THE ROLE OF RISK AVERSION IN THE DETERMINATION OF EQUILIBRIUM STOCK PRICES AND THEIR VARIABILITY

by

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April 1985

Revised March 1986

* I am grateful to the participants of the Finance Seminar at Wharton for their helpful discussions. I am especially indebted to Irwin Friend, Richard Kihlstrom, Andrew Lo, George Pennacchi, Randolph Westerfield and Stephen Zeldes for their valuable comments. Any remaining errors are my own responsibility.

ABSTRACT

In this paper, it is shown that risk aversion plays a critical role in the determination of the equilibrium stock prices and their variability in a one-asset pure exchange economy. Specifically, it is argued that the variance of equilibrium stock prices is a strictly increasing convex function of the Arrow-Pratt measure of relative risk aversion, γ, if γ is greater than one. Furthermore, it is shown that the inequality underlying variance bounds tests can be reversed in our model with risk aversion. Therefore, it is concluded that the high volatility of stock prices relative to dividends may imply a rejection of risk neutrality rather than a failure of stock market efficiency.

ESSAY II

THE ROLE OF RISK AVERSION IN THE DETERMINATION OF EQUILIBRIUM STOCK PRICES AND THEIR VARIABILITY

SECTION I

INTRODUCTION

The recent empirical work by LeRoy-Porter [1981] and Shiller [1981] which is based on a variance bound test has cast a serious doubt on the validity of stock market efficiency on the ground that stock prices are too volatile to be justified in the context of market efficiency. They argue that their findings are strongly against market efficiency, since the volatility of stock prices significantly violates its implied upper bound.

Marsh-Merton in their recent paper [1985] argue that it is not the case that empirical findings using a variance bound test imply the inefficiency of stock market, since a variance bound test is actually a joint test of the equilibrium asset pricing model and the underlying stochastic process of dividends. They show that the inequality implied by the stationary process of dividends is reversed in every sample if the stochastic process of dividends is assumed to be non-stationary.[1]

Mankiw-Romer-Shapiro [M-R-S, 1985] argue that their new unbiased tests find evidence contradicting market efficiency. It should be noted that their tests are immune to criticisms by Marsh-Merton [1985] since the dividend process need not follow a stationary time series in their tests.

Motivated by the observation that the present-value equation for stock prices is valid only under risk neutrality,[2] LeRoy-Lacivita [1981] claim that the high volatility of stock prices can be consistent with stock market efficiency under high risk aversion. They show in a simple two-state finite-horizon model that the variability of stock prices is strictly increasing in the Arrow-Pratt measure of relative risk aversion [1971, 1964].

Michener [1982] presents an excellent example which explores the possibility that the general equilibrium asset pricing model can be consistent with the high volatility of stock prices for extreme risk aversion and the martingale property of stock prices. He demonstrates (in a one-asset exchange economy in which a representative consumer has log utility and the stochastic process of dividends is characterized by an AR (1)) that the variance bounds may be violated by a significant margin at the same time the stock price is almost a random walk.

This essay is an extension of theoretical findings by Michener [1982] and LeRoy-LaCivita [1981].[3] It is shown that risk aversion plays a critical role in the determination of equilibrium stock prices and their variability in a one-asset pure exchange economy.

Specifically, it is argued that the variance of the equilibrium stock prices is a strictly increasing convex function of the relative risk aversion measure, γ, if γ is greater than one. Furthermore, it is shown that the inequality underlying variance bounds tests can be reversed in our model with risk aversion. It should be noted that the discount rate moves around in general equilibrium, and with certain utility functions this can reverse the inequality.

This essay is organized as follows. In Section II, the basic model is constructed along the lines of Lucas [1978] and Michener [1982]. In Section III, the equilibrium stock price function is derived under some assumptions, and the variance of equilibrium stock prices as a function of the relative risk aversion measure is analyzed. We explore the critical role of risk aversion in the determination of equilibrium stock prices and their variability. In Section IV, the comparative statics is used to find the relationship between underlying parameters (such as a consumption shock, a discount factor) and the variance of equilibrium stock prices. In Section V, the variance bounds implied by our model are examined, and some implications for the empirical tests for stock market efficiency are discussed. Section VI concludes the paper, and suggests some unsolved problems.

SECTION II

THE BASIC MODEL

Our model is a special case of Lucas' model [1978]. We consider an exchange economy with a representative consumer (for a large number of identical consumers) who maximizes an additively separable utility of the form:

$$E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right], \quad 0 < \beta < 1, \qquad (1)$$

where c_t is consumption of a single good at time t,

β is a discount factor,

$u(\cdot)$ is a current period utility function,

$E[\cdot]$ is an expectations operator.

We assume that the utility function of the representative consumer exhibits constant relative aversion (CRRA)[4] of the form, $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, $\gamma \neq 1$ or $u(c_t) = \ln c_t$.

The consumption good is produced on one productive unit. The unit produces its nonstorable consumption good, y, according to the following first-order Markov process or random walk:[5]

$$\ln y' = \alpha \ln y + \varepsilon_t, \tag{2}$$

where $0 \leq \alpha \leq 1$ and $\varepsilon_t \sim N(0,\sigma^2)$.

Ownership in the productive unit is determined each period in a competitive stock market. The unit has outstanding one perfectly divisible equity share. A share entitles its owner as of the beginning of t to all of the unit's output in period t. Shares are traded, after payments of real dividends, at a competitively determined equilibrium price, p_t.

Suppose that z_t, x_t are the consumer's beginning-of-period and end-of-period stock shares respectively. Then, the consumer's budget constraint is given by

$$c + p(y) \cdot x \leq y \cdot z + p(y) \cdot z, \tag{3}$$

where p(y) is the equilibrium stock price in terms of y. The value function, V(z,y), is defined by

$$V(z,y) = \max_{(c,x)} \left[u(c) + \beta \int V(x,y')\, dF(y'|y) \right] \tag{4}$$

subject to (3).

In this economy, it is trivial to determine the equilibrium price function, p(y), and the value function, V(z,y). In equilibrium (if it exists), all output will be consumed and all shares will be held, i.e., $c_t = y_t \cdot z_t$

and $z_t = x_t = 1$ for all t. However, sufficient care should be taken in determining equilibrium, since CRRA utility is, in general, not bounded.

Assuming the existence and uniqueness of equilibrium[6] for the time being, we focus on the derivation of the equilibrium price function, p(y). In equilibrium, p(y) must satisfy the following stochastic Euler equation:

$$u'(y)\ p(y) = \beta \int u'(y')\ (y' + p(y'))\ dF(y'|y). \quad (5)$$

Define
$$g(y) \equiv \beta \int u'(y')\ y' dF(y'|y),$$
$$f(y) \equiv u'(y)\ p(y). \quad (6)$$

Then, $f(y) = g(y) + \beta \int f(y')\ dF(y'|y)$,
$$p(y) = \frac{f(y)}{u'(y)},$$

where $u'(y) = y^{-\gamma},\ \gamma > 0$,
$$\ell n\ y'|\ell n\ y \sim N(\alpha \ell n\ y, \sigma^2),\ 0 \leq \alpha \leq 1.$$

SECTION III

THE DERIVATION OF EQUILIBRIUM STOCK PRICES AND THEIR VARIANCE

Our objective is to find p(y) satisfying the following functional equation:

$$f(y) = g(y) + \beta \int f(y')\ dF(y'|y), \quad (7)$$

$$p(y) = \frac{f(y)}{u'(y)}, \quad (8)$$

where $u'(y) = y^{-\gamma}, \gamma > 0$,

$\ln y' \mid \ln y \sim N(\alpha \ln y, \sigma^2),\ 0 \leq \alpha \leq 1$,

$f(y)$ and $g(y)$ are defined by (6).

