alternative ways:

$$(6.1) \qquad K_t = (1+x)^t[K_0 + n \sum_{i=1}^{t} ((1+n)(1+g)/(1+x))^i]$$

$$(6.2) \qquad K_t = (1+x)^tK_0 + [(1+n)(1+g)]^t[n \sum_{j=0}^{t-1} ((1+x)/(1+n)(1+g))^j].$$

When $(1+x) > (1+n)(1+g)$, the summation term in (6.1) converges as $t\to\infty$, and aggregate capital grows asymptotically at the rate x. When $(1+x) <$ $(1+n)(1+g)$, the summation term in (6.2) converges, and the behavior of K_t is dominated asymptotically by the term $[(1+n)(1+g)]^t$. (When $(1+x)=(1+n)(1+g)$, $K_t = (1+x)^t(K_0 + nt)$.) We conclude that when $n>0$, the asymptotic growth rate of aggregate capital is equal to the maximum of x and (approximately) n+g.

Recall that rational Ponzi games can exist in this economy if and only if aggregate wealth grows at least as fast as the interest rate. Since r must exceed x for a meaningful solution to agents' optimization problems, it follows that with $n > 0$ and $r > x$, the condition for rational Ponzi games in this economy is $(1+r) \leq (1+n)(1+g)$, as stated in Section 4.2 of the paper.

Appendix 5.1

Price and quantity paths for outstanding debt in a rational Ponzi game

(Proof of Proposition 5.1)

In this appendix, we describe the time path for the price, quantity, and total value for outstanding debt when the government is perpetually rolling over all interest and principal. We examine both one period discount bonds and consols that pay coupon R.

Bond prices

The perfect foresight assumption implies that the one period holding return on all assets must be equal. This implies the following arbitrage equation:

(1) $$(P_{t+1} + R)/P_t = 1 + r_{t+1},$$

where P_{t+1} is the price of the government bond at time t+1 (after the time t+1 coupon is paid) and r_{t+1} is the equilibrium interest rate between t and t+1 in the economy with the government debt. (Introduction of the debt, recall, can change the equilibrium interest rate.)

For a one period discount bond, the coupon is equal to one, and P_{t+1} is equal to zero, due to our assumption that the bond self-destructs at the end of t+1. Therefore the price at time t is given by:

(2) $$P_t = 1/(1 + r_{t+1}).$$

For a consol, the price sequence must satisfy the following difference equation:

(3) $$P_{t+1} - P_t = r_{t+1}P_t - R.$$

This equation does not rule out "bubble" price paths for the asset. It does say that for a given interest rate, the capital gains on the consol must be larger the higher the time t price of the consol. Tirole (1985) shows that when there is free disposal of assets (i.e. the price must always be non-negative) the price of any asset must be at least as great as the present value of all cash flows associated with the asset (its "fundamental"). If this were not true, the price path above would at some point imply a price less than zero. This tells us:

$$(4) \qquad P_t \leq \sum_{s=1}^{\infty} \Gamma(t,t+s)R = P_F.$$

Equations (2) and (4) give us condition (5.2) in Proposition 5.1.

The quantity of bonds outstanding

Define X_t as the quantity of government bonds outstanding at time t. Let the government roll over all interest and principal on the bonds by issuing new bonds. The quantity of new bonds that the government must issue will depend on the current price of the bonds. Look first at one period discount bonds. Since there are X_t of these bonds outstanding at time t (each of which promises to pay one good at time t+1), the government must raise $X_t \cdot 1$ goods by selling bonds at t+1. This means that the outstanding quantity of bonds at t+1 will equal $(X_t \cdot 1)/P_{t+1}$, implying the following difference equation for the stock of outstanding bonds:

$$(5) \qquad X_{t+1} = X_t/P_{t+1}.$$

Next, examine consols. If there are X_t outstanding consols at time t, then the government must issue new consols in order to raise X t R goods to pay the coupon payments in t+1. This means that X_tR/P_{t+1} new consols must be issued in period t+1. Since the consols never disappear, the difference equation for the quantity of consols outstanding is:

$$X_{t+1} = X_t + X_t R/P_{t+1}. \tag{6}$$

The outstanding value of government debt

Combining the above equations for price and quantity, we can derive the difference equation for the outstanding value of government debt. For one period discount bonds we have:

$$\text{(7a)} \quad X_{t+1}P_{t+1} = (X_t/P_{t+1})(P_{t+1}) = (X_tP_t)/P_t = X_tP_t/(1/(1+r_{t+1}))$$

$$= (X_tP_t)(1+r_{t+1}),$$

i.e., the value of outstanding debt grows at the rate of interest. For consols:

$$\text{(7b)} \quad X_{t+1}P_{t+1} = (X_t + RX_t/P_{t+1})(P_{t+1}) = X_tP_{t+1} + RX_t$$

$$= X_t(P_{t+1} + R) = X_t(P_t(1+r_{t+1}) - R + R)$$

$$= (X_tP_t)(1+r_{t+1}),$$

i.e., the aggregate value of the consols rises at the rate of interest, <u>independent of R and P_t</u>. In other words, whether or not the consols are priced at their "fundamental", the outstanding value of debt rises at the rate of interest.

The total value of outstanding debt in a rational Ponzi game must grow at the rate of interest. Tirole's (1985) asset bubbles also grow at the rate of interest and have no associated cash flows. Rational Ponzi games are therefore identical to asset bubbles, with the exception of conditions (5.2). Therefore, for the case when both bubbles and rational Ponzi games exist simultaneously, the set of equilibria is characterized by Tirole's conditions applied to the <u>sum</u> of the initial bubble and initial Ponzi game, plus equation 5.2.

Appendix 5.2

Proofs of Propositions in 5.2 - 5 6

Variables denoted by * are equilibrium values in the monetary economy.

Proof of Proposition 5.2

From Proposition 5.1, equilibria are indexed by the value of $B_o = B_o^D + B_o^M$. Since any existing debt must be finite valued and any new debt is assumed to be finitely lived and therefore must be finite valued, trading an equal value of money for Ponzi game debt (or vice-versa) will leave B_o unchanged at the original sequence of interest rates. Therefore the original interest rate sequence continues to constitute an equilibrium.

Proof of Proposition 5.3

Let P_o, X_o be the equilibrium with the Ponzi game, with $X_oP_o = B_o$. Choose initial quantity of money X_{o*}. $P_{o*} = P_o$ is a feasible monetary equilibria, since $P_{o*} > 0$ and $X_{o*}\, P_{o*} = B_o$.

Proof of Proposition 5.4

For one period debt, $P_o = 1/(1+r_1) < \infty$, by equation (2), Appendix 5.1. For multiperiod, but finitely lived debt, $P_o = P_{F*} < \infty$, by equation (4), Appendix 5.1. Choose $X_o = B_{o*}/P_o$. This implies $X_oP_o = B_o = B_{o*}$.

Proof of Proposition 5.5, part (i)

$P_{F*} = \Sigma\, r_*(t)R < \infty$. Choose X_0 such that $0 < X_o \leq B_{o*}/P_{F*}$, so $0 < X_oP_F \leq B_{o*}$. Therefore there exists a $P_o \geq P_F$ such that $X_oP_o = B_{o*}$. Since $P_o \geq P_F$, it is feasible.

Proof of Proposition 5.5, part (ii)

For any $R > 0$, $P_{F*} = \infty$, and $P_o \geq P_F = \infty$. Therefore there does not exist an X_o s.t. $X_o P_o = B_{o*}$. If positive coupon consols existed in this economy, they would be infinitely priced, regardless of the coupon size. Therefore there is no quantity of consols small enough such that the initial value of consols could be equal to B_{o*}.

Proof of Proposition 5.6

Let P_o, X_o, B_o, be the equilibrium quantities in the equilibrium with the "bubble" on the consol, i.e., $P_o > P_F$. Choose $X_o = X_o P_o / P_F$. Then $P_o = P_F$ will be an equilibrium because $P_o \geq P_F$ and $P_o X_o = B_o$.

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