Successive substitution of function $f(\cdot)$ in (7) yields

$$f(y) = g(y) + \beta\, E[g(y') \mid y] + \beta^2\, E\{E[g(y'') \mid y'] \mid y\} + \ldots .$$

From the definition of $g(y)$ and properties of lognormal distributions,[7]

$$g(y) = \beta \int u'(y')\, y'\, dF(y' \mid y) = \beta\, e^{\frac{1}{2}(1-\gamma)^2\sigma^2}\, y^{\alpha(1-\gamma)},$$

$$E[g(y') \mid y] = \beta\, e^{\frac{1}{2}(1+\alpha^2)\ (1-\gamma)^2\sigma^2}\, y^{\alpha^2(1-\gamma)},$$

$$E\{E[g(y'') \mid y'] \mid y\} = \beta\, e^{\frac{1}{2}(1+\alpha^2+\alpha^4)\ (1-\gamma)^2\sigma^2}\, y^{\alpha^3(1-\gamma)} . \qquad (9)$$

Therefore,

$$f(y) = \beta\, e^{\frac{1}{2}(1-\gamma)^2\sigma^2}\, y^{\alpha(1-\gamma)} + \beta^2\, e^{\frac{1}{2}(1+\alpha^2)\ (1-\gamma)^2\sigma^2}\, y^{\alpha^2(1-\gamma)}$$

$$+ \beta^3\, e^{\frac{1}{2}(1+\alpha^2+\alpha^4)\ (1-\gamma)^2\sigma^2}\, y^{\alpha^3(1-\gamma)} + \ldots$$

$$= \beta\, e^{\frac{1}{2}(1-\gamma)^2\sigma^2}\, y^{\alpha(1-\gamma)} \{1 + \beta\, e^{\frac{1}{2}\alpha^2(1-\gamma)^2\sigma^2}\, y^{-\alpha(1-\alpha)(1-\gamma)}$$

$$+ \beta^2\, e^{\frac{1}{2}(\alpha^2+\alpha^4)(1-\gamma)^2\sigma^2}\, y^{\alpha(1+\alpha)(1-\alpha)(1-\gamma)} + \ldots\}. \qquad (10)$$

It is noted that the attempt to find an explicit solution to f(y) is frustrated by the fact that f(y) might not converge in general. To find the explicit solution, additional assumptions on γ or α are necessary.

It is interesting to observe that log utility of a representative consumer ($\gamma = 1$) yields the equilibrium price function, p(y), which is proportional to y for all $\alpha \in [0,1]$, as is shown by Michener [1982] and Brock [1982]. We now investigate the general set of assumptions on the utility function of the representative consumer and the stochastic process of dividends that yields such a proportional equilibrium price function.

Proposition 1. Consider two pure exchange economies which are identical except the utility function of a representative consumer and the stochastic process of dividends.

Economy I (Michener Economy): $u(y) = \ell n\ y$,

$$\ell n\ y' = \alpha \ell n\ y + \varepsilon_t,\ 0 \le \alpha \le 1,\ \varepsilon_t \sim N(0,\sigma^2).$$

Economy II: $u(y) = \frac{1}{1-\gamma} y^{1-\gamma},\ \gamma > 0,\ \gamma \neq 1,$

$$y'^{1-\gamma} = \alpha\ y^{1-\gamma} + \varepsilon_t,\ 0 \le \alpha \le 1,\quad \varepsilon_t \sim N(0,\sigma^2).$$ [8]

Then, the equilibrium price functions in both economies are proportional to current dividends:

$$p(y) = \frac{\beta}{1-\beta}\, y \text{ in Economy I,}$$

$$p(y) = \frac{\alpha\beta}{1-\alpha\beta}\, y \text{ in Economy II.}$$ Proof: See Appendix 1.

We observe the following facts from Proposition 1 along the lines of equation (20) of Lucas [1978]. First, in Economy I the income effect $(\frac{yf'(y)}{f(y)})$ is 1, but the information effect $(-\frac{yu''(y)}{u'(y)})$ is 0. This is evident from the observation that $p(y) = \frac{\beta}{1-\beta}\, y$, i.e., p(y) is independent of α. Second, in Economy II, the income effect is γ, while the information effect is $1-\gamma$. Therefore, the information effect is positive (negative), if and only if γ is less (greater) than 1.

From Proposition 1, we can say that equilibrium stock prices reflect available information in the economy, but do not know yet the role of risk aversion in the determination of equilibrium stock prices. To investigate the role of risk aversion, we assume that the representative consumer has CRRA utility, and that the stochastic process of dividends is characterized by the form, $\ln y_t = \varepsilon_t \sim N(0,\sigma^2)$ or $\ln\frac{y'}{y} = \varepsilon_t \sim N(0,\sigma^2)$. We show in the following propositions that the variance of equilibrium stock prices is a strictly increasing convex function of the Arrow-Pratt measure of relative risk aversion if the measure is greater than one.

<u>Proposition 2</u>. Consider a pure exchange economy in which a representative consumer has CRRA utility and the log's of

dividends are serially independent identically distributed normal random variables, i.e., $\ell n\, y_t = \varepsilon_t \sim N(0,\sigma^2)$ for all t. Then, equilibrium stock prices and their variance are given by the following:

$$p(y) = \frac{\beta}{1-\beta} e^{\frac{1}{2}(1-\gamma)^2\sigma^2} y^{\gamma},$$

$$\text{Var}[p(y)] = (\frac{\beta}{1-\beta})^2 \{e^{(3\gamma^2-2\gamma+1)\sigma^2} - e^{(2\gamma^2-2\gamma+1)\sigma^2}\}$$

for $0 < \beta < 1$, $\gamma > 0$.

Furthermore, Var[p(y)] is a strictly increasing convex function of $\gamma \varepsilon [\frac{2}{5}, \infty)$.

Proof: See Appendix 2.

Proposition 2 shows that risk aversion of a representative consumer plays a critical role in the determination of equilibrium stock prices and their variance. Therefore, the variance of equilibrium stock prices can be very large, if high risk aversion is assumed.

Some interesting observations on this economy can be made. First, the income effect is γ, while the information effect is 0 by construction. Second, the equilibrium stock prices follow lognormal distributions with means $\ell n\frac{\beta}{1-\beta} + \frac{1}{2}(2\gamma^2-2\gamma+1)\sigma^2$ and variances $(\frac{\beta}{1-\beta})^2\{e^{(3\gamma^2-2\gamma+1)\sigma^2}\}$ for all t. It is interesting to note that the lognormal distribution of the dividend process is preserved in equilibrium stock prices.

We now turn to an interesting case in which the assumption on the stochastic process of dividends implies the martingale property of equilibrium stock prices.

Proposition 3. Consider a pure exchange economy in which a representative consumer has CRRA utility and the stochastic process of dividends is characterized by the form,

$\ell n \frac{y'}{y} = \varepsilon_t \sim N(0,\sigma^2)$, where $0 < \sigma^2 < \frac{\ell n \beta^{-2}}{(1-\gamma)^2}$. Then, the equilibrium stock prices and their variance (conditional on all information available a finite amount of time before date t) are given by the following:

$$p(y) = \frac{\beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2}}{1-\beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2}} \, y,$$

$$\mathrm{Var}[p(y)] = \frac{\beta^2 e^{(1-\gamma)^2\sigma^2}}{[1-\beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2}]^2} \, \mathrm{var}(y), \quad 0 < \beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2} < 1, \gamma > 0.$$

Furthermore, Var[p(y)] is a strictly increasing convex function of $\gamma \in (1, 1 + \frac{\ell n\beta^{-2}}{\sigma})$ and a strictly decreasing convex function of $\gamma \in (\max(0, 1 - \frac{\ell n\beta^{-2}}{\sigma}), 1)$.

Proof: See Appendix 3.

The most interesting feature of proposition 3 is that the equilibrium stock price is proportional to the current dividend. This can be compared with Economy I of Proposition 1 in which p(y) is proportional to the current dividend y, but is independent of $\alpha \in [0,1]$.

Proposition 3 enables us to draw many interesting economic interpretations. First, equilibrium stock prices are minimized when a representative consumer has log utility function. The information effect is $1-\gamma$, while the income effect is γ. Therefore, the information effect is 0, if and only if $\gamma = 1$.

Second, the equilibrium stock prices are symmetric in the sense that the stock prices associated with the relative risk aversion measure 1-e are the same as those associated with 1+e, if [1-e,1+e] is a subinterval of

$$\gamma \in (\max(0, 1 - \frac{\ln \beta^{-2}}{\sigma}), 1 + \frac{\ln \beta^{-2}}{\sigma}).$$

Third, it is somewhat surprising to observe that the variability of the equilibrium stock prices is minimized when the representative consumer has log utility.[9] Fourth, the stochastic process of dividends in this economy implies that the equilibrium stock prices follow a discrete time version of a geometric Brownian motion which is one of the most common assumptions on stock returns in the finance literature.[10] In other words, the martingale property[11] of the equilibrium stock prices exactly holds.

Our findings are somewhat different from those by LeRoy-LaCivita [1981]. They argue in a two-state finite-horizon model that the volatility of the equilibrium stock prices unambiguously increases with risk aversion. In our model, the variance of the stock prices is not only a strictly increasing function of the relative risk aversion measure γ, but also a convex function of γ (Proposition 2,3), if γ is greater than 1. However, in the case that γ is less than 1, the variance of the stock prices is a strictly decreasing convex function of γ (Proposition 3), or the effect of γ on the variability is ambiguous (Proposition 2).

It seems that these differences come from the different characterization of the underlying stochastic process of dividends. In any case, the variability of equilibrium stock prices can be large in both their model and our model, if the relative risk aversion measure is nontrivial.

SECTION IV

THE EFFECT OF UNDERLYING PARAMETERS ON THE VARIABILITY OF EQUILIBRIUM STOCK PRICES

The economic interpretation of underlying parameters (α, β, and σ^2) is given, before major results are discussed. First, α is a parameter which represents information on the future dividends. Therefore, it is natural to attribute the variability of the equilibrium stock prices to the change in α. In investigating the effect of α, we make the assumption that the log of the dividend is an AR(1), i.e., $\ell n\, y' = \alpha\, \ell n\, y + \varepsilon_t$, where $0 < \alpha < 1$, $\varepsilon_t \sim N(0, \sigma^2)$. Second, the discount factor, β, is a parameter which represents time preference. Finally,

σ^2 is a parameter which measures the variance of dividend shocks. The introduction of a new dividend shock in our model means the increase in σ^2. It should be noted that we consider only highly stylized economies defined in Proposition 1, 2, and 3. The observation that the equilibrium price function is hard to find under general set of conditions compromises our restrictive analysis.

Proposition 4. Consider a pure exchange economy in which a representative consumer has log utility and the log of the dividend is an AR(1), i.e., $\ell n\ y' = \alpha\ \ \ell n\ y + \varepsilon_t$, where $0 < \alpha < 1$, $\varepsilon_t \sim N(0,\sigma^2)$. Then, the unconditional variance of equilibrium stock prices is a strictly increasing convex function of α, β and σ^2.

Proof: See Appendix 4.

It is intuitively clear that the variability of equilibrium stock prices increases if the discount factor increases or the variance of dividends increases. However, it might be surprising that the variability increases with the information parameter, α. The explanation is that the variability of equilibrium stock prices increases with α, since the increase in α results in the increase in the variance of ℓny for the stochastic process of dividends in this economy.

Proposition 5. Consider a pure exchange economy defined in Proposition 2. Then, the variance of equilibrium stock prices is a strictly increasing convex function of β and σ^2.

<u>Proposition 6</u>. Consider a pure exchange economy defined in Proposition 3. Then, the conditional variance of the equilibrium stock prices is a strictly increasing convex function of β and σ^2.

Proof: See Appendix 6.

SECTION V

VARIANCE BOUNDS IMPLIED BY THE MODEL

In this section, we show that the high volatility of stock prices found by Shiller [1981], LeRoy-Porter [1981] and Mankiw-Romer-Shapiro [1985] basically based on a variance bound test is not necessarily attributable to stock market inefficiency in our economy with risk aversion.

As shown by Marsh-Merton [1985], the key assumptions underlying a variance bound test can be summarized as follows:

(A.1) $p(y_t) = E_t[p^*(y_t)|I_t]$, where I_t is the information set available to markets at time t, $p^*(y_t) = \sum_{i=1}^{\infty} \rho^i y_{t+i}$.

(A.2) $\rho(= \frac{1}{1+r})$ is constant over time, $0 < \rho < 1$, r is the real expected rate of return on the stock market.

(A.3) $\{y_t\}_{t=0}^{\infty}$ follows a stationary time series.

Since $p(y_t)$ is an optimal forecast of $p^*(y_t)$, the null hypothesis to test is as follows:

$$H_0: \text{Var}[p(y_t)] \leq \text{Var}[p^*(y_t)].$$

However, it should be emphasized that the null hypothesis is correct if the key assumptions are all satisfied.

Various authors have questioned the validity of a variance bound test for a variety of theoretical and empirical reasons. Flavin [1983] shows that in small samples the variance bound test tends to be biased, often severely, toward a rejection of the null hypothesis of market efficiency. Based on this observation, she argues that the apparent violation of market efficiency may be reflecting the sampling properties of the volatility measures rather than a failure of the market efficiency hypothesis itself.

Marsh-Merton [1985] show that under the assumption of a non-stationary dividend process the variance bound test finds the high volatility of stock prices relative to dividends in every sample, even if the efficient market hypothesis, (A.1), is correct. Based on this reasoning, they conclude that the excess volatility of stock prices relative to dividends might imply a rejection of a stationary dividend process, (A.3), rather than a failure of market efficiency.

However, Mankiw-Romer-Shapiro in their recent paper [1985] argue that their new unbiased tests are immune to criticisms by Flavin and Marsh-Merton, since their volatility measures are unbiased and the dividend process need not follow a stationary time series in their tests. They conclude that their new tests find evidence contradicting market efficiency while they do not find evidence as striking as that Shiller reports.

The Mankiw-Romer-Shapiro [M-R-S] bound test can be summarized as follows in the context of our model: Let $p^0(y_t)$ be some naive forecast of $p^*(y)$,

i.e., $p^0(y_t) = F_t[p^*(y_t)|I_t] \equiv \frac{\rho}{1-\rho} y_t$, where F_t denotes a naive forecast of $p^*(y_t)$. Then (A.1) and (A.2) imply that

$$E[p^*(y_t) - p^0(y_t)]^2 = E[p^*(y_t) - p(y_t)]^2 + E[p(y_t) - p^0(y_t)]^2,$$

since the optimal forecast error is uncorrelated with information available at time t. Therefore, the null hypothesis is that $E[p^*(y_t) - p^0(y_t)]^2 \geq E[p^*(y_t) - p(y_t)]^2$ and

$E[p^*(y_t) - p^0(y_t)]^2 \geq E[p(y_t) - p^0(y_t)]^2$.

The observation that the present-value equation, (A.1), is valid only under risk neutrality enables us to claim that (A.1) is not true in our model with risk aversion. We argue that the empirical evidence found by LeRoy-Porter [1981], Shiller [1981] and Mankiw-Romer-Shapiro [1985] might imply a rejection of risk neutrality rather than a failure of market efficiency, since the inequality is reversed in our model with risk aversion.

We now start by stating the following proposition which was shown by Michener [1982].

<u>Proposition 7</u>. Consider both a risk neutral economy where the

stochastic process of dividends is characterized by the form $y' = (1-\alpha) + \alpha y + \varepsilon_t,\ 0 \leq \alpha < 1, \varepsilon_t \sim N(0,\sigma^2)$, and Michener Economy defined in Proposition 1. Then, $\mathrm{Var}[p^*(y)] \geq \mathrm{Var}[p(y)]$ in a risk neutral economy, and $\mathrm{Var}[p^*(y)] \leq \mathrm{Var}[p(y)]$ in Michener Economy.

Proof: See Appendix 7.

Proposition 8. Consider a pure exchange economy defined in Proposition 2. Then, $E[p^*(y_t) - p(y_t)]^2 \geq E[p^*(y_t) - p^0(y_t)]^2$ for $\gamma \in [1,\infty)$. Furthermore, $h(\gamma) \equiv E[p^*(y_t)-p(y_t)]^2 - E[p^*(y_t)-p^0(y_t)]^2 = (\frac{\beta}{1-\beta})^2 e^{2(\gamma^2-\gamma+1)\sigma^2}\{e^{(\gamma^2-1)\sigma^2}-1\}$ is a strictly increasing convex function of γ, if γ is greater than 1.

Proof: See Appendix 8.

In general equilibrium, the discount rate moves around and with certain utility functions this can reverse the inequalities underlying the M-R-S bounds tests.

It is interesting to observe that risk aversion plays a critical role in violating the M-R-S bounds in our model. As is shown by the following table and figure, the M-R-S bounds are uniformly violated in our economy, if γ is greater than one.

$\beta = 0.942, \ e^{\sigma^2} = 1.1$

γ	$h(\gamma)$
1	0
2	154
3	1,145
4	9,988

Table 1

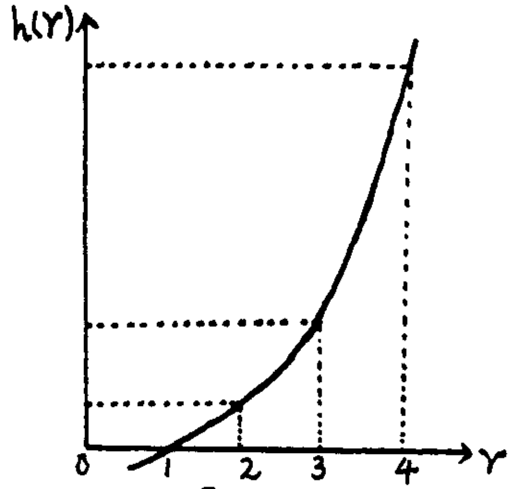

Figure 1

Proposition 9. Consider an economy defined in Proposition 3. Then, $E[p^*(y_t)-p(y_t)]^2 \geq E[p^*(y_t)-p^0(y_t)]^2$ for

$$\max(0, 1 - \frac{\ln\beta^{-2}}{\sigma}) < \gamma < 1 + \frac{\ln\beta^{-2}}{\sigma}.$$

Proof: See Appendix 9.

SECTION VI

CONCLUSION

In this essay, we have shown that risk aversion plays a critical role in the determination of equilibrium stock prices and their variability in a one-asset exchange economy.

Specifically, we have proved that the variance of equilibrium stock prices is a strictly increasing convex function of the Arrow-Pratt relative risk aversion measure, γ, if γ is greater than one. It is our contention that stock prices are volatile because risk averse agents make unsuccessful attempt to smooth their consumption stream over time.

In this setting, the effect of underlying parameters (information on future dividends, the discount factor, consumption shocks) on the variability of equilibrium stock prices has been analyzed by using the comparative statics.

We also have shown that the inequality underlying variance bounds tests by LeRoy-Porter [1981], Shiller [1981] and Mankiw-Romer-Shapiro [1985] can be reversed in our model with risk aversion. The observation that the present-value equation holds only under risk neutrality enables us to claim that the variance bounds tests are not relevant in general equilibrium asset pricing models with risk aversion.

We conclude that the high volatility of stock prices relative to dividends may imply a rejection of risk neutrality rather than a failure of stock market efficiency. It might be interesting to estimate[12] the Arrow-Pratt measure of relative risk aversion, γ, which is consistent with the high volatility of stock prices, in the context of general equilibrium asset pricing models with risk aversion. How sensitive our results are to the assumption of exchange economy is also an open question.

This paper has some unsolved problems. First, we failed to find the equilibrium stock prices in a one-asset pure exchange economy where a representative consumer has CRRA utility and the stochastic process of dividends is characterized by the general form, $\ln y' = \alpha \ln y + \varepsilon_t$, $0 \leq \alpha \leq 1$, $\varepsilon_t \sim N(0,\sigma^2)$.

Second, it might be an interesting research topic to derive a testable hypothesis about stock market efficiency from general equilibrium asset pricing models with risk aversion.

Third, we note that our model can be applied to the term structure of interest rates.

FOOTNOTES

1. For this issue, see Marsh-Merton [1985] , especially Theorem I and II.

2. This is shown by Lucas [1978] , namely equation (14).

3. This paper is different from Michener [1982], Grossman-Shiller [1981] and LeRoy-Lacivita [1981] on the following grounds:
   First, Michener shows that risk aversion of agents (log utility) can lead to violations of the Shiller variance bounds. However, his model is not sufficient to show that the Mankiw-Romer-Shapiro bounds [M-R-S bounds] can be violated in a risk averse economy. This paper seeks a set of sufficient conditions which can lead to the violations of the M-R-S bounds.
   Second, Grossman-Shiller estimate the relative risk aversion measure to reconcile stock price movements with the volatility of dividends. However, this paper theoretically derives a functional relationship between the stock price volatility and the risk aversion measure.
   Third, LeRoy-Lacivita consider a two-state finite-horizon model to show that the variability of stock prices is increasing in the risk aversion measure. However, this paper uses an infinite-state infinite-horizon model to show that the volatility of stock prices is not only strictly increasing in risk aversion but also strictly convex in risk aversion. More importantly, this paper explicitly considers the M-R-S bounds.

4. Brown-Gibbons [1985] claim that CRRA utility is a natural choice because of its desirable theoretical properties. Among other things, CRRA utility satisfies aggregation theorem by Rubinstein [1974], and it displays decreasing absolute risk aversion emphasized by Arrow [1971].

5. For notational convenience, we define $y_t, y_{t+1}, y_{t+2}, \ldots$ as $y, y', y'', \ldots$. Note also that $p \equiv p_t$, $c \equiv c_t$, $x \equiv x_t$ and $z \equiv z_t$.

6. For the existence and uniqueness of equilibrium, see Lucas [1978], especially Proposition 1 and 3.

7. Suppose that $\log x \sim N(\mu, \sigma^2)$. Then, $E(X^n) = e^{n\mu + \frac{1}{2}n^2\sigma^2}$

for any real number, n. For a complete discussion of lognormal distributions, see Aitchison-Brown [1963].

8. There is no specific reason to believe that this dividend process is a reasonable characterization of a real world. It also should be noted that Economy II is subject to the problem that dividends can be negative. I am grateful to Stephen Zeldes for this point.

9. Intuition might have suggested that the risk neutral case yielded the lowest variance. We believe that this counter-intuitive result comes from the information effect because the information effect is 0 if and only if $\gamma = 1$.

10. The stochastic process of stock returns assumed in the Black-Scholes option pricing formula is that $ds = \mu s\, dt + \sigma s\, dw$, where $dw \sim N(0,dt)$. Let $Y = \ln s$.

    Then, by Itô's Lemma, $dY = (\mu - \frac{1}{2}\sigma^2)\, dt + \sigma dw$.

    $Y(t) = Y(0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma w_t$, assuming $w_0 = 0$.

    Therefore, $\ln \frac{s(t)}{s(0)} = (\mu - \frac{1}{2}\sigma^2)\, t + \sigma w_t \sim N[(\mu - \frac{1}{2}\sigma^2)\, t, \sigma^2 t]$.

    For a complete discussion of Itô stochastic processes, see Gihman-Skorohod [1972].

11. For the martingale property of stock prices in asset pricing models, see LeRoy [1973], Lucas [1978] and Michener [1982].

12. Friend-Blume [1975] estimate γ to be in excess of 2 and roughly constant across different levels of wealth, using cross-sectional surveys of portfolio holdings of individuals. Grossman-Shiller [1981] claim γ is 4 based on a graphical examination of stock price volatility, while Hansen-Singleton [1983] report that γ is between 0 and 1.

APPENDICES

Appendix 1: Proof of Proposition 1

Proof: Economy I: We will not reproduce here, since it was shown by Michener [1982] that $p(y) = \frac{\beta}{1-\beta} y$ and $V(z,y) = \frac{1}{1-\beta}\ln z + \frac{1}{1-\alpha\beta} \ln y$. We only observe that p(y) can be derived easily from (10). If $\gamma = 1$, then (10) reduces to

$$f(y) = \beta + \beta^2 + \beta^3 + \ldots$$

$$= \frac{\beta}{1-\beta} .$$

By definition, $p(y) = \frac{f(y)}{u'(y)} = \frac{\beta}{1-\beta} y.$

Economy II: We derive p(y) first and verify that p(y) is the equilibrium price function and V(z,y) is well-defined.

$$g(y) = \beta \int y' \; u'(y') \; dF(y'|y) = \beta[E \; y'^{1-\gamma}|y] = \alpha\beta y^{1-\gamma},$$

$$E[g(y')|y] = E[y'^{1-\gamma}|y] = \alpha^2\beta y^{1-\gamma},$$

$$\ldots\ldots\ldots\ldots$$

$$\text{Therefore, } f(y) = \alpha\beta y^{1-\gamma} + \alpha^2\beta^2 y^{1-\gamma} + \ldots$$

$$= \frac{\alpha\beta}{1-\alpha\beta} y^{1-\gamma}.$$

By definition, $p(y) = \frac{f(y)}{u'(y)} = \frac{\alpha\beta}{1-\alpha\beta} y.$

From (4),

$$V(z,y) = \max_{(c,x)} [\, u(c) + \beta \int V(x,y') \; dF(y'|y)]$$

$$\text{s.t. } c + p(y)\cdot x \leq y\cdot z + p(y)\cdot z,$$

where $u(c) = \frac{1}{1-\gamma} c^{1-\gamma}, \ \gamma > 0, \gamma \neq 1,$

$$y'^{1-\gamma} = \alpha y^{1-\gamma} + \varepsilon_t, \ 0 \leq \alpha \leq 1, \ \varepsilon_t \sim N(0,\sigma^2),$$

$$0 < \beta < 1.$$

Suppose that $p(y) = k_1 y$, $V(z,y) = k_2 + k_3 \frac{1}{1-\gamma}(yz)^{1-\gamma}$, where k_1, k_2 and k_3 are constants.

Then, $$k_2 + \frac{k_3}{1-\gamma}(yz)^{1-\gamma} = \max_x \left[\frac{1}{1-\gamma}\{yz + k_1 y(z-x)\}^{1-\gamma} + \beta \int \{k_2 + \frac{k_3}{1-\gamma}(xy')^{1-\gamma}\} \, dF(y'|y)\right].$$

$$0 = -k_1 y\{yz + k_1 y(z-x)\}^{-\gamma} + \beta \int k_3 y'(xy')^{-\gamma} dF(y'|y)$$

$$= -k_1 y\{yz + k_1 y(z-x)\}^{-\gamma} + \alpha\beta k_3 x^{-\gamma} y^{1-\gamma}.$$

Since in equilibrium $x = z$ by construction,

$$k_1 (xy)^{-\gamma} = k_3 \alpha\beta (xy)^{-\gamma}.$$

Therefore, $k_1 = \alpha\beta k_3$.

$$k_2 + \frac{k_3}{1-\gamma}(yz)^{1-\gamma} = \frac{1}{1-\gamma}(yz)^{1-\gamma} + \beta k_2 + \frac{\beta k_3}{1-\gamma} z^{1-\gamma} E[y'^{1-\gamma}|y].$$

$$k_2 + \frac{k_3}{1-\gamma}(yz)^{1-\gamma} = \beta k_2 + \left(\frac{1}{1-\gamma} + \frac{\alpha\beta k_3}{1-\gamma}\right)(yz)^{1-\gamma}, \text{ since}$$

$E[y'^{1-\gamma}|y] = \alpha y^{1-\gamma}$.

Since the above equation must hold for all y and z,

$$k_2 = \beta k_2, \frac{k_3}{1-\gamma} = \frac{1}{1-\gamma}(1+\alpha\beta k_3).$$

It follows that $k_2 = 0$ and $k_3 = \frac{1}{1-\alpha\beta}$.

Therefore, $p(y) = \frac{\alpha\beta}{1-\alpha\beta}\, y$ and $V(z,y) = \frac{1}{(1-\alpha\beta)(1-\gamma)}\,(yz)^{1-\gamma}$

for $\gamma > 0,\ \gamma \neq 1$. Q.E.D.

Appendix 2: Proof of Proposition 2

Proof: From (10), $f(y) = \beta\, e^{\frac{1}{2}(1-\gamma)^2\sigma^2}(1 + \beta + \beta^2 + \ldots)$

$$= \frac{\beta}{1-\beta}\, e^{\frac{1}{2}(1-\gamma)^2\sigma^2}.$$

By definition, $p(y) = \frac{f(y)}{u'(y)} = \frac{\beta}{1-\beta}\, e^{\frac{1}{2}(1-\gamma)^2\sigma^2}\, y^{\gamma}$.

In similar manner to Proposition 1, it can be shown that $p(y)$ is an equilibrium price function and the value function, $V(z,y)$, is well-defined.

$$\ell n\, p(y) = \ell n \frac{\beta}{1-\beta} + \frac{1}{2}(1-\gamma)^2\sigma^2 + \gamma \ell n\, y \sim N[\ell n \frac{\beta}{1-\beta} + \frac{1}{2}(1-\gamma)^2\sigma^2, \gamma^2\sigma^2].$$

Using properties of lognormal distributions,

$$\mathrm{Var}[p(y)] = e^{2\ell n\frac{\beta}{1-\beta} + (1-\gamma)^2\sigma^2 + \gamma^2\sigma^2}(e^{\gamma^2\sigma^2}-1)$$

$$= (\frac{\beta}{1-\beta})^2\, e^{(1-\gamma)^2\sigma^2 + \gamma^2\sigma^2}(e^{\gamma^2\sigma^2}-1)$$

$$= (\frac{\beta}{1-\beta})^2\, \{e^{(3\gamma^2 - 2\gamma + 1)\sigma^2} - e^{(2\gamma^2 - 2\gamma + 1)\sigma^2}\}.$$

Suppose that $\mathrm{Var}[p(y)]$ is twice differentiable with respect to γ.

$$\frac{\partial Var[p(y)]}{\partial \gamma} = (\frac{\beta}{1-\beta})^2\sigma^2\{2(3\gamma-1)\ e^{(3\gamma^2-2\gamma+1)\sigma^2} - 2(2\gamma-1)e^{(2\gamma^2-2\gamma+1)\sigma^2}\}$$

$$= 2(\frac{\beta}{1-\beta})^2\sigma^2\ e^{(2\gamma^2-2\gamma+1)\sigma^2}\{(3\gamma-1)e^{\gamma^2\sigma^2}-(2\gamma-1)\}$$

$$> 2(\frac{\beta}{1-\beta})^2\sigma^2\ e^{(2\gamma^2-2\gamma+1)\sigma^2}(3\gamma-1)(e^{\gamma^2\sigma^2}-1)$$

$$> 0 \text{ if } \gamma \geq \frac{1}{3}.$$

$$\frac{\partial^2 Var[p(y)]}{\partial \gamma^2} = 2(\frac{\beta}{1-\beta})^2\sigma^2\{3\ e^{(3\gamma^2-2\gamma+1)^2\sigma^2} + 2(3\gamma-1)^2\sigma^2\ e^{(3\gamma^2-2\gamma+1)\sigma^2}$$

$$-2e^{(2\gamma^2-2\gamma+1)\sigma^2} -2(2\gamma-1)^2\sigma^2 e^{(2\gamma^2-2\gamma+1)\sigma^2}\}$$

$$> 2(\frac{\beta}{1-\beta})^2\sigma^2\{2(3\gamma-1)^2\sigma^2\ e^{(3\gamma^2-2\gamma+1)\sigma^2} -2(2\gamma-1)^2\sigma^2 e^{(2\gamma^2-2\gamma+1)\sigma^2}\}$$

$$= 4(\frac{\beta}{1-\beta})^2\sigma^4\{(3\gamma-1)^2\ e^{(3\gamma^2-2\gamma+1)\sigma^2} -(2\gamma-1)^2 e^{(2\gamma^2-2\gamma+1)\sigma^2}\}$$

$$> 4\ (\frac{\beta}{1-\beta})^2\sigma^4\{(3\gamma-1)^2\ e^{(3\gamma^2-2\gamma+1)\sigma^2} -(2\gamma-1)^2\ e^{(3\gamma^2-2\gamma+1)\sigma^2}\}$$

$$= 4\ (\frac{\beta}{1-\beta})^2\sigma^4\ e^{(3\gamma^2-2\gamma+1)\sigma^2}\gamma(5\gamma-2)$$

$$\geq\ 0 \text{ if } \gamma \geq \frac{2}{5}\ .$$

Appendix 3: Proof of Proposition 3

Proof: From (10),

$$f(y) = \beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2}\ y^{1-\gamma}\{1+\beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2} + \beta^2\ e^{(1-\gamma)^2\sigma^2}$$

$$+ \ . \ . \ . \}\ .$$

Assume that $0 < \beta\, e^{\frac{1}{2}(1-\gamma)^2\sigma^2} < 1$, i.e., $0 < \sigma^2 < \frac{\ln\beta^{-2}}{(1-\gamma)^2}$.

Then
$$f(y) = \frac{\beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2}}{1-\beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2}}\, y^{1-\gamma}.$$

By definition,
$$p(y) = \frac{f(y)}{u'(y)} = \frac{\beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2}}{1-\beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2}}\, y.$$

In a similar manner to Proposition 1, it can be shown that the above p(y) is an equilibrium stock price and the value function is well-defined, i.e.,

$$V(z,y) = \frac{1}{(1-\gamma)\{1-\beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2}\}}(yz)^{1-\gamma} \text{ for } \gamma > 0,\ \gamma \neq 1,$$

$$V(z,y) = \frac{1}{1-\beta}\ln(yz) \text{ for } \gamma = 1.$$

$$\mathrm{Var}[p(y)] = \frac{\beta^2 e^{(1-\gamma)^2\sigma^2}}{[1-\beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2}]^2}\ \mathrm{var}(y), \text{ if } 0 < \mathrm{var}(y) < \infty .$$

Suppose that Var[p(y)] is a twice differentiable function of $\gamma \in (\max(0, 1-\frac{\ln\beta^{-2}}{\sigma}), 1 + \frac{\ln\beta^{-2}}{\sigma})$. Then,

$$\frac{\partial \mathrm{Var}\,[p(y)]}{\partial\gamma} = \mathrm{Var}(y)\frac{1}{[1-\beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2}]^4}\{-2\beta^2(1-\gamma)\sigma^2 e^{(1-\gamma)^2\sigma^2} \times$$

$$(1-\beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2})^2 - \beta^2 e^{(1-\gamma)^2\sigma^2} \times 2(1-\beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2}) \times$$

$$\beta(1-\gamma)\sigma^2 e^{\frac{1}{2}(1-\gamma)^2\sigma^2}\}$$

$$= \mathrm{Var}(y)\frac{1}{[1-\beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2}]^3} \times$$

$$\{-2\beta^3(1-\gamma)\sigma^2 e^{(1-\gamma)^2\sigma^2}(1-\beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2})$$

$$-2\beta^3(1-\gamma)\sigma^2 e^{\frac{3}{2}(1-\gamma)^2\sigma^2}\}$$

$$= \frac{-2(1-\gamma)\beta^2\sigma^2 e^{(1-\gamma)^2\sigma^2}}{[1-\beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2}]^3}\,\mathrm{Var}(y).$$

Therefore, $\frac{\partial \mathrm{Var}[p(y)]}{\partial\gamma} > 0$ if $\gamma \in (1, 1 + \frac{\ln\beta^{-2}}{\sigma^2})$,

$= 0$ if $\gamma = 1$,

< 0 if $\gamma \in (\max(0, 1 - \frac{\ln\beta^{-2}}{\sigma^2}), 1)$.

$$\frac{\partial^2 \mathrm{Var}[p(y)]}{\partial\gamma^2} = \frac{1(1-\gamma)^2\beta^2\sigma^4 e^{(1-\gamma)^2\sigma^2}}{[1-\beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2}]^4}\{\beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2} + 2\} > 0$$

for $\max(0, 1 - \frac{\ln\beta^{-2}}{\sigma}) < \gamma < 1 + \frac{\ln\beta^{-2}}{\sigma}$, $\gamma = 1$. Q.E.D.

Appendix 4: Proof of Proposition 4

Proof: From Proposition 1, $p(y) = \frac{\beta}{1-\beta} y$ and $Var[p(y)] = (\frac{\beta}{1-\beta})^2 Var(y)$. Since the log of the dividend is an AR(1) and $\varepsilon_t \sim N(0,\sigma^2)$, $\ell n\, y \sim N(0, \frac{\sigma^2}{1-\alpha^2})$.

Using properties of lognormal distributions,

$$Var\ (y) = e^{\frac{\sigma^2}{1-\alpha^2}} (e^{\frac{\sigma^2}{1-\alpha^2}} - 1) = e^{\frac{2\sigma^2}{1-\alpha^2}} - e^{\frac{\sigma^2}{1-\alpha^2}} .$$

Therefore, $Var[p(y)] = (\frac{\beta}{1-\beta})^2 (e^{\frac{2\sigma^2}{1-\alpha^2}} - e^{\frac{\sigma^2}{1-\alpha^2}})$.

Suppose that $Var[p(y)]$ is a twice differentiable function of α, β and σ^2.

Then,

$$\frac{\partial Var[p(y)]}{\partial \alpha} = (\frac{\beta}{1-\beta})^2 \{ \frac{4\alpha\sigma^2}{(1-\alpha^2)^2} e^{\frac{2\sigma^2}{1-\alpha^2}} - \frac{2\alpha\sigma^2}{(1-\alpha^2)^2} e^{\frac{\sigma^2}{1-\alpha^2}} \}$$

$$> (\frac{\beta}{1-\beta})^2 \frac{2\alpha\sigma^2}{(1-\alpha^2)} e^{\frac{2\sigma^2}{1-\alpha^2}} > 0.$$

$$\frac{\partial^2 Var[p(y)]}{\partial \alpha^2} = (\frac{\beta}{1-\beta})^2 \{ \frac{4\sigma^2(1-\alpha^2)^2 + 16\alpha^2(1-\alpha^2)\sigma^2}{(1-\alpha^2)^4} e^{\frac{2\sigma^2}{1-\alpha^2}}$$

$$+ \frac{16\alpha^2\sigma^4}{(1-\alpha^2)^4} e^{\frac{2\sigma^2}{1-\alpha^2}} - \frac{2\sigma^2(1-\alpha^2)^2 + 8\alpha^2(1-\alpha)\sigma^2}{(1-\alpha^2)^4} e^{\frac{\sigma^2}{1-\alpha^2}}$$

$$- \frac{4\alpha^2\sigma^4}{(1-\alpha^2)^4} e^{\frac{2\sigma^2}{1-\alpha^2}} \}$$

$$> (\frac{\beta}{1-\beta})^2 \{ \frac{2\sigma^2(1-\alpha^2)^2 + 8\alpha^2(1-\alpha^2)\sigma^2}{(1-\alpha^2)^4} e^{\frac{2\sigma^2}{1-\alpha^2}}$$

$$+ \frac{12\alpha^2\sigma^4}{(1-\alpha^2)^4} e^{\frac{2\sigma^2}{1-\alpha^2}} \} > 0.$$

$$\frac{\partial \mathrm{Var}[p(y)]}{\partial \beta} = (e^{\frac{2\sigma^2}{1-\alpha^2}} - e^{\frac{\sigma^2}{1-\alpha^2}}) \frac{1}{(1-\beta)^4} \{ 2\beta(1-\beta)^2 + \beta^2 \times 2(1-\beta) \}$$

$$= \frac{2\beta}{(1-\beta)^3} (e^{\frac{2\sigma^2}{1-\alpha^2}} - e^{\frac{\sigma^2}{1-\alpha^2}}) > 0.$$

$$\frac{\partial^2 \mathrm{Var}[p(y)]}{\partial \beta^2} = (e^{\frac{2\sigma^2}{1-\alpha^2}} - e^{\frac{\sigma^2}{1-\alpha^2}}) \frac{1}{(1-\beta)^6} \{ 2(1-\beta)^3 + 2\beta \quad 3(1-\beta)^2 \}$$

$$= \frac{2(1+2\beta)}{(1-\beta)^4} (e^{\frac{2\sigma^2}{1-\alpha^2}} - e^{\frac{\sigma^2}{1-\alpha^2}}) > 0.$$

$$\frac{\partial \mathrm{Var}\,[\,p(y)\,]}{\partial \sigma^2} = \left(\frac{\beta}{1-\beta}\right)^2 \left\{ \frac{2}{1-\alpha^2}\, e^{\frac{2\sigma^2}{1-\alpha^2}} - \frac{1}{1-\alpha^2}\, e^{\frac{\sigma^2}{1-\alpha^2}} \right\}$$

$$> \left(\frac{\beta}{1-\beta}\right)^2 \left(\frac{1}{1-\alpha^2}\right) e^{\frac{2\sigma^2}{1-\alpha^2}} > 0.$$

$$\frac{\partial^2 \mathrm{Var}\,[\,p(y)\,]}{\partial (\sigma^2)^2} = \left(\frac{\beta}{1-\beta}\right)^2 \left\{ \left(\frac{2}{1-\alpha^2}\right)^2 e^{\frac{2\sigma^2}{1-\alpha^2}} - \left(\frac{1}{1-\alpha^2}\right)^2 e^{\frac{\sigma^2}{1-\alpha^2}} \right\}$$

$$> \left(\frac{\beta}{1-\beta}\right)^2 \left\{ \left(\frac{2}{1-\alpha^2}\right)^2 - \left(\frac{1}{1-\alpha^2}\right)^2 \right\} e^{\frac{2\sigma^2}{1-\alpha^2}}$$

$$= \left(\frac{\beta}{1-\beta}\right)^2 \frac{3}{(1-\alpha^2)^2}\, e^{\frac{2\sigma^2}{1-\alpha^2}} > 0. \qquad \text{Q.E.D.}$$

Appendix 5: Proof of Proposition 5

Proof: From Proposition 2, $p(y) = \frac{\beta}{1-\beta}\, e^{\frac{1}{2}(1-\gamma)^2\sigma^2}\, y^{\gamma}$ and

$\mathrm{Var}[\,p(y)] = \left(\frac{\beta}{1-\beta}\right)^2 \{e^{(3\gamma^2-2\gamma+1)\sigma^2}\}$. $\frac{\partial \mathrm{Var}[\,p(y)]}{\partial \beta} > 0$,

$\frac{\partial^2 \mathrm{Var}[\,p(y)]}{\partial \beta^2} > 0$ from Proposition 4.

$$\frac{\partial Var[p(y)]}{\partial \sigma^2} = (\frac{\beta}{1-\beta})^2 \{(3\gamma^2-2\gamma+1)\ e^{(3\gamma^2-2\gamma+1)\sigma^2}$$

$$- (2\gamma^2-2\gamma+1)\ e^{(2\gamma^2-2\gamma+1)\sigma^2}\}$$

$$> (\frac{\beta}{1-\beta})^2\ \gamma^2\ e^{(3\gamma^2-2\gamma+1)\sigma^2} > 0.$$

$$\frac{\partial^2 Var[p(y)]}{\partial (\sigma^2)^2} = (\frac{\beta}{1-\beta})^2 \{(3\gamma^2-2\gamma+1)^2\ e^{(3\gamma^2-2\gamma+1)\sigma^2}$$

$$- (2\gamma^2-2\gamma+1)^2\ e^{(2\gamma^2-2\gamma+1)\sigma^2}\}$$

$$> (\frac{\beta}{1-\beta})^2 \{(3\gamma^2-2\gamma+1)^2 - (2\gamma^2-2\gamma+1)^2\} e^{(3\gamma^2-2\gamma+1)\sigma^2}$$

$$= (\frac{\beta}{1-\beta})^2 \gamma^2 \{(\gamma-1)^2+2\gamma^2\}^2 + \{(\gamma-1)^2+\gamma^2\}^2 \times$$

$$e^{(3\gamma^2-2\gamma+1)\sigma^2} > 0. \qquad Q.E.D.$$

Appendix 6: Proof of Proposition 6

Proof: From Proposition 3,

$$Var[p(y)] = \frac{\beta^2\ e^{(1-\gamma)^2\sigma^2}}{[1-\beta\ e^{\frac{1}{2}(1-\gamma)^2\sigma^2}]^2} Var(y).$$

It can be easily shown that

$$\frac{\partial Var[p(y)]}{\partial \beta} > 0,\ \frac{\partial^2 Var[p(y)]}{\partial \beta^2} > 0 \text{ and } \frac{\partial Var[p(y)]}{\partial \sigma^2} > 0,$$

$$\frac{\partial^2 \mathrm{Var}[p(y)]}{\partial(\sigma^2)^2} > 0.$$

Q.E.D.

Appendix 7: Proof of Proposition 7

Proof: Michener Economy: See Michener [1982].

Risk neutral economy:

$$p(y) = \sum_{s=1}^{\infty} \beta^s E[y_{t+s}|y_t = y] = \sum_{s=1}^{\infty} [\beta^s \alpha^s (y-1) + \beta^s]$$

$$= \frac{\alpha\beta}{1-\alpha\beta} y + \frac{\beta}{1-\beta} - \frac{\alpha\beta}{1-\alpha\beta} .$$

$$\mathrm{Var}[p(y)] = \left(\frac{\alpha\beta}{1-\alpha\beta}\right)^2 \mathrm{Var}(y) = \left(\frac{\alpha\beta}{1-\alpha\beta}\right)^2 \frac{\sigma^2}{1-\alpha^2} .$$

$$\mathrm{Var}[p^*(y)] = \left(\frac{1}{1+r}\right)^2 \left\{ \frac{1+ \alpha/(1+r)}{1- \alpha/(1+r)} \right\} \frac{\sigma^2}{(I-\alpha^2)(1-[1/(1+r)]^2} ,$$

as was shown by Michener [1982][1]. Suppose that the real interest rate, r, can be estimated by the average over time of the dividend-price ratio. Then,

$$r = \frac{E[y]}{E[p(y)]} = \frac{1-\beta}{\beta} , \quad \mathrm{Var}[p^*(y)] = \frac{\beta^2(1+\alpha\beta)}{1-\alpha\beta} \frac{\sigma^2}{(1-\alpha^2)(1-\beta^2)} .$$

$$\mathrm{Var}[p^*(y)] - \mathrm{Var}[p(y)] = \frac{\beta^2\sigma^2}{(1-\alpha\beta)(1-\alpha^2)} \left(\frac{1+\alpha\beta}{1-\beta^2} - \frac{\alpha^2}{1-\alpha\beta}\right)$$

$$= \frac{\beta^2\sigma^2}{(1-\alpha\beta)^2(1-\alpha^2)(1-\beta^2)} \{(1+\alpha\beta)(1-\alpha\beta) - \alpha^2(1-\beta^2)\}$$

$$= \frac{\beta^2\sigma^2}{(1-\alpha\beta)^2(1-\beta^2)} > 0. \qquad \text{Q.E.D.}$$

Appendix 8: Proof of Proposition 8

Proof: $p(y_t) = \frac{\beta}{1-\beta} e^{\frac{1}{2}(1-\gamma)^2\sigma^2} y_t^{\gamma}$ from

Proposition 2,

$$p^0(y_t) = \frac{\rho}{1-\rho} y_t = \sum_{i=1}^{\infty} \rho^i y_t,$$

$$p^*(y_t) = \sum_{i=1}^{\infty} \rho^i y_{t+i} \text{ by definition.}$$

$$E[p^*(y_t) - p(y_t)]^2 = \mathrm{Var}[p^*(y_t) - p(y_t)] + \{E[p^*(y_t) - p(y_t)]\}^2$$

$$= \mathrm{Var}[p^*(y_t)] + \mathrm{Var}[p(y_t)] + \{E[p^*(y_t)] - E[p(y_t)]\}^2$$

$$= \frac{\rho^2}{1-\rho^2}(e^{2\sigma^2} - e^{\sigma^2}) + \left(\frac{\beta}{1-\beta}\right)^2 \{e^{(3\gamma^2-2\gamma+1)\sigma^2} - e^{(2\gamma^2-2\gamma+1)\sigma^2}\} + \{\frac{\rho}{1-\rho}e^{\frac{1}{2}\sigma^2} - \frac{\beta}{1-\beta}e^{\frac{1}{2}(2\gamma^2-2\gamma+1)\sigma^2}\}^2$$

$$= \frac{\rho^2}{1-\rho^2}(e^{2\sigma^2} - e^{\sigma^2}) + \left(\frac{\rho}{1-\rho}\right)^2 e^{\sigma^2} + \left(\frac{\beta}{1-\beta}\right)^2 e^{(3\gamma^2-2\gamma+1)\sigma^2} - 2\frac{\rho}{1-\rho}\frac{\beta}{1-\beta}e^{(\gamma^2-\gamma+1)\sigma^2}.$$

$$E[p^*(y_t)-p^0(y_t)]^2 = \mathrm{Var}[p^*(y_t)] + \mathrm{Var}[p^0(y_t)]$$

$$= \frac{\rho^2}{1-\rho^2} + \left(\frac{\rho}{1-\rho}\right)^2 (e^{2\sigma^2} - e^{\sigma^2}).$$

Suppose that r $= \dfrac{E[y_t]}{E[p(y_t)]} = \dfrac{1-\beta}{\beta} e^{\gamma(1-\gamma)\sigma^2}.$

Then, $\rho = \dfrac{1}{1+r} = \dfrac{\beta}{\beta + (1-\beta)e^{\gamma(1-\gamma)\sigma^2}}$.

$$E[p^*(y_t)-p(y_t)]^2 = \frac{\beta^2(e^{2\sigma^2} - e^{\sigma^2})}{2\beta(1-\beta)e^{\gamma(1-\gamma)\sigma^2} + (1-\beta)^2 e^{2\gamma(1-\gamma)\sigma^2}}$$

$$+ \left(\frac{\beta}{1-\beta}\right)^2 e^{(2\gamma^2-2\gamma+1)\sigma^2} + \left(\frac{\beta}{1-\beta}\right)^2 e^{(3\gamma^2-2\gamma+1)\sigma^2}$$

$$- 2\left(\frac{\beta}{1-\beta}\right)^2 e^{(2\gamma^2-2\gamma+1)\sigma^2}$$

$$= \frac{\beta^2(e^{2\sigma^2} - e^{\sigma^2})}{2\beta(1-\beta)e^{\gamma(1-\gamma)\sigma^2} + (1-\beta)^2 e^{2\gamma(1-\gamma)\sigma^2}}$$

$$+ \left(\frac{\beta}{1-\beta}\right)^2 e^{(2\gamma^2-2\gamma+1)\sigma^2} (e^{\gamma^2\sigma^2} - 1).$$

$$E[p^*(y_t) - p^0(y_t)]^2 = \frac{\beta^2(e^{2\sigma^2} - e^{\sigma^2})}{2\beta(1-\beta)e^{\gamma(1-\gamma)\sigma^2} + (1-\beta)^2 e^{2\gamma(1-\gamma)\sigma^2}}$$

Therefore, $E[p^*(y_t)-p(y_t)]^2 - E[p^*(y_t)-p^0(y_t)]^2 \equiv h(\gamma)$

$$= \left(\frac{\beta}{1-\beta}\right)^2 e^{2\gamma(\gamma-1)\sigma^2}\{e^{(\gamma^2+1)\sigma^2} - e^{\sigma^2} - e^{2\sigma^2} + e^{\sigma^2}\}$$

$$= \left(\frac{\beta}{1-\beta}\right)^2 e^{2\sigma^2} e^{2\gamma(\gamma-1)\sigma^2}\{e^{(\gamma^2-1)\sigma^2} - 1\}$$

$$= \left(\frac{\beta}{1-\beta}\right)^2 e^{2(\gamma^2-\gamma+1)\sigma^2}\{e^{(\gamma^2-1)\sigma^2} - 1\}$$

≥ 0 if $\gamma \geq 1$,

< 0 if $0 < \gamma < 1$.

Suppose that $h(\gamma)$ is twice differentiable with respect to γ. Then, it can be shown that $h'(\gamma) > 0$ and $h''(\gamma) > 0$ for $\gamma \in (1, \infty)$. Q.E.D.

Appendix 9: Proof of Proposition 9

Proof: $p(y_t) = \frac{\beta'}{1-\beta'} y_t$, where $\beta' = \beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2}$ from Proposition 3,

$$p^0(y_t) = \frac{\rho}{1-\rho} y_t = \sum_{i=1}^{\infty} \rho^i y_t,$$

$$p^*(y_t) = \sum_{i=1}^{\infty} \rho^i y_{t+1} \text{ by definition.}$$

Let E denote the expectation[2] conditional on the initial conditions.

Then,
$$E[p^*(y_t)-p^0(y_t)]^2 = E[\sum_{i=1}^{\infty} \rho^i(y_{t+i}-y_t)]^2.$$

$$E[p^*(y_t)-p(y_t)]^2 = E[\sum_{i=1}^{\infty} (\rho^i y_{t+i}-\beta'^i y_t)]^2$$

$$= E[\sum_{i=1}^{\infty} \rho^i(y_{t+i}-y_t)]^2 + (\frac{\rho}{1-\rho} - \frac{\beta'}{1-\beta'})^2 E[y_t^2].$$

Therefore,
$$E[p^*(y_t)-p(y_t)]^2 - E[p^*(y_t) - p^0(y_t)]^2$$

$$= (\frac{\rho}{1-\rho} - \frac{\beta'}{1-\beta'})^2 E[y_t^2] \geq 0.$$

The equality[3] holds when $\rho = \beta' \equiv \beta e^{\frac{1}{2}(1-\gamma)^2\sigma^2}$. It should be noted that $\rho=\beta'$ if the real interest rate is estimated by the average over time of the dividend-to-price ratio.

Q.E.D.

FOOTNOTES TO APPENDICES

1. He derives Var[p*(y)] by observing that the special density of p*(y) is the same as that of an AR(2). For the derivation, see Michener [1982] and Box-Jenkins [1976].

2. Non-stationary of the dividend process poses no difficulties for the existence of these conditional expectations, as long as the expectations are taken conditional on all information available a finite amount of time before date t. For this issue, see Mankiw-Romer-Shapiro [1985] and Marsh-Merton [1985].

3. The importance of the correct discount factor was emphasized by Andrew Abel in an appendix to Mankiw-Romer-Shapiro [1985]. If the correct discount factor can be estimated from sample, the null hypothesis will be as follows:

   $E[p^*(y_t)-p(y_t)]^2 = E[p^*(y_t)-p^0(y_t)]^2$. However, empirical findings by Mankiw-Romer-Shapiro [1985]

   show that $E[p^*(y_t)-p(y_t)]^2 = 0.156$ and

   $E[p^*(y_t)-p^0(y_t)]^2 = 0.139$ when $r = 6\%$.

